Size Improvements In Unit Root Tests For Time Series With Serially Correlated Errors

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Abstract

This paper proposes a few nonparametric tests to reduce severe size distortions in the Phillips-Perron tests. It is shown that these lead to noticeable improvements by including terms that slowly converge to zero in asymptotic approximations.

1 Introduction

The unit root tests proposed by Phillips (1987) and Phillips and Perron (1988) (the pp tests) are widely used to cope with data generating processes (DGPs) with serially correlated errors. At the same time, many articles including Perron and Ng (1996) and Haldrup and Jansson (2008) have pointed out that their sizes are significantly distorted even under a sample size as large as 500. It is clearly recognized that a kernel estimator is needed in the test construction and that its slow convergence to the true value is the main cause of such distortions. Some modifications using the parametric kernels in Perron and Ng (1996) above are restrictive, in the sense that the results were not generally satisfactory.

The purpose of this paper is to propose testing methods that reduce such size distortions in wide-ranging DGPs. The tests proposed are elaborated so that the distributions of the terms that involve slow convergence to zero, as in the pp tests, become nuisance parameter free. Adopting the asymptotic distributions evaluated by the inclusion of these, instead of the limiting distributions, more accurate approximations to the true finite distributions are obtained, resulting in noticeable size improvements. The properties in relation to the asymptotic powers are similarly derived. Simulation results indicate that the tests exhibit desirable size performances in many DGPs.

2 DGP and test statistics

Let $\{u_t\}$ be a stochastic process generated by

$$u_t = \epsilon_t + \sum_{j=1}^{\infty} \psi_j \, \epsilon_{t-j}, \qquad \sum_{j=1}^{\infty} j^{\nu} \, |\psi_j| < 0 \qquad \exists \, \nu > 3, \tag{1}$$

where $\{\varepsilon_t\}$ is i.i.n. $(0, \sigma^2)$ with $\sigma > 0$. Following the convention for unit root testing, the DGP is given as

$$y_t = y_{t-1} + \mu_1 + u_t, \qquad \forall \, t \geq 1, \tag{2}$$

under the null (H_0) and

$$y_t = \rho\, y_{t-1} + \mu_0 + t\,\mu_1 + u_t, \qquad 0 < |\rho| < 1, \qquad \forall\, t \geq 1, \tag{3}$$

under the alternative (H_1), where μ_i are constants, and let $y_1, \cdots, y_T$ be observations with the assumption that $y_0 = O_p(1)$.

For the construction of the test statistics, put

$$\hat{\mu} = \sum_{t=2}^{T} \Delta y_t / T, \qquad \tilde{\mu} = \sum_{t=2}^{T} y_{t-1}/T,$$

$$(\hat{\mu}_0,\, \hat{\mu}_1) = \left(\sum_{t=2}^{T} \Delta y_t g_t' / T \right) G_T^{-1}, \quad (\tilde{\mu}_0,\, \tilde{\mu}_1) = \left(\sum_{t=2}^{T} y_{t-1} g_t' / T \right) G_T^{-1},$$

where $g'_t = (1, t-1)$ and $G_T = \Sigma_{t=2}^{T} g_t g_t'/T$, and put

$$\Delta \tilde{y}_{t;i} = \Delta y_t - \hat{c}_{i;t}, \quad \tilde{y}_{t-1;i} = y_{t-1} - \tilde{c}_{i;t}, \qquad i = 1, 2, 3,$$

where $\hat{c}_{1;t} = \tilde{c}_{1;t} = 0$, $\hat{c}_{2;t} = \hat{\mu}$, $\tilde{c}_{2;t} = \tilde{\mu}$, $\hat{c}_{3;t} = \hat{\mu}_0 + (t-1)\hat{\mu}_1$ and $\tilde{c}_{3;t} = \tilde{\mu}_0 + (t-1)\tilde{\mu}_1$. Also, let $\hat{\rho}_i$, $i = 1, 2, 3$, be the OLS coefficients of y_{t-1} from the three standard regression models for y_t in the unit root tests, corresponding to the no drift, demeaned, and detrended cases respectively, and put

$$\begin{aligned} \hat{Q}_{i;T} \;=\; & \sum_{t=3}^{T} (y_t - \hat{\rho}_i\, y_{t-1} - (1 - \hat{\rho}_i)\, \tilde{c}_{i;t} - \hat{c}_{i;t})\, \Delta \tilde{y}_{t-1;i}/T \\ & - \hat{\rho}_i\, (1 - \hat{\rho}_i) \sum_{t=3}^{T} \Delta \tilde{y}_{t-1;i} \tilde{y}_{t-1;i}/T. \end{aligned}$$

Moreover, we need a type of kernel estimator with a bandwidth parameter as in the pp tests. Let $\{S_T\}$ be a sequence of positive integers, such that $\lim_{T\to\infty} S_T/T^{1/2} = 0$ and $\lim_{T\to\infty} T^{1/4}/S_T = 0$, and define $\hat{P}_{i;T;S_T}$ as

$$\hat{P}_{i;T;S_T} = \sum_{j=2}^{S_T} \sum_{t=j+2}^{T} \Delta \tilde{y}_{t;i} \Delta \tilde{y}_{t-j;i}/T.$$

The test statistics proposed are now given as

$$\hat{M}_{i;T;S_T} = \frac{\sum_{t=2}^{T} \Delta \tilde{y}_{t;i} \tilde{y}_{t-1;i}/T - \hat{P}_{i;T;S_T} - \hat{Q}_{i;T}}{\left(\sum_{t=2}^{T} \Delta \tilde{y}_{t;i}^2/T + 2\,\hat{P}_{i;T:S_T} + 2\,\hat{Q}_{i;T} \right)^{1/2} \left(\sum_{t=2}^{T} \tilde{y}_{t-1;i}^2/T^2 \right)^{1/2}}.$$

$\hat{M}_{1;T;S_T}$ is defined only for the case $\mu_1 = 0$ under H_0 and $\mu_0 = \mu_1 = 0$ under H_1, and $\hat{M}_{2;T:S_T}$ is not defined unless $\mu_1 = 0$, whereas $\hat{M}_{3;T;S_T}$ is defined without assuming such a restriction. Apart from $\hat{Q}_{i;T}$, the forms of $\hat{M}_{i;T;S_T}$ are similar to those of the pp test statistics if the residuals are replaced by their corresponding differences (including the demeaned and detrended variants) and the truncated kernel is adopted.

3 Asymptotics

Define the statistics $\bar{N}_{i;T;S_T}$ as

$$\bar{N}_{i;T;S_T} = \frac{\sum_{t=2}^{T} \epsilon_t \eta_{t-1}/T \; - \; \bar{P}_{T;S_T} \; - \; \tilde{R}_{i;T}}{\left(\sum_{t=2}^{T} \epsilon_t^2/T \; + \; 2\,\tilde{P}_{T:S_T}\right)^{1/2} \left(\sum_{t=2}^{T} \eta_{t-1}^2/T^2 \; - \; \hat{R}_{i;T}\right)^{1/2}},$$

where $\eta_{t-1} = \sum_{h=1}^{t-1} \epsilon_h$, $\bar{P}_{T;S_T} = \sum_{j=1}^{S_T-1} \sum_{t=j+1}^{T} \epsilon_t \epsilon_{t-j}/T$, $\tilde{R}_{1;T} = \hat{R}_{1;T} = 0$,

$$\tilde{R}_{i;T} = \left(\sum_{t=2}^{T} \epsilon_t g'_{i;t}/T^{1/2}\right) \left(\sum_{t=2}^{T} g_{i;t} g'_{i;t}/T\right)^{-1} \left(\sum_{t=2}^{T} g_{i;t} \eta_{t-1}/T^{3/2}\right),$$

$$\hat{R}_{i;T} = \left(\sum_{t=2}^{T} \eta_{t-1} g'_{i;t}/T^{3/2}\right) \left(\sum_{t=2}^{T} g_{i;t} g'_{i;t}/T\right)^{-1} \left(\sum_{t=2}^{T} g_{i;t} \eta_{t-1}/T^{3/2}\right),$$

with $g_{2;t} = 1$ and $g'_{3\,;t} = (1, (t-1)/T)$. Also, put $\rho_y = Cov(y_t, y_{t-1})/V(y_t)$ under H_1. We shall now establish some asymptotics.

Theorem 1: *Suppose that y_t is generated by (2)/(3). Then:*

(i) *Under H_0,*

$$\hat{M}_{i;T;S_T} = \bar{N}_{i;T;S_T} + O_p(T^{-1/2}), \qquad i = 1, 2, 3$$

(ii) *Under H_1,*

$$T^{-1/2}\,\hat{M}_{i;T;S_T} = -\,(1/\sqrt{2})\,(1-\rho)\,(1-\rho_y) + O_p(T^{-1/2}), \qquad i = 1, 2, 3$$

The proof of the theorem is in the Appendix.

We propose to adopt the lower percentiles of the finite sample distributions of $\bar{N}_{i;T;S_T}$ at each T and S_T for the critical points of the tests by $\hat{M}_{i;T;S_T}$, noting that S_T is given and that the finite sample distributions are nuisance parameter free.[1] This provides accurate approximations to those of the true distributions. As seen in (ii) of Theorem, the tests are also consistent.

For the case where ε_t is not distributed as Gaussian, it is easily shown that the asymptotic distributions of $S_T^{-1/2} T^{1/2} \bar{P}_{T;S_T}$ become $N(0, \sigma^4)$ since $\sum_{t=j_1+1}^{T} \varepsilon_t \varepsilon_{t-j_1}$ and $\sum_{t=j_2+1}^{T} \varepsilon_t \varepsilon_{t-j_2}$ are independent as $j_1 \neq j_2$. In many cases, from this and the well-known results based on Brownian motions, the distributions of $\bar{N}_{i;T;S_T}$ from Gaussian ε_t will be expected to be good approximations to those derived from non-Gaussian ε_t.

We do not theoretically argue the issue of determining an optimal S_T, given that the automatic criteria in Andrews (1991) etc. are not applicable to the truncated kernel. However, letting $\tilde{S}_{i;T}$ be S_T minimizing

$$\left| \sum_{t=S_T+3}^{T} \Delta\tilde{y}_{t;i} \Delta\tilde{y}_{t-S_T-1;i}/(T - S_T - 3) \right|$$

over $\mathcal{F}$ as a set of one satisfying the condition given above, it may lead to a practical selection. In the simulation results in the subsequent section, we will also include the results for such $\tilde{S}_T$.

The asymptotics under local alternatives are also not dealt with since they are not simply formulated and should be approached only by altering ρ or by ψ_j, noting that the behavior of y_t is close to that of an $I(0)$ if (2) is true but $|1 + \sum_{j=1}^{\infty} \psi_j|$ is small.

4 Simulation

In this section, we suppose that u_t is generated by $ARMA(1, 1)$, i.e., $u_t = bu_{t-1} + \varepsilon_t + c\varepsilon_{t-1}$ with $\sigma = 1$ and $y_0 = u_0 = 0$, and we calculate the sizes and powers of the tests proposed in several DGPs. All calculations were made in Gauss, and the number of iterations was 70,000.

Only a part of the results for $\hat{M}_{1;T;S_T}$ is reported.[2] $\mathcal{F}$ is set to $\{5,\cdots, 10\}$ when $T = 200$ and to $\{9,\cdots, 14\}$ when $T = 500$.

