

Canonical Correlation Analysis of Mongolian Teachers' Mathematical Knowledge for Teaching with a Focus on Concept Image and Concept Definition in Secondary Geometry

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Abstract

Teacher mathematical knowledge for teaching (MKT) (Ball et al, 2008) is a key to student achievement in school mathematics. In this paper, Mongolian secondary school teachers' mathematical knowledge for teaching is analyzed using canonical correlation analysis (Hotelling, 1935) focusing on mathematical theory of concept image and concept definition (Tall & Vinner, 1981) of planes shapes in secondary geometry. The term "analysis" is conceptualized as identifying interrelations among sub-domains of MKT and between MKT in concept image (MKTCI) and concept definition (MKTCD) aspects. The study selected 57 secondary school teachers in Mongolia and administered a questionnaire on MKTCI and MKTCD of plane shapes. In order to identify within and between relationships, the canonical correlation analysis is conducted. A result of the analysis indicates that there is a general positive relationship between MKTCI and MKTCD of the teachers, although some variables related to Mongolian secondary school geometry curriculum and teaching play a significant role in the relationship.

1. Introduction

Results of international and national evaluation indicated that Mongolian secondary school students' achievement is particularly low in school geometry which is "a subject with an important role in school mathematics, and development of students' spatial ability and logical reasoning skills" (French, 2004). A cause of the low achievement in geometry is identified as teachers with poor knowledge of teaching this subject (Javzmaa, 2009). It has already been documented that teachers' certain lack of knowledge of mathematics is associated with less successful teaching and lower student attainment (Ball et al, 2008; Askew, 2008). For a success in learning of geometry, teachers need to be equipped with adequate mathematical knowledge for teaching geometry, in particular, geometrical concepts such as the plane shapes. It is conceptualized that the investigating on Mongolian secondary school teachers' mathematical knowledge for teaching geometry can make a significant contribution to improving students' learning of geometry. Then, this statement is set as an aim of the research.

2. Geometry in the Mongolian School Curriculum Today

Geometry is one of the domains in school mathematics curriculum from primary through grade 12 in Mongolia. Traditionally, school geometry emphasizes the plane shape concepts and their formal definitions exercising various geometrical problems. At primary level, by the curriculum, a main concept of geometry is intended to be the plane shape focusing on a square, rectangle, quadrangle and triangle including measurements. By the geometry curriculum, students at lower primary grades recognize shapes according to how they look like using visual prototypes in surrounding environment such as doors and desks. Their thinking of the shapes is characterized at visual level via standard examples. At upper primary grades, it is expected that the level of thinking involves aspects of the descriptive level, so, students deal with some properties of the shapes by observing, measuring and drawing. However, by the curriculum and textbooks, it can be seen that students still work with the prototype, standard-orientation examples, but, no opportunity to deal with different shapes, for example in non-standard orientations. At the end of the primary school, students' all mental attributes of the shapes are characterized by these examples. At lower secondary grades, an essential concept is still the plane shapes including more examples of quadrilaterals and polygons. Students identify properties of the shapes through

geometrical constructions, and recognize a shape using its properties. Certain forms of mental images of the shapes are expected to be established at this level, so, “students’ prior experiences with the concept embody this concept image” (Vinner & Hershkowitz, 1980). Based on this image, it is also expected in the lower secondary curriculum that informal definitions for the shapes can be developed as outputs of the geometrical constructions. Indeed, if the concept image is not properly developed, it cannot be coherent with the definition, and may cause conflicts between the concept image and definition (Tall & Vinner, 1981). At upper secondary level, students are introduced formal definitions of the shape through language-based instruction, deal with necessary and sufficient conditions for the concept and apply in proofs of theorems and axioms as well as the classification of the shapes. Formal definitions for the shapes are expected to be developed based on students’ informal definitions built on prior grades, so, understanding of, for instance, equivalent definitions are crucial in the instructions.

Current practice of geometry teaching and learning the shape indicate that concept image of the plane shapes is improperly established in Mongolian children, then, their understanding cannot fully developed or advanced for the formal concept definition later on. It is recognized as a “tension” which exists between the concept image and concept definition in geometry (Bingolbali & Monaghan, 2008; Even & Tirosh, 1995; Tall & Vinner, 1981; Tirosh et al., 2011).

3. Concept Image and Concept Definition in Geometry

By Clements (2011), students often use concept images rather than definitions of concepts in their reasoning, and the concept images are adversely affected by inappropriate instruction that teachers bring. One of the strongest mathematics theory that is able to deal with the “tension” can be the concept image and concept definition (CICD) theory (Tall & Vinner, 1981). Tall and Vinner (1981) developed CICD theory; they described the term concept image as “the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes”. All mental pictures (pictorial, symbolic and others), all mental attributes (conscious, or unconscious) and associated processes are included in the concept image (Semadeni, 2008). The concept definition refers to “a form of words used to specify that concept” (Tirosh et al., 2011). A formal concept definition is a definition accepted by the mathematical community whereas personal concept definition may be formed by the individual and change with time and circumstance. The portion of the concept image which is activated at a particular time is the evoked concept image. At different times, seemingly conflicting images may be evoked. There is a part of the concept image or concept definition which may conflict with another part of the concept image or concept definition; it is a potential conflict factor. Such factors need never be evoked in circumstances which cause actual cognitive conflicts but if they are so evoked the factors concerned will be then called cognitive conflict factors. CICD has weathered the years well and one of the most cited theory in the literature (Bingolbali & Monaghan, 2008). By Tirosh et al (2011), knowing aspects of CICD theory enlightens teachers to work with the concept image and definition in geometry effectively, and deal with the difficulties that may raise during students’ successful learning of geometry.

4. Literature review and research aim

4.1 Teacher mathematical knowledge for teaching

An important contribution to teacher subject matter knowledge has been Shulman’s concept of “pedagogical content knowledge” (PCK) (Ball, 2002, p.4). This is the special amalgam of content and pedagogy that is uniquely the province of teachers, their own special form of professional understanding (Shulman, 1986). PCK has received the most attention in the mathematics education research and professional development literature in recent years (Chick et al., 2006; Hill, Rowan & Ball, 2005). Based on Shulman’s conception of PCK, Ball et al (2008) developed “mathematical knowledge for teaching” (MKT) framework, as shown in **Figure 1**, and defined it as the particular form of mathematical knowledge which is useful for and usable in the work that teachers do as they teach mathematics to their students (Ball & Bass, 2000). The framework consists of two domains that demonstrate both “subject matter knowledge” (SMK) and PCK. In **Figure 1**, the left side of the oval, labeled SMK that is pure mathematical content knowledge that is necessary for teaching mathematics. It contains three strands that lie outside Shulman’s popular conceptualization of SMK. *Common content knowledge (CCK)*, roughly described as

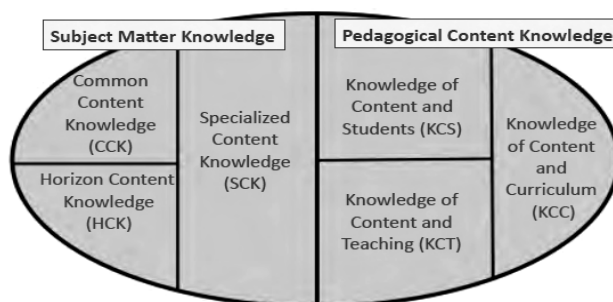


Figure 1. Mathematical Knowledge for Teaching Framework Adapted from Hill, Ball, and Schilling (2008)

