広島大学学位請求論文

The conservation law for nonlinear Schrödinger equations with non-vanishing boundary conditions at spatial infinity

(空間遠方で消滅しない境界条件をもつ
非線型シュレディンガー方程式の
保存則について)

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主論文

The conservation law for nonlinear Schrödinger equations with non-vanishing boundary conditions at spatial infinity

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Chapter 1

Introduction

1.1 Outline of this thesis

The nonlinear Schrödinger equation

$$\begin{cases} i\partial_t u + \Delta u = \lambda |u|^{p\ \ 1} u, \quad t \ni [0, \in\ ),\ x \ni \mathbb{R}^n, \\ u(0,x) = u_0(x), \quad x \ni \mathbb{R}^n. \end{cases} \tag{1.1.1}$$

appears as relevant model in great various physical phenomena: for example, nonlinear waves such as propagation of a laser beam, water wave, plasma wave (e.g. [22]).

On the other hand, (1.1.1) has been extensively studied in the mathematical literatures (e.g. [3]). In genaral, since Schrödinger equation is often ill-posed in other than L^2 based spaces, in fact, the linear Schrödinger equation is ill-posed in $C^{\in}$ (see Section 5.2 in [15]), most of the mathematical literatures for (1.1.1) are investigated under L^2 based spaces.

The important property of Schrödinger equations is that Schrödinger operator $U(t) = e^{it\Delta}$ has a smoothing effect of a solution. The effect is described as L^p-L^{p^∞}estimate

$$\backslash U(t)\phi\backslash_{L^p} \geq Ct^{\ \ n(\frac{1}{2}\ \ \frac{1}{p})}\backslash\phi\backslash_{L^{p^\infty}}$$

for $1 \geq p^\infty \geq p \geq \in$. Hence, from L^p-L^{p^∞}estimate, we obtain so-called "Strichartz's estimate" to play an important role to show the existence of the solution for (1.1.1) (see Section 2.1).

Moreover, (1.1.1) has some of conservation laws. The typical conservation laws are as follows:

The conservation law of the mass

$$\backslash u(t)\backslash_{L^2} = \backslash u_0\backslash_{L^2},$$

The conservation law of the energy

$$E(u(t)) := \backslash\ \ u(t)\backslash_{L^2} + \frac{2\lambda}{p+1}\backslash u(t)\backslash_{L^{p+1}}^{p+1} = E(u_0),$$

The conservation law of the momentum

$$P(u(t)) := \mathrm{Im}\bigcap_{\mathbb{R}^n} \bar{u}(t)\ \ u(t)dx = P(u_0),$$

The pseudo conformal conservation law

$$
\begin{aligned}
&\|(x+2it\nabla)u(t)\|_{L^2}^2+\frac{8t^2\lambda}{p+1}\|u(t)\|_{p+1}^{p+1}\\
&=\|xu_0\|_{L^2}^2-\frac{4\lambda(n(p-1)-4)}{p+1}\int_0^t s\|u(s)\|_{p+1}^{p+1}ds.
\end{aligned}
$$

The aim of our study is to investigate how conservaton laws play a role in a behavior of the solution of various nonlinear Schrödinger equations.

Generally, we take two steps to construct a time global solution for the Cauchy problem of (1.1.1) (see [3]). The first step is to construct a time local solution to Duhamel's integral equation by combining a contraction argument with Strichartz's estimate. The next step is to extend the solution to the time global solution by using conservation laws of the mass and the energy.

Moreover, by applying the conservation law of the momentum, we can get a observation of a variance $\|xu\|_{L^2}^2$ (see Section 2.5.4). Also, the pseudo conformal conservation law is essential for a observation of the asymptotic behavior of the solution of (1.1.1) (See Section 2.5.3). Hence, to investigate (1.1.1), it is very important to obtain conservation laws.

For example, we obtain formally the conservation law of energy by multiplying the equation (1.1.1) by $\bar{u}_t$, integrating over $\mathbb{R}^n$, and taking the real part. There are basically two methods to justify the procedure above. One is that solutions is approximated by a sequence of regular solutions, using the continuous dependence of solutions on the initial data. The other is to use a sequence of regularized equations of (1.1.1) whose solutions have enough regularities to perform the procedure above (see Section 2.4). However, these two methods involve a limiting procedure on approximate solutions. Instead, for (1.1.1), Ozawa [19] derives conservation laws of the mass and the energy by using additional properties of solutions provided by Strichartz's estimates.

Next, we consider defocusing nonlinear Schrödinger equations in dimension $n \geq 4$.

$$
\begin{cases}
i\partial_t u+\Delta u=f(|u|^2)u, & t\in[0,\infty),\ x\in\mathbb{R}^n,\\
u(0,x)=u_0(x), & x\in\mathbb{R}^n,
\end{cases}
\tag{1.1.2}
$$

The unknown function u has the following non-vanishing boundary condition:

$$|u(x)|^2\to\rho_0 \quad \text{as} \quad |x|\to\infty,$$

where $\rho_0>0$. The nonlinear term f is assumed to be defocusing as follows:

$$f(\rho_0)=0, \quad f'(\rho_0)>0. \tag{$\mathbf{H}_f$}$$

(1.1.2) describes various physical backgrounds such as Bose-Einstein condensation, superfluidity, and nonlinear topics (dark soliton, optical vortices) (see [17], [20]). Because of the boundary condition, we can not consider (1.1.2) in L^2 based spaces, namely it is difficult to investigate (1.1.2). Two important model cases for (1.1.2) have been extensively studied both in the physical and mathematical literatures: the Gross-Pitaevskii equation (where $f(r)=r-1$, $\rho_0=1$) and the so-called "cubic-quintic" Schrödinger equation (where $f(r)=(r-\rho_0)(3r-2a-\rho_0)$, $0<a<\rho_0$).

Here, we focus on Gross-Pitaevskii equation

$$\begin{cases} i\partial_t u + \Delta u = (|u|^2 - 1)u, \quad t \in [0,\infty),\ x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), \quad x \in \mathbb{R}^n. \end{cases} \tag{1.1.3}$$

Since Béthnel-Saut [2] proves that the Caucy problem (1.1.3) is globally well-posed in $1 + H^1$ for $n = 2, 3$, many mathematican have been studied (1.1.3). In particular, for $n = 2, 3$, Gérard [21] proves the global wellposeness of (1.1.3) for large data in

$$E_{\rho_0} = \{u \in H^1_{loc}(\mathbb{R}^n); \quad \nabla u \in L^2(\mathbb{R}^n), |u|^2 - \rho_0 \in L^2(\mathbb{R}^n)\}$$

with $p_0 = 1$.

Furthermore, Gallo [8] proves that for $n \geq 4$, (1.1.2) under suitable assumptions on f is globally wellposed in E_{ρ_0}.

In this thesis, for the equation (1.1.2) in $n = 2, 3, 4$, we derive the conservation law for time local solutions without approximating procedure. Instead of that, we use Ozawa's idea [19]. Note that when $n = 1$, because $H^1 \hookrightarrow L^\infty$, Gallo [8] derived it without approximating procedure, and that for $n \geq 2$, Gallo [8] derives it using the approximate argument (see a proof of Threorem 3.1.4). We follow Ozawa's idea, however, we can not derive the conservation law only by Ozawa's idea, due to the nonlinear term and the space of solutions. We derive the conservation law to combine Ozawa's idea with decomposing the nonlinear term $f(|u|^2)u$ as

$$f(|u|^2)u = \chi(D_x)(f(|u|^2)u) + \sum_{j=1}^{n} (1 - \chi(D_x))P_j(D_x)\partial_{x_j}(f(|u|^2)u),$$

where $\chi \in C_0^\infty(\mathbb{R}^n)$ is a cutoff function and $P_j(\xi) = i\xi_j/|\xi|^2$, by applying the method for the decomposition of Schrödinger operator in Gérard [9] (See Lemma 3.1.3). Moreover, note that we can decompose $u_0 \in E_{\rho_0}$ as $u_0 = \phi + w_0$ such that $\phi \in E$ satisfying the following condition (1.1.4) and $w_0 \in H^1$ (see Lemma 3.1.4):

$$\phi \in C_b^\infty(\mathbb{R}^n), \qquad \nabla\phi \in H^\infty(\mathbb{R}^n)^n, \quad |\phi|^2 - \rho_0 \in L^2(\mathbb{R}^n). \tag{1.1.4}$$

Our main result in this thesis is as follows:

Theorem 1.1.1. *Let $n = 2, 3, 4$. Let $\rho_0 > 0$, and $f \in C^2(\mathbb{R}_+)$ satisfying $(\mathbf{H}_f)$. Moreover, we assume that there exist $\alpha_1 \geq 1$, with a supplementary condition $\alpha_1 < \alpha_1^*$ if $n = 3, 4$ ($\alpha_1^* = 3$ if $n = 3$, $\alpha_1^* = 2$ if $n = 4$) such that*

$$\exists C_0 > 0,\ s.t.\ \forall r \geq 1,\ |f^{(k)}(r)| \leq C_0 r^{\alpha_1 - 1 - k}\ (k = 1, 2). \tag{$\mathbf{H}^\infty_{\alpha_1}$}$$

Let ϕ be a function satisfying

$$\phi \in C_b^2(\mathbb{R}^n), \qquad \nabla\phi \in H^2(\mathbb{R}^n)^n, \quad |\phi|^2 - \rho_0 \in L^2(\mathbb{R}^n). \tag{$\mathbf{H}^\infty_\phi$}$$

Let $w \in C([0,T], H^1(\mathbb{R}^n))$ be a mild solution of the integral equation

$$w(t) = U(t)w_0 - i\int_0^t U(t - t')F(w(t'))dt'$$

for some $w_0 \ni H^1$ *and* $T > 0$*, where* $F(w) := \quad \Delta\phi + f(|\phi + w|^2)(\phi + w)$*.*

Then $\mathcal{F}(w(t)) = \mathcal{F}(w_0)$ *for all* $t \ni [0, T]$*, where*

$$\mathcal{F}(w) := \bigcap_{\mathbb{R}^n} | \quad (\phi + w)|^2 dx + \bigcap_{\mathbb{R}^n} V(|\phi + w|^2) dx,$$

and

$$V(r) := \bigcap_{\rho_0}^{r} f(s) ds.$$

Moreover, as a corollary to the main result, we can deduce a globally well-posedness of (1.1.2). Due to Theorem 1.1.1, we can remove a technical assumption of the nonlinear term. We have the following result:

Corolary 1.1.1. *Let* $n = 2, 3, 4$*. We assume that* f *and* ϕ *satisfy the same assumptions as in Theorem 1.1.1, with a supplementary assumption as* f *satisfying* $(\mathbf{H}_{\alpha_2})$ *for some* $\alpha_2 \ni \mathbb{R}$ *with* $\alpha_1 \quad \alpha_2 \geq 1/2$*. Then (1.1.2) is globally well-posed in* $\phi + H^1(\mathbb{R}^n)$*. That is, for any* $w_0 \ni H^1(\mathbb{R}^n)$*, there exist a unique* $w \ni C(\mathbb{R}, H^1(\mathbb{R}^n))$ *such that* $\phi + w$ *solves (1.1.2) with the initial data* $w(0) = w_0$*. Moreover, for any* $T > 0$*, the flow map* $w_0 \not\forall \ w \ (H^1 \uparrow \ C([0, T], H^1))$ *is Lipschitz continuous on the bounded sets of* $H^1(\mathbb{R}^n)$*. The energy* $\mathcal{F}(w)$ *is conserved by the flow.*

This thesis is organized as follows: In Chapter 2, we present previous works of the nonlinear Schrödinger equation (1.1.1). First, we give the representation of a solution of the linear Schrödinger equation and Strichartz estimates. Next, we consider a local wellposedness of (1.1.1) in some of L^2 based spaces. Moreover, we derive exactly various conservation laws of (1.1.1). Finally, we state global behaviors for solutions of (1.1.1), given by applying conservation laws.