As observed in Tables 2 and 3, the tests proposed exhibited entirely satisfactory performances, although the power performances tend to be worse as S_T increases. We also note that the case where b is close to -1 led to results such as those in a local alternative case.

TABLE 1

Lower 5 % Points of

the Finite Sample Distributions of $\bar{N}_{1;T;S_T}$

$T=200$		$T=500$	
$S_T=5$	$S_T=10$	$S_T=5$	$S_T=10$
-1.801543	-1.695338	-1.874812	-1.818321

5 Concluding remarks

We have been established that the nonparametric tests proposed in this paper achieve noticeable size improvements in the unit root tests with serially correlated errors. The test statistics proposed are constructed on the basis of the t-statistic, and those on the basis of the normalized OLS statistic will be easily constructed as in the pp tests, and will be shown to possess similar asymptotics. We will also leave the issue on how kernels other than the truncated one can construct similar tests and achieve similar results to future research.

TABLE 2

Size of the tests (5% nominal size) as $\mu_1 = 0$

Test \ DGP		$b=-0.2$, $c=0.8$	$b=-0.2$, $c=-0.2$	$b=-0.5$, $c=0.2$	$b=0$, $c=0.5$	$b=0.5$, $c=0.2$	$b=0.8$, $c=-0.5$
				$T=200$			
	$S_T=5$	0.05	0.054	0.048	0.052	0.05	0.026
$\hat{M}_{1;T;S_T}$	$S_T=10$	0.049	0.056	0.051	0.049	0.052	0.047
	$S_T=\tilde{S}_{1;T}$	0.049	0.051	0.047	0.05	0.052	0.044
				$T=500$			
$\hat{M}_{1;T;S_T}$	$S_T=5$	0.052	0.049	0.048	0.051	0.05	0.023
	$S_T=10$	0.052	0.05	0.051	0.051	0.053	0.044
	$S_T=\tilde{S}_{1;T}$	0.051	0.048	0.05	0.05	0.053	0.049

TABLE 3

Power of the tests (5% nominal size) as $\mu_0 = \mu_1 = 0$

Test \ DGP		$b = -0.2$, $c = 0.8$	$b = -0.2$, $c = -0.2$	$b = -0.5$, $c = 0.2$	$b = 0$, $c = 0.5$	$b = 0.5$, $c = 0.2$	$b = 0.8$, $c = -0.5$
		$T = 200$ and $\rho = 0.95$					
$\hat{M}_{1;T;S_T}$	$S_T = 5$	0.636	0.608	0.587	0.641	0.668	0.475
	$S_T = 10$	0.478	0.511	0.482	0.481	0.523	0.543
	$S_T = \tilde{S}_{1;T}$	0.524	0.555	0.531	0.526	0.57	0.545
		$T = 200$ and $\rho = 0.92$					
$\hat{M}_{1;T;S_T}$	$S_T = 5$	0.845	0.811	0.788	0.852	0.894	0.768
	$S_T = 10$	0.572	0.658	0.598	0.576	0.642	0.721
	$S_T = \tilde{S}_{1;T}$	0.66	0.751	0.699	0.658	0.71	0.758
		$T = 500$ and $\rho = 0.95$					
$\hat{M}_{1;T;S_T}$	$S_T = 5$	0.997	0.984	0.989	0.997	0.998	0.987
	$S_T = 10$	0.959	0.921	0.94	0.961	0.98	0.987
	$S_T = \tilde{S}_{1;T}$	0.902	0.914	0.914	0.903	0.926	0.973
		$T = 500$ and $\rho = 0.92$					
$\hat{M}_{1;T;S_T}$	$S_T = 5$	1.0	0.996	0.998	1.0	1.0	1.0
	$S_T = 10$	0.962	0.949	0.951	0.966	0.988	0.998
	$S_T = \tilde{S}_{1;T}$	0.877	0.947	0.919	0.877	0.906	0.981

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FOOTNOTES

[1] These depend upon S_T only via $\bar{P}_{T;S_T}$ as the term being concerned with the slow convergence to zero (see Table 1), and the limiting distributions are the same as those of the Dickey-Fuller test.

[2] The full report is also available from the author upon request.

Appendix

Proof of Theorem 1: First, for $i = 1, 2, 3$, put

$$\begin{aligned} \tilde{P}_{i;T;S_T} &= \sum_{j=1}^{S_T} \sum_{t=j+2}^{T} \Delta\tilde{y}_{t;i} \Delta\tilde{y}_{t-j;i}/T, \\ \tilde{Q}_{i;T} &= (1 - \hat{\rho}_i)^2 \sum_{t=3}^{T} \Delta\tilde{y}_{t-1;i} \tilde{y}_{t-1;i}/T. \end{aligned}$$

Also, note that

$$\begin{aligned} & \sum_{t=3}^{T} (y_t - \hat{\rho}_i\, y_{t-1} - (1 - \hat{\rho}_i)\, \tilde{c}_{i;t} - \hat{c}_{i;t})\, \Delta\tilde{y}_{t-1;i}/T \\ = & \sum_{t=3}^{T} \Delta\tilde{y}_{t;i} \Delta\tilde{y}_{t-1;i}/T + (1 - \hat{\rho}_i) \sum_{t=3}^{T} \Delta\tilde{y}_{t-1;i} \tilde{y}_{t-1;i}/T, \end{aligned}$$

since

$$\begin{aligned} & y_t - \hat{\rho}_i\, y_{t-1} - (1 - \hat{\rho}_i)\, \tilde{c}_{i;t} - \hat{c}_{i;t} \\ = & \Delta y_t + (1 - \hat{\rho}_i)\, y_{t-1} - (1 - \hat{\rho}_i)\, \tilde{c}_{i;t} - \hat{c}_{i;t} \\ = & \Delta y_t - \hat{c}_{i;t} + (1 - \hat{\rho}_i)\, (y_{t-1} - \tilde{c}_{i;t}) \\ = & \Delta\tilde{y}_{t;i} + (1 - \hat{\rho}_i)\, \tilde{y}_{t-1:i} \end{aligned}$$

by the definitions of $\Delta\tilde{y}_{t;i}$ and $\tilde{y}_{t-1;i}$. Based on this, we can rewrite $\hat{M}_{i;T;S_T}$ as

$$\hat{M}_{i;T;S_T} = \frac{\sum_{t=2}^{T} \Delta\tilde{y}_{t;i} \tilde{y}_{t-1;i}/T - \tilde{P}_{i;T;S_T} - \tilde{Q}_{i;T}}{\left(\sum_{t=2}^{T} \Delta\tilde{y}_{t;i}^2/T + 2\, \tilde{P}_{i;T:S_T} + 2\, \tilde{Q}_{i;T}\right)^{1/2} \left(\sum_{t=2}^{T} \tilde{y}_{t-1;i}^2/T^2\right)^{1/2}}$$

under both H_0 and H_1. Moreover, put

$$\psi = 1 + \sum_{h=1}^{\infty} \psi_h, \qquad \psi_h^{(1)} = -\sum_{j=h+1}^{\infty} \psi_j, \qquad v_t = \sum_{h=0}^{\infty} \psi_h^{(1)} \epsilon_{t-h}$$

with respect to ψ_j in (1). (1) can be then converted to

$$u_t = \psi\, \epsilon_t + v_t - v_{t-1}, \tag{A.1}$$

which in turn leads to

$$\sum_{h=1}^{t-1} u_h = \psi\, \eta_{t-1} + v_{t-1} - v_0. \tag{A.2}$$

(i) It follows immediately from (2) that

$$y_{t-1} = \sum_{h=1}^{t-1} u_h + (t-1)\mu_1 + y_0 \qquad \forall\, t \geq 1. \tag{A.3}$$

It is now obvious from the ergodicity property of u_t and the assumption $y_0 = O_p(1)$ or $O(1)$(i.e., the initial condition) that

$$\sum_{t=2}^{T} u_t y_0/T = T^{-1/2}\, y_0 \left(\sum_{t=2}^{T} u_t/T^{1/2}\right) = O_p(T^{-1/2}).$$

For the case $\mu_1 = 0$, it is easily seen that

$$\sum_{t=2}^{T} \Delta\tilde{y}_{t;1} \tilde{y}_{t-1;1}/T = \sum_{t=2}^{T} u_t (\sum_{h=1}^{t-1} u_h)/T + O_p(T^{-1/2}), \tag{A.4}$$

noting that

$$\Delta\tilde{y}_{t;1} = \Delta y_t = u_t, \qquad \tilde{y}_{t-1;1} = y_{t-1} = \sum_{h=1}^{t-1} u_h \, + \, y_0,$$

$$\sum_{t=2}^{T} u_t y_0 / T = T^{-1/2}\, y_0 \left(\sum_{t=2}^{T} u_t / T^{1/2} \right) = O_p(T^{-1/2}).$$

We also have

$$\sum_{t=2}^{T} \Delta\tilde{y}_{t;i}\tilde{y}_{t-1;i}/T = \sum_{t=2}^{T} u_t (\sum_{h=1}^{t-1} u_h)/T \, - \, \tilde{K}_{i;T}, \qquad i = 2, 3, \tag{A.5}$$

$$\sum_{t=2}^{T} \tilde{y}^2_{t-1;i}/T^2 = \sum_{t=2}^{T}(\sum_{h=1}^{t-1} u_h)^2/T^2 \, - \, \hat{K}_{i;T}, \qquad i = 1, 2, 3, \tag{A.6}$$

where

$$\begin{aligned}
\tilde{K}_{2;T} &= \left(\sum_{t=2}^{T} u_t / T^{1/2} \right) \{(T-1)/T\} \, \{\sum_{t=2}^{T}(\sum_{h=1}^{t-1} u_h)/T^{3/2}\}, \\
\hat{K}_{2;T} &= \{\sum_{t=2}^{T}(\sum_{h=1}^{t-1} u_h)/T^{3/2}\}^2 \, \{(T-1)/T\}, \\
\tilde{K}_{3;T} &= \{\sum_{t=2}^{T} u_t \left(1, \, \frac{t-1}{T}\right) / T^{1/2}\} \, \{\sum_{t=2}^{T} \begin{pmatrix} 1 \\ \frac{t-1}{T} \end{pmatrix} (1, \, t-1) \, / T\}^{-1} \\
&\quad \times \{\sum_{t=2}^{T} \begin{pmatrix} 1 \\ \frac{t-1}{T} \end{pmatrix} (\sum_{h=1}^{t-1} u_h)/T^{3/2}\}, \\
\hat{K}_{3;T} &= \{\sum_{t=2}^{T}(\sum_{h=1}^{t-1} u_h)\left(1, \, \frac{t-1}{T}\right) / T^{3/2}\} \, \{\sum_{t=2}^{T} \begin{pmatrix} 1 \\ \frac{t-1}{T} \end{pmatrix} \left(1, \, \frac{t-1}{T}\right) / T\}^{-1} \\
&\quad \times \{\sum_{t=2}^{T} \begin{pmatrix} 1 \\ \frac{t-1}{T} \end{pmatrix} (\sum_{h=1}^{t-1} u_h)/T^{3/2}\},
\end{aligned}$$

These are derived from (2), (A.3) and the facts that $\Delta\tilde{y}_{t;2}$ and $\tilde{y}_{t-1;2}$ are residuals from the regression models

$$\begin{aligned}
\Delta y_t &= \bar{\mu} \, + \, e_{t;2}, \qquad t = 2, \cdots, T, \\
y_{t-1} &= \breve{\mu} \, + \, \bar{e}_{t;2}, \qquad t = 2, \cdots, T,
\end{aligned}$$

respectively and similarly $\Delta\tilde{y}_{t;3}$ and $\tilde{y}_{t-1;3}$ are residuals from the regression models

$$\begin{aligned}
\Delta y_t &= \bar{\mu}_0 \, + \, (t-1)\bar{\mu}_1 \, + \, e_{t;3}, \qquad t = 2, \cdots, T, \\
y_{t-1} &= \breve{\mu}_0 \, + \, (t-1)\breve{\mu}_1 \, + \, \bar{e}_{t;3}, \qquad t = 2, \cdots, T,
\end{aligned}$$

respectively. Recall again that $\Delta\tilde{y}_{t;0;i}$ and $\tilde{y}_{t;1;i}$, $i = 1, 2$, are not defined unless $\mu_1 = 0$. Also, for $i = 2, 3$, we regard $\Sigma_{h=1}^{t-1} u_h$ in (A.5) or (A.6) as one satisfying

$$\sum_{h=1}^{t-1} u_h = \psi \, \eta_{t-1} \, + \, v_{t-1} \tag{A.2$'$}$$

since any regression above vsnishes v_0.