knowledge that is used in the work of teaching in ways in common with how it is used in many other professions or occupations that also use mathematics. *Specialized content knowledge* (SCK) is mathematical knowledge that allows teachers to engage in particular teaching including “to accurately and correctly choose mathematical ideas, provide mathematical explanations for common rules and procedures which statement best explains, and examine and understand unusual solution methods and ideas of problems” (Hill, Ball, & Schilling, 2008). Third strand of SMK is *horizon content knowledge* (HCK); it is interpreted as “a kind of mathematical ‘peripheral vision’ needed in teaching, a view of the larger mathematical landscape that teaching requires” (Hill, Ball, & Schilling, 2008). It is an awareness of how mathematical topics are related to over the span of mathematics included in the curriculum (Ball et al, 2008). The right side of the oval represents strands associated with Shulman’s proposed PCK, it contains *knowledge of content and students* (KCS), *knowledge of content and teaching* (KCT), and *knowledge of content and curriculum* (KCC) (Hill, Ball, & Schilling, 2008). KCS is knowledge related to students and content which centrally includes students’ common misconceptions, difficulties about particular mathematics content, emerging incomplete ideas as expressed in the ways that students use language and students’ anticipation of what students likely to think during a particular mathematics content (Ball et al, 2008). KCT brings knowing about teaching and knowing about mathematics together (Hill, Ball, & Schilling, 2008). It is teachers’ knowledge of teaching particular content that includes how to select appropriate representations to illustrate the content, evaluate instructional advantages and disadvantages of the chosen representations, sequence particular content, choose examples and non-examples for a particular content (Ball et al, 2008). KCC is the curricular knowledge needed for teaching. It includes knowledge of articulating the strands of the curriculum, knowing students’ prior and after knowledge in the curriculum, and determining learning goals about a particular topic for a particular activity (Ball, 2014). The framework has been appreciated as it provides a clearer sense of categories of content knowledge for teaching that might inform the design of support materials for teachers as well as teacher education and professional development (Ball et al, 2008).

However, the framework has some weaknesses or missing points. *Firstly*, researchers who used the framework interpreted the sub-domains of the MKT in same and different ways. The most common features of the interpretations stand on CCK, SCK, KCT and KCS, yet, the most controversies are in HCK. It may have two reasons. One reason is due to its vague interpretation. Another is related to how it linked to what Shulman initially conceptualized. Shulman (1981) introduced two types of curricular knowledge - vertical and lateral curriculum. HCK refers vertical curriculum knowledge which is familiarity with the topics and issues during the preceding and later years (Fernandez & Figueiras, 2014). This interpretation overlaps one of the ideas in KCC which interpreted by Ball (2014) - “knowing students’ prior and after knowledge in the curriculum”. Except, researchers usually do not consider KCC. This is, probably, a fact that curriculum situation is varied in different education contexts. *Second*, MKT framework does not acknowledge interrelations among the sub-domains. Teacher subject knowledge for teaching is complex and interrelated to each other in nature. For example, teachers’ SCK is strongly related to KCS and KCT those are foundations for recognizing students’ misconceptions and identifying advantages and disadvantages of the representation. *Third*, for this framework, most commonly used areas of lower secondary mathematics are number and algebra, but, few on geometry; and the range of lower secondary mathematical topics is far more diverse compared to primary mathematics. Somavajulu (2012) remarked that research on teacher MKT secondary geometry was still lacking in geometry areas. In addition, the framework has not been explicitly combined with the more general mathematics theory. By Tirosh et al (2011), the mathematics education research community should consider how to combine theories of mathematics in order to provide a more comprehensive theory for investigating ... teachers’ MKT. The above weaknesses or missing points lead the research aim of the present study to complement the MKT framework by taking into account of interrelations among the sub-domains, context of mathematics curriculum, and mathematical theory that has a potential to deal with problematic situation in geometry learning through enhancing teachers’ MKT.

4.2 Concept image and concept definition (CICD) and MKT

In the context of teacher knowledge, Tirosh et al (2011) investigated CICD theory in order to describe how combining theories of teachers’ MKT with theory of CICD may be used as a tool to build kindergarten teachers’ MKT referring on triangle concept. In this study, Tirosh et al (2011) focused on CCK, SCK, KCS and KCT sub-domains of the MKT with combination of CICD of triangle; and interpreted each of them as cells (**Table 1**).

Table 1. Concept image and concept definition framework

	Domains of teacher knowledge			
Domains of mathematical thinking	CCK	SCK	KCS	KCT
Concept image	Cell 1	Cell 2	Cell 3	Cell 4
Concept definition	Cell 5	Cell 6	Cell 7	Cell 8

Resource from (Tirosh et al., 2011)

Tirosh et al (2011) interpreted that Cell 1 is teachers' common knowledge of concept image includes the drawing examples and non-examples of triangles. Cell 2 is teachers' specialized content knowledge that includes a rich concept image of triangles incorporating different triangles with different orientations. It also includes a broad image of non-examples of triangles beyond circles and squares. Cell 3 is teachers' knowledge related to students and concept images including a prototypical triangle and that young children may not identify as a triangle. Cell 4 is teachers' knowledge related to teaching and concept images that includes knowing which examples and non-examples to present to students which will broaden their concept image of a triangle considering triangles with different orientations. Cell 5 is teachers' common knowledge that a triangle may be defined as a polygon. Cell 6 is teachers' specialized knowledge of a concept definition. It includes knowing three different definitions for the triangle. Cell 7 is teachers' knowledge related to students and concept definitions. It is the knowing that a minimalist definition may not be appropriate for young students at the first or second van Hiele level. Cell 8 is teachers' knowledge related to teaching and concept definitions. It includes knowing which examples and non-examples of a triangle to present to encourage children's use of concept definitions and promote their advancement along the van Hiele levels (p.235).

Cell-based above interpretations are mainly focused on examples and non-examples of a triangle, and structure of minimalist definition for the triangle concept. In the concept definition, it emphasizes how attributes of the shape are reflected in necessary and sufficient condition of the concept definition. Their study was very qualitative and only used responses of interviews. They found out that in order to strengthen teachers' MKT, a tool with combination of MKT and CICD could be effectively used. However, in the combination, the mathematical context used to illustrate of this tool and the grade-level at which teachers taught must be taken into consideration (Tirosh et al, 2011). This study concludes and recommends that:

“Regarding mathematical context, geometry is a natural venue for discussing images and definitions. In addition, the combination of the theories has potential to be used for investigating and promoting teachers' MKT in both elementary and secondary grades, but not high grades. Knowing aspects of CICD theory enlightens teachers to work with the concept image and concept definition in geometry effectively and deal with the difficulties that may raise during the learning (p 241).

However, this study did not investigate how teachers' MKT in concept image may relate and support MKT in concept definition of the triangle concept.

5. Research methodology

5.1 Research questions and framework

Based on the problem and findings from previous research, aim of this research is to identify Mongolian secondary school teachers' mathematical knowledge for teaching focusing on concept image and concept definition for the plane shape in secondary geometry. The aim of the research leads to the following questions, and demands to be answered:

1. What is Mongolian secondary teachers' mathematical knowledge for teaching concept image of the plane shapes?
2. What is Mongolian secondary teachers' mathematical knowledge for teaching concept definition of the plane shapes?
3. What are relationships among sub-domains of teachers' MKT and mutual relations between the concept image and concept definition in MKT referring to the plane shapes in lower secondary geometry?

In accordance to the above research questions, a research framework is developed considering the points: (1) this research focuses on the plane shapes of lower secondary geometry in Mongolia. (2) Teachers' MKT in this research does not include HCK. Because HCK refers to “teachers' familiarity with the topics and issues during the preceding and later years” (Fernandez & Figueiras, 2014). Indeed, by Hill, Ball and Shilling (2008), “KCC is also some extent deals with teachers' knowing of how a particular mathematics content are taught preceding years and will be taught later years”. From these interpretations, for circumstance of this research, overlapping ideas of these two sub-domains can be investigated under one of them, KCC, when teachers are asked to explain why a particular content is appropriate with particular grade students. (3) The research includes KCC sub-domain because the research context is specific to Mongolia where national level geometry curriculum exists and emphasize in geometry teaching and learning at schools. (4) For this research, concept image refers all mental images of the plane shapes and related concepts. “Formal concept definition” stands for a definition reflected in secondary school geometry curriculum and students' textbook of Mongolia. (5) Sub-domains of MKT of CICD of the planes shapes do not directly adapt Tirosh et al (2011) interpretation, because of facts that their research is contextualized in kindergarten geometry, and single triangle concept is stressed in the content. However, some essential ideas from Tirosh et al (2011) are applied in the development of CICD in this research. Two important ideas in their interpretation are inclusion of examples and non-examples of the shapes for promoting not only concept image but also concept definitions for the shapes. By Tall (1988), in order to encourage student concept image, they must be given richer experiences so that they are able to form a more coherent concept; it involves a balance between the variety of

examples and non-examples necessary to gain coherent images and the complexity which may increase the cognitive demand. Another idea is that what attributes of the shape must be reflected in necessary and sufficient condition of the concept definition. Zazkis and Leikin (2008) consider that definitions of mathematical concepts, the underlying structures of the definitions and the process of defining are fundamental components. In addition, this research focuses on not only the triangle but also other shapes with supporting concepts such as angle, symmetry. It means that levels of teacher' MKT, geometry content and students learning are broader than those in kindergarten. Therefore, the research interprets MKT sub-domains in CICD as it is shown in **Table 2**.