Chapter 3 is devoted to present previous works for defocusing nonlinear Schrödinger equations with non-vanishing boundary conditions (1.1.2). First, we state that for $n = 2, 3$, (1.1.3) is globally wellposed in $1 + H^1$ with a large initial data, proven by [2]. Secondly, we present that [9] proves the existence of energy solution for (1.1.3) with a large initial data. In particular, we introduce methods to decompose the element of E_{ρ_0}, and observe an action of Schrödinger operator in E_{ρ_0}. Finally, we show that for $n \geq 4$, (1.1.2) under suitable assumptions on f is globally wellposed in E_{ρ_0} by [8].

In Chapter 4, we introduce a new method to derive conservation laws of the mass and the energy for (1.1.1) by using additional properties of solutions provided by Strichartz's estimates proven by [19].

Chapter 5 is devoted to introduce the main result of this thesis. First, we present the main result, and that we can improve the result of [8] by applying the main result. Next, we give estimates of the nonlinear term and results of the time-derivative term needed for the proof of the main result, respectively. Finally, we prove the main result.

In Appendix, we present the notation and some of results, used in this thesis.

Chapter 2

Basic results of nonlinear Schrödinger equations with power type nonlinearity

2.1 Fundamental properties of linear Schrödinger equations

In this section, we consider the Cauchy problem for the linear Schrödinger equation

$$\begin{cases} i\partial_t u + \Delta u = 0, \quad t \ni [0, \in\ ),\ x \ni \mathbb{R}^n, \\ u(0,x) = u_0(x), \quad x \ni \mathbb{R}^n, \end{cases} \tag{2.1.1}$$

where $u(t,x) : [0, \in\ ) * \mathbb{R}^n \uparrow\ \mathbb{C}$ and the initial data u_0 is a complex valued function on $\mathbb{R}^n$. $S(t)$ denotes the fundamental solution of (2.1.1), that is

$$S(t) = \frac{1}{(4\pi it)^{n/2}} e^{\ \frac{i\ x\ ^2}{4t}}. \tag{2.1.2}$$

The solution of (2.1.1) is described as

$$u = S(t) \bullet u_0 = U(t)u_0,$$

where $U(t)$ is Schrödinger operator $e^{it\Delta}$. For (2.1.1), we have the following result:

Theorem 2.1.1 (e.g. [23])**.** *Let $s \ni \mathbb{R}$ and $u_0 \ni H^s(\mathbb{R}^n)$. Then there exists a unique solution $u = U(t)u_0$ of (2.1.1) with*

$$u \ni C([0, \in\ ), H^s(\mathbb{R}^n)) \{\ C^1([0, \in\ ), H^{s\ \ 2}(\mathbb{R}^n))$$

Note that for some $f \ni C(\mathbb{R}, H^s)$, a solution of the Cauchy problem

$$\begin{cases} i\partial_t u + \Delta u = f, \quad t \ni \mathbb{R},\ x \ni \mathbb{R}^n, \\ u(0,x) = 0, \quad x \ni \mathbb{R}^n \end{cases} \tag{2.1.3}$$

is described as

$$u(t) = \quad i \bigcap_0^t U(t \quad s) f(s) ds.$$

Thus, for the Cauchy problem

$$\begin{cases} i\partial_t u + \Delta u = f, & t \ni [0, \in), \ x \ni \mathbb{R}^n, \\ u(0,x) = u_0, & x \ni \mathbb{R}^n, \end{cases} \tag{2.1.4}$$

we obtain the following result:

Theorem 2.1.2 (e.g. [23])**.** *Let $s \ni \mathbb{R}$ and $u_0 \ni H^1(\mathbb{R}^n)$. Let $f \ni C(\mathbb{R}, H^s)$. Then there exists a unique solution*

$$u = U(t)u_0 \quad i \bigcap_0^t U(t \quad s) f(s) ds$$

of (2.1.4) with

$$u \ni C([0, \in), H^s(\mathbb{R}^n)) \{ \ C^1([0, \in), H^{s \quad 2}(\mathbb{R}^n)).$$

We present smoothing properties for the solution of Schrödinger equations.

Proposition 2.1.1 (e.g. [3])**.** *Let $p \ni [2, \in)$. Then*

$$\backslash U(t)\varphi \backslash_{L^p(\mathbb{R}^n)} \geq (4\pi |t|)^{\ n(\frac{1}{2} \ \frac{1}{p})} \backslash \varphi \backslash_{L^{p^\infty}(\mathbb{R}^n)}$$

for all $\varphi \ni L^{p^\infty}(\mathbb{R}^n)$ and $t \neq 0$.

Combining Proposition 2.1.1 with Hardy-Littlewood-Sobolev inequality and duality argument, we get some of estimates called Strichartz's estimates. We need the following definitions to mention them:

Definition 2.1.1. *(i) A positive exponent p^∞ is called the dual exponent of p if p and p^∞ satisfy $1/p + 1/p^\infty = 1$.*

(ii) A pair of two exponents (p, q) is called an admissible pair if $(p.q)$ satisfies $2/p + n/q = n/2$, $p \approx 2$ and $(p, q) \neq (2, \in)$.

Strichartz's estimates are described as the following Theorem:

Theorem 2.1.3 (Strichartz's estimates, e.g. [3])**.** *The following properties holds:*

(i) For every $\varphi \ni L^2(\mathbb{R}^n)$, the function $t \forall \ U(t)\varphi$ belongs to

$$C(\mathbb{R}, L^2(\mathbb{R}^n)) \{ \ L^q(\mathbb{R}, L^r(\mathbb{R}^n))$$

for any admissible pair (q, r). Furthermore, there exists a positive constant C such that

$$\backslash U(\cdot)\varphi \backslash_{L^q(\mathbb{R}, L^r)} \geq C \backslash \varphi \backslash_{L^2}$$

for all $\varphi \ni L^2(\mathbb{R}^n)$.

(ii) *Let $I \to \mathbb{R}$ be an interval. (p_1, q_1) and (p_2, q_2) denote admissible pairs. Let $t_0 \ni \bar{I}$. For any $f \ni L^{p_1^\infty}(I, L^{q_1^\infty})$, the function*

$$t \not\forall \ \Phi_f(t) = \bigcap_{t_0}^{t} U(t \quad s) f(s) ds$$

belongs to

$$C(\bar{I}, L^2(\mathbb{R}^n)) \{ \ L^{p_2}(I, L^{q_2}(\mathbb{R}^n)).$$

Furthermore, there exists a positive constant C not depending on I such that

$$\backslash \Phi_f \backslash_{L^{p_2}(I, L^{q_2}(\mathbb{R}^n))} \geq C \backslash f \backslash_{L^{p_1^\infty}(I, L^{q_1^\infty}(\mathbb{R}^n))}$$

for all $f \ni L^{p_1^\infty}(I, L^{q_1^\infty}(\mathbb{R}^n))$.

Corolary 2.1.1 (e.g. [3])**.** *Let $s \ni \mathbb{R}$ and $I \to \mathbb{R}$ be an interval. (p_1, q_1) and (p_2, q_2) denote admissible pairs. Let $t_0 \ni \bar{I}$. Then*

(i) *for any $\varphi \ni H^s(\mathbb{R}^n)$,*

$$\backslash U(t)\varphi \backslash_{L^{|} \ (\mathbb{R}, H^s(\mathbb{R}^n))} \geq C \backslash \varphi \backslash_{H^s(\mathbb{R}^n)},$$

$$\backslash U(t)\varphi \backslash_{L^{p_1}(\mathbb{R}, B^s_{q_1,2}(\mathbb{R}^n))} \geq C \backslash \varphi \backslash_{H^s(\mathbb{R}^n)}.$$

(ii) *for any $f \ni L^1(I, H^s(\mathbb{R}^n))$,*

$$\left(\bigcap_{t_0}^{t} U(t \quad \tau) f(\tau) d\tau \right)_{L^{|} \ (I, H^s(\mathbb{R}^n))} \geq C \backslash f \backslash_{L^1(I, H^s(\mathbb{R}^n))}.$$

(iii) *for all $f \ni L^{p_1^\infty}(I, B^s_{q_1^\infty,2}(\mathbb{R}^n))$,*

$$\left(\bigcap_{t_0}^{t} U(t \quad \tau) f(\tau) d\tau \right)_{L^{p_2}(I, B^s_{q_2,2}(\mathbb{R}^n))} \geq C \backslash f \backslash_{L^{p_1^\infty}(I, B^s_{q_1^\infty,2}(\mathbb{R}^n))}.$$

$\mathscr{S}(\mathbb{R}^n) = \mathscr{S}$ denotes a Schwartz space on $\mathbb{R}^n$. We state a factorization of $U(t)$ called Dollard decomposition.

Proposition 2.1.2 (e.g. [3])**.** *For any $\phi \ni \mathscr{S}$,*

$$U(t)\phi = M(t)D(t)\mathcal{S}M(t)\phi,$$

where

$$M(t) = e^{\frac{i \ x \ ^2}{4t}}, \quad D(t)\psi = \frac{1}{(2it)^{n/2}} \psi \left) \frac{x}{2t} \right(.$$

Proof. Using

$$\mathcal{S} \quad ^1[\exp(\quad it|\xi|^2)] = \frac{1}{(2it)^{n/2}} e^{\frac{i \ x \ ^2}{4t}}, \quad \mathcal{S} \quad ^1[f \times g] = \frac{1}{(2\pi)^{n/2}} \check{f} \bullet \check{g},$$

we calculate

$$U(t)\phi = \mathcal{S} \quad ^1 \exp(\quad it|\xi|^2)\mathcal{S}\phi$$

$$
\begin{aligned}
&= (2\pi)^{\ \ n/2}\mathcal{S}^{\ \ 1}[\exp(\ \ it|\xi|^2)] \bullet \phi \\
&= \frac{1}{(4\pi it)^{n/2}} e^{\frac{i\ x\ ^2}{4t}} \bullet \phi \\
&= \frac{1}{(4\pi it)^{n/2}} \bigcap_{\mathbb{R}^n} e^{\frac{i\ x\ \ y\ ^2}{4t}} \phi(y)dy \\
&= \frac{1}{(4\pi it)^{n/2}} \bigcap_{\mathbb{R}^n} e^{\frac{i(\ x\ ^2\ \ 2x*y+\ y\ ^2)}{4t}} \phi(y)dy \\
&= \frac{1}{(4\pi it)^{n/2}} \bigcap_{\mathbb{R}^n} e^{\frac{i\ x\ ^2}{4t}} \times e^{\ \ \frac{ix*y}{2t}} \times e^{\frac{y\ ^2}{4t}} \phi(y)dy \\
&= e^{\frac{i\ x\ ^2}{4t}} \frac{1}{(2it)^{n/2}} \frac{1}{(2\pi)^{n/2}} \bigcap_{\mathbb{R}^n} e^{\ \ i\left(\frac{x}{2t}\right) \times y} \times \Big) e^{\frac{y\ ^2}{4t}} \phi(y) \Big[\ dy \\
&= e^{\frac{i\ x\ ^2}{4t}} \frac{1}{(2it)^{n/2}} \mathcal{S}\left[e^{\frac{y\ ^2}{4t}}\phi\right] \Big) \frac{x}{2t} \Big(\\
&= M(t)D(t)\mathcal{S}M(t)\phi.
\end{aligned}
$$