Next, turn to the derivation of some asymptotics. It can be easily checked that

$$\sum_{t=2}^{T}(v_t - v_{t-1})/T^{1/2} = O_p(T^{-1/2}),$$
$$\sum_{t=2}^{T}(t-1)\,(v_t - v_{t-1})/T^{3/2} = O_p(T^{-1/2}), \tag{A.7}$$

$$\begin{aligned}
&\sum_{t=2}^{T}(v_t - v_{t-1})\,\eta_{t-1}/T \\
=\;& \sum_{t=2}^{T} v_t\eta_{t-1}/T - \sum_{t=1}^{T-1} v_t\eta_t/T \\
=\;& \sum_{t=2}^{T-1} v_t\eta_{t-1}/T - \sum_{t=2}^{T-1} v_t\eta_{t-1}/T + v_T\eta_{T-1}/T - \sum_{t=1}^{T-1} v_t\epsilon_t/T - v_2\eta_1/T \\
=\;& v_{T-1}\eta_{T-1}/T - \sum_{t=1}^{T-1} v_t\epsilon_t/T - v_2\eta_1/T \\
=\;& -\sum_{t=1}^{T-1} v_t\epsilon_t/T + O_p(T^{-1/2}),
\end{aligned}$$

i.e.,

$$\sum_{t=2}^{T}(v_t - v_{t-1})\,\eta_{t-1}/T = -\sum_{t=1}^{T-1} v_t\epsilon_t/T + O_p(T^{-1/2}), \tag{A.8}$$

with the notice that $Ev_t\varepsilon_t = \psi_0^{(1)}\sigma^2$ (and therefore $\Sigma_{t=1}^{T-1}v_t\varepsilon_t/T = \psi_0^{(1)}\sigma^2 + O_p(T^{-1/2})$,

$$\begin{aligned}
&\sum_{t=3}^{T}(v_{t-1} - v_{t-2})\,\eta_{t-1}/T \\
=\;& \sum_{t=3}^{T} v_{t-1}\eta_{t-1}/T - \sum_{t=2}^{T-1} v_{t-1}\eta_t/T \\
=\;& \sum_{t=3}^{T-1} v_{t-1}\eta_{t-1}/T - \sum_{t=3}^{T-1} v_{t-1}\eta_{t-1}/T + v_{T-1}\eta_{T-1}/T - \sum_{t=2}^{T-1} v_{t-1}\epsilon_t/T - v_1\eta_1/T \\
=\;& v_{T-1}\eta_{T-1}/T - \sum_{t=2}^{T-1} v_{t-1}\epsilon_t/T - v_1\eta_1/T = O_p(T^{-1/2}),
\end{aligned}$$

i.e.,

$$\sum_{t=3}^{T}(v_{t-1} - v_{t-2})\,\eta_{t-1}/T = O_p(T^{-1/2}). \tag{A.9}$$

Note that some asymptotics for overdifferenced series are different from those for $I(0)$ series. It is also trivial that

$$\sum_{t=2}^{T} v_0/T^{3/2} = O_p(T^{-1/2}), \qquad \sum_{t=2}^{T}(t-1)\,v_0/T^{5/2} = O_p(T^{-1/2}). \tag{A.10}$$

Moreover, we have

$$\sum_{t=2}^{T} \eta_{t-1}/T^{3/2} = O_p(1), \qquad \sum_{t=2}^{T}(t-1)\,\eta_{t-1}/T^{5/2} = O_p(1), \tag{A.11}$$

$$\sum_{t=2}^{T} v_{t-1}/T^{3/2} = O_p(T^{-1}), \qquad \sum_{t=2}^{T}(t-1)\, v_{t-1}/T^{5/2} = O_p(T^{-1}). \tag{A.12}$$

$$\hat{\rho}_i - 1 = O_p(T^{-1}), \qquad i = 1, 2, 3, \tag{A.13}$$

noting that these are the well-known asymptotics for $I(0)$ and $I(1)$ series and the results derived in connection with the pp tests (see Hamilton (1994, p. 486 and pp. 504-512) e.g.).

From (A.1) and $(A.2)'$, (A.7) and (A.12), it is led to that

$$\begin{aligned}
\tilde{K}_{2;T} &= \psi^2 \{\sum_{t=2}^{T} \epsilon_t/T^{1/2}\}\{(T-1)/T\}\{\sum_{t=2}^{T} \eta_{t-1}/T^{3/2}\} + O_p(T^{-1/2}), \\
\hat{K}_{2;T} &= \psi^2 \{\sum_{t=2}^{T} \eta_{t-1}/T^{3/2}\}\{(T-1)/T\}\{\sum_{t=2}^{T} \eta_{t-1}/T^{3/2}\} + O_p(T^{-1}), \\
\tilde{K}_{3;T} &= \psi^2 \{\sum_{t=2}^{T} \epsilon_t \left(1,\, \frac{t-1}{T}\right)/T^{1/2}\}\{\sum_{t=2}^{T} \begin{pmatrix} 1 \\ \frac{t-1}{T} \end{pmatrix} (1,\, t-1)\, /T\}^{-1} \\
&\quad \times \{\sum_{t=2}^{T} \begin{pmatrix} 1 \\ \frac{t-1}{T} \end{pmatrix} \eta_{t-1}/T^{3/2}\} + O_p(T^{-1/2}), \\
\hat{K}_{3;T} &= \psi^2 \{\sum_{t=2}^{T} \eta_{t-1} \left(1,\, \frac{t-1}{T}\right)/T^{3/2}\}\{\sum_{t=2}^{T} \begin{pmatrix} 1 \\ \frac{t-1}{T} \end{pmatrix} \left(1,\, \frac{t-1}{T}\right)/T\}^{-1} \\
&\quad \times \{\sum_{t=2}^{T} \begin{pmatrix} 1 \\ \frac{t-1}{T} \end{pmatrix} \eta_{t-1}/T^{3/2}\} + O_p(T^{-1}),
\end{aligned}$$

i.e.,

$$\tilde{K}_{i;T} = \tilde{R}_{i;T} + O_p(T^{-1/2}), \quad \hat{K}_{i;T} = \hat{R}_{i;T} + O_p(T^{-1/2}), \quad i = 1, 2, 3. \tag{A.14}$$

and it follows immediately from (A.1), (A.2), (A.8) and (A.10) that

$$\begin{aligned}
&\sum_{t=2}^{T} u_t (\sum_{h=1}^{t-1} u_h)/T \\
=\ & \sum_{t=2}^{T} (\psi\epsilon_t + v_t - v_{t-1})\,(\psi\eta_{t-1} + v_{t-1} - v_0)\,/T \\
=\ & \psi^2 \sum_{t=2}^{T} \epsilon_t\eta_{t-1}/T + \psi \sum_{t=2}^{T} \epsilon_t v_{t-1}/T \\
& + \psi \sum_{t=2}^{T} (v_t - v_{t-1})\,\eta_{t-1}/T - T^{-1/2}\, v_0 \{\sum_{t=2}^{T} (\psi\epsilon_t + v_t - v_{t-1})\,/T^{1/2}\} \\
& + \sum_{t=2}^{T} (v_t - v_{t-1}) v_{t-1}/T \\
=\ & \psi^2 \sum_{t=2}^{T} \epsilon_t\eta_{t-1}/T - \psi \sum_{t=1}^{T-1} v_t\epsilon_t/T + \sum_{t=2}^{T} (v_t - v_{t-1}) v_{t-1}/T + O_p(T^{-1/2}),
\end{aligned}$$

$$
\begin{aligned}
&\sum_{t=2}^{T}(\sum_{h=1}^{t-1} u_h)^2/T^2 \\
=\;& \sum_{t=2}^{T} (\psi\eta_{t-1} + v_{t-1} - v_0)^2 /T^2 \\
=\;& \psi^2 \sum_{t=2}^{T} \eta_{t-1}^2/T^2 - T^{-1/2}\, 2\psi\, v_0 \left(\sum_{t=2}^{T} \eta_{t-1}/T^{3/2}\right) + O_p(T^{-1}) \\
=\;& \psi^2 \sum_{t=2}^{T} \eta_{t-1}^2/T^2 + O_p(T^{-1/2})
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
\sum_{t=2}^{T} u_t(\sum_{h=1}^{t-1} u_h)/T \;&=\; \psi^2 \sum_{t=2}^{T} \epsilon_t \eta_{t-1}/T - \psi \sum_{t=1}^{T-1} v_t \epsilon_t/T \\
&\quad + \sum_{t=2}^{T} (v_t - v_{t-1}) v_{t-1}/T + O_p(T^{-1/2}),
\end{aligned}
\qquad (A.15)
$$

$$
\sum_{t=2}^{T}(\sum_{h=1}^{t-1} u_h)^2/T^2 = \psi^2 \sum_{t=2}^{T} \eta_{t-1}^2/T^2 + O_p(T^{-1/2}), \qquad (A.16)
$$

with the notice that $\Sigma_{h=1}^{t-1} u_h$ following from $\tilde{y}_{t-1;2}$ or $\tilde{y}_{t-1;3}$ in (A.5) or (A.6) satisfies

$$
\sum_{t=2}^{T}(\sum_{h=1}^{t-1} u_h)^2/T^2 = \psi^2 \sum_{t=2}^{T} \eta_{t-1}^2/T^2 + O_p(T^{-1}),
$$

which follows from the cancellation of v_0 mentioned in connection with $(A.2)'$.