Table 2. Interpretative definitions for sub-domains of MKT in CICD

MKT	Concept Image (CI)	Concept Definition (CD)
CCK	CCKCI: Common content knowledge of concept images of quadrilateral when symmetry is involved	CCKCD: Common content knowledge of concept definition that rectangle is formally defined as a parallelogram.
SCK	SCKCI: Specialized content knowledge of concept images polygons with particular symmetrical properties that is not commonly discussed and knowing if the given statements about the polygon images are mathematically true or not. It also includes knowledge of critical attributes of the polygon.	SCKCD: Specialized knowledge of choosing mathematically correct formal concept definition of rectangle and recognizing what is involved (excluded or included classifications of shapes) in the various definitions. It also includes knowledge of structure of (necessary and sufficient condition) a formal definition of the shape concept.
KCS	KCSCI: Knowledge of students' common misconception related to quadrilateral concept images. It also includes causes of students' misconception on inner angles of quadrilaterals.	KCSCD: Knowledge about content and students of concept definition. It includes knowing what is confusing in their ideas related to the formal definition of inscribed angles and students' incomplete interpretation of this definition.
KCT	KCTCI: Knowledge related to teaching and content - triangle concept images - that includes knowing advantages and disadvantages the given representations. The given representations consist of examples and non-examples of a triangle that highlight critical attributes of the shape. It also includes choice of the representations to teach the triangle concept image.	KCTCD: Knowledge related to content and teaching of concept definition for a triangle. It includes selecting the most appropriate representation to illustrate triangle concept definition and reasons beyond the chosen representation. It also includes knowledge of how to use examples and non-examples in the representation to define the triangle concept.
KCC	KCCCI: Knowledge of curriculum and content of concept image that what grade students should be taught the symmetrical property of triangle through geometrical construction and what learning goals can be set for this activity of construction.	KCCCD: Knowledge of curriculum and formal concept definition of symmetry. It includes knowing at what grade level students are typically taught the formal definition of symmetry and students' familiarity (previous and after knowledge related to definition) with the definitions.

In **Table 2**, the interpretation of the sub-domains focus on not only quadrilaterals but also symmetry and related concepts. For the concept image, critical attributes, examples and non-examples of the shapes and symmetrical properties of the shapes are emphasized. Meantime, concept definition mainly prefers necessary and sufficient condition for the definition, and how classification of the quadrilaterals reflects inclusive and exclusive definitions.

The research process and variables are illustrated in **Figure 2** as a framework. First of all, the sample teachers' MKT of concept image (furthermore, MKTCI) and MKT of concept definition (furthermore, MKTCD) are investigated, then, characteristics of them are separately identified. Relationships among sub-domains of teachers' MKT and mutual relations between MKTCI and MKTCD referring to the plane shapes in secondary geometry are investigated and analyzed using the Canonical Correlation Analysis (CCA). It is expected that the digging into the relationships among the sub-domains may have a contribution to development of MKT framework and understanding about Mongolian teachers' MKT.

5.2 Canonical correlation analysis

Canonical correlation analysis (CCA) is a general linear model (Thompson, 2000) on statistics; it has often been conceptualized as a unified approach to many univariate and multivariate parametric statistical procedures (Fan & Konold, 2010).

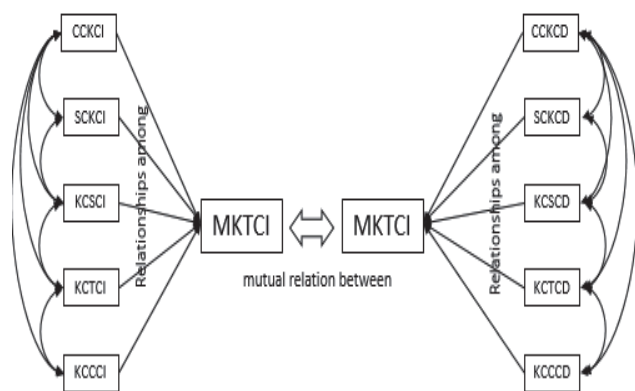


Figure 2. Research framework

In CCA, two sets of variables are examined with the goal of understanding the multivariate relational patterns between the two sets as more parsimoniously operationalized by the canonical correlation. The illustration of sets of variables are presented in **Figure 3**. One set of observed variables is considered as teacher MKTCI, and it consists of 5 sub-variables namely CCKCI, SCKCI, KCSCI, KCTCI, and KCCCI. The other set of observed variables is teacher MKTCD consisting of sub-variables of CCKCD, SCKCD, KCSCD, KCTCD and KCCCD. The single headed arrows from the observed variables to the unobserved canonical variates (for example. MKTCH1* and MKTCD1*, and MKTCI2* and MKTCD2*) denote the presumed direction of influence, and the canonical variates are derived by using canonical weights or function coefficients to linearly combine the observed variables. The curved double headed arrow linking the pair of canonical variates represent the canonical correlation. The total number of canonical correlations possible is equal to the number of observed variables in the smaller of the two sets, though not all of the canonical correlations may be statistically/practically meaningful (Fan & Konold, 2010). In Figure, R_{C1} , R_{C2} , R_{C3} , R_{C4} , and R_{C5} are possible. Last, the double headed arrows, linking the observed variables in each of the two sets, reflect that the correlations among them are taken into account in the derivation of the canonical function coefficients. The Pearson product-moment correlation coefficient between the two canonical variates is the first canonical correlation coefficient R_{C1} , which is the maximum of all possible canonical correlation coefficients that can be extracted from the variables. In a similar vein, the remaining pairs of canonical variates can be constructed and the canonical correlation coefficients from R_{C2} to R_{C5} can be obtained as shown in **Figure 3**. The construction of the remaining pairs of canonical variates are subject to the orthogonality condition: the second canonical variates are not correlated with either of the canonical variates in the first canonical variates, etc. In other words, all correlations across the variates are zero. If additional pairs of canonical variates can be extracted, as is the case when more observed variables are included in the design, all subsequent pairs of canonical variates are subject to this orthogonality condition relative to all previously extracted canonical pairs. In **Figure 3**, not all, but some examples of relationships are illustrated by lines.