This completes the proof. □

2.2 Local wellposedness of nonlinear Schrödinger equations with power type nonlinearity

In this section, we consider the Cauchy problem for the nonlinear Schrödinger equation

$$
\Big\}\begin{array}{l} i\partial_t u + \Delta u = \lambda|u|^{p\ \ 1}u, \quad t \ni [0, \in\ ),\ x \ni \mathbb{R}^n, \\ u(0,x) = u_0(x), \quad x \ni \mathbb{R}^n, \end{array} \tag{2.2.1}
$$

where, $\lambda \ni \mathbb{R}$, $p > 1$, $u(t,x) : [0, \in\ ) * \mathbb{R}^n \uparrow\ \mathbb{C}$ and the initial data u_0 is a complex valued function in $\mathbb{R}^n$. In what follows, $f(u)$ denotes $\lambda|u|^{p\ \ 1}u$. We present some of local wellposeness results of (2.2.1)

Theorem 2.2.1 ([24])**.** *Let $1 < p < 1+4/n$. Let (q,r) be some admissible pair. For any $u_0 \ni L^2$, there exists $T > 0$ such that there exists a unique solution*

$$
u \ni C([0,T];L^2)\ \{\ L^q_{loc}((0,T);L^r)
$$

of (2.2.1). Moreover, u depends continously on u_0 in L^2. Namely, there exists $T_0 > 0$ depending only on $\backslash u_0\backslash_{L^2}$ such that if $\}u_{0,n}\langle^{\in}_{n=1} \to L^2$ satisfying $u_{0,n} \uparrow u_0$ in L^2 as $n \uparrow\ \in$, then there exist $u_n \ni C([0,T_0],L^2)$ such that corresponding solutions of (2.2.1) with $u(0) = u_{0,n}$ for n large enough, satisfying

$$
\sup_{t\mathcal{D}[0,T_0]} \backslash u_n(t)\quad u(t)\backslash_{L^2} \geq C\backslash u_{0,n}\quad u_0\backslash_2 \uparrow\ \ 0
$$

as $n \uparrow\ \in$, where C is a positive constant not depending on $\backslash u_0\backslash_{L^2}$.

Proof. We show that the map

$$
\Phi(u) = U(t)u_0 \quad i\lambda \bigcap_0^t U(t\quad s)(|u|^{p\ \ 1}u)(s)ds
$$

is a contraction mapping in

$$
\begin{aligned}
X_T = \}v \ni C([0,T],L^2) \{\ L^\sigma((0,T),L^{p+1});\\
&\qquad \backslash v \backslash_{L^\mid_T L^2} + \backslash v \backslash_{L^\sigma_T L^{p+1}} \geq 3C_0 \backslash u_0 \backslash_{L^2} =: M \langle,\\
d(v_1,v_2) = \backslash v_1 \quad v_2 \backslash_{L^\sigma_T L^{p+1}}
\end{aligned}
$$

with $\sigma = 4(p+1)/n(p \quad 1)$. First, using Strichartz estimate and Hölder inequality, we estimate that for all $u \ni X_T$,

$$
\begin{aligned}
&\backslash \Phi(u) \backslash_{L^\mid_T L^2} + \backslash \Phi(u) \backslash_{L^\sigma_T L^{p+1}}\\
&\geq \backslash U(t)u_0 \backslash_{L^\mid_T L^2} + \left(\bigcap_0^t U(t \quad s)(|u|^{p \quad 1} u)(s) ds \right)_{L^\mid_T L^2}\\
&\quad + \backslash U(t)u_0 \backslash_{L^\sigma_T L^{p+1}} + \left(\bigcap_0^t U(t \quad s)(|u|^{p \quad 1} u)(s) ds \right)_{L^\sigma_T L^{p+1}}\\
&\geq 2C_0 \backslash u_0 \backslash_{L^2} + C \left(|u|^{p \quad 1} u \right)_{L^{\sigma \infty}_T L^{(p+1)/p}}\\
&\geq 2C_0 \backslash u_0 \backslash_{L^2} + C \backslash u \backslash^p_{L^{\sigma \infty p}_T L^{p+1}}\\
&\geq 2C_0 \backslash u_0 \backslash_{L^2} + CT^\delta M^p \quad (\delta := \frac{n}{4} \Big) 1 + \frac{4}{n} \quad p \Big[\ )\\
&\geq M
\end{aligned}
$$

if T is sufficiently small depending only on $\backslash u_0 \backslash_{L^2}$, which $\Phi(v) \ni X_T$ holds. Similarly, we see that for any u, $v \ni X_T$,

$$
\begin{aligned}
d(\Phi(u),\Phi(v)) &= \backslash \Phi(u) \quad \Phi(v) \backslash_{L^\sigma_T L^{p+1}}\\
&\geq \left(\bigcap_0^t U(t \quad s) \} (|u|^{p \quad 1} u)(s) \quad (|v|^{p \quad 1} v)(s) \langle ds \right)_{L^\sigma_T L^{p+1}}\\
&\geq C \left(|u|^{p \quad 1} u \quad |v|^{p \quad 1} v \right)_{L^{\sigma \infty}_T L^{(p+1)/p}}\\
&\geq C \left((|u|^{p \quad 1} + |v|^{p \quad 1}) |u \quad v| \right)_{L^{\sigma \infty}_T L^{(p+1)/p}}\\
&\geq CT^\delta (\backslash u \backslash^{p \quad 1}_{L^\sigma_T L^{p+1}} + \backslash v \backslash^{p \quad 1}_{L^\sigma_T L^{p+1}}) \backslash u \quad v \backslash_{L^\sigma_T L^{p+1}}\\
&\geq CT^\delta M^{p \quad 1} d(u,v)\\
&\geq \frac{1}{2} d(u,v)
\end{aligned}
$$

if T is sufficiently small depending only on $\backslash u_0 \backslash_{L^2}$, which Φ is the contraction mapping in X_T. In conclusion, we get a local L^2 solution u of (3.1.5).

We show that for all admissible pair (q,r), $u \ni L^q_T L^r$. Using Strichartz's estimate and

$$
u = U(t)u_0 \quad i\lambda \bigcap_0^t U(t \quad s)(|u|^{p \quad 1} u)(s) ds,
$$

we can compute

$$
\begin{aligned}
\backslash u \backslash_{L^q_T L^r} &\geq \backslash U(t)u_0 \backslash_{L^q_T L^r} + \left(\lambda \bigcap_0^t U(t \quad s)(|u|^{p \quad 1} u)(s) ds \right)_{L^q_T L^r}\\
&\geq C_0 \backslash v_0 \backslash_{L^2} + CT^\delta \backslash u \backslash^p_{L^\sigma_T L^{p+1}}
\end{aligned}
$$

$$\geq C_0\backslash v_0\backslash_{L^2} + CT^\delta M^{p\quad 1}\backslash u\backslash_{L^\sigma_T L^{p+1}},$$

which implies $u \ni L^q_T L^r$.

Next, we show the uniqueness of the solution. We denote a corresponding solution of (2.2.1) with $v(0) = u_0$ by $v \ni C([0,T]; L^2) \{ \ L^\sigma_{loc}((0,T); L^{p+1})$. We define t_1 by

$$t_1 = \sup\}t \ni [0,T], u(s) = v(s) \ a.e.\ for\ all\ s \ni [0,t]\langle.$$

If $t_1 = T$, then we get the desired cliam. We assume $t_1 < T$. From Strichartz's estimate and Sobolev embedding, it holds that

$$\begin{aligned}
&\backslash u \quad v\backslash_{L^\sigma([t_1,t_2],L^{p+1})} \\
&\geq \Big(\lambda \bigcap_0^t U(t \quad s)(|u|^{p\quad 1}u \quad |v|^{p\quad 1}v)(s)ds\Big(_{L^\sigma([t_1,t_2],L^{p+1})} \\
&\geq C \Big(|u|^{p\quad 1}u \quad |v|^{p\quad 1}v\Big(_{L^{\sigma'}([t_1,t_2],L^{(p+1)/p})} \\
&\geq C(t_1 \quad t_2)^\delta(\backslash u\backslash^{p\quad 1}_{L^\sigma([t_1,t_2],L^{p+1})} + \backslash v\backslash^{p\quad 1}_{L^\sigma(([t_1,t_2],L^{p+1})})\backslash u \quad v\backslash_{L^\sigma([t_1,t_2],L^{p+1})}
\end{aligned}$$

for all $t_2 \ni (t_1, T]$. If t_2 satisfies

$$C(t_1 \quad t_2)^\delta(\backslash u\backslash^{p\quad 1}_{L^\sigma([t_1,t_2],L^{p+1})} + \backslash v\backslash^{p\quad 1}_{L^\sigma(([t_1,t_2],L^{p+1})}) < 1,$$

then we get

$$\backslash u \quad v\backslash_{L^\sigma([t_1,t_2],L^{p+1})} = 0,$$

which yields $u = v$ a.e. on $[t_1, t_2]$. This contradicts the definition of t_1.

Finally, we show a continuous dependence. Let $\}u_{0,n}\langle_{n=1}^{\in} \to L^2$ such that $u_{0,n} \uparrow \ u_0$ in L^2 as $n \uparrow \ \in$. Then, there exists $n_0 \ni \mathbb{N}$ such that if $n \approx n_0$, then

$$\backslash u_{0,n}\backslash_{L^2} \geq 2\backslash u_0\backslash_{L^2} \tag{2.2.2}$$

Hence, from (2.2.2), it follows that there exists $T_0 > 0$ uniformly for n such that $u_n \ni C([0,T_0], L^2)$ of (2.2.1) with $u_n(0) = u_{0,n}$. Note that there exists a constant $C > 0$ such that

$$\backslash u\backslash_{L^\sigma_{T_0} L^{p+1}} + \sup_{n \to n_0} \backslash u_n\backslash_{L^\sigma_{T_0} L^{p+1}} \geq C\backslash u_0\backslash_{L^2}. \tag{2.2.3}$$

Furthermore, using Strichartz's estimate and (2.2.3), we estimate

$$\begin{aligned}
&\backslash u_n \quad u\backslash_{L^{\natural}_{\tilde T} L^2} + \backslash u_n \quad u\backslash_{L^\sigma_{\tilde T} L^{p+1}} \\
&\geq 2C_0\backslash u_{0,n} \quad u_0\backslash_{L^2} \\
&\quad + C\widetilde{T}^\delta(\backslash u_n\backslash^{p\quad 1}_{L^\sigma_{\tilde T} L^{p+1}} + \backslash u\backslash^{p\quad 1}_{L^\sigma_{\tilde T} L^{p+1}})\backslash u_n \quad u\backslash_{L^\sigma_{\tilde T} L^{p+1}} \\
&\geq 2C_0\backslash u_0 \quad \widetilde{u_0}\backslash_{L^2} \\
&\quad + C(\backslash u_0\backslash_{L^2})\widetilde{T}^\delta(\backslash u_n \quad u\backslash_{L^{\natural}_{\tilde T} L^2} + \backslash u_n \quad u\backslash_{L^\sigma_{\tilde T} L^{p+1}}).
\end{aligned}$$