Note that (A.14) to (A.16) are also the standard results derived from the asymptotics for $I(0)$ and $I(1)$ above. Similarly,

$$
\begin{aligned}
\sum_{t=2}^{T} \Delta\tilde{y}_{t;1}^2/T \;&=\; \sum_{t=2}^{T} u_t^2/T, \\
\sum_{t=2}^{T} \Delta\tilde{y}_{t;2}^2/T \;&=\; \sum_{t=2}^{T} u_t^2/T - T^{-1} \left(\sum_{t=2}^{T} u_t/T^{1/2}\right)^2 \{(T-1)/T\} \\
&=\; \sum_{t=2}^{T} u_t^2/T + O_p(T^{-1}), \\
\sum_{t=2}^{T} \Delta\tilde{y}_{t;3}^2/T \;&=\; \sum_{t=2}^{T} u_t^2/T - T^{-1} \{\sum_{t=2}^{T} u_t \left(1, \frac{t-1}{T}\right) /T^{1/2}\} \\
&\quad \times \{\sum_{t=2}^{T} \begin{pmatrix} 1 \\ \frac{t-1}{T} \end{pmatrix} \left(1, \frac{t-1}{T}\right) /T\}^{-1} \{\sum_{t=2}^{T} \begin{pmatrix} 1 \\ \frac{t-1}{T} \end{pmatrix} u_t/T^{1/2}\} \\
&=\; \sum_{t=2}^{T} u_t^2/T + O_p(T^{-1}),
\end{aligned}
$$

i.e.,

$$
\sum_{t=2}^{T} \Delta\tilde{y}_{t;i}^2/T = \sum_{t=2}^{T} u_t^2/T + O_p(T^{-1}), \qquad i = 1, 2, 3. \qquad (A.17)
$$

In view of the regression models above and (A.7), it is also straightforward to establish that

$$\hat{\mu} = T^{-1/2}\,\hat{z} \,+\, O_p(T^{-1}) \qquad\qquad \tilde{\mu} = y_0 \,+\, T^{1/2}\,\hat{w} \,+\, O_p(T^{-1/2}), \tag{A.18}$$

under the assumption of $\mu_1 = 0$. Moreover, noting that

$$\{\sum_{t=2}^{T} \begin{pmatrix} 1 \\ \frac{t-1}{T} \end{pmatrix} \left(1,\, \frac{t-1}{T}\right) /T\}^{-1} = \begin{pmatrix} 4 & -6 \\ -6 & 12 \end{pmatrix} + O(T^{-1}),$$

it is derived similarly that

$$\begin{aligned} &\hat{\mu}_0 = \mu_1 \,+\, T^{-1/2}\,\hat{z}_0 \,+\, O_p(T^{-1}), \\ &\hat{\mu}_1 = T^{-3/2}\,\hat{z}_1 \,+\, O_p(T^{-2}), \\ &\tilde{\mu}_0 = y_0 \,+\, T^{1/2}\,\hat{w}_0 \,+\, O_p(T^{-1/2}), \\ &\tilde{\mu}_1 = \mu_1 \,+\, T^{-1/2}\,\hat{w}_1 \,+\, O_p(T^{-3/2}), \end{aligned} \tag{A.19}$$

where

$$\begin{aligned} \hat{z} &= \psi \sum_{s=2}^{T} \epsilon_s / T^{1/2}, \\ \hat{w} &= \psi \sum_{s=2}^{T} \eta_{s-1} / T^{3/2}, \\ \hat{z}_0 &= \psi\,\{4 \sum_{s=2}^{T} \epsilon_s / T^{1/2} \,-\, 6 \sum_{s=2}^{T} (s-1)\,\epsilon_s / T^{3/2}\}, \\ \hat{z}_1 &= \psi\,\{12 \sum_{s=2}^{T} (s-1)\,\epsilon_s / T^{3/2} \,-\, 6 \sum_{s=2}^{T} \epsilon_s / T^{1/2}\}, \\ \hat{w}_0 &= \psi\,\{4 \sum_{s=2}^{T} \eta_{s-1} / T^{3/2} \,-\, 6 \sum_{s=2}^{T} (s-1)\,\eta_{s-1} / T^{5/2}\}, \\ \hat{w}_1 &= \psi\,\{12 \sum_{s=2}^{T} (s-1)\,\eta_{s-1} / T^{5/2} \,-\, 6 \sum_{s=2}^{T} \eta_{s-1} / T^{3/2}\}. \end{aligned}$$

As another standard asymptotic for $I(0)$ and $I(1)$, we have

$$\sum_{t=3}^{T} \Delta\tilde{y}_{t-1;i}\tilde{y}_{t-1;i}/T = O_p(1) \qquad\qquad i = 1, 2, 3. \tag{A.20}$$

(A.20) will be shown as follows: Recalling (2), (A.3) and the definitions of $\Delta\tilde{y}_{t-1;i}$ and $\tilde{y}_{t-1;i}$, i.e.,

$$\Delta\tilde{y}_{t-1;1} = \Delta y_{t-1} = u_{t-1}, \quad \tilde{y}_{t-1;1} = y_{t-1} = \sum_{h=1}^{t-1} u_h \,+\, y_0,$$

$$\Delta\tilde{y}_{t-1;2} = \Delta y_{t-1} \,-\, \hat{\mu} = u_{t-1} \,-\, \hat{\mu}, \quad \tilde{y}_{t-1;2} = y_{t-1} \,-\, \tilde{\mu} = \sum_{h=1}^{t-1} u_h \,+\, y_0 \,-\, \tilde{\mu},$$

$$\Delta\tilde{y}_{t-1;3} = \Delta y_{t-1} \,-\, \hat{\mu}_0 \,-\, (t-2)\,\hat{\mu}_1 = u_{t-1} \,+\, \mu_1 \,-\, \hat{\mu}_0 \,-\, (t-2)\,\hat{\mu}_1,$$

$$\tilde{y}_{t-1;3} = y_{t-1} \,-\, \tilde{\mu}_0 \,-\, (t-1)\,\tilde{\mu}_1 = \sum_{h=1}^{t-1} u_h \,+\, y_0 \,+\, (t-1)\,\mu_1 \,-\, \tilde{\mu}_0 \,-\, (t-1)\,\tilde{\mu}_1,$$

and using (A.1), (A.2)/$(A.2)'$, (A.7), (A.9), (A.10) to (A.12) and (A.18) to (A.19), it is led to that

$$\begin{aligned}
&\sum_{t=3}^{T} \Delta\tilde{y}_{t-1;1}\tilde{y}_{t-1;1}/T \\
=\;& \sum_{t=3}^{T} u_{t-1}\left(\sum_{h=1}^{t-1} u_h \,+\, y_0\right)/T \\
=\;& \sum_{t=3}^{T} (\psi\epsilon_{t-1} \,+\, v_{t-1} \,-\, v_{t-2})\,(\psi\eta_{t-1} \,+\, v_{t-1} \,-\, v_0 \,+\, y_0)\,/T \\
=\;& \psi^2 \sum_{t=3}^{T} \epsilon_{t-1}\eta_{t-1}/T \,+\, \psi \sum_{t=3}^{T} \epsilon_{t-1}v_{t-1}/T \,+\, \sum_{t=3}^{T}(v_{t-1} \,-\, v_{t-2})v_{t-1}/T \\
&+\psi \sum_{t=3}^{T} (v_{t-1} \,-\, v_{t-2})\,\eta_{t-1}/T \,+\, \{\sum_{t=3}^{T} (\psi\epsilon_{t-1} \,+\, v_{t-1} \,-\, v_{t-2})\,/T\}\,(y_0 \,-\, v_0) \\
=\;& \psi^2 \sum_{t=3}^{T} \epsilon_{t-1}\eta_{t-1}/T \,+\, \psi \sum_{t=3}^{T} \epsilon_{t-1}v_{t-1}/T \,+\, \sum_{t=3}^{T}(v_{t-1} \,-\, v_{t-2})v_{t-1}/T \,+\, O_p(T^{-1/2}) \\
=\;& O_p(1),
\end{aligned}$$

$$\begin{aligned}
&\sum_{t=3}^{T} \Delta\tilde{y}_{t-1;2}\tilde{y}_{t-1;2}/T \\
=\;& \sum_{t=3}^{T} (u_{t-1} \,-\, \hat{\mu})\left(\sum_{h=1}^{t-1} u_h \,+\, y_0 \,-\, \tilde{\mu}\right)/T \\
=\;& \sum_{t=3}^{T} \left(u_{t-1} \,-\, T^{-1/2}\,\hat{z} \,+\, O_p(T^{-1})\right)\left(\sum_{h=1}^{t-1} u_h \,-\, T^{1/2}\,\hat{w} \,+\, O_p(T^{-1/2})\right)/T \\
=\;& \sum_{t=3}^{T} u_{t-1}(\sum_{h=1}^{t-1} u_h)/T \,-\, \hat{w} \sum_{t=3}^{T} u_{t-1}/T^{1/2} \,-\, \hat{z} \sum_{t=3}^{T}(\sum_{h=1}^{t-1} u_h)/T^{3/2} \\
&+\sum_{t=3}^{T} \hat{z}\hat{w}/T \,+\, O_p(T^{-1}) \sum_{t=3}^{T}(\sum_{h=1}^{t-1} u_h)/T \,-\, O_p(T^{-1})\,T^{1/2}\,\hat{w} \sum_{t=3}^{T}(1/T) \,+\, O_p(T^{-1}) \\
=\;& \sum_{t=3}^{T} u_{t-1}(\sum_{h=1}^{t-1} u_h)/T \,-\, \hat{w} \sum_{t=3}^{T} u_{t-1}/T^{1/2} \,-\, \hat{z} \sum_{t=3}^{T}(\sum_{h=1}^{t-1} u_h)/T^{3/2} \\
&+\hat{z}\hat{w} \,+\, O_p(T^{-1/2}) \\
=\;& \sum_{t=3}^{T} (\psi\epsilon_{t-1} \,+\, v_{t-1} \,-\, v_{t-2})\,(\psi\eta_{t-1} \,+\, v_{t-1})\,/T \\
&-\psi\,\hat{w} \sum_{t=3}^{T} \epsilon_{t-1}/T^{1/2} \,-\, \psi\,\hat{z} \sum_{t=3}^{T} \eta_{t-1}/T^{3/2} \,+\, \hat{z}\hat{w} \,+\, O_p(T^{-1/2}) \\
=\;& O_p(1),
\end{aligned}$$

$$\begin{aligned}
&\sum_{t=3}^{T} \Delta\tilde{y}_{t-1;3}\tilde{y}_{t-1;3}/T \\
=\;& \sum_{t=3}^{T} (u_{t-1} \,+\, \mu_1 \,-\, \hat{\mu}_0 \,-\, (t-2)\,\hat{\mu}_1) \\
&\times \left(\sum_{h=1}^{t-1} u_h \,+\, y_0 \,+\, (t-1)\,\mu_1 \,-\, \tilde{\mu}_0 \,-\, (t-1)\,\tilde{\mu}_1\right)/T
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{t=3}^{T} \Big(u_{t-1} - T^{-1/2}\,\hat{z}_0 - T^{-3/2}\,(t-2)\,\hat{z}_1 + O_p(T^{-1}) \Big) \\
&\quad \times \left(\sum_{h=1}^{t-1} u_h - T^{1/2}\,\hat{w}_0 - T^{-1/2}\,(t-1)\,\hat{w}_1 + O_p(T^{-1/2}) \right) /T \\
&= \sum_{t=3}^{T} u_{t-1}(\sum_{h=1}^{t-1} u_h)/T - \hat{w}_0 \sum_{t=3}^{T} u_{t-1}/T^{1/2} - \hat{w}_1 \sum_{t=3}^{T}(t-1)u_{t-1}/T^{3/2} \\
&\quad - \hat{z}_0 \sum_{t=3}^{T}(\sum_{h=1}^{t-1} u_h)/T^{3/2} - \hat{z}_1 \sum_{t=3}^{T}(t-2)(\sum_{h=1}^{t-1} u_h)/T^{5/2} + O_p(T^{-1/2}) \\
&\quad + \sum_{t=3}^{T} \Big(T^{-1/2}\,\hat{z}_0 + T^{-3/2}\,(t-1)\,\hat{z}_1 \Big) \Big(T^{1/2}\,\hat{w}_0 + T^{-1/2}\,(t-1)\,\hat{w}_1 \Big) /T \\
&= \sum_{t=3}^{T} (\psi\epsilon_{t-1} + v_{t-1} - v_{t-2})\,(\psi\eta_{t-1} + v_{t-1})\,/T \\
&\quad - \psi\,\hat{w}_0 \sum_{t=3}^{T} \epsilon_{t-1}/T^{1/2} - \psi\,\hat{w}_1 \sum_{t=3}^{T}(t-1)\epsilon_{t-1}/T^{3/2} \\
&\quad - \psi\,\hat{z}_0 \sum_{t=3}^{T} \eta_{t-1}/T^{3/2} - \psi\,\hat{z}_1 \sum_{t=3}^{T}(t-2)\eta_{t-1}/T^{5/2} + O_p(T^{-1/2}) \\
&\quad + \hat{z}_0\hat{w}_0 + (1/2)\,(\hat{z}_0\hat{w}_1 + \hat{z}_1\hat{w}_0) + (1/3)\,\hat{z}_1\hat{w}_1 + O_p(T^{-1/2}) \\
&= O_p(1).
\end{aligned}
$$