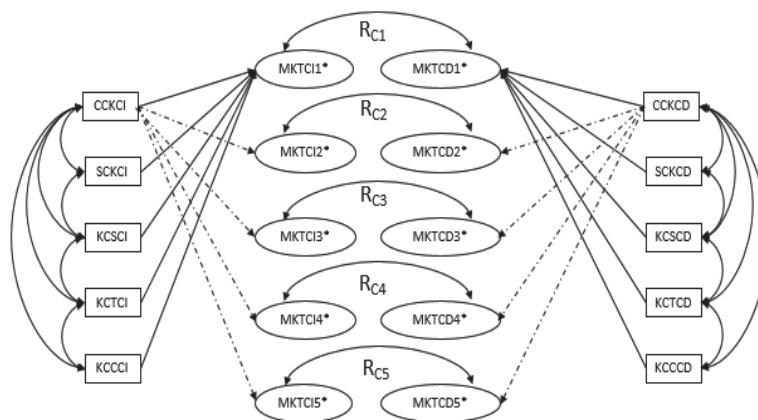


Figure 3. Path diagram of Canonical Correlation Analysis
Adapted from Fan & Konold (2010)

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5.3 Research sample

The research sampled 57 secondary mathematics teachers of Mongolia. For the sampling, at first, 8 secondary schools in Mongolia are selected. To select the schools, how to maximize opportunities to uncover variations among teachers’ MKT is mainly focused. Schools are selected from urban, semi-urban and rural areas of Mongolia. A reason of this geographical variety is that urban schools have teachers who graduated from pre-service university with better grades, and had more opportunity to attend centralized professional development trainings. While, rural schools have teachers whose situation is a quite different; and they do not have the same professional trainings opportunity as urban teachers have due to financial burden on their shoulders. All secondary mathematics teachers at the selected schools are sampled.

Table 3. Sample teacher demographics

Teachers (female)	Teacher age (age interval)	Teaching experience (year interval)	Education level (degree)
57 (78.9%)	22.8% (23-28)	21.1% (1-5)	19.3% (Diploma) 54.4% (Bachelor) 26.3% (Master)
	12.3% (29-34)	17.9% (6-11)	
	14% (35-40)	17.9% (12-17)	
	21.1% (41-46)	8.9% (18-23)	
	24.6% (47-52)	25% (24-29)	
	5.3% (53 and above)	8.9% (30 and above)	

Table 3 presents common demographic features of the sample teachers. Most (78.9%) teachers are female, aged more than 35 years old (65%). They had mathematics including geometry teaching experience in secondary schools more than 11 years (60.7%).

Most (73.7%) teachers including those with diploma graduated from teacher pre-service education institutions and earned the degree of bachelor in mathematics teaching. It must be explained that diploma teachers graduated from 5-year pre-service teacher education institutions before 1990. During this time, bachelor and master degrees were not introduced to higher education system of Mongolia, thus, the diploma is considered as an equal degree to bachelor in secondary mathematics teaching. It may be important to note that these demographic features indeed reflect typical characteristics of secondary school mathematics teachers in Mongolia.

5.4 Research instrument

In the research, in order to investigate teacher MKTCI and MKTCD of the plane shapes, a questionnaire (see APPENDIX 1) is developed as a research instrument. Each sub-domain of MKTCI and MKTCD has 2 - 4 items, in total, 18 items are developed (Table 4) in accordance to Table 2. This instrument aims to reveal Mongolian secondary school teachers MKT by referring the plane shape concept in lower secondary geometry. Questionnaire items must focus on CICD theory of the plane shapes. In addition, it needs to emphasize what knowledge should be reflected in each sub-domain of MKT. In order to develop MKT questionnaire items, two aspects are concerned. Those are the setting of an item which concerns what context should be given, and how it can be asked; and content of an item which is about geometry content with consideration to its level. Mongolian secondary mathematics teachers' familiarity with the item setting is an issue for the validity. In addition, geometry content of each item must fit with secondary geometry curriculum. The most significant concern is appropriateness to developed interpretations (Table 2) of sub-domains of MKT in CICD.

Table 4. Relevance between items and curriculum including originality of items

MKT	Item	Curriculum relevance	Resource
CCKCI	Q1	Quadrilaterals and interrelations among the quadrilaterals	(LMT, 2008)
SCKCI	Q2-Q5	Polygons with symmetrical properties	(LMT, 2008)
KCSCI	Q6	Inner and outer angles of irregular quadrilaterals	(LMT, 2008)
KCTCI	Q7-Q8	Triangles and its attributes	Self-developed referencing from teacher and student textbooks
KCCCI	Q9-Q11	Triangle and symmetry; geometrical construction through practical work	Self-developed referencing from teacher and student textbooks
CCKCD	Q12	Rectangle definition and critical attributes for the definition	Self-developed referencing from Silfverberg and Matsuo (2008)
SCKCD	Q13	Quadrilateral classification (hierarchical and partitional) and related definitions	(LMT, 2008)
KCSCD	Q14	Definition of the inscribed angle supported by an idea of triangle	(Somayajulu, 2012)
KCTCD	Q15-Q16	Triangle definition and understanding of structure of this definition (necessary and sufficient condition)	Self-developed referencing from teacher and student textbooks
KCCCD	Q17-Q18	Symmetry definition and features of the definition	Self-developed referencing from teacher and student textbooks

Table 4 presents relevance between questionnaire items and curriculum, and originality of the items. In general, all item settings are adapted from the released items of Learning Mathematics for Teaching (LMT) project conducted by Michigan University researchers in 2008. A reason for adapting item settings is perceived as that (1) all school teachers are familiar with context given in the item setting, (2) it precisely reflects MKT aspects, and (3) by secondary mathematics curriculum and teacher textbook, settings described in items are already familiar with Mongolian secondary school mathematics teachers. Ways of questioning is also experienced by the sample teachers during their study in pre-service teacher education institutions.

Except adapting the item setting, 8 items are directly used from the LMT Project released items and studies by other researchers. Because their content fits with secondary school geometry curriculum and textbooks. The remaining (10) items are developed by the researcher referencing from Silfverberg and Matsuo (2008) and school geometry curriculum including textbooks. Q8, Q10, Q11, Q14, Q16 and Q18 are open-response items, the rest of them are closed-multiple choice items. It means, there will be quantitative and qualitative data.

In order to ensure the content validity of the questionnaire, peer-review is conducted. In addition, Mark Scheme (see

APPENDIX 2) can provide more evidences for content validity. All expected responses from the sample teachers are intended in the mark scheme, thus, the fit of teacher actual responses to the expected responses in mark scheme can ensure the validity of the instrument.

Using this instrument, data was collected during December 2014 when geometry is taught to secondary students. Teachers' responses to the questionnaire items provided quantitative and qualitative data on teacher MKT in CICD of plane shapes. Data analysis was done using SPSS 22.0. In order to apply quantitative method in the data analysis, at first, all qualitative data are translated into quantitative data using ideas of the microanalysis technique adapted from the grounded theory developed by Strauss and Corbin (1998). The microanalysing technique is based on an inductive method to deal with the qualitative data, and it applies a combination of open and axial codings. This research applies to the open coding to identify conceptual categories of the data using data to data comparison. After the open coding took place, the axial coding is utilized for establishing categories of the data organizing the properties into conceptual ones. The clusters enable to give marks according to the mark scheme. For example, Q18 is open response item. One of the teachers responded that "*Definition A*" was the most appropriate for grade 7. How open and axial coding are done for this response is illustrated in **Figure 4**.

Figure 4. Q18 item and teacher's response

Q18. Why do you think your selected definition is appropriate with grade 7?

Definition A: *If line "a" crosses through the midpoint of AB segment; and this line is perpendicular with the segment, points A and B will be the symmetrical along the line "a". Line "a" is called as a mirror line of the symmetry.*

Teacher's actual response: *Because, I know that this topic is intended in the grade 7 curriculum. Also, the given definition is a basic definition for the symmetry.*

By **Figure 4**, teacher's actual response is analyzed line by line, and labeled ideas in the response. This is a simple case to code the response openly. Here, three clear concepts mentioned, thus, categories are created as "*curriculum intention*", "*level of the concept definition*", and "*definition for the symmetry*". These 3 categories are compared to other responses and analyzed taking into consideration of aspects in research framework. In order to conduct axial coding, 3 categories need to be linked categories at the level of properties. Obviously, "*level of the concept definition*", and "*definition for the symmetry*" categories need to be analyzed to identify how they are linked. In context of secondary school geometry, a level of concept definition is related to a level of content of a concept that reflected in a definition. These 2 categories are emerged under "*a level of the symmetry content given in the definition*". Now, there are 2 clusters as "*grade 7 curriculum intention of the symmetry*" and "*a level of the symmetry content given in the definition*". By the mark scheme, these clusters are given 1 mark respectively, and this response awards, in total, 2 marks (see APPENDIX 2). The quantification of other open item responses are taken place the same as Q18 item.