If $\widetilde{T}$ is sufficiently small depending only on $\backslash u_0\backslash_{L^2}$, then it follows that

$$\backslash u_n \quad u\backslash_{L^{\natural}_{\tilde T} L^2} + \backslash u_n \quad u\backslash_{L^\sigma_{\tilde T} L^{p+1}} \geq 2C_0\backslash u_{0,n} \quad u_0\backslash_{L^2}.$$

Repeating the above procedure, we get

$$\|u_n - u\|_{L^\infty_{T_0} L^2} \le C\|u_{0,n} - u_0\|_{L^2} \to 0$$

as $n \to \infty$. Note that we can recover the interval $[0, T_0]$ thanks to (2.2.3). This completes the proof. □

Remark 2.2.1. *Exactly, in the above proof, we show that the map Φ is the contraction mapping in*

$$\begin{aligned} \widetilde{X_T} = \{ v \in L^\infty((0,T), L^2) \cap L^\sigma((0,T), L^{p+1});\\ \|v\|_{L^\infty_T L^2} + \|v\|_{L^\sigma_T L^{p+1}} \le 3C_0\|u_0\|_{L^2} =: M\},\\ d(v_1, v_2) = \|v_1 - v_2\|_{L^\sigma_T L^{p+1}}. \end{aligned}$$

By Theorem 2.1.3, we can prove that the solution u belongs to $C([0,T], L^2) \cap L^\sigma((0,T), L^{p+1})$.

From now on, we put $\alpha(n) = 1 + \frac{4}{n-2}$ if $n \ge 3$ and $\alpha(n) = \infty$ if $n = 1, 2$.

Theorem 2.2.2 ([16])**.** *Let $1 < p < \alpha(n)$. Let (q, r) be a some admissible pair. For any $u_0 \in H^1$, there exists $T > 0$ such that there exists a unique solution*

$$u \in C([0,T]; H^1) \cap L^q_{loc}((0,T); W^{1,p+1})$$

of (2.2.1). Moreover, u depends continously on u_0 in H^1

Proof. We show that the map

$$\Phi(u) = U(t)u_0 - i\lambda \int_0^t U(t-s)(|u|^{p-1}u)(s)ds$$

is a contraction mapping in

$$\begin{aligned} X_T = \{ v \in C([0,T], H^1) \cap L^\sigma((0,T), W^{1,p+1});\\ \|v\|_{L^\infty_T L^2} + \|v\|_{L^\sigma_T W^{1,p+1}} \le 4C_0\|u_0\|_{H^1} =: M\},\\ d(v_1, v_2) = \|v_1 - v_2\|_{L^\sigma_T L^{p+1}} \end{aligned}$$

with $\sigma = 4(p+1)/n(p-1)$. First, combining Strichartz estimate with Hölder inequality and Sobolev embedding, we deduce that for all $u \in X_T$,

$$\begin{aligned} &\|\Phi(u)\|_{L^\infty_T L^2} + \|\Phi(u)\|_{L^\sigma_T L^{p+1}}\\ &\le \|U(t)u_0\|_{L^\infty_T L^2} + \left\| \int_0^t U(t-s)(|u|^{p-1}u)(s)ds \right\|_{L^\infty_T L^2}\\ &\quad + \|U(t)u_0\|_{L^\sigma_T L^{p+1}} + \left\| \int_0^t U(t-s)(|u|^{p-1}u)(s)ds \right\|_{L^\sigma_T L^{p+1}}\\ &\le 2C_0\|u_0\|_{L^2} + C\left\| |u|^{p-1}u \right\|_{L^{\sigma'}_T L^{(p+1)/p}}\\ &\le 2C_0\|u_0\|_{L^2} + C\|u\|^p_{L^{\sigma' p}_T L^{p+1}}\\ &\le 2C_0\|u_0\|_{L^2} + CT^{1/\sigma' p}\|u\|^p_{L^\infty_T H^1} \end{aligned}$$

$$\begin{aligned}
&\geq 2C_0 \backslash u_0 \backslash_{L^2} + CT^{1/\sigma \infty p} M^p \\
&\geq M/2
\end{aligned}$$

if T is sufficiently small depending only $\backslash u_0 \backslash_{H^1}$. Hence, it follows from Strichartz estimate, Hölder inequality and Sobolev embedding that

$$\begin{aligned}
&\backslash \quad \Phi(u) \backslash_{L^{\infty}_T L^2} + \backslash \quad \Phi(u) \backslash_{L^{\sigma}_T L^{p+1}} \\
&\geq \backslash U(t) \quad u_0 \backslash_{L^{\infty}_T L^2} + \left\| \int_0^t U(t \quad s) \quad (|u|^{p \quad 1} u)(s) ds \right\|_{L^{\infty}_T L^2} \\
&\quad + \backslash U(t) \quad u_0 \backslash_{L^{\sigma}_T L^{p+1}} + \left\| \int_0^t U(t \quad s) \quad (|u|^{p \quad 1} u)(s) ds \right\|_{L^{\sigma}_T L^{p+1}} \\
&\geq 2C_0 \backslash \quad u_0 \backslash_{L^2} + C \left\| |u|^{p \quad 1} \quad u \right\|_{L^{\sigma \infty}_T L^{(p+1)/p}} \\
&\geq 2C_0 \backslash \quad u_0 \backslash_{L^2} + C \backslash u \backslash^{p \quad 1}_{L^{\infty}_T L^{p+1}} \backslash \quad u \backslash_{L^{\sigma \infty p}_T L^{p+1}} \\
&\geq 2C_0 \backslash \quad u_0 \backslash_{L^2} + CT^{\delta} \backslash u \backslash^{p \quad 1}_{L^{\infty}_T H^1} \backslash \quad u \backslash_{L^{\sigma}_T L^{p+1}} \\
&\geq 2C_0 \backslash \quad u_0 \backslash_{L^2} + CT^{\delta} M^p \ (\delta := \frac{n+2 \quad (n \quad 2)p}{2(p+1)}) \\
&\geq M/2
\end{aligned}$$

if T is sufficiently small depending only on $\backslash u_0 \backslash_{H^1}$. Thus $\Phi(v) \ni X_T$ holds. Similary, it holds that for all u, $v \ni X_T$,

$$\begin{aligned}
d(\Phi(u), \Phi(v)) &= \backslash \Phi(u) \quad \Phi(v) \backslash_{L^{\sigma}_T L^{p+1}} \\
&\geq \left\| \int_0^t U(t \quad s) \} (|u|^{p \quad 1} u)(s) \quad (|v|^{p \quad 1} v)(s) \langle ds \right\|_{L^{\sigma}_T L^{p+1}} \\
&\geq C \left\| |u|^{p \quad 1} u \quad |v|^{p \quad 1} v \right\|_{L^{\sigma \infty}_T L^{(p+1)/p}} \\
&\geq C \left\| (|u|^{p \quad 1} + |v|^{p \quad 1}) |u \quad v| \right\|_{L^{\sigma \infty}_T L^{(p+1)/p}} \\
&\geq CT^{\delta} (\backslash u \backslash^{p \quad 1}_{L^{\infty}_T L^{p+1}} + \backslash v \backslash^{p \quad 1}_{L^{\infty}_T L^{p+1}}) \backslash u \quad v \backslash_{L^{\sigma}_T L^{p+1}} \\
&\geq CT^{\delta} (\backslash u \backslash^{p \quad 1}_{L^{\infty}_T H^1} + \backslash v \backslash^{p \quad 1}_{L^{\infty}_T H^1}) \backslash u \quad v \backslash_{L^{\sigma}_T L^{p+1}} \\
&\geq CT^{\delta} M^{p \quad 1} d(u, v) \\
&\geq \frac{1}{2} d(u, v)
\end{aligned}$$

if T is sufficiently small depending only $\backslash u_0 \backslash_{H^1}$, that is Φ is the contraction mapping in X_T. In conclusion, we get a local H^1 solution u of (3.1.5). remaining assertions follow easy from way similar to Theorem 2.2.1. □

Remark 2.2.2. *For H^1 solution $u \ni C([0,T], H^1)$ with initial data $u_0 \ni H^1$, it holds that $u \ni C^1([0,T], H^{\ 1})$. Indeed, putting $t_0 \ni [0,T]$, using Sobolev embedding and a continous dependence, we obtain that*

$$\begin{aligned}
\backslash f(u(t)) \quad f(u(t_0)) \backslash_{H^{\ 1}} &\geq \backslash f(u(t)) \quad f(u(t_0)) \backslash_{L^{\frac{p+1}{p}}} \\
&\geq C(\backslash u(t) \backslash^{p \quad 1}_{L^{p+1}} + \backslash u(t_0) \backslash^{p \quad 1}_{L^{p+1}}) \backslash u(t) \quad u(t_0) \backslash_{L^{p+1}} \\
&\geq C(\backslash u \backslash^{p \quad 1}_{L^{\infty}_T H^1} + \backslash u \backslash^{p \quad 1}_{L^{\infty}_T H^1}) \backslash u(t) \quad u(t_0) \backslash_{H^1}
\end{aligned}$$

$$\uparrow \ 0$$

as $t \uparrow \ t_0$*, and*

$$\backslash \Delta u(t) \quad \Delta u(t_0) \backslash_{H^{\ 1}} \geq \backslash u(t) \quad u(t_0) \backslash_{H^1} \uparrow \ 0$$

as $t \uparrow \ t_0$*. Hence, it follows from the equation (2.2.1) that*

$$\begin{aligned} \backslash \partial_t u(t) \quad \partial_t u(t_0) \backslash_{H^{\ 1}} &\geq \backslash f(u(t)) \quad f(u(t_0)) \backslash_{H^{\ 1}} + \backslash \Delta u(t) \quad \Delta u(t_0) \backslash_{H^{\ 1}} \\ &\uparrow \ 0 \end{aligned}$$

as $t \uparrow \ t_0$*, which implies* $u \ni C^1([0,T], H^{\ 1})$*.*

Remark 2.2.3 (see [3] and [16])**.** *For* H^1 *solution* $u \ni C([0,T], H^1)$ *with initial data* $u_0 \ni H^1$*, the following results hold:*

(i) if $u_0 \ni H^2$*, then* $u \ni C([0,T], H^2)$*,*

(ii) if $u_0 \ni \Sigma$*, then* $u \ni C([0,T], \Sigma)$*,*

where $\Sigma = \}f \ni H^1;\ xf \ni L^2\langle$*.*

Next, we present a locall wellposedness result for H^s solutions of (2.2.1).