(A.20), together with (A.13), ensures that

$$
\tilde{Q}_{i;T} = O_p(T^{-2}), \qquad i = 1, 2, 3. \tag{A.21}
$$

On the other hand, it is easily shown that

$$
\begin{aligned}
&\sum_{j=1}^{S_T} \sum_{t=j+2}^{T} (u_t + u_{t-j})\,/T = O_p(S_T T^{-1/2}), \\
&\sum_{j=1}^{S_T} \sum_{t=j+2}^{T} \hat{z}^2/T = O_p(S_T), \\
&T^{-1} \sum_{j=1}^{S_T} \sum_{t=j+2}^{T} (t-1)\,(u_t + u_{t-j})\,/T = O_p(S_T T^{-1/2}), \\
&\sum_{j=1}^{S_T} \sum_{t=j+2}^{T} \left(\hat{z}_0 + \frac{t-1}{T}\,\hat{z}_1 \right)^2 /T = O_p(S_T).
\end{aligned}
$$

Recalling again that

$$
\begin{aligned}
\Delta\tilde{y}_{t-j;1} &= u_{t-j}, \\
\Delta\tilde{y}_{t-j;2} &= u_{t-j} - \hat{\mu} = u_{t-j} - T^{-1/2}\,\hat{z} + O_p(T^{-1}), \\
\Delta\tilde{y}_{t-j;3} &= u_{t-j} + \mu_1 - \hat{\mu}_0 - (t-j-1)\hat{\mu}_1 \\
&= u_{t-j} - T^{-1/2}\,\hat{z}_0 - T^{-1/2} \left(\frac{t-j-1}{T} \right) \hat{z}_1 + O_p(T^{-1}),
\end{aligned}
$$

these results lead to

$$
\begin{aligned}
&\sum_{j=1}^{S_T}\sum_{t=j+2}^{T} \Delta\tilde{y}_{t;1}\Delta\tilde{y}_{t-j;1}/T \\
=\;& \sum_{j=1}^{S_T}\sum_{t=j+2}^{T} u_t u_{t-j}/T,
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{j=1}^{S_T}\sum_{t=j+2}^{T} \Delta\tilde{y}_{t;2}\Delta\tilde{y}_{t-j;2}/T \\
=\;& \sum_{j=1}^{S_T}\sum_{t=j+2}^{T} u_t u_{t-j}/T \;-\; T^{-1/2}\,\hat{z}\,\{\sum_{j=1}^{S_T}\sum_{t=j+2}^{T} (u_t \,+\, u_{t-j})\,/T\} \\
&+T^{-1}\left(\sum_{j=1}^{S_T}\sum_{t=j+2}^{T} \hat{z}^2/T\right) \,+\, O_p(T^{-1}),
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{j=1}^{S_T}\sum_{t=j+2}^{T} \Delta\tilde{y}_{t;3}\Delta\tilde{y}_{t-j;3}/T \\
=\;& \sum_{j=1}^{S_T}\sum_{t=j+2}^{T} u_t u_{t-j}/T \;-\; T^{-1/2}\,\hat{z}_0\,\{\sum_{j=1}^{S_T}\sum_{t=j+2}^{T} (u_t \,+\, u_{t-j})\,/T\} \\
&-T^{-3/2}\,\hat{z}_1\,\{\sum_{j=1}^{S_T}\sum_{t=j+2}^{T} ((t-j-1)\,u_t \,+\, (t-1)\,u_{t-j})\,/T\} \\
&+T^{-1}\,\{\sum_{j=1}^{S_T}\sum_{t=j+2}^{T}\left(\hat{z}_0 \,+\, \frac{t-1}{T}\,\hat{z}_1\right)\left(\hat{z}_0 \,+\, \frac{t-j-1}{T}\,\hat{z}_1\right)/T\} \\
&+O_p(T^{-1}),
\end{aligned}
$$

i.e.,

$$
\tilde{P}_{i;T;S_T} = \sum_{j=1}^{S_T}\sum_{t=j+2}^{T} u_t u_{t-j}/T \,+\, O_p(S_T T^{-1}) \qquad i=1,2,3. \tag{A.22}
$$

Recall that $\lim_{T\to\infty} S_T/T^{1/2} = 0$, which implies that $O_p(S_T T^{-1})$ is smaller than $O_p(T^{-1/2})$. We also see from (A.1) that

$$
\begin{aligned}
&\sum_{j=1}^{S_T}\sum_{t=j+2}^{T} u_t u_{t-j}/T \\
=\;& \psi^2 \sum_{j=1}^{S_T}\sum_{t=j+2}^{T} \epsilon_t\epsilon_{t-j}/T \,+\, \psi \sum_{j=1}^{S_T}\sum_{t=j+2}^{T} \epsilon_t(v_{t-j} \,-\, v_{t-j-1})/T \\
&+\psi \sum_{j=1}^{S_T}\sum_{t=j+2}^{T} (v_t \,-\, v_{t-1})\epsilon_{t-j}/T \\
&+\sum_{j=1}^{S_T}\sum_{t=j+2}^{T} (v_t \,-\, v_{t-1})(v_{t-j} \,-\, v_{t-j-1})/T.
\end{aligned} \tag{A.23}
$$

It is obvious that

$$\epsilon_3 v_1/T + \epsilon_4 v_1/T + \cdots + \epsilon_{S_T} v_1/T = O_p(S_T^{1/2}T^{-1}),$$

$$v_T\epsilon_{T-1}/T + v_T\epsilon_{T-2}/T + \cdots + v_T\epsilon_{T-S_T+1}/T = O_p(S_T^{1/2}T^{-1}).$$

Recalling that $v_t = \sum_{h=0}^{\infty}\psi_h^{(1)}\varepsilon_{t-h}$ with

$$\psi_h^{(1)} = (-1)\sum_{i=h+1}^{\infty}\psi_i, \qquad \sum_{h=1}^{\infty}h^{\nu}|\psi_h| < \infty \qquad for\ \nu > 3,$$

which implies that $\psi_h^{(1)} = o(h^2)$ for sufficiently large h, it is asserted that

$$\begin{aligned}
&\sum_{t=S_T}^{T} v_t\epsilon_{t-S_T}/T \\
=\ & E\, v_t\epsilon_{t-S_T} + O_p(T^{-1/2}) = \psi_{S_T}\sigma + O_p(T^{-1/2}) \\
=\ & o(S_T^{-2}) + O_p(T^{-1/2}) = O_p(T^{-1/2}),
\end{aligned}$$

$$\begin{aligned}
&\sum_{t=S_T}^{T} v_t(v_{t-S_T} - v_{t-S_T-1})/T \\
=\ & E\, v_t(v_{t-S_T} - v_{t-S_T-1}) + O_p(T^{-1/2}) = O(S_T^{-2}) + O_p(T^{-1/2}) = O_p(T^{-1/2}),
\end{aligned}$$

which are in turn followed by

$$\begin{aligned}
&\psi\sum_{j=1}^{S_T}\sum_{t=j+2}^{T}\epsilon_t(v_{t-j} - v_{t-j-1})/T \\
=\ & \psi\sum_{t=3}^{T}\epsilon_t v_{t-1}/T - \psi\sum_{t=3}^{T}\epsilon_t v_{t-2}/T \\
& +\psi\sum_{t=4}^{T}\epsilon_t v_{t-2}/T - \psi\sum_{t=4}^{T}\epsilon_t v_{t-3}/T \\
& +\psi\sum_{t=5}^{T}\epsilon_t v_{t-3}/T - \psi\sum_{t=5}^{T}\epsilon_t v_{t-4}/T + \cdots \\
& +\psi\sum_{t=S_T+1}^{T}\epsilon_t v_{t-S_T+1}/T - \psi\sum_{t=S_T+1}^{T}\epsilon_t v_{t-S_T}/T \\
& +\psi\sum_{t=S_T+2}^{T}\epsilon_t v_{t-S_T}/T - \psi\sum_{t=S_T+2}^{T}\epsilon_t v_{t-S_T-1}/T \\
=\ & \psi\sum_{t=3}^{T}\epsilon_t v_{t-1}/T - \psi\sum_{t=S_T+2}^{T}\epsilon_t v_{t-S_T-1}/T \\
& -\psi\,(\epsilon_3 v_1/T + \epsilon_4 v_1/T + \cdots + \epsilon_{S_T+1}v_1/T) \\
=\ & O_p(T^{-1/2}),
\end{aligned}$$

$$\begin{aligned}
&\psi\sum_{j=1}^{S_T}\sum_{t=j+2}^{T}(v_t - v_{t-1})\epsilon_{t-j}/T \\
=\ & \psi\sum_{t=3}^{T}v_t\epsilon_{t-1}/T - \psi\sum_{t=2}^{T-1}v_t\epsilon_t/T
\end{aligned}$$

$$
\begin{aligned}
& + \psi \sum_{t=4}^{T} v_t \epsilon_{t-2}/T \,-\, \psi \sum_{t=3}^{T-1} v_t \epsilon_{t-1}/T \\
& + \psi \sum_{t=5}^{T} v_t \epsilon_{t-3}/T \,-\, \psi \sum_{t=4}^{T-1} v_t \epsilon_{t-2}/T \,+\, \cdots \\
& + \psi \sum_{t=S_T+1}^{T} v_t \epsilon_{t-S_T+1}/T \,-\, \psi \sum_{t=S_T}^{T-1} v_t \epsilon_{t-S_T+2}/T \\
& + \psi \sum_{t=S_T+2}^{T} v_t \epsilon_{t-S_T}/T \,-\, \psi \sum_{t=S_T+1}^{T-1} v_t \epsilon_{t-S_T+1}/T \\
= \;& -\psi \sum_{t=2}^{T-1} v_t \epsilon_t/T \,+\, \psi \sum_{t=S_T+2}^{T} v_t \epsilon_{t-S_T}/T \\
& + \psi \left(v_T \epsilon_{T-1}/T \,+\, v_T \epsilon_{T-2}/T \,+\, \cdots \,+\, v_T \epsilon_{T-S_T+1}/T \right) \\
= \;& -\psi \sum_{t=2}^{T-1} v_t \epsilon_t/T \,+\, O_p(T^{-1/2}),
\end{aligned}
$$