6. Results and discussions

In order to identify sample teachers' MKTCI and MKTCD, descriptive statistics are calculated. General behavior of the data is described in **Table 5** as of means, standard deviation, and normality check using Shapiro-Wilk. The normality check provides opportunity to apply the appropriate statistical methods in data analysis. The reliability of the questionnaire is estimated as Cronbach alpha coefficient. In **Table 5**, CI stands for the confidence interval.

Table 5. Descriptive statistics of the data

Min	Max	Mean	Standard deviation	Shapiro-Wilk Test			Reliability
				Statistics	df	Sig.	
14	31	21.79 CI (20.53; 23.05)	4.337 CI (20.6; 22.9)	.969	56	.234	.603

The descriptive statistics in **Table 5** summarizes the data behavior. It is evidenced by Shapiro-Wilk statistics (.969 at p value of .234) that the test is normally distributed. The reliability of the test is estimated by Cronbach coefficient at .603 which indicates the moderate reliability, however, it can be interpreted by less sample size. In general, for the research context, this level of reliability can be considered as acceptable. By the mark scheme, the full mark is calculated as 51. All teachers are marked between 14 to 31, and no one did approach the highest mark. Teachers' average mark is estimated as 21.79 marks.

6.1 Teachers' MKTCI

Teacher responses to MKTCI items are marked and its means are calculated item by item (**Table 6**). In order to interpret the results in **Table 6**, means of the multiple items under the same sub-domain are averaged; and higher value means indicate more correct or better knowledge for a particular sub-domain. For example, to interpret SCKCI sub-domain result, means of Q2 through Q5 are averaged, and estimated average of the means is interpreted.

Table 6. Expected maximum marks and descriptive statistics of items

MKT	CCKCI	SCKCI				KCSCI	KCTCI		KCCCI		
Item	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	Q11
Expected max mark	2	1	1	1	1	5	3	5	2	4	4
Mean	1.1	.33	.79	.88	.27	3.98	2.15	2.8	1.27	3.47	2.9
SD	.39	.48	.41	.33	.45	1.1	.97	0.8	.84	0.57	0.59

By **Table 6**, in overall, means of teachers' marks are higher than others in Q6, Q7, Q8, Q9, Q10, and Q11 items. Higher means indicate better knowledge of the sub-domains of MKTCI. These items represent KCSCI, KCTCI and KCCCI as it described in **Table 6**. It means that the sample teachers rely on mainly these three sub-domains of MKTCI to teach the concept image of the shape and related properties, however, they lack of CCKCI and SCKCI. Q6 represents KCSCI; and more than half (53.1%) of teachers know the most reasonable appraisal for students' common misconception related to the image of quadrilaterals and meaning of its interior angles. Meantime, for KCTCI, half (51.1%) of the teachers selected the most appropriate representation that only includes prototype examples and non-examples of a triangle for teaching this concept. Following Q7 item, open question - Q8 - is asked to reveal reasons of why they selected a particular representation to teach the triangle concept. The most common response (41.8%) to Q8 is that the representation includes examples and non-examples of triangles that can promote students' understanding of critical attributes of a triangle. This response conveys that in order to select the representation, sample teachers focus on critical attributes through examples and non-examples of a triangle. In the sub-domain of KCCCI, 61.1% of the sample teachers know that what grade level students should learn the content in item Q9. Following Q10 and Q11 reveal teachers' knowing on why this content should be taught to this grade. By responses to Q10 and Q11, most of them do not explain why this practical work is intended in a particular grade curriculum, and how the learning objectives should be. However, as for Q10, teachers mentioned the intention of the geometry curriculum. It seems that teachers know what geometry content is exactly intended in the curriculum, but, they do not know why a particular content and tasks are there.

However, means of teachers' marks are lower in CCKCI and SCKCI items. It can be explained that those teachers do not have sufficient mathematical knowledge that is supposed to be developed from their school years through the teaching. This is due to a reason that high school graduates, who are not as capable in mathematics as others, choose teacher universities in Mongolia. Those more capable in mathematics tend to pursue higher education in economics, engineering and law.

6.2 Teacher MKTCD

Teachers' MKTCD is analyzed the same as MKTCI. Descriptive statistics of the items are estimated (**Table 7**). Interpretation of results of **Table 7** applies the same procedure as **Table 6**.

Table 7. Expected maximum marks and descriptive statistics of items

MKT	CCKCD	SCKCD	KCSCD	KCTCD		KCCCD	
Item	Q12	Q13	Q14	Q15	Q16	Q17	Q18
Expected max marks	4	3	2	4	2	2	5
Mean	3	1.38	.27	2.56	1.39	3.23	3.44
SD	1.14	1.12	.49	.96	0.59	.83	0.53

Table 7 presents trends of marks in teachers' MKTCD. Means of teachers' marks are higher in Q12, Q15, Q16, Q17 and Q18. It means that teachers have sound knowledge in CCKCD, KCTCD and KCCCD sub-domains. For MKTCD, teachers lack of

knowledge in SCKCD and KCSCD sub-domains. Q12 item reflects teachers' knowledge that rectangle is formally defined as a parallelogram. By secondary school mathematics textbook grade 9, a rectangle is formally defined as "a parallelogram which has one right angle". Q15 item reflects teachers' KCTCD that includes selecting the most appropriate representation to illustrate triangle concept definition and examples and non-examples for defining the concept. Most (56.1%) of the sample teachers selected a representation that consists of examples of various triangles; but do not include non-examples. Q16 reveals why they think that the selected representation is the most appropriate one. By teachers' responses to Q16, more than half of them mentioned that the selected representation included examples which exactly showed shapes comprised three points (do not lie on the same plane) and three straight lines (connects these three points). By secondary mathematics textbook grade 8, it is a necessary and sufficient condition in triangle definition. Q17 reflects teachers' KCCCD that is about knowledge of what grade students should learn a particular content. More than half (77.9%) of the teachers correctly responded to the item. However, when they are asked Q18 in which why this content should be taught to grade 7 students, they mentioned following reasons:

- At grade 7, in order to learn the coordinate system, students deal with how to find symmetrical points, and symmetry transformation, this work must be done before the coordinate system - This explains how the topic is intended in the curriculum, and how these topics are integrated;
- This concept or topic will be explained in relation to the coordinate system which explains how grade 8 textbook structures this content and how it related to other geometrical topics;
- During study of the positional relationship between points and lines in grades 6 and 7, understanding of the symmetry must be given which deals with students' understanding and knowledge and integration of the topics;
- The definition B used simple language that could be one of the characteristics of the definition;

The above description presents that the sample teachers' MKTCD greatly rely on what they learned at schools, and what curriculum and textbook intend about the shapes.

Moreover, these teachers do have lower mean marks in SCKCD. A reason of this low mean can be explained the similar to SCKCI. In addition, interestingly, these teachers did not respond well on KCSCD items. It can be superficially interpreted that student learning is not significantly emphasized in teacher preparation courses and school context. However, more research is needed in this direction.

6.3 Interrelations in MKTCI and MKTCD

Interrelations means relationships among sub-domains of MKT and mutual relations between MKTCI and MKTCD; and it is analyzed using the CCA. Before running the CCA, appropriateness of the data to the CCA must be considered. By Tabachnick and Fidell (1989), there are several conditions for use of the CCA. As for the sample size, the CCA can be applied in even in smaller sample size (Stevens, 1986), as well as, data does not have the outliers. Multivariate normality is also required but not specifically, however, variables are normally distributed. In general, there is not complication in the calculation of the CCA. Importantly, there should be a reasonable expectation that the two sets of variables are substantively related. As for this research context, MKTCI and MKTCD sets of variables are logically related, so, no restriction exists for use of the CCA. Another concern for the CCA, variables' collinearity and it needs to be assessed by the correlation matrix. It must be noted that CCA results are summarized based on recommendations by Alissa & Robin (2005). In order to estimate statistics in **Table 8**, data values under the same sub-domains are merged. For example, for KCTCI, data from Q7 and Q8 are merged.