Theorem 2.2.3 ([4])**.** *Let* $0 < s < \min\}1, n/2\langle$ *and* $1 < p < 1 + 4/(n \quad 2s)$*. Let* (γ, ρ) *be a admissible pair defined by*

$$\rho = \frac{n(p+1)}{n + s(p \quad 1)}, \quad \gamma = \frac{4(p+1)}{(p \quad 1)(n \quad 2s)}. \tag{2.2.4}$$

For any $u_0 \ni H^s$*, there exists* $T > 0$ *such that there exists a unique solution*

$$u \ni C([0,T]; H^s) \{ \ L^{\gamma}_{loc}((0,T); B^s_{\rho,2})$$

of (2.2.1). Moreover, u *depends continously on* u_0 *in* H^s*.*

Proof. First, note that the following Lemmas holds:

Lemma 2.2.1 ([4])**.** *Let* $0 < s < 1$*. Let* $\rho > 1$ *satisfying (2.2.4).* $f(u)$ *denotes* $\lambda|u|^{p \ 1}u$*. Then*

$$\backslash f(u) \backslash_{B^s_{\rho',2}} \geq C \backslash u \backslash^p_{B^s_{\rho,2}} \tag{2.2.5}$$

$$\backslash f(u) \quad f(v) \backslash_{L^{\rho'}} \geq C(\backslash u \backslash^{p \ 1}_{B^s_{\rho,2}} + \backslash v \backslash^{p \ 1}_{B^s_{\rho,2}}) \backslash u \quad v \backslash_{L^\rho} \tag{2.2.6}$$

for all u*,* $v \ni B^s_{\rho,2}(\mathbb{R}^n)$*.*

Proof. Noting that $\sigma = (p \quad 1)\rho\rho'/(\rho \quad \rho') = n(p+1)/(n \quad 2s)$, Hölder inequality implies

$$\backslash |u|^{p \ 1} v \backslash_{L^{\rho'}} \geq \backslash u \backslash^{p \ 1}_{L^\sigma} \backslash v \backslash_{L^\rho}. \tag{2.2.7}$$

From (2.2.7), it follows that for any $y \ni \mathbb{R}^n$,

$$\begin{aligned} \backslash f(u)(\times \ y) \quad f(u)(\times) \backslash_{L^{\rho'}} &\geq C(\backslash u(\times \ y) \backslash^{p \ 1}_{L^\sigma} + \backslash u \backslash^{p \ 1}_{L^\sigma}) \backslash u(\times \ y) \quad u(\times) \backslash_{L^\rho} \\ &\geq C \backslash u \backslash^{p \ 1}_{L^\sigma} \backslash u(\times \ y) \quad u(\times) \backslash_{L^\rho}. \end{aligned}$$

Therefore, using Lemma 6.2.5 (ii), we get

$$\|f(u)\|_{\dot{B}^s_{\rho',2}} \le C\|u\|^{p-1}_{L^\sigma}\|u\|_{\dot{B}^s_{\rho,2}}. \tag{2.2.8}$$

Thus, Combining (2.2.7) with Lemma 6.2.5 (i), (2.2.8), we compute

$$\begin{aligned}
\|f(u)\|_{B^s_{\rho',2}} &\le C(\|f(u)\|_{L^{\rho'}} + \|f(u)\|_{\dot{B}^s_{\rho',2}}) \\
&\le C(\|u\|^{p-1}_{L^\sigma}\|u\|_{L^\rho} + \|u\|^{p-1}_{L^\sigma}\|u\|_{\dot{B}^s_{\rho,2}}) \\
&\le C\|u\|^{p-1}_{L^\sigma}\|u\|_{B^s_{\rho,2}}. \qquad (2.2.9)
\end{aligned}$$

Here, $s < n/2$ implies $\rho \approx 2$. Thus, Lemma 6.2.4 yields $B^s_{\rho,2}(\mathbb{R}^n) \hookrightarrow H^{s,\rho}(\mathbb{R}^n)$. Since $s < n/2$ implies $s\rho < n$, by using Gagliardo-Nirenbreg inequality, we obtain $H^{s,\rho}(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n)$ for all $p \in [\rho, n(\alpha+2)/(n-2s)]$. These imply

$$B^s_{\rho,2}(\mathbb{R}^n) \hookrightarrow L^{n(p+1)/(n-2s)}(\mathbb{R}^n). \tag{2.2.10}$$

Combining (2.2.9) with (2.2.10), we get (2.2.5). Similarly, it follows from Hölder inequality, (2.2.10) and (2.2.7) that

$$\begin{aligned}
\|f(u) - f(v)\|_{L^{\rho'}} &\le C(\|u\|^{p-1}_{L^{n(p+1)/(n-2s)}} + \|v\|^{p-1}_{L^{n(p+1)/(n-2s)}})\|u - v\|_{L^\rho} \\
&\le C(\|u\|^{p-1}_{B^s_{\rho,2}} + \|v\|^{p-1}_{B^s_{\rho,2}})\|u - v\|_{L^\rho}.
\end{aligned}$$

This complete the proof of (2.2.6). □

We back to the proof of Theorem 2.2.3. We show that the map

$$\Phi(u) = U(t)u_0 - i\lambda\int_0^t U(t-s)(|u|^{p-1}u)(s)ds$$

is a contraction mapping in

$$\begin{aligned}
X_T = \{u \in C([0,T], H^s) \cap L^\gamma((0,T), B^s_{\rho,2}) \\
\|u\|_{L^\infty_T H^s} \le M, \quad \|u\|_{L^\gamma_T B^s_{\rho,2}} \le M\}, \\
d(u,v) := \|u - v\|_{L^\infty_T L^2} + \|u - v\|_{L^\gamma_T L^\rho},
\end{aligned}$$

where, $M = 2C_1\|u_0\|_{H^s}$. First, combining Strichartz estimate with Hölder inequality, Sobolev embedding, we deduce that for all $u \in X_T$,

$$\begin{aligned}
&\|\Phi(u)\|_{L^\gamma_T B^s_{\rho,2}} \\
&\le \|U(t)u_0\|_{L^\gamma_T B^s_{\rho,2}} + \left\|\lambda\int_0^t U(t-s)(|u|^{p-1}u)(s)ds\right\|_{L^\gamma_T B^s_{\rho,2}} \\
&\le C_1\|U(t)u_0\|_{L^\gamma_T H^{s,\rho}} + \left\|\lambda\int_0^t U(t-s)(|u|^{p-1}u)(s)ds\right\|_{L^\gamma_T B^s_{\rho,2}} \\
&\le C_1\|u_0\|_{H^s} + C\||u|^{p-1}u\|_{L^{\gamma'}_T B^s_{\rho',2}} \\
&\le C_1\|u_0\|_{H^s} + CT^{(4-(p-1)(n-2s))/4}\|u\|^p_{L^\gamma_T B^s_{\rho,2}} \\
&\le M
\end{aligned}$$

if T is sufficiently small depending only on $\backslash u_0 \backslash_{H^s}$. Similary, since

$$\backslash \Phi(u) \backslash_{L^\gamma_T B^s_{\rho,2}} \geq M,$$

$\Phi(v) \ni X_T$ holds. Moreover, it follows that for all u, $v \ni X_T$,

$$\begin{aligned}
& d(\Phi(u), \Phi(v)) \\
& = \backslash \Phi(u) \quad \Phi(v) \backslash_{L^\infty_T L^2} + \backslash \Phi(u) \quad \Phi(v) \backslash_{L^\gamma_T L^\rho} \\
& \geq \left\| \lambda \bigcap_0^t U(t \quad s) \{ (|u|^{p \quad 1} u)(s) \quad (|v|^{p \quad 1} v)(s) \} ds \right\|_{L^\infty_T L^2} \\
& \quad + \left\| \lambda \bigcap_0^t U(t \quad s) \{ (|u|^{p \quad 1} u)(s) \quad (|v|^{p \quad 1} v)(s) \} ds \right\|_{L^\gamma_T L^\rho} \\
& \geq C \left\| |u|^{p \quad 1} u \quad |v|^{p \quad 1} v \right\|_{L^{\gamma'}_T L^{\rho'}} \\
& \geq C \left\| (|u|^{p \quad 1} + |v|^{p \quad 1}) |u \quad v| \right\|_{L^{\gamma'}_T L^{\rho'}} \\
& \geq C T^\delta (\backslash u \backslash^{p \quad 1}_{L^\gamma_T B^s_{\rho,2}} + \backslash v \backslash^{p \quad 1}_{L^\gamma_T B^s_{\rho,2}}) \backslash u \quad v \backslash_{L^\gamma_T L^\rho} \quad (\delta = (4 \quad \alpha(n \quad 2s))/4) \\
& \geq C T^\delta M^{p \quad 1} d(u, v) \\
& \geq \frac{1}{2} d(u, v)
\end{aligned}$$

if T is sufficiently small depending only on $\backslash u_0 \backslash_{H^s}$, which Φ is the contraction mapping in X_T. In conclusion, we get a local H^s solution u of (3.1.5). Remaining assertions follow easy from way similar to Theorem 2.2.1.

□

2.3 The derivation of various conservation laws for nonlinear Schrödinger equations with power type nonlinearity

We present a method to derive formally the various conservation laws for (2.2.1).

We can obtain formally the conservation law of the mass $\backslash u \backslash_{L^2}$ by multiplying the equation (2.2.1) by $\overline{u}$, integrating over $\mathbb{R}^n$, and taking the imaginary part as follows:

$$\begin{aligned}
0 &= 2 \operatorname{Im}(i\partial_t u + \Delta u \quad \lambda |u|^{p \quad 1} u, u)_{L^2} \\
&= 2 \operatorname{Im}(i\partial_t u, u)_{L^2} \\
&= 2 \operatorname{Re}(\partial_t u, u)_{L^2} \\
&= \frac{d}{dt} \backslash u(t) \backslash^2_{L^2}.
\end{aligned}$$

Next, We can obtain formally the conservation law of the energy $E(u)$ by multiplying the equation (2.2.1) by $\overline{\partial_t u}$, integrating over $\mathbb{R}^n$, and taking the real part as follows:

$$\begin{aligned}
0 = &\quad 2 \operatorname{Re}(i\partial_t u + \Delta u \quad \lambda |u|^{p \quad 1} u, \partial_t u)_{L^2} \\
= &\quad 2 \operatorname{Re}(\Delta u, \partial_t u)_{L^2} + 2 \operatorname{Re}(\lambda |u|^{p \quad 1} u, \partial_t u)_{L^2}
\end{aligned}$$

$$
\begin{aligned}
&= 2\,\mathrm{Re}(\nabla u, \partial_t \nabla u)_{L^2} + 2\,\mathrm{Re}(\lambda|u|^{p-1}u, \partial_t u)_{L^2} \\
&= \frac{d}{dt}E(u(t)),
\end{aligned}
$$

where

$$
E(u) = \|\nabla u\|_{L^2}^2 + \frac{2\lambda}{p+1}\|u\|_{L^{p+1}}^{p+1}.
$$

Also, we can obtain formally the conservation law of the moumentum $P(u)$ by multiplying the equation (2.2.1) by $\nabla \bar{u}$, integrating over $\mathbb{R}^n$, and taking the real part as follows:

$$
\begin{aligned}
0 &= 2\,\mathrm{Re}\left\langle i\partial_t u + \Delta u - \lambda|u|^{p-1}u, \nabla u\right\rangle_{L^2} \\
&= 2\,\mathrm{Re}(i\partial_t u, \nabla u)_{L^2} - 2\,\mathrm{Re}(\lambda|u|^{p-1}u, \nabla u)_{L^2} = \frac{d}{dt}P(u(t)),
\end{aligned}
$$

where

$$
P(u) = \mathrm{Im}\int_{\mathbb{R}^n} u \nabla \bar{u}\, dx.
$$

Finally, we present a method to obtain formally the conservation law of the pseudo conformal conservation law

$$
\begin{aligned}
&\|(x + 2it\nabla)u(t)\|_{L^2}^2 + \frac{8t^2\lambda}{p+1}\|u(t)\|_{p+1}^{p+1} \\
&\quad = \|xu_0\|_{L^2}^2 - \frac{4\lambda(n(p-1)-4)}{p+1}\int_0^t s\|u(s)\|_{p+1}^{p+1}\, ds. \qquad (2.3.1)
\end{aligned}
$$

