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In the efficiency measurement of the production process that involves byproduction of undesirable outputs, those conventional methodologies that treat undesirable factors as ad hoc inputs do not correctly reflect the true production process. Färe et. al (1989) used the inversed efficiency multiplier for undesirable outputs, modifying the BCC model into a non-linear programming problem at a sacrifice of linearity.[1]

Instead, Seiford and Zhu (2002) applies a linear, monotone, decreasing transformation to undesirable outputs by "reversing" them. Their approach thus preserves the linearity and convexity, and is readily interpreted as the standard output-oriented DEA. Novelty of their approach is therefore that it treats undesirable outputs differently from desirable outputs or inputs, and still direct application of linear programming is possible as in the ordinary DEA, just like the BCC model. Obtained efficiency scores, however, depend on the choice of potential ceiling amount of undesirable outputs, or *where* the undesirable outputs are reversed. The translation as a result retains only the classification invariance, and it is not solution invariant or even ordering invariant.

As another strong alternative, Chung et. al (1997), followed by Färe and Grosskopf (2004) and others, proposed the directional distance function (DDF) approach. DDF measures the efficiency in the direction that the desirable outputs are increased and undesirable outputs are decreased. Literature is in search of identity between these two stream of methodological evolutions; Seiford and Zhu (2005) discuss briefly about the "link" between the DDF approach and their reversing method. However, it is still left to this short communication to identify the exact conditions under which these two attractive methods become identical.

Directional Distance Function Approach

DDF approach measures the efficiency of the ith decision making unit (DMU) say θ_i as

$$\theta_i = 1 - \beta_i \frac{\|g\|}{\|(y_i, u_i)\|} \tag{1}$$

[1]See Banker et. al (1984) for the BCC model.

or, the inefficiency β_i as

$$\begin{aligned} \beta_i &= \max \beta \\ s.t. \quad & \left(y_i + \beta g^y, u_i - \beta g^u\right) \in P\left(x_i\right), \end{aligned} \tag{2}$$

where $g = (g^y, -g^u)$ is the direction vector, y is a K-vector of desirable outputs, u is an M-vector of undesirable outputs, x is an N-vector of inputs, and P is a production set such that

$$\begin{aligned} P\left(x\right) &= \{(y,u)| \sum_{j=1}^{J} z_j y_{jk} \geqq y_k, \quad k = 1, \ldots, K, \\ & \sum_{j=1}^{J} z_j u_{jm} = u_m, \qquad m = 1, \ldots, M, \\ & \sum_{j=1}^{J} z_j x_{jn} \leqq x_n, \qquad n = 1, \ldots, N, \\ & z_k \geqq 0, \qquad k = 1, \ldots, K\} \end{aligned}$$

with subscript $j \in \{1, \ldots, J\}$ representing the jth DMU. The equality constraint in the second line implies the weak disposability of undesirable outputs.

The "Reversing" Method

Seiford and Zhu (2002) propose the following treatment of undesirable outputs, with which the inefficiency score for the ith DMU, say $\bar{\beta}_i$ is measured through the conventional DEA framework as follows:

$$\begin{aligned} \bar{\beta}_i &= \max \beta \\ s.t. \quad & \sum_{j=1}^{J} z_j y_{jk} \geqslant (1+\beta)\, y_{ik}, \qquad k = 1, \ldots, K, \\ & \sum_{j=1}^{J} z_j \bar{u}_{jm} \geqslant (1+\beta)\, \bar{u}_{im}, \qquad m = 1, \ldots, M, \\ & \sum_{j=1}^{J} z_j x_{jn} \leqslant x_{in}, \qquad n = 1, \ldots, N, \\ & \sum_{j=1}^{J} z_j = 1, \\ & z_j \geqslant 0, \qquad \forall j = 1, \ldots, J, \end{aligned} \tag{3}$$

where $\bar{u}_j = w - u_j$ for some w for all $j = 1, \ldots, J$. Seiford and Zhu sets the ceiling vector w to be at a level that is large enough so that $\bar{u}_{jm}$ is positive for any j and m. That is, w is common for all

DMUs in the data set. This arbitrariness in the choice of w is the cause of the discrepancy between their method and DDF.