Table 8. Summary statistics of two sets of variables (n=57)

MKTCI										
CCKCI	1.00									
SCKCI	-.201	1.00								
KCSCI	-.076	.196	1.00							
KCTCI	-.093	-.082	.137	1.00						
KCCCI	-.145	-.099	.170	.267*	1.00					
MKTCD										
CCKCD	.082	.04	.114	-.196	.152	1.00				
SCKCD	.013	-.076	.053	.090	.019	.142	1.00			
KCSCD	-.154	.031	.296*	.127	.092	.172	.015	1.00		
KCTCD	-.083	-.187	.078	.382**	.061	-.339**	.018	.060	1.00	
KCCCD	.003	-.09	.009	.259	.535**	-.100	.173	.153	.147	1.00
Std	.285	1.005	1.22	1.369	1.52	1.302	1.142	.464	1.237	1.191

*. Correlation is significant at the 0.05 level (2-tailed);

** . Correlation is significant at the 0.01 level (2-tailed)

Table 8 shows that there are some correlations among the variables, because p values for * and ** indicated results show statistically significant at the .05 level (2-tailed). Following the correlation matrix, CCA has run for MKTCI and MKTCD sets consisted of 5 sub-variables. The analysis provided 5 canonical functions with squared canonical correlations of .373, .239, .093, .017 and .000 for each successive function.

First of all, by full evaluation of the canonical model, full model is statistically significant using Wilk's λ criterion. Wilk's λ of .425, $F(25, 176.10)=1.823$, $p<.001$, which presents a possible relationship. However, the statistical significance does provide some "swinging" results for large or small sample sizes, it is important to interpret effect size alongside p values to determine the practical significance. Inclusion of the effect size also strongly recommended by American Psychology Association (APA) Task Force on Statistical Inference (1999) in the research papers. Wilk's λ represents the amount of variance not shared between the variable sets. Therefore, an overall effect of $1-.425=.575=R_c^2$ for the full model. This effect size can be considered as large, and it indicates that the full model has 57% of the variance shared between the variable sets. By the dimension reduction analysis, the full model (Function I to V) are statistically significant as well as function II to V is also statistically significant ($F(16, 147.28)=1.246$, $p<.001$). Other functions do not have statistical significances. Canonical correlations of the functions are .373, .239, .093, .017 and .000 respectively. It indicates that first two functions are considered essential (37.3% and 23.9% of shared variance) for this study. Last three functions have 9.3%, 1.7% and 0% of the remaining variance in the variable sets after the extraction of first two functions.

Table 9. CCA Results

	Function I			Function II			
	<i>Coef</i>	r_s	r_s^2 (%)	<i>Coef</i>	r_s	r_s^2 (%)	h^2
MKTCI							
CCKCI	.260	.109	1.19	.309	.297	8.82	10.01
SCKCI	.076	-.118	1.39	.404	.323	10.43	11.82
KCSCI	-.198	-.026	0.07	-.327	-.347	12.04	12.11
KCTCI	.197	.236	5.57	-.808	-.859	73.79	<u>79.36</u>
KCCCI	1.023	<u>.949</u>	90.06	.206	-.150	2.25	<u>92.31</u>
<i>Adequacy</i>			19.66			21.47	
<i>Redundancy</i>			2.75			1.29	
R_c		.373	.14		.239	.06	
MKTCD							
<i>Adequacy</i>			17.16			24.69	
<i>Redundancy</i>			2.4			1.48	
CCKCD	.438	.251	6.3	.248	.397	15.76	22.06
SCKCD	-.218	.015	0.02	-.239	-.230	5.29	5.31
KCSCD	-.227	0.000	0	-.429	-.441	19.45	19.45
KCTCD	.049	.030	0.09	-.746	-.866	74.99	<u>75.08</u>
KCCCD	1.001	<u>.891</u>	79.38	-.039	-.282	7.95	<u>87.33</u>

Table 9 provides the standardized canonical function coefficients (*Coef*), structure coefficients (r_s), squared structure coefficients (r_s^2) and the communalities (h^2) for the canonical functions I and II. In Table 11, structured coefficients above .45 are underlined, as well as communalities above 45% are underlined to show the most useful variable in the model. As for the function I, KCCCI variable has more contribution to the MKTCI variable, in other words it shares a large amount of variance with MKTCI variate. This result is supported by the squared structure coefficients. KCCCI variable has also larger canonical function coefficient. As well, SCKCI, KCSCI and KCTCI variables have low function and structured coefficients, so, they share little with the MKTCI variate. Moreover, even though CCKCI, SCKCI, KCSCI, and KCTCI have smaller structure coefficients, they have some degrees of inverse and positive relations. CCKCI, KCTCI and KCCCI variables have the same signs, thus, they have positive relations. SCKCI and KCSCI variables have the same signs, thus, they also have positive relations, yet, these two variables inversely related to other three variables. CCKCI sub-domain has moderate function coefficients, in contrast, smaller structured coefficient. This may be resulted from the supression effect. MKTCD and KCCCD variable are the only contributor to this. The structure coefficient for KCCCD variable is positive. This variable also positively related to KCCCI variable of MKTCI. The same as MKTCI variate, by the function and structured coefficients, SCKCD, KCSCD and KCTCD variables share little with MKTCD variate. The same interpretation as CCKCI can be provided for CCKCD variable. Based on the above interpretation, first canonical function presents a general positive relationship between MKTCI and MKTCD variables through KCCCI and KCCCD variables those play more important role in defining the function I.

As for function II, the only one relevant variable to MKTCI variable is the KCTCI with structured coefficient is -0.859 . This variable is inversely related to function II. As MKTCD variable, the KCTCD is dominant variable, and it also inversely related. From the structured coefficient, KCTCI and KCTCD are positively related because they have the same signs. Their canonical coefficients are also larger in their sets. Communalities in **Table 9** also shows relationships between MKTCI and MKTCD through KCTCI, KCCCI, KCTCD and KCCCD variables.

7. Conclusions and further issues

Conclusions are drawn in accordance to the research questions. The sample Mongolian secondary teachers' MKTCI is characterized by their KCSCI, KCTCI and KCCCI. They know a reasonable appraisal for students' common misconception in concept image of quadrilateral, and related concept of inner angles. Their representation to teach triangle concept rely on examples and non-examples focusing on critical attributes of the shape. The sample teachers know what topic or content should be taught to a particular grade, however, they lack of knowledge why the particular content and tasks are intended for a particular grade. It seems that the sample teacher simply implement what is intended in the curriculum without having specialized knowledge about the geometry content and tasks. This is an indication of lack of SCKCI.

For MKTCD, the sample teachers have sound CCKCD, KCTCD and KCCCD. These sub-domains of knowledge is a reflection of what secondary geometry curriculum and textbooks tell. The sample teachers rely on what concept definition they learned while they were students and what concept definitions are intended in the curriculum and textbooks. In the representation to teach the concept definition, teachers emphasize only examples of the shapes because it clearly presents what condition is illustrated in the concept definition. They do not consider non-examples in the concept definition. They lack of SCKCD and KCSCD which include choosing mathematically correct definition, understanding how the definition is related to classification of the shapes, what is confusing and difficulty in students ideas related to the concept definition. It may be one of the alternatives to explain a tension that exists between concept image and concept definition in geometry specific to Mongolian secondary schools.

By the canonical correlation analysis, there is a positive relationship between teachers' MKTCI and MKTCD, and this relationship is principally determined through their KCC and KCT in CICC. In addition, their MKT is mainly characterized by the KCC and KCT, but not other domains of MKT. This may be due to the fact that Mongolia is one of the countries where national curriculum is developed, and its implementation is emphasized in every aspects of teaching. It also can be deduced that KCT is more related to KCC in context where national curriculum exists. Thus, structure of the sub-domains of MKT framework might be different, for example, Mongolian context. In addition, inclusion of CICC in MKT enables opportunity to investigate a tension between concept image and concept definition more systematically, and identify where a cause of the tension exists.