First, by multiplying the equation (2.2.1) by $|x|^2\bar{u}$, integrating over $\mathbb{R}^n$, and taking the imaginary part, we deduce that

$$
\begin{aligned}
0 &= 2\,\mathrm{Im}(i\partial_t u + \Delta u - \lambda|u|^{p-1}u, |x|^2 u)_{L^2} \\
&= 2\,\mathrm{Im}(i\partial_t u, |x|^2 u)_{L^2} + 2\,\mathrm{Im}(\Delta u, |x|^2 u)_{L^2} \\
&= \frac{d}{dt}\|xu(t)\|_{L^2}^2 - 2\,\mathrm{Im}\int_{\mathbb{R}^n} \nabla u \cdot \nabla(|x|^2\bar{u})\, dx \\
&= \frac{d}{dt}\|xu(t)\|_{L^2}^2 - 2\,\mathrm{Im}\int_{\mathbb{R}^n} \nabla u \cdot (2x\bar{u} + |x|^2\nabla\bar{u})\, dx \\
&= \frac{d}{dt}\|xu(t)\|_{L^2}^2 - 4\,\mathrm{Im}\int_{\mathbb{R}^n} \nabla u \cdot x\bar{u}\, dx. \qquad (2.3.2)
\end{aligned}
$$

Combining

$$
\nabla\partial_t u \cdot x\bar{u} = \nabla\cdot((\partial_t u)x\bar{u}) - n(\partial_t u)\bar{u} - (\partial_t u)x \cdot \nabla\bar{u}
$$

with divergence Theorem, we compute

$$
\begin{aligned}
&\frac{d}{dt}\,\mathrm{Im}\int_{\mathbb{R}^n} \nabla u \cdot x\bar{u}\, dx \\
&= \mathrm{Im}\int_{\mathbb{R}^n} \nabla u \cdot x\overline{\partial_t u}\, dx + \mathrm{Im}\int_{\mathbb{R}^n} \nabla\partial_t u \cdot x\bar{u}\, dx
\end{aligned}
$$

$$= \operatorname{Im}\int_{\mathbb{R}^n} \nabla u \cdot x\overline{\partial_t u}dx + \operatorname{Im}\int_{\mathbb{R}^n} \nabla\cdot((\partial_t u)x\overline{u})dx$$
$$- n\operatorname{Im}\int_{\mathbb{R}^n} (\partial_t u)\overline{u}dx - \operatorname{Im}\int_{\mathbb{R}^n} (\partial_t u)x\cdot\nabla\overline{u}dx$$
$$= 2\operatorname{Im}\int_{\mathbb{R}^n} \nabla u\cdot x\overline{\partial_t u}dx - n\operatorname{Im}\int_{\mathbb{R}^n} (\partial_t u)\overline{u}dx. \tag{2.3.3}$$

Note that combining

$$2\operatorname{Re}\int_{\mathbb{R}^n} \nabla(\nabla u\cdot x)\cdot\nabla\overline{u}dx$$
$$= 2\operatorname{Re}\int_{\mathbb{R}^n} \sum_{k=1}^{n} \partial_k \left(\sum_{j=1}^{n} x_j\partial_j u\right)\overline{\partial_k u}dx$$
$$= 2\operatorname{Re}\int_{\mathbb{R}^n} \sum_{k=1}^{n} \left(\partial_k u + \sum_{j=1}^{n} x_j\partial_{j,k}u\right)\overline{\partial_k u}dx$$
$$= 2\operatorname{Re}\int_{\mathbb{R}^n} \sum_{k=1}^{n} |\partial_k u|^2 dx + \sum_{k,j=1}^{n} x_j\partial_{j,k}u\cdot\overline{\partial_k u}dx$$
$$= 2\|\nabla u\|_{L^2}^2 + 2\operatorname{Re}\int_{\mathbb{R}^n} \sum_{k,j=1}^{n} x_j\partial_{j,k}u\cdot\overline{\partial_k u}dx$$

with

$$A := 2\operatorname{Re}\int_{\mathbb{R}^n} \sum_{k,j=1}^{n} x_j\partial_{j,k}u\cdot\overline{\partial_k u}dx$$
$$= -2\operatorname{Re}\int_{\mathbb{R}^n} \sum_{k,j=1}^{n} \partial_k u\cdot\partial_j\left(x_j\overline{\partial_k u}\right)dx$$
$$= -2\operatorname{Re}\sum_{j=1}^{n}\int_{\mathbb{R}^n} \sum_{k=1}^{n} |\partial_k u|^2 dx - 2\operatorname{Re}\int_{\mathbb{R}^n} \sum_{k,j=1}^{n} \partial_k u\cdot x_j\overline{\partial_{j,k}u}\,dx$$
$$= -2n\|\nabla u\|_{L^2}^2 - A,$$

we deduce that

$$2\operatorname{Re}\int_{\mathbb{R}^n} \nabla(\nabla u\cdot x)\cdot\nabla\overline{u}dx = (2-n)\|\nabla u\|_{L^2}^2.$$

Hence, using (2.3.3) together with

$$2\operatorname{Im}\int_{\mathbb{R}^n} \nabla u\cdot x\overline{\partial_t u}dx$$
$$= 2\operatorname{Re}\int_{\mathbb{R}^n} \nabla u\cdot x\overline{(\Delta u - \lambda|u|^{p-1}u)}dx$$
$$= 2\operatorname{Re}\int_{\mathbb{R}^n} \nabla u\cdot x\overline{\Delta u}dx - 2\operatorname{Re}\int_{\mathbb{R}^n} \lambda\nabla u\cdot x|u|^{p-1}\overline{u}dx$$
$$= -2\operatorname{Re}\int_{\mathbb{R}^n} \nabla(\nabla u\cdot x)\cdot\nabla\overline{u}dx - \frac{\lambda}{p+1}\int_{\mathbb{R}^n} x\cdot\nabla(|u|^{p+1})dx$$

$$= (2-n)\|\nabla u\|_{L^2}^2 - \frac{2n\lambda}{p+1}\|u\|_{L^{p+1}}^{p+1}$$

and

$$\begin{aligned}
n\,\mathrm{Im}\int_{\mathbb{R}^n}(\partial_t u)\overline{u}dx &= n\,\mathrm{Im}\int_{\mathbb{R}^n} i(\Delta u - \lambda|u|^{p-1}u)\overline{u}dx \\
&= n\,\mathrm{Re}\int_{\mathbb{R}^n}(\Delta u - \lambda|u|^{p-1}u)\overline{u}dx \\
&= -n\,\mathrm{Re}\int_{\mathbb{R}^n}\nabla u \cdot \nabla\overline{u}dx - n\,\mathrm{Re}\int_{\mathbb{R}^n}\lambda|u|^{p-1}u\overline{u}dx \\
&= -n\|\nabla u\|_{L^2}^2 - n\lambda\|u\|_{L^{p+1}}^{p+1},
\end{aligned}$$

we obtain

$$\begin{aligned}
&\frac{d}{dt}\mathrm{Im}\int_{\mathbb{R}^n}\nabla u\cdot x\overline{u}dx \\
&= (2-n)\|\nabla u\|_{L^2}^2 - \frac{2n\lambda}{p+1}\|u\|_{p+1}^{p+1} - (-n\|\nabla u\|_{L^2}^2 - n\lambda\|u\|_{p+1}^{p+1}) \\
&= 2\|\nabla u\|_{L^2}^2 + \frac{\lambda n(p-1)}{p+1}\|u\|_{p+1}^{p+1} && (2.3.4) \\
&= 2\left(\|\nabla u\|_{L^2}^2 + \frac{2\lambda}{p+1}\|u\|_{p+1}^{p+1}\right) - \frac{2\lambda}{p+1}\|u\|_{p+1}^{p+1} + \frac{\lambda n(p-1)}{p+1}\|u\|_{p+1}^{p+1} \\
&= 2E(u_0) + \frac{\lambda(n(p-1)-4)}{p+1}\|u\|_{p+1}^{p+1}. && (2.3.5)
\end{aligned}$$

Concatenating (2.3.5) and

$$\begin{aligned}
h(t) &:= \|(x+2it\nabla)u(t)\|_{L^2}^2 + \frac{8t^2\lambda}{p+1}\|u(t)\|_{p+1}^{p+1} \\
&= \|xu(t)\|_{L^2}^2 + 4t^2\|\nabla u(t)\|_{L^2}^2 + 2\,\mathrm{Re}\int_{\mathbb{R}^n} 2it\nabla u(t)\cdot x\overline{u(t)}dx + \frac{8t^2\lambda}{p+1}\|u(t)\|_{p+1}^{p+1} \\
&= \|xu(t)\|_{L^2}^2 + 4t^2E(u_0) - 4t\,\mathrm{Im}\int_{\mathbb{R}^n}\nabla u(t)\cdot x\overline{u(t)}dx,
\end{aligned}$$

we get

$$\begin{aligned}
h'(t) &= \frac{d}{dt}\|xu(t)\|_{L^2}^2 + 8tE(u_0) - 4\,\mathrm{Im}\int_{\mathbb{R}^n}\nabla u(t)\cdot x\overline{u(t)}dx \\
&\quad - 4t\frac{d}{dt}\mathrm{Im}\int_{\mathbb{R}^n}\nabla u(t)\cdot x\overline{u(t)}dx \\
&= 8tE(u_0) - 4t\frac{d}{dt}\mathrm{Im}\int_{\mathbb{R}^n}\nabla u(t)\cdot x\overline{u(t)}dx \\
&= 8tE(u_0) - 4t\left(2E(u_0) + \frac{\lambda(n(p-1)-4)}{p+1}\|u(t)\|_{p+1}^{p+1}\right) \\
&= -\frac{4\lambda(n(p-1)-4)}{p+1}t\|u(t)\|_{p+1}^{p+1}. && (2.3.6)
\end{aligned}$$

In conclusion, we see that

$$\|(x+2it\nabla)u(t)\|_{L^2}^2 + \frac{8t^2\lambda}{p+1}\|u(t)\|_{p+1}^{p+1}$$

$$= \backslash x u_0 \backslash_{L^2}^2 \quad \frac{4\lambda(n(p \quad 1) \quad 4)}{p+1} \bigcap_0^t s \backslash u(s) \backslash_{p+1}^{p+1} ds.$$

2.4 Justification of the derivation of conservation laws

In this section, we justify the method to derive conservation laws of the mass and the energy for H^1 solutions of nonlinear Schrödinger equation (2.2.1) as in Section 2.3. To Justify the method, we present two approximating arguments.

2.4.1 Justification of the derivation of conservation laws by Sequence of regularized equations whose solutions have enough regularities

In what follows, let $\rho \ni C_0(\mathbb{R}^n)$ with $0 \geq \rho \geq 1$, $\rho(x) = 0$ if $\|x\| \approx 1$, $\sum_{\mathbb{R}^n} \rho(x)dx = 1$. We denote $\rho_\varepsilon(x) = \frac{1}{\varepsilon^n}\rho \;\; \frac{x}{\varepsilon}\langle$ by ρ_ε.