$$
\begin{aligned}
& \sum_{j=1}^{S_T} \sum_{t=j+2}^{T} (v_t \,-\, v_{t-1})(v_{t-j} \,-\, v_{t-j-1})/T \\
= \;& \sum_{t=3}^{T} v_t (v_{t-1} \,-\, v_{t-2})/T \,-\, \sum_{t=2}^{T-1} v_t (v_t \,-\, v_{t-1})/T \\
& + \sum_{t=4}^{T} v_t (v_{t-2} \,-\, v_{t-3})/T \,-\, \sum_{t=3}^{T-1} v_t (v_{t-1} \,-\, v_{t-2})/T \\
& + \sum_{t=5}^{T} v_t (v_{t-3} \,-\, v_{t-4})/T \,-\, \sum_{t=4}^{T-1} v_t (v_{t-2} \,-\, v_{t-3})/T \,+\, \cdots \\
& + \sum_{t=S_T+1}^{T} v_t (v_{t-S_T+1} \,-\, v_{t-S_T})/T \,-\, \sum_{t=S_T}^{T-1} v_t (v_{t-S_T+2} \,-\, v_{t-S_T+1})/T \\
& + \sum_{t=S_T+2}^{T} v_t (v_{t-S_T} \,-\, v_{t-S_T-1})/T \,-\, \sum_{t=S_T+1}^{T-1} v_t (v_{t-S_T+1} \,-\, v_{t-S_T})/T \\
= \;& -\sum_{t=2}^{T-1} v_t (v_t \,-\, v_{t-1})/T \,+\, \sum_{t=S_T+2}^{T} v_t (v_{t-S_T} \,-\, v_{t-S_T-1})/T \\
& + \{ v_T (v_{T-1} - v_{T-2})/T \,+\, v_T (v_{T-2} - v_{T-3})/T \,+\, \cdots \\
& \qquad + v_T (v_{T-S_T+1} - v_{T-S_T})/T \} \\
= \;& -\sum_{t=2}^{T-1} v_t (v_t \,-\, v_{t-1})/T \,+\, \sum_{t=S_T+2}^{T} v_t (v_{t-S_T} \,-\, v_{t-S_T-1})/T \\
& + v_T (v_{T-1} - v_{T-S_T})/T \\
= \;& -\sum_{t=2}^{T-1} v_t (v_t \,-\, v_{t-1})/T \,+\, O_p(T^{-1/2}),
\end{aligned}
$$

These results, together with (A.23), lead to

$$
\begin{aligned}
& \sum_{j=1}^{S_T} \sum_{t=j+2}^{T} u_t u_{t-j}/T \\
= \; & \psi^2 \sum_{j=1}^{S_T} \sum_{t=j+2}^{T} \epsilon_t \epsilon_{t-j}/T \\
& - \psi \sum_{t=2}^{T-1} v_t \epsilon_t/T - \sum_{t=2}^{T-1} v_t (v_t - v_{t-1})/T + O_p(T^{-1/2}). \qquad (A.24)
\end{aligned}
$$

By combining (A.24) with (A.15), we attain to

$$
\begin{aligned}
& \sum_{t=2}^{T} u_t (\sum_{h=1}^{t-1} u_h)/T - \sum_{j=1}^{S_T} \sum_{t=j+2}^{T} u_t u_{t-j}/T \\
= \; & \psi^2 \sum_{t=2}^{T} \epsilon_t \eta_{t-1}/T - \psi \sum_{t=1}^{T-1} v_t \epsilon_t/T + \sum_{t=2}^{T} (v_t - v_{t-1}) v_{t-1}/T + O_p(T^{-1/2}) \\
& - \psi^2 \sum_{j=1}^{S_T} \sum_{t=j+2}^{T} \epsilon_t \epsilon_{t-j}/T \\
& + \psi \sum_{t=2}^{T-1} v_t \epsilon_t/T + \sum_{t=2}^{T-1} v_t (v_t - v_{t-1})/T + O_p(T^{-1/2}) \\
= \; & \psi^2 \sum_{t=2}^{T} \epsilon_t \eta_{t-1}/T - \psi^2 \sum_{j=1}^{S_T} \sum_{t=j+2}^{T} \epsilon_t \epsilon_{t-j}/T \\
& + v_1 \epsilon_1/T + v_T v_{T-1}/T - v_1^2/T + O_p(T^{-1/2}) \\
= \; & \psi^2 \sum_{t=2}^{T} \epsilon_t \eta_{t-1}/T - \psi^2 \sum_{j=1}^{S_T} \sum_{t=j+2}^{T} \epsilon_t \epsilon_{t-j}/T + O_p(T^{-1/2}),
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
& \sum_{t=2}^{T} u_t (\sum_{h=1}^{t-1} u_h)/T - \sum_{j=1}^{S_T} \sum_{t=j+2}^{T} u_t u_{t-j}/T \\
= \; & \psi^2 \sum_{t=2}^{T} \epsilon_t \eta_{t-1}/T - \psi^2 \sum_{j=1}^{S_T} \sum_{t=j+2}^{T} \epsilon_t \epsilon_{t-j}/T + O_p(T^{-1/2}). \qquad (A.25)
\end{aligned}
$$

In view of (A.1) again,

$$
\begin{aligned}
& \sum_{t=2}^{T} u_t^2/T \\
= \; & \sum_{t=2}^{T} (\psi \epsilon_t + v_t - v_{t-1})^2 /T \\
= \; & \psi^2 \sum_{t=2}^{T} \epsilon_t^2/T + \sum_{t=2}^{T} (v_t - v_{t-1})^2/T + 2\psi \sum_{t=2}^{T} \epsilon_t (v_t - v_{t-1})/T \\
= \; & \psi^2 \sum_{t=2}^{T} \epsilon_t^2/T + 2 \sum_{t=2}^{T} v_t^2/T - 2 \sum_{t=2}^{T} v_t v_{t-1}/T + 2\psi \sum_{t=2}^{T} \epsilon_t v_t + O_p(T^{-1/2}).
\end{aligned}
$$

Combining this with (A.24) leads to

$$
\begin{aligned}
&\sum_{t=2}^{T} u_t^2/T \,+\, 2\sum_{j=1}^{S_T}\sum_{t=j+2}^{T} u_t u_{t-j}/T \\
=\;& \psi^2 \sum_{t=2}^{T} \epsilon_t^2/T \,+\, 2\sum_{t=2}^{T} v_t^2/T \,-\, 2\sum_{t=2}^{T} v_t v_{t-1}/T \,+\, 2\,\psi \sum_{t=2}^{T} \epsilon_t v_t \,+\, O_p(T^{-1/2}) \\
&+2\,\psi^2 \sum_{j=1}^{S_T}\sum_{t=j+2}^{T} \epsilon_t \epsilon_{t-j}/T \\
&-2\,\psi \sum_{t=2}^{T-1} v_t \epsilon_t/T \,-\, 2\sum_{t=2}^{T-1} v_t(v_t \,-\, v_{t-1})/T \,+\, O_p(T^{-1/2}) \\
=\;& \psi^2 \sum_{t=2}^{T} \epsilon_t^2/T \,+\, 2\,\psi^2 \sum_{j=1}^{S_T}\sum_{t=j+2}^{T} \epsilon_t \epsilon_{t-j}/T \,+\, O_p(T^{-1/2}),
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
&\sum_{t=2}^{T} u_t^2/T \,+\, 2\sum_{j=1}^{S_T}\sum_{t=j+2}^{T} u_t u_{t-j}/T \\
=\;& \psi^2 \sum_{t=2}^{T} \epsilon_t^2/T \,+\, 2\,\psi^2 \sum_{j=1}^{S_T}\sum_{t=j+2}^{T} \epsilon_t \epsilon_{t-j}/T \,+\, O_p(T^{-1/2}).
\end{aligned}
\tag{A.26}
$$

Using (A.4) to (A.6), (A.14), (A.16), (A.17), (A.21), (A.22), (A.25) and (A.26), we can easily derive that

$$
\begin{aligned}
&\sum_{t=2}^{T} \Delta\tilde{y}_{t;i}\tilde{y}_{t-1;i}/T \,-\, \tilde{P}_{i;T;S_T} \,-\, \tilde{Q}_{i;T} \\
=\;& \psi^2 \sum_{t=2}^{T} \epsilon_t \eta_{t-1}/T \,-\, \psi^2 \sum_{j=1}^{S_T}\sum_{t=j+2}^{T} \epsilon_t \epsilon_{t-j}/T \,-\, \psi^2\, \tilde{R}_{i;T} \,+\, O_p(T^{-1/2}) \\
=\;& \psi^2 \,\{\sum_{t=2}^{T} \epsilon_t \eta_{t-1}/T \,-\, \bar{P}_{T;S_T} \,-\, \tilde{R}_{i;T}\}, \\
&\left(\sum_{t=2}^{T} \Delta\tilde{y}_{t;i}^2/T \,+\, 2\,\tilde{P}_{i;T:S_T} \,+\, 2\,\tilde{Q}_{i;T}\right)^{1/2} \left(\sum_{t=2}^{T} \tilde{y}_{t-1;i}^2/T^2\right)^{1/2} \\
=\;& \psi^2 \left(\sum_{t=2}^{T} \epsilon_t^2/T \,+\, 2\sum_{j=1}^{S_T}\sum_{t=j+2}^{T} \epsilon_t \epsilon_{t-j}/T\right)^{1/2} \left(\sum_{t=2}^{T} \eta_{t-1}^2/T^2 \,-\, \hat{R}_{i;T}\right)^{1/2} \\
&+O_p(T^{-1/2}) \\
=\;& \psi^2 \left(\sum_{t=2}^{T} \epsilon_t^2/T \,+\, 2\,\bar{P}_{T:S_T}\right)^{1/2} \left(\sum_{t=2}^{T} \eta_{t-1}^2/T^2 \,-\, \hat{R}_{i;T}\right)^{1/2} \,+\, O_p(T^{-1/2}).
\end{aligned}
$$

Considering the forms of $\hat{M}_{i;T;S_T}$ in the first part of the proof, the proof for (i) is completed.