It will be more interesting to dig into why certain sub-domains are well performed and dominant in teachers' MKT in concept image and concept definition of plane shapes. This may be due to what pre-service teacher education curriculum focus, and how school context shapes teachers' MKT. More research is needed in this direction.

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APPENDIX 1

Questionnaire for teachers' mathematical knowledge for teaching

Q1. Ms. Tsetseg found the following problem in the textbook she was using:

What do you call all quadrilaterals whose two diagonals are both lines of symmetry?

Which of the following is the correct answer for this problem? (Select ONE answer).

- A. Squares; B. Rectangles; C. Parallelograms; D. Rhombi; E. Trapezoids

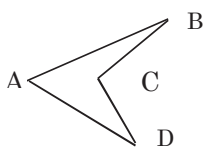
Q2-5. In a lesson on symmetry, Ms. Bayasgalan asked his class to generate polygons with at least one line of symmetry and to make observations about symmetric polygons. For each of the following claims, decide whether or not it is mathematically true. (Select TRUE, FALSE, or I AM NOT SURE for each).

	TRUE T	FALSE F	I'M NOT SURE U
Q2. If a line of symmetry cuts through a side then it makes a right angle with that side			
Q3. If a line of symmetry passes through a vertex, then it bisects the angle at that vertex			
Q4. The areas on each side of the line of symmetry are equal			
Q5. If a quadrilateral has exactly two lines of symmetry, then it must be a rectangle			

Q6. Ms. Ariunaa's students know that the sum of the angles in a triangle is 180° . She states that the sum of the angles in a quadrilateral is 360° and illustrates this with three examples - a rectangle, a parallelogram, and an irregular quadrilateral. She then asks the class to check other examples. Bayar, a student, raises his hand and says that he has a counterexample. When Ms. Ariunaa asks him to show it to the class, he draws the figure below:

Bayar argues that angle A is about a right angle, angle C is only slightly larger, and angles B and D are very small, so the sum $A+B+C+D$ cannot be the same as four right angles.

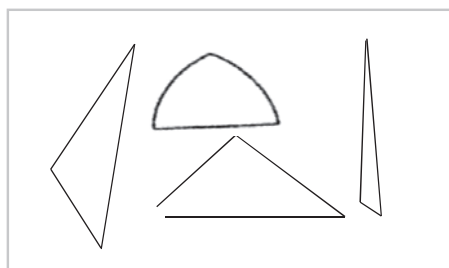
Which of the following is the most reasonable appraisal of this situation? (select ONE answer)



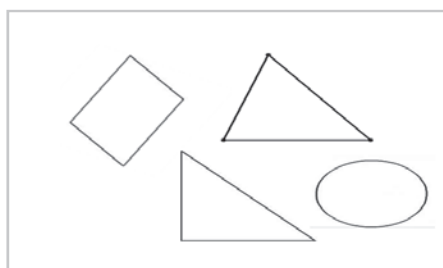
1. The angle sum formula applies only to convex quadrilaterals;
2. Bayar's argument is not convincing because it is based on inexact estimates;
3. Bayar does not seem to understand the meaning of interior angles in the case of non-convex polygons;
4. Bayar does not understand what a counterexample is;
5. The figure Bayar drew is not a quadrilaterals;

Q7-8. At professional development workshop, teachers are given assignment to develop representations to teach a triangle to students. They have developed the following two different representations (example and non-example set) on the topic.

Representation 1



Representation 2



Q7. Please answer which representation would you choose (Tick as ✓)?

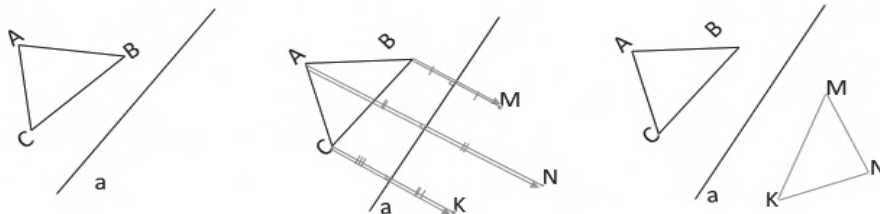
- A. Representation 1; B. Representation 2; C. Both are equally important; D. I am not sure

Q8. Please write up all advantages and disadvantages for each representation in the following Table.

	Advantage	Disadvantage
Representation 1		
Representation 2		

Q9-11. In school mathematics textbook, there are several practical works and one of them is given as follow:

Practical work



1. Draw line "a" and ABC triangle as first figure.
2. Construct symmetrical points of A, B, and C along the line "a"; and note the symmetrical points as N, M and K respectively.
3. Connect M, N and K points by line segments.
4. What if ABC triangle is folded as "a" line, do ABC and MNK triangles overlap?

Please answer what grade would you use this practical work and what would be learning goals about a triangle shape for this practical work? Please write on the following space.

Q9. I would use this practical work for grade (*Choose one of the following responses*):
 A. Grade 8; B. Grade 7; C. Grade 9; D. Grade 6; E. Any grade; F. I am not sure

Q10. Why is it appropriate with this grade

.....

Q11. If you use this practical work for teaching triangle concept, what learning objectives would you set up?

.....

Q12. Which of the following definitions could be a definition for a rectangle? (Circle up ONES that can be a definition for a rectangle)

- A. A rectangle has four right angles, four straight sides, two equal diagonals and two equal and parallel opposite sides;
- B. A rectangle is one of the geometrical shapes;
- C. A rectangle is a quadrangle, which has four right angles;
- D. A rectangle is a quadrangle, which has four right angles and which adjacent sides are different lengths;
- E. A rectangle is a quadrangle, whose opposite sides are parallel;
- F. A rectangle is a parallelogram, which has one right angle;
- G. A rectangle looks like a stretched square;

Q13. Ms. Ankhaa is preparing to teach a lesson on quadrilaterals. She sees that her textbook uses a different definition of trapezoid from the one that was in her college math method book.

Her teacher's edition defines "a trapezoid as a quadrilaterals with exactly one pair of parallel sides. (Definition I)"

Her college math methods book defines "a trapezoid as a quadrilateral with at least one pair of parallel sides. (Definition II)"

Ms. Ankhaa thinks the choice of definition might affect how one classifies shapes. Which of the following is true? (Mark the correct ONE as ✓)

-
- A. A rectangle is a trapezoid according to Definition II but not according to Definition I
 - B. A rectangle is a trapezoid according to Definition I but not according to Definition II
 - C. A rectangle is a trapezoid by both definitions
 - D. All quadrilaterals are trapezoid according to Definition II
 - E. The definitions are really the same
 - F. I am not sure
-

Q14. Mr. Bat was teaching about inscribed angles to students; and he introduced a conjecture that "inscribed angles that share a common chord have the same measure" is true. One of his students explained as follow:

Student: *I noticed the two angles take up the same amount of the circumference so their openings must be the same.*





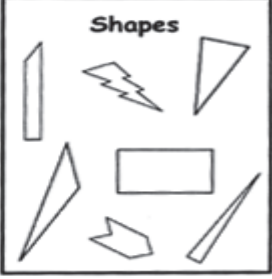
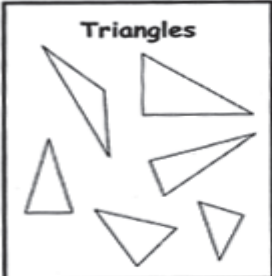



Please write up what would be a complicating idea for this student statement.

Complicating idea would be:

.....

.....

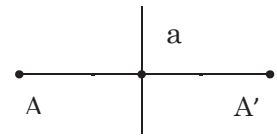
Q15. Ms. Bayarmaa wants her students to understand the structure of definition for triangle, and then improve their understanding. To help them, she wants to give them some shapes using the representations. She goes to the store to look for a visual aid to help with this lesson. Which of the following is most likely to help students improve their definition? (Circle ONE answer.) Please explain your choice.