We consider the Cauchy problem for the nonlinear Schrödinger equation

$$\begin{cases} i\partial_t u_\varepsilon + \Delta u_\varepsilon = \rho_\varepsilon \bullet f(\rho_\varepsilon \bullet u_\varepsilon), \quad t \ni [t_0 \quad T, t_0 + T],\ x \ni \mathbb{R}^n, \\ u_\varepsilon(t_0, x) = (\rho_\varepsilon \bullet u_0)(x), \quad x \ni \mathbb{R}^n. \end{cases} \tag{2.4.1}$$

For (2.4.1), we have the following result:

Theorem 2.4.1 (e.g. [23])**.** *Let $1 < p < \alpha(n)$ and $0 < \varepsilon < 1$. For all $u_0 \ni H^1$, there exists $T > 0$ such that there exists unique solution*

$$u_\varepsilon \ni \bigcap_{j=1}^{\in} C^1([t_0 \quad T, t_0 + T], H^j) \tag{2.4.2}$$

of (2.4.1) with

$$\backslash u_\varepsilon \backslash_{L^{|}([t_0 \quad T, t_0+T], H^1)} \geq 2 \backslash u_0 \backslash_{H^1}. \tag{2.4.3}$$

Moreover, denoting a solution of (2.2.1) replacing u_0 to $u(t_0)$ by $u \ni C([t_0 \quad T, t_0 + T], H^1)$, for $2 \geq q \geq \alpha(n) + 1$,

$$\sup_{t \mathcal{D} I_T} \backslash u_\varepsilon(t) \quad u(t) \backslash_{L^q} \uparrow \ 0 \tag{2.4.4}$$

as $\varepsilon \uparrow\ 0$.

Proof. From now on, $L^p_{t_0} X$ denotes the Banach space $L^p([t_0 \quad T, t_0 + T], X)$ for $p \ni [1, \in]$ and a Banach space X.

We show that the map

$$\Phi(u_\varepsilon) = U(t)(\rho_\varepsilon \bullet u_0) \quad i \bigcap_0^t U(t \quad s)\rho_\varepsilon \bullet f(\rho_\varepsilon \bullet u_\varepsilon)(s) ds$$

is a contraction mapping in

$$X_{t_0} = \}v \ni C([t_0 \quad T, t_0 + T], H^1) \{ \ L^\sigma((t_0 \quad T, t_0 + T), W^{1,p+1});$$

$$\|v\|_{L^\infty_{t_0}H^1} + \|v\|_{L^\sigma_{t_0}W^{1,p+1}} \leq 4C_0\|v_0\|_{H^1} =: M\},$$

$$d(v_1, v_2) = \|v_1 - v_2\|_{L^\sigma_{t_0}L^{p+1}}$$

with $\sigma = 4(p+1)/n(p-1)$. By Housdorff Young's inequality, since we have

$$\|\rho_0 * u_0\|_{H^1} \leq \|u_0\|_{H^1},$$
$$\|\rho_\varepsilon * v\|_{L^q} \leq \|v\|_{L^q}$$

for all $v \in L^q$ with $q \geq 1$, we can show that Φ is the contraction map in the way similar to Theorem 2.2.1. Indeed, combining Strichartz's estimate with Hölder's inequality and Sobolev embedding, we deduce that for all $u_\varepsilon \in X_T$,

$$\begin{aligned}
&\|\Phi(u_\varepsilon)\|_{L^\infty_{t_0}L^2} + \|\Phi(u_\varepsilon)\|_{L^\sigma_{t_0}L^{p+1}} \\
&\leq \|U(t)\rho_\varepsilon * u_0\|_{L^\infty_{t_0}L^2} + \left\|\int_0^t U(t-s)(\rho_\varepsilon * f(\rho_\varepsilon * u_\varepsilon))(s)ds\right\|_{L^\infty_{t_0}L^2} \\
&\quad + \|U(t)\rho_\varepsilon * u_0\|_{L^\sigma_{t_0}L^{p+1}} + \left\|\int_0^t U(t-s)(\rho_\varepsilon * f(\rho_\varepsilon * u_\varepsilon))(s)ds\right\|_{L^\sigma_{t_0}L^{p+1}} \\
&\leq 2C_0\|\rho_\varepsilon * u_0\|_{L^2} + C\|\rho_\varepsilon * f(\rho_\varepsilon * u_\varepsilon)\|_{L^{\sigma'}_{t_0}L^{(p+1)/p}} \\
&\leq 2C_0\|u_0\|_{L^2} + C\|f(\rho_\varepsilon * u_\varepsilon)\|_{L^{\sigma'}_{t_0}L^{(p+1)/p}} \\
&\leq 2C_0\|u_0\|_{L^2} + C\|\rho_\varepsilon * u_\varepsilon\|^p_{L^{\sigma'p}_{t_0}L^{p+1}} \\
&\leq 2C_0\|u_0\|_{L^2} + C\|u_\varepsilon\|^p_{L^{\sigma'p}_{t_0}L^{p+1}} \\
&\leq 2C_0\|u_0\|_{L^2} + CT^{1/\sigma'p}\|u_\varepsilon\|^p_{L^\infty_{t_0}H^1} \\
&\leq 2C_0\|u_0\|_{L^2} + CT^{1/\sigma'p}M^p \\
&\leq M/2
\end{aligned}$$

if T satisfies $CT^{1/\sigma'p}M^p \leq 2C_0\|\nabla u\|_{L^2}$. Hence, it follows from Strichartz's estimate, Hölder's inequality and Sobolev embedding that

$$\begin{aligned}
&\|\nabla\Phi(u_\varepsilon)\|_{L^\infty_{t_0}L^2} + \|\nabla\Phi(u_\varepsilon)\|_{L^\sigma_{t_0}L^{p+1}} \\
&\leq \|U(t)\nabla(\rho_\varepsilon * u_0)\|_{L^\infty_{t_0}L^2} + \left\|\int_0^t U(t-s)\nabla(\rho_\varepsilon * f(\rho_\varepsilon * u_\varepsilon))(s)ds\right\|_{L^\infty_{t_0}L^2} \\
&\quad + \|U(t)\nabla(\rho_\varepsilon * u_0)\|_{L^\sigma_{t_0}L^{p+1}} + \left\|\int_0^t U(t-s)\nabla(\rho_\varepsilon * f(\rho_\varepsilon * u_\varepsilon))(s)ds\right\|_{L^\sigma_{t_0}L^{p+1}} \\
&\leq 2C_0\|\nabla(\rho_\varepsilon * u_0)\|_{L^2} + C\|\nabla(\rho_\varepsilon * f(\rho_\varepsilon * u_\varepsilon))\|_{L^{\sigma'}_{t_0}L^{(p+1)/p}} \\
&\leq 2C_0\|\rho_\varepsilon * \nabla u_0\|_{L^2} + C\|\rho_\varepsilon * \nabla f(\rho_\varepsilon * u_\varepsilon)\|_{L^{\sigma'}_{t_0}L^{(p+1)/p}} \\
&\leq 2C_0\|\nabla u_0\|_{L^2} + C\|\nabla f(\rho_\varepsilon * u_\varepsilon)\|_{L^{\sigma'}_{t_0}L^{(p+1)/p}} \\
&\leq 2C_0\|\nabla u_0\|_{L^2} + C\left\||\rho_\varepsilon * u_\varepsilon|^{p-1}\nabla(\rho_\varepsilon * u_\varepsilon)\right\|_{L^{\sigma'}_{t_0}L^{(p+1)/p}} \\
&\leq 2C_0\|\nabla u_0\|_{L^2} + C\|\rho_\varepsilon * u_\varepsilon\|^{p-1}_{L^\infty_{t_0}L^{p+1}}\|\rho_\varepsilon * \nabla u_\varepsilon\|_{L^{\sigma'p}_{t_0}L^{p+1}} \\
&\leq 2C_0\|\nabla u_0\|_{L^2} + C\|u_\varepsilon\|^{p-1}_{L^\infty_{t_0}L^{p+1}}\|\nabla u_\varepsilon\|_{L^{\sigma'p}_{t_0}L^{p+1}}
\end{aligned}$$

$$
\begin{aligned}
&\geq 2C_0\backslash \quad u_0\backslash_{L^2} + CT^{\delta}\backslash u_\varepsilon\backslash^{p \quad 1}_{L^{\natural}_{t_0} H^1}\backslash \quad u_\varepsilon\backslash_{L^{\sigma}_{t_0} L^{p+1}} \\
&\geq 2C_0\backslash \quad u_0\backslash_{L^2} + CT^{\delta}M^p \ (\delta = \frac{n+2 \quad (n \quad 2)p}{2(p+1)}) \\
&\geq M/2
\end{aligned}
$$

if T satisfies $CT^{\delta}M^p \geq 2C_0\backslash u\backslash_{L^2}$. Thus $\Phi(u_\varepsilon) \ni X_T$ holds. Similary, it holds that for all u_ε, $v_\varepsilon \ni X_T$,

$$
\begin{aligned}
&d(\Phi(u_\varepsilon), \Phi(v_\varepsilon)) \\
&= \backslash\Phi(u_\varepsilon) \quad \Phi(v_\varepsilon)\backslash_{L^{\sigma}_{t_0} L^{p+1}} \\
&\geq \left(\bigcap_0^t U(t \quad s)\rho_\varepsilon \bullet \} f(\rho_\varepsilon \bullet u_\varepsilon) \quad f(\rho_\varepsilon \bullet v_\varepsilon)\langle ds\right)_{L^{\sigma}_{t_0} L^{p+1}} \\
&\geq \left(\bigcap_0^t U(t \quad s)\}(\|\rho_\varepsilon \bullet u_\varepsilon\|^{p \quad 1}\rho_e \bullet u_\varepsilon)(s) \quad (\|\rho_\varepsilon \bullet v_\varepsilon\|^{p \quad 1}\rho_\varepsilon \bullet v_\varepsilon)(s)\langle ds\right)_{L^{\sigma}_{t_0} L^{p+1}} \\
&\geq C\left(\|\rho_\varepsilon \bullet u_\varepsilon\|^{p \quad 1}\rho_\varepsilon \bullet u_\varepsilon \quad \|\rho_{\ \varepsilon} \bullet v_\varepsilon\|^{p \quad 1}\rho_\varepsilon \bullet v_\varepsilon\right)_{L^{\sigma\infty}_{t_0} L^{(p+1)/p}} \\
&\geq C\left((\|\rho_\varepsilon \bullet u_\varepsilon\|^{p \quad 1} + \|\rho_\varepsilon \bullet v_\varepsilon\|^{p \quad 1})\|\rho_\varepsilon \bullet u_\varepsilon \quad \rho_\varepsilon \bullet v_\varepsilon\|\right)_{L^{\sigma\infty}_{t_0} L^{(p+1)/p}} \\
&\geq CT^{\delta}(\backslash\rho_\varepsilon \bullet u_\varepsilon\backslash^{p \quad 1}_{L^{\natural}_{t_0} L^{p+1}} + \backslash\rho_\varepsilon \bullet v_\varepsilon\backslash^{p \quad 1}_{L^{\natural}_{t_0} L^{p+1}})\backslash\rho_\varepsilon \bullet (u_\varepsilon \quad v_\varepsilon)\backslash_{L^{\sigma}_{t_0} L^{p+1}} \\
&\geq CT^{\delta}(\backslash u_\varepsilon\backslash^{p \quad 1}_{L^{\natural}_{t_0} L^{p+1}} + \backslash v_\varepsilon\backslash^{p \quad 1}_{L^{\natural}_{t_0} L^{p+1}})\backslash u_\varepsilon \quad v_\varepsilon\backslash_{L^{\sigma}_{t_0} L^{p+1}} \\
&\geq CT^{\delta}(\backslash u_\varepsilon\backslash^{p \quad 1}_{L^{\natural}_{t_0} H^1} + \backslash v_\varepsilon\backslash^{p \quad 1}_{L^{\natural}_{t_0} H^1})\backslash u_\varepsilon \quad v_\varepsilon\backslash_{L^{\sigma}_{t_0} L^{p+1}} \\
&\geq CT^{\delta}M^{p \quad 1}d(u_\varepsilon, v_\varepsilon) \\
&\geq \frac{1}{2}d(u_\varepsilon, v_\varepsilon)
\end{aligned}
$$

if T satisfies $CT^{\delta}M^{p \quad 1} \geq 1/2$, that is Φ is the contraction mapping in X_T. In conclusion, we get a local solution u of (2.4.1). Uniqueness of the solution follows easy from way similar to Theorem 2.2.1.