(ii) Put

$$
\begin{aligned}
\bar{\mu}_0 \;&=\; \frac{1}{1\,-\,\rho}\,\mu_0 \,-\, \frac{\rho}{(1\,-\,\rho)^2}\,\mu_1 \qquad\qquad \bar{\mu}_1 = \frac{1}{1\,-\,\rho}\,\mu_1, \\
\breve{v}_t \;&=\; \frac{-\,\rho}{(1\,-\,\rho)\,(1\,-\,\rho\,B)}\,\epsilon_t \,+\, \frac{1}{1\,-\,\rho\,B}\,v_t, \\
w_t \;&=\; \frac{\psi}{1\,-\,\rho}\,\epsilon_t \,+\, \breve{v}_t \,-\, \breve{v}_{t-1},
\end{aligned}
$$

where B stands for the backward operator, i.e., $Bv_t = v_{t-1}$. Since $(1 - B)\,1 = 0$, $(1 - B)\,t = 1$ and

$$\frac{1}{1-\rho B} = \frac{1}{1-\rho} + \frac{-\rho(1-B)}{(1-\rho)(1-\rho B)},$$

it follows from (3) and (A.1) that

$$\begin{aligned} y_t &= \frac{1}{1-\rho B}(\mu_0 + t\mu_1 + u_t) \\ &= \frac{1}{1-\rho}\mu_0 + t\frac{1}{1-\rho}\mu_1 + \frac{-\rho(1-B)}{(1-\rho)(1-\rho B)}t\mu_1 \\ &\quad + \frac{1}{1-\rho B}(\psi\epsilon_t + (1-B)v_t) \\ &= \frac{1}{1-\rho}\mu_0 + t\frac{1}{1-\rho}\mu_1 - \frac{\rho}{(1-\rho)^2}\mu_1 \\ &\quad + \frac{1}{1-\rho B}(\psi\epsilon_t + (1-B)v_t) \\ &= \frac{1}{1-\rho}\mu_0 + t\frac{1}{1-\rho}\mu_1 - \frac{\rho}{(1-\rho)^2}\mu_1 \\ &\quad + \frac{\psi}{1-\rho}\epsilon_t + (1-B)\{\frac{-\rho}{(1-\rho)(1-\rho B)} + \frac{1}{1-\rho B}\}v_t, \end{aligned}$$

i.e.,

$$y_t = \bar{\mu}_0 + t\bar{\mu}_1 + w_t, \tag{A.27}$$

therefore,

$$\Delta y_t = \bar{\mu}_1 + \Delta w_t. \tag{A.28}$$

Since w_t is mean zero, weakly stationary and ergodic by definition, it follows that for $j = 0, 1, \cdots, S_T$,

$$\sum_{t=2}^{T}(\Delta w_{t-j} + w_{t-1})/T^{1/2} = O_p(1), \qquad \sum_{t=2}^{T}(t-1)(\Delta w_{t-j} + w_{t-1})/T^{3/2} = O_p(1),$$

similarly to (A.12). This ensures that

$$\sum_{t=2}^{T}\Delta\tilde{y}_{t;1}\tilde{y}_{t-1;1}/T = \sum_{t=2}^{T}\Delta w_t w_{t-1}/T + O_p(T^{-1/2}), \tag{A.29}$$

$$\sum_{t=2}^{T}\Delta\tilde{y}_{t;i}\tilde{y}_{t-1;i}/T = \sum_{t=2}^{T}\Delta w_t w_{t-1}/T + O_p(T^{-1}), \qquad i = 2, 3, \tag{A.30}$$

$$\sum_{t=2}^{T}\tilde{y}_{t-1;i}^2/T = \sum_{t=2}^{T}w_{t-1}^2/T + O_p(T^{-1}), \qquad i = 1, 2, 3, \tag{A.31}$$

in view of (A.27) and (A.28) and recalling that $\Delta\tilde{y}_{t;2}$ and $\tilde{y}_{t-1;2}$ are residuals from the regression models

$$\begin{aligned} \Delta y_t &= \breve{\mu}_0 + \breve{e}_{t;0;2}, \qquad t = 2, \cdots, T, \\ y_{t-1} &= \breve{\mu}_1 + \breve{e}_{t;1;2}, \qquad t = 2, \cdots, T, \end{aligned}$$

respectively and similarly $\Delta\tilde{y}_{t;3}$ and $\tilde{y}_{t-1;3}$ are residuals from the regression models

$$\begin{aligned}\Delta y_t &= \breve{\mu}_{0;0} + (t-1)\breve{\mu}_{0;1} + \breve{e}_{t;0;3},\\ y_{t-1} &= \breve{\mu}_{1;0} + (t-1)\breve{\mu}_{1;1} + \breve{e}_{t;1;3}, \qquad t = 2,\cdots,T,\end{aligned}$$

respectively. Recall again that $\Delta\tilde{y}_{t;i}$ and $\tilde{y}_{t-1;i}$, $i=1, 2$, are not defined unless $\mu_1=0$. Using similar arguments to those used for (A.20) and (A.22) in the proof of (i), we also have

$$\sum_{t=3}^{T}\Delta\tilde{y}_{t-1;i}\tilde{y}_{t-1;i}/T = \sum_{t=3}^{T}\Delta w_{t-1}w_{t-1}/T + O_p(T^{-1}), \qquad i = 1,2,3, \tag{A.32}$$

$$\tilde{P}_{i;T;S_T} = \sum_{j=1}^{S_T}\sum_{t=j+2}^{T}\Delta w_t\Delta w_{t-j}/T + O_p(S_T T^{-1}) \qquad i = 1,2,3. \tag{A.33}$$

Moreover, put $R_y(0) = V(y_t)$ and $R_y(1) = Cov(y_t, y_{t-1})$, and it is obvious from (A.27) that $R_y(0) = Ew_t^2$ and $R_y(1) = Ew_tw_{t-1}$. By the weak stationarity and ergodicity of w_t again,

$$\sum_{t=3}^{T}\Delta w_{t-1}w_{t-1}/T = R_y(0) - R_y(1) + O_p(T^{-1/2}), \tag{A.34}$$

$$\sum_{t=2}^{T}w_{t-1}^2/T = R_y(0) + O_p(T^{-1/2}), \tag{A.35}$$

$$\hat{\rho}_i = \rho + O_p(T^{-1/2}), \qquad i = 1,2,3. \tag{A.36}$$

Recalling the definitions of $\tilde{Q}_{i;T}$ in the first part of the proof and putting (A.32), (A.34) and (A.36) together, it is derived that

$$\tilde{Q}_{i;T} = (1-\rho)^2\{R_y(0) - R_y(1)\} + O_p(T^{-1/2}), \qquad i = 1,2,3. \tag{A.37}$$

On the other hand, it is straightforward to check that

$$\sum_{t=2}^{S_T+2}\Delta w_t w_{t-1}/T = O_p(S_T T^{-1}),$$

$$\sum_{t=2}^{S_T+2}\Delta w_t^2/T = O_p(S_T T^{-1}),$$

$$\sum_{t=T-S_T}^{T}\Delta w_t^2/T = O_p(S_T T^{-1}),$$

$$
\begin{aligned}
&\sum_{j=1}^{S_T}\sum_{t=j+2}^{S_T+2} \Delta w_t \Delta w_{t-j}/T \\
= \; & \sum_{t=3}^{S_T+2} \Delta w_t \Delta w_{t-1}/T + \sum_{t=4}^{S_T+2} \Delta w_t \Delta w_{t-2}/T + \sum_{t=5}^{S_T+2} \Delta w_t \Delta w_{t-3}/T \\
& + \cdots + \sum_{t=S_T+2}^{S_T+2} \Delta w_t \Delta w_{t-S_T}/T \\
= \; & \Delta w_3 \Delta w_2/T + \Delta w_4(\Delta w_3 + \Delta w_2)/T + \Delta w_5(\Delta w_4 + \Delta w_3 + \Delta w_2)/T \\
& + \cdots + \Delta w_{S_T+2}(\Delta w_{S_T+1} + \cdots + \Delta w_2)/T \\
= \; & \Delta w_3(w_2 - w_1)/T + \Delta w_4(w_3 - w_1)/T + \Delta w_5(w_4 - w_1)/T \\
& + \cdots + \Delta w_{S_T+2}(w_{S_T+1} - w_1)/T \\
= \; & O_p(S_T T^{-1})
\end{aligned}
$$

noting again that $\lim_{T\to\infty} S_T/T^{1/2} = 0$. These lead to

$$
\begin{aligned}
&\sum_{t=2}^{T} \Delta w_t w_{t-1}/T \\
= \; & \sum_{t=S_T+3}^{T} \Delta w_t w_{t-1}/T + \sum_{t=2}^{S_T+2} \Delta w_t w_{t-1}/T \\
= \; & \sum_{t=S_T+3}^{T} \Delta w_t w_{t-1}/T + O_p(S_T T^{-1}),
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{j=1}^{S_T}\sum_{t=j+2}^{T} \Delta w_t \Delta w_{t-j}/T \\
= \; & \sum_{j=1}^{S_T}\sum_{t=S_T+3}^{T} \Delta w_t \Delta w_{t-j}/T + \sum_{j=1}^{S_T}\sum_{t=j+2}^{S_T+2} \Delta w_t \Delta w_{t-j}/T \\
= \; & \sum_{j=1}^{S_T}\sum_{t=S_T+3}^{T} \Delta w_t \Delta w_{t-j}/T + O_p(S_T T^{-1}) \\
= \; & \sum_{t=S_T+3}^{T} \Delta w_t \left(\sum_{j=1}^{S_T} \Delta w_{t-j}\right)/T + O_p(S_T T^{-1}) \\
= \; & \sum_{t=S_T+3}^{T} \Delta w_t (w_{t-1} - w_{t-S_T-1})/T + O_p(S_T T^{-1}).
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
&\sum_{t=2}^{T} \Delta w_t w_{t-1}/T - \sum_{j=1}^{S_T}\sum_{t=j+2}^{T} \Delta w_t \Delta w_{t-j}/T \\
= \; & \sum_{t=S_T+3}^{T} \Delta w_t w_{t-1}/T - \sum_{t=S_T+3}^{T} \Delta w_t (w_{t-1} - w_{t-S_T-1})/T + O_p(S_T T^{-1}) \\
= \; & \sum_{t=S_T+3}^{T} \Delta w_t w_{t-S_T-1}/T + O_p(S_T T^{-1}) \\
= \; & E\,\Delta w_t w_{t-S_T-1} + O_p(T^{-1/2}) + O_p(S_T T^{-1}) \\
= \; & o(S_T^2) + O_p(T^{-1/2}) + O_p(S_T T^{-1}) = O_p(T^{-1/2}),
\end{aligned}
$$

i.e.,

$$\sum_{t=2}^{T} \Delta w_t w_{t-1}/T - \sum_{j=1}^{S_T} \sum_{t=j+2}^{T} \Delta w_t \Delta w_{t-j}/T = O_p(T^{-1/2}). \tag{A.38}$$