The Identity Conditions

The above set up by Seiford and Zhu does not yield the identity that the literature is looking for. Instead, under the following conditions, the "reversing" method (3) becomes identical to the DDF given in (2).

First, DDF implies that the ceiling vector w in (3) above is different among all DMUs unlike what is proposed by Seiford and Zhu. Let us define w_i be the ceiling vector for the ith DMU, then for these two methods to be identical, it must be that

$$w_i = (\mathrm{I}_M + \Gamma_i^u)\, u_i$$

where Γ_i^u is an $M \times M$ diagonal matrix with the mth diagonal element being $\gamma_{im}^u = u_{im}/g_m^u$ and off-diagonal elements being all zeros. I_M is an $M \times M$ identity matrix.

Second, we linearly translate the output vectors y_j and u_j into $\bar{y}_j$ and $\bar{u}_j$ such that

$$\bar{u}_j = w_i - \Gamma_i^u u_j$$

i.e., translated undesirable outputs are again reversed after appropriate scaling, and

$$\bar{y}_j = \Gamma_i^y y_j + (\mathrm{I}_K - \Gamma_i^y)\, y_i$$

for all j where Γ_i^y is an $K \times K$ diagonal matrix with the kth diagonal element being $\gamma_{ik}^y = y_{ik}/g_k^y$ and again off-diagonal elements being all zeros, and I_K is an $K \times K$ identity matrix. Translated desirable outputs are linear combinations of the ith and jth DMUs'. Obviously we have $\bar{y}_i = y_i$ and $\bar{u}_i = u_i$.

Third, in order to capture the weak disposability that is assumed in DDF, a hypothetical DMU that we refer to as the 0th DMU say, is added to the data set. Output and input vectors of the 0th DMU, each denoted by y_0, u_0, and x_0 are set as follows:

$$\begin{pmatrix} y_0 \\ u_0 \\ x_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ x_i \end{pmatrix}$$

and we translate here again as other DMUs that $\bar{y}_0 = \Gamma_i^y y_0 + (\mathrm{I}_K - \Gamma_i^y)\, y_i = (\mathrm{I}_K - \Gamma_i^y)\, y_i$ and $\bar{u}_0 = w_i - \Gamma_i^u u_0 = w_i$.[2]

This translation of outputs and inputs does not preclude negative elements in $\bar{y}_j$ and $\bar{u}_j$. However, $\bar{y}_i$ and $\bar{u}_i$ are always positive by construction, thus measuring efficiency just for the ith DMU is still feasible. This is due to the fact that, when one interprets DDF in the ordinary DEA framework, the

[2] Note here that when $(g^y, g^u) = (y_i, u_i)$ as typically assumed in the literature, Γ_i^y and Γ_i^u are identity matrices and hence it becomes that $\bar{y}_j = y_j$ and $\bar{u}_j = w_i - u_j$ where $w_i = 2u_i$.

production frontier to which the efficiency is measured is not the same for all DMUs; that is, one production frontier is used only to estimate the efficiency of one DMU.

Using these variables in the set up by Seiford and Zhu above in (3) gives β_i, the inefficiency score of DDF via DEA as

$$\begin{aligned} \beta_i &= \max \beta \qquad (4) \\ s.t. \quad & \sum_{j=0}^{J} z_j \bar{y}_{jk} \geqslant (1+\beta)\, \bar{y}_{ik}, \qquad k = 1, \ldots, K, \\ & \sum_{j=0}^{J} z_j \bar{u}_{jm} \geqslant (1+\beta)\, \bar{u}_{im}, \qquad m = 1, \ldots, M, \\ & \sum_{j=0}^{J} z_j x_{jn} \leqslant x_{in}, \qquad n = 1, \ldots, N, \\ & \sum_{j=0}^{J} z_j = 1, \\ & z_j \geqslant 0, \qquad \forall j = 0, \ldots, J \end{aligned}$$

for $i = 1, \ldots, J$. We can then retrieve the efficiency score θ_i just as in (1).

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