Next, by Duhamel principle, we can transform the equation (2.4.1) to a integral equation

$$
u_\varepsilon = U(t)(\rho_\varepsilon \bullet u_0) \quad i\bigcap_0^t U(t \quad s)\rho_\varepsilon \bullet f(\rho_\varepsilon \bullet u_\varepsilon)(s)ds. \tag{2.4.5}
$$

Multiplying (2.4.5) by ∂_x^α for all multi-index α, we get

$$
\partial_x^\alpha u_\varepsilon = U(t)(\partial_x^\alpha \rho_\varepsilon \bullet u_0) \quad i\bigcap_0^t U(t \quad s)(\partial_x^\alpha \rho_\varepsilon \bullet f(\rho_\varepsilon \bullet u_\varepsilon))(s)ds,
$$

which impies

$$
u_\varepsilon \ni \overset{\infty}{\underset{j=1}{}} C([t_0 \quad T, t_0 + T], H^j). \tag{2.4.6}
$$

By the equation (2.4.1), this yields (2.4.2) (see Remark 2.2.2).

Next, we show (2.4.3). Using (2.4.5), we obtain

$$
\backslash u_\varepsilon\backslash_{L^{\natural}_{t_0} H^1} \geq \backslash U(t)(\rho_\varepsilon \bullet u_0)\backslash_{L^{\natural} \ H^1} + \left(\bigcap_0^t U(t \quad s)\rho_\varepsilon \bullet f(\rho_\varepsilon \bullet u_\varepsilon)(s)ds\right)_{L^{\natural}_{t_0} H^1}
$$

$$\geq \backslash \rho_\varepsilon \bullet u(t_0)\backslash_{H^1} + C_0 \backslash \rho_\varepsilon \bullet f(\rho_\varepsilon \bullet u_\varepsilon)\backslash_{L^{\sigma\infty}_{t_0} W^{1,\frac{p+1}{p}}}$$
$$\geq \backslash u(t_0)\backslash_{H^1} + C_0 \backslash f(\rho_\varepsilon \bullet u_\varepsilon)\backslash_{L^{\sigma\infty}_{t_0} W^{1,\frac{p+1}{p}}}$$
$$\geq \backslash u(t_0)\backslash_{H^1} + CT^{1/\sigma\infty p} M^p + CT^\delta M^p$$
$$\geq 2\backslash u(t_0)\backslash_{H^1},$$

if T satisfies $CT^{1/\sigma\infty p} M^p + CT^\delta M^p < \backslash u(t_0)\backslash_{H^1}$.

Finally, we show (2.4.4). we estimate that

$$\begin{aligned}
&\backslash u \quad u_\varepsilon \backslash_{L^\sigma_{t_0} L^{p+1}} \\
&\geq \backslash U(t)(u_0 \quad \rho_\varepsilon \bullet u_0)\backslash_{L^\sigma_{t_0} L^{p+1}} \\
&\quad + \Big(\bigcap_0^t U(t \quad s)(f(u) \quad \rho_\varepsilon \bullet f(\rho_\varepsilon \bullet u_\varepsilon))(s)ds\Big)_{L^\sigma_{t_0} L^{p+1}} \\
&\geq \backslash u \quad \rho_\varepsilon \bullet u_0 \backslash_{L^2} + \backslash f(u) \quad \rho_\varepsilon \bullet f(\rho_\varepsilon \bullet u_\varepsilon)\backslash_{L^{\sigma\infty}_{t_0} L^{\frac{p+1}{p}}} \\
&\geq \backslash u \quad \rho_\varepsilon \bullet u_0 \backslash_{L^2} \\
&\quad + \backslash f(u) \quad \rho_\varepsilon \bullet f(u)\backslash_{L^{\sigma\infty}_{t_0} L^{\frac{p+1}{p}}} + \backslash \rho_\varepsilon \bullet f(u) \quad \rho_\varepsilon \bullet f(\rho_\varepsilon \bullet u_\varepsilon)\backslash_{L^{\sigma\infty}_{t_0} L^{\frac{p+1}{p}}} \\
&\geq \backslash u \quad \rho_\varepsilon \bullet u_0 \backslash_{L^2} \\
&\quad + \backslash f(u) \quad \rho_\varepsilon \bullet f(u)\backslash_{L^{\sigma\infty}_{t_0} L^{\frac{p+1}{p}}} + \backslash f(u) \quad f(\rho_\varepsilon \bullet u_\varepsilon)\backslash_{L^{\sigma\infty}_{t_0} L^{\frac{p+1}{p}}} \\
&\geq \backslash u \quad \rho_\varepsilon \bullet u_0 \backslash_{L^2} \\
&\quad + \backslash f(u) \quad \rho_\varepsilon \bullet f(u)\backslash_{L^{\sigma\infty}_{t_0} L^{\frac{p+1}{p}}} \\
&\quad + CT^\delta(\backslash u \backslash^{p\ \ 1}_{L^{|}_{t_0} H^1} + \backslash u_\varepsilon \backslash^{p\ \ 1}_{L^{|}_{t_0} H^1})\backslash u \quad \rho_\varepsilon \bullet u_\varepsilon \backslash_{L^\sigma_T L^{p+1}} \\
&\geq \backslash u \quad \rho_\varepsilon \bullet u_0 \backslash_{L^2} \\
&\quad + \backslash f(u) \quad \rho_\varepsilon \bullet f(u)\backslash_{L^{\sigma\infty}_{t_0} L^{\frac{p+1}{p}}} \\
&\quad + CT^\delta(\backslash u \backslash^{p\ \ 1}_{L^{|}_{t_0} H^1} + \backslash u_\varepsilon \backslash^{p\ \ 1}_{L^{|}_{t_0} H^1})\backslash u \quad \rho_\varepsilon \bullet u \backslash_{L^\sigma_T L^{p+1}} \\
&\quad + CT^\delta(\backslash u \backslash^{p\ \ 1}_{L^{|}_{t_0} H^1} + \backslash u_\varepsilon \backslash^{p\ \ 1}_{L^{|}_{t_0} H^1})\backslash u \quad u_\varepsilon \backslash_{L^\sigma_T L^{p+1}} \\
&\geq \backslash u \quad \rho_\varepsilon \bullet u_0 \backslash_{L^2} \\
&\quad + \backslash f(u) \quad \rho_\varepsilon \bullet f(u)\backslash_{L^{\sigma\infty}_{t_0} L^{\frac{p+1}{p}}} \\
&\quad + CT^\delta(\backslash u \backslash^{p\ \ 1}_{L^{|}_{t_0} H^1} + 2^{p\ \ 1}\backslash u(t_0)\backslash^{p\ \ 1}_{H^1})\backslash u \quad \rho_\varepsilon \bullet u \backslash_{L^\sigma_{t_0} L^{p+1}} \\
&\quad + CT^\delta(\backslash u \backslash^{p\ \ 1}_{L^{|}_{t_0} H^1} + 2^{p\ \ 1}\backslash u(t_0)\backslash^{p\ \ 1}_{H^1})\backslash u \quad u_\varepsilon \backslash_{L^\sigma_{t_0} L^{p+1}}.
\end{aligned}$$

Therefore, if T is small enough depending on $\backslash u \backslash_{L^{|}_{t_0} H^1}$ and $\backslash u(t_0)\backslash_{H^1}$, then we obtain

$$\begin{aligned}
\backslash u \quad u_\varepsilon \backslash_{L^\sigma_{t_0} L^{p+1}} &\geq \backslash u \quad \rho_\varepsilon \bullet u_0 \backslash_{L^2} \\
&\quad + CT^\delta(\backslash u \backslash^{p\ \ 1}_{L^{|}_{t_0} H^1} + 2^{p\ \ 1}\backslash u(t_0)\backslash^{p\ \ 1}_{H^1})\backslash u \quad \rho_\varepsilon \bullet u \backslash_{L^\sigma_{t_0} L^{p+1}} \\
&\uparrow \ 0
\end{aligned}$$

as $\varepsilon \uparrow \ 0$. Applying Strichartz's estimate and Gagliardo-Nirenberg's inequality, this implies (2.4.4). ☐

Using Theorem 2.4.1, we can prove exactly the following result:

Proposition 2.4.1 (e.g. [23])**.** *Let $u_0 \ni H^1$. (q, r) denotes some admissible pair. Let $T > 0$. Assume that u is a corresponding solution of (2.2.2) satisfying*

$$u \ni C([0, T]; H^1) \{ \ L^q_{loc}((0, T); W^{1,r}).$$

Then, it holds that

$$E(u(t)) = E(u_0),$$
$$\backslash u(t) \backslash_{L^2} = \backslash u_0 \backslash_{L^2}$$

for all $t \ni [0, T]$, where

$$E(u) = \backslash \quad u \backslash^2_{L^2} + \frac{2\lambda}{p+1} \backslash u \backslash^{p+1}_{L^{p+1}}.$$

Proof. We take $t_0 \ni [0, T]$. By applying Theorem 2.4.1, there exists $\widetilde{T} \ni (0, T)$ such that there exists a solution

$$u_\varepsilon \ni \bigcap_{j=1}^{\infty} C^1([t_0 \quad \widetilde{T}, t_0 + \widetilde{T}], H^j)$$

of (2.4.1) replacing u_0 to $u(t_0)$. Namely, u_ε satisfies (2.4.3) and (2.4.4). In what follows, we put $I_{\widetilde{T}} = [t_0 \quad \widetilde{T}, t_0 + \widetilde{T}]$. Multiplying the equation (2.4.1) by $\overline{u}_\varepsilon$, integrating over $\mathbb{R}^n$, and taking imaginary part, we caluculate

$$\begin{aligned} 0 &= 2\,\mathrm{Im}(i\partial_t u_\varepsilon + \Delta u_\varepsilon \quad \rho_\varepsilon \bullet f(\rho_\varepsilon \bullet u_\varepsilon), u_\varepsilon)_{L^2} \\ &= 2\,\mathrm{Im}(i\partial_t u_\varepsilon, u_\varepsilon