<p>a)</p> <div style="border: 1px solid black; padding: 5px; width: fit-content;"> <p style="text-align: center;">Shapes</p> <div style="display: flex; justify-content: space-around;"> <div style="text-align: center;">  square </div> <div style="text-align: center;">  triangle </div> </div> <div style="display: flex; justify-content: space-around; margin-top: 10px;"> <div style="text-align: center;">  circle </div> <div style="text-align: center;">  rectangle </div> </div> </div>	<p>b)</p> <div style="border: 1px solid black; padding: 5px; width: fit-content;"> <p style="text-align: center;">Shapes</p>  </div>
<p>c)</p> <div style="border: 1px solid black; padding: 5px; width: fit-content;"> <p style="text-align: center;">Triangles</p>  </div>	<p>d)</p> <div style="border: 1px solid black; padding: 5px; width: fit-content;"> <p style="text-align: center;">Triangles</p> <div style="display: flex; justify-content: space-around; margin-bottom: 5px;">    </div> <p style="font-size: small;">A triangle has 3 corners, 1 on the top and 2 on the bottom.</p> <p style="font-size: small;">A triangle is a polygon.</p> <p style="font-size: small;">A clown's hat is like a triangle.</p> </div>

Q16. Please explain why you did choose the representation

Q17-18. In mathematics textbooks, two different definitions for different grades are given as follow:

Definition A: *If line “a” crosses through the midpoint of AB segment; and this line is perpendicular with the segment, points A and B will be the symmetrical along the line “a”. Line “a” is called as a mirror line of the symmetry.*



Definition B: *If the line “a” crosses through the midpoint of AA’ segment, and perpendicular with the segment, AA’ is symmetrical point along the line “a”. All points on this line are symmetrical along itself.*

Which of the definition for grade 7 and why do you think so? Please state the grade in first column and give a reason in next column of Table.

Q17. Which definition is taught to grade 7 students?

- A. Definition B B. Definition A C. I am not sure

Q18. Why do you think your selected definition is appropriate with grade 7?

.....

APPENDIX 2

Mark scheme for teacher response

MKT	Item	Expected response	Marking scheme	Min	Max	Descriptions of the scheme
CCKCI	Q1	C	If: C - 2; A, B, D - 1; E - 0	0	2	C - This response requires two kinds of knowledge: (a) <i>Properties of symmetry for quadrilaterals</i> ; (b) <i>Quadrilateral classification</i> . B, A, D - these responses requires knowledge of properties of symmetry for quadrilaterals. E - Lack of both kinds of knowledge.
SCKCI	Q2	F	If: Correct - 1; Incorrect - 0;	0	1	Correct T & F - It requires mathematically true knowledge of (a) properties of quadrilaterals, and (b) symmetry for quadrilaterals, however, incorrect T & F - deals with some lack of both knowledge, U - indicates no knowledge of the both.
	Q3	T		0	1	
	Q4	T		0	1	
	Q5	F		0	1	
KCTCD	Q15	D	If: D - 4; A - 3; C - 2; B - 1;	0	4	A - requires knowledge of prototype example and intuitive non-example of the shape (it deals with <u>some</u> attributes for the definition); B - it requires knowledge of prototype examples and non-essential attributes for the definition; C - demands knowledge of the shape examples in various orientations, but do not promote the definition (just for recognition of the shape) maybe for triangle classifications; D - examples and non-examples (intuitive and non-intuitive) for essential attributes of the shape (promote <u>meaning of the definition</u>)
	Q16	Open	If two points are mentioned - 2; one of the points - 1; no points - 0;	0	2	The explanation can reveal teacher knowledge of how the chosen representation can promote student understanding of (a) <i>main attributes of triangle for its definition</i> , and (b) <i>necessary and sufficient condition for the definition</i> as advantages of the representations;
KCSCI	Q6	3	If: 3 - 5; 1 - 4; 2 - 3; 4 - 2; 5 - 1;	0	5	The response to this question requires knowledge of (a) inner angles of the convex quadrilaterals, and (b) student misconception about the sum of inner angles. If response is 1 - it deals with only the inner angles of the convex quadrilateral; 2 - measurement of the angles; 3 - sum of inner angles of the convex quadrilateral and student misconception; 4 - contrary example for the sum of inner angles; 5 - quadrilaterals;
KCSCD	Q14	Open	If three points are mentioned - 3; two points - 2; one of the points - 1; no points - 0;	0	3	The item deals with teacher knowledge related to the definition for inscribed angles and student complicating ideas about the definition. Two points can be discussed: (a) <i>the same chord and the same amount of the circumference</i> ; and (b) <i>the opening - mathematical term</i> ; (c) <i>inscribed angle</i>
CCKCD	Q12	F	If: F - 4; A - 3; E, C, D - 2; B, G - 1;	0	4	A - requires knowledge of all attributes of the shape, however no understanding of the concept definition; E, C, D - some unnecessary attributes of the shape and not concept definition; F - essential attributes of the shape for its definition and structure of the concept definition; B, G - No attributes of the shape;
KCCCD	Q17	B	If: B - 2; A - 1; C - 0;	0	2	B - Exact knowledge of curriculum intention about the definition; A - Lack of knowledge of curriculum intention of the definition; C - No knowledge;
	Q18	Open	If five points are mentioned - 5; any four - 4; any three - 3; any two - 2; any one - 1; no - 0;	0	5	The response should deal with three points: (a) <i>symmetry concept intention in grade 7 curriculum</i> , (b) <i>content integration in different grade curriculum</i> , (c) <i>specific features of the definitions (grade 8, the definition includes a characteristic of the symmetry line itself)</i> , (d) <i>textbook content</i> ; (e) <i>student intended proficiencies in the curriculum</i> ;
KCTCI	Q7	A	If: A - 3; C - 2; B - 1; D - 0;	0	3	A - reflects teacher knowledge of essential attributes of the shape for promoting proper image of the shape, however B - lack of proper image of the triangle shape, C - indicates some knowledge of the above points; D - no consideration of the above knowledge;
	Q8	Open	If four points are mentioned - 4; any three - 3; any two - 2; any one - 1; no - 0;	0	4	Advantages and disadvantage of the representations must include (a) <i>consideration of student deeper understanding of the triangle concept image</i> , (b) <i>emphasis of attributes of the shape</i> , (c) <i>examples including prototype of a triangle & non-examples (intuitive and non-intuitive) of a triangle</i> , (d) <i>student common misconceptions</i> ;
KCCCI	Q9	B	If: A - 1; B - 2; others - 0;	0	2	A - lack of understanding of practical work application in grade 8; B - Exact correct knowledge of how geometry content is allocated in the curriculum; others - no knowledge or mis-understanding;
	Q10	Open	If four points are mentioned - 4; any three - 3; any two - 2; any one - 1; no - 0;	0	4	The reason should include: (a) <i>intention of practical work in grade 7 curriculum including textbook</i> ; (b) <i>topic (geometrical construction of symmetry) intention in grade 7 curriculum including content integration through grades 6-9</i> ; (c) <i>nature of the last question in the practical work - congruent triangles</i> ; (d) <i>particular proficiencies of grade 7 students</i> ;
	Q11	Open	If four points are mentioned - 4; any three - 3; any two - 2; any one - 1; no - 0;	0	4	According to the curriculum, the learning goals need to consider student knowledge and skills related to: (a) <i>geometrical constructions of symmetrical triangles using the tools</i> ; (b) <i>understanding of the reflection</i> ; (c) <i>properties of the congruent triangles</i> ; (d) <i>developing informal definitions for the symmetry</i> ;
SCKCD	Q13	A	If: A - 3; D - 2; B, C, E - 1; F - 0;	0	3	The response reflects teacher knowledge of inclusive and exclusive definitions for the quadrilaterals as well as their classification. A - mathematically correct knowledge of two types of definitions as well as their classification; D - lack of mathematically correct: for only convex quadrilaterals but not the concave ones; B, C, E - not correct; F - no knowledge at all
				0	51	