Similarly,

$$\sum_{t=2}^{S_T+2} \Delta w_t^2/T + \sum_{t=T-S_T}^{T} \Delta w_t^2/T = O_p(S_T T^{-1}),$$

$$\begin{aligned}
&\sum_{j=1}^{S_T} \sum_{t=j+2}^{S_T+2} \Delta w_t \Delta w_{t-j}/T + \sum_{j=1}^{S_T} \sum_{t=T-S_T}^{T} \Delta w_t \Delta w_{t-j}/T \\
= \;& \sum_{t=3}^{S_T+2} \Delta w_t \Delta w_{t-1}/T + \sum_{t=4}^{S_T+2} \Delta w_t \Delta w_{t-2}/T + \sum_{t=5}^{S_T+2} \Delta w_t \Delta w_{t-3}/T \\
&+ \cdots + \sum_{t=S_T+2}^{S_T+2} \Delta w_t \Delta w_{t-S_T}/T + \sum_{t=T-S_T}^{T} \Delta w_t \Delta w_{t-1}/T \\
&+ \sum_{t=T-S_T}^{T} \Delta w_t \Delta w_{t-2}/T + \sum_{t=T-S_T}^{T} \Delta w_t \Delta w_{t-3}/T \\
&+ \cdots + \sum_{t=T-S_T}^{T} \Delta w_t \Delta w_{t-S_T}/T \\
= \;& \Delta w_3 \Delta w_2/T + \Delta w_4(\Delta w_3 + \Delta w_2)/T + \Delta w_5(\Delta w_4 + \Delta w_3 + \Delta w_2)/T \\
&+ \cdots + \Delta w_{S_T+2}(\Delta w_{S_T+1} + \cdots + \Delta w_2)/T \\
&+ \sum_{t=T-S_T}^{T} \Delta w_t(w_{t-1} - w_{t-S_T-1})/T \\
= \;& \Delta w_3(w_2 - w_1)/T + \Delta w_4(w_3 - w_1)/T + \Delta w_5(w_4 - w_1)/T \\
&+ \cdots + \Delta w_{S_T+2}(w_{S_T+1} - w_1)/T + \sum_{t=T-S_T}^{T} \Delta w_t(w_{t-1} - w_{t-S_T-1})/T \\
= \;& O_p(S_T T^{-1}),
\end{aligned}$$

$$\begin{aligned}
&\sum_{j=1}^{S_T} \sum_{t=j+2}^{S_T+j+2} \Delta w_t \Delta w_{t-j}/T + \sum_{j=1}^{S_T} \sum_{t=T-S_T+j+2}^{T} \Delta w_t \Delta w_{t-j}/T \\
= \;& \sum_{t=3}^{S_T+3} \Delta w_t \Delta w_{t-1}/T + \sum_{t=4}^{S_T+4} \Delta w_t \Delta w_{t-2}/T \\
&+ \sum_{t=5}^{S_T+5} \Delta w_t \Delta w_{t-3}/T + \cdots + \sum_{t=S_T+2}^{2S_T+2} \Delta w_t \Delta w_{t-S_T}/T \\
&+ \sum_{t=T-S_T+1}^{T} \Delta w_t \Delta w_{t-1}/T + \sum_{t=T-S_T+2}^{T} \Delta w_t \Delta w_{t-2}/T \\
&+ \sum_{t=T-S_T+3}^{T} \Delta w_t \Delta w_{t-3}/T + \cdots + \sum_{t=T}^{T} \Delta w_t \Delta w_{t-S_T}/T
\end{aligned}$$

$$
\begin{aligned}
&= \Delta w_3 \Delta w_2/T + \Delta w_4(\Delta w_3 + \Delta w_2)/T + \Delta w_5(\Delta w_4 + \Delta w_3 + \Delta w_2)/T \\
&\quad + \cdots + \Delta w_{S_T+2}(\Delta w_{S_T+1} + \cdots + \Delta w_2)/T \\
&\quad + \Delta w_{S_T+3}(\Delta w_{S_T+2} + \cdots + \Delta w_3)/T + \Delta w_{S_T+4}(\Delta w_{S_T+2} + \cdots + \Delta w_4)/T \\
&\quad + \Delta w_{S_T+5}(\Delta w_{S_T+2} + \cdots + \Delta w_5)/T + \cdots + \Delta w_{2S_T+2}\Delta w_{S_T+2}/T \\
&\quad + \Delta w_{T-S_T+1}\Delta w_{T-S_T}/T + \Delta w_{T-S_T+2}(\Delta w_{T-S_T+1} + \Delta w_{T-S_T})/T \\
&\quad + \Delta w_{T-S_T+3}(\Delta w_{T-S_T+2} + \Delta w_{T-S_T+1} + \Delta w_{T-S_T})/T \\
&\quad + \cdots + \Delta w_T(\Delta w_{T-1} + \cdots + \Delta w_{T-S_T})/T \\
&= \Delta w_3(w_2 - w_1)/T + \Delta w_4(w_3 - w_1)/T + \Delta w_5(w_4 - w_1)/T \\
&\quad + \cdots + \Delta w_{S_T+2}(w_{S_T+1} - w_1)/T + \Delta w_{S_T+3}(w_{S_T+2} - w_2)/T \\
&\quad + \Delta w_{S_T+4}(w_{S_T+2} - w_3)/T + \Delta w_{S_T+5}(w_{S_T+2} - w_4)/T \\
&\quad + \cdots + \Delta w_{2S_T+2}(w_{S_T+2} - w_{S_T+1})/T \\
&\quad + \Delta w_{T-S_T+1}(w_{T-S_T} - w_{T-S_T-1})/T + \Delta w_{T-S_T+2}(w_{T-S_T+1} - w_{T-S_T-1})/T \\
&\quad + \Delta w_{T-S_T+3}(w_{T-S_T+2} - w_{T-S_T-1})/T + \cdots + \Delta w_T(\Delta w_{T-1} - w_{T-S_T-1})/T \\
&= O_p(S_T T^{-1}),
\end{aligned}
$$

leading to

$$
\begin{aligned}
&\sum_{t=2}^{T} \Delta w_t^2/T \\
&= \sum_{t=S_T+3}^{T-S_T-1} \Delta w_t^2/T + \sum_{t=2}^{S_T+2} \Delta w_t^2/T + \sum_{t=T-S_T}^{T} \Delta w_t^2/T \\
&= \sum_{t=S_T+3}^{T-S_T-1} \Delta w_t^2/T + O_p(S_T T^{-1}) \\
&= \sum_{t=S_T+3}^{T-S_T-1} \Delta w_t w_t/T - \sum_{t=S_T+3}^{T-S_T-1} \Delta w_t w_{t-1}/T + O_p(S_T T^{-1}),
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{j=1}^{S_T} \sum_{t=j+2}^{T} \Delta w_t \Delta w_{t-j}/T \\
&= \sum_{j=1}^{S_T} \sum_{t=S_T+3}^{T-S_T-1} \Delta w_t \Delta w_{t-j}/T + \sum_{j=1}^{S_T} \sum_{t=j+2}^{S_T+2} \Delta w_t \Delta w_{t-j}/T \\
&\quad + \sum_{j=1}^{S_T} \sum_{t=T-S_T}^{T} \Delta w_t \Delta w_{t-j}/T \\
&= \sum_{j=1}^{S_T} \sum_{t=S_T+3}^{T-S_T-1} \Delta w_t \Delta w_{t-j}/T + O_p(S_T T^{-1}) \\
&= \sum_{t=S_T+3}^{T-S_T-1} \Delta w_t \left(\sum_{j=1}^{S_T} \Delta w_{t-j} \right) /T + O_p(S_T T^{-1}) \\
&= \sum_{t=S_T+3}^{T-S_T-1} \Delta w_t (w_{t-1} - w_{t-S_T-1})/T + O_p(S_T T^{-1}),
\end{aligned}
$$

$$
\begin{aligned}
& \sum_{j=1}^{S_T} \sum_{t=j+2}^{T} \Delta w_t \Delta w_{t-j}/T \\
= \;& \sum_{j=1}^{S_T} \sum_{t=S_T+j+3}^{T-S_T-1+j} \Delta w_t \Delta w_{t-j}/T \,+\, \sum_{j=1}^{S_T} \sum_{t=j+2}^{S_T+j+2} \Delta w_t \Delta w_{t-j}/T \\
& + \sum_{j=1}^{S_T} \sum_{t=T-S_T+j+2}^{T} \Delta w_t \Delta w_{t-j}/T \\
= \;& \sum_{j=1}^{S_T} \sum_{t=S_T+j+3}^{T-S_T-1+j} \Delta w_t \Delta w_{t-j}/T \,+\, O_p(S_T T^{-1}) \\
= \;& \sum_{j=1}^{S_T} \sum_{t=S_T+j+3}^{T-S_T-1+j} (w_t \,-\, w_{t-1}) \Delta w_{t-j}/T \,+\, O_p(S_T T^{-1}) \\
= \;& \sum_{j=1}^{S_T} \sum_{t=S_T+3}^{T-S_T-1} \Delta w_t (w_{t+j} \,-\, w_{t+j-1})/T \,+\, O_p(S_T T^{-1}) \\
= \;& \sum_{t=S_T+3}^{T-S_T-1} \Delta w_t (w_{t+S_T} \,-\, w_t)/T \,+\, O_p(S_T T^{-1}).
\end{aligned}
$$

From these results we have

$$
\begin{aligned}
& \sum_{t=2}^{T} \Delta w_t^2/T \,+\, 2 \sum_{j=1}^{S_T} \sum_{t=j+2}^{T} \Delta w_t \Delta w_{t-j}/T \\
= \;& \sum_{t=S_T+3}^{T-S_T-1} \Delta w_t w_t/T \,-\, \sum_{t=S_T+3}^{T-S_T-1} \Delta w_t w_{t-1}/T \,+\, O_p(S_T T^{-1}) \\
& + \sum_{t=S_T+3}^{T-S_T-1} \Delta w_t (w_{t-1} \,-\, w_{t-S_T-1})/T \,+\, O_p(S_T T^{-1}) \\
& + \sum_{t=S_T+3}^{T-S_T-1} \Delta w_t (w_{t+S_T} \,-\, w_t)/T \,+\, O_p(S_T T^{-1}) \\
= \;& \sum_{t=S_T+3}^{T-S_T-1} \Delta w_t (w_{t+S_T} \,-\, w_{t-S_T-1})/T \,+\, O_p(S_T T^{-1}) \\
= \;& E\, \Delta w_t (w_{t+S_T} \,-\, w_{t-S_T-1}) \,+\, O_p(T^{-1/2}) \,+\, O_p(S_T T^{-1}) \\
= \;& o(S_T^{-2}) \,+\, O_p(T^{-1/2}) \,+\, O_p(S_T T^{-1}) \\
= \;& O_p(T^{-1/2}),
\end{aligned}
$$

i.e.,

$$
\sum_{t=2}^{T} \Delta w_t^2/T \,+\, 2 \sum_{j=1}^{S_T} \sum_{t=j+2}^{T} \Delta w_t \Delta w_{t-j}/T = O_p(T^{-1/2}). \qquad (A.39)
$$

Putting (A.29) to (A.31), (A.33), (A.35) and (A.37) to (A.39) together, it is easily established that

$$
\begin{aligned}
& \sum_{t=2}^{T} \Delta \tilde{y}_{t;i} \tilde{y}_{t-1;i}/T \,-\, \tilde{P}_{i;T;S_T} \,-\, \tilde{Q}_{i;T} \\
= \;& -(1 \,-\, \rho)^2 \{R_y(0) \,-\, R_y(1)\} \,+\, O_p(T^{-1/2}),
\end{aligned}
$$

$$
\begin{aligned}
& \sum_{t=2}^{T} \Delta \tilde{y}_{t;i}^2 / T \;+\; 2\, \tilde{P}_{i;T:S_T} \;+\; 2\, \tilde{Q}_{i;T} \\
= \;& (1 \,-\, \rho)^2 \left\{ R_y(0) \,-\, R_y(1) \right\} \;+\; O_p(T^{-1/2}),
\end{aligned}
$$

therefore,

$$
\begin{aligned}
& T^{1/2} \left(\sum_{t=2}^{T} \Delta \tilde{y}_{t;i}^2 / T \;+\; 2\, \tilde{P}_{i;T:S_T} \;+\; 2\, \tilde{Q}_{i;T} \right)^{1/2} \left(\sum_{t=2}^{T} \tilde{y}_{t-1;i}^2 / T^2 \right)^{1/2} \\
= \;& \sqrt{2}\, (1 \,-\, \rho) \left\{ R_y(0) \,-\, R_y(1) \right\}^{1/2} \left\{ R_y(0) \right\}^{1/2} \;+\; O_p(T^{-1/2}).
\end{aligned}
$$

In view of the forms of $\hat{M}_{i;T;S_T}$ and recalling that $\rho_y = R_y(1)/R_y(0)$, it is easy to establish the results required for (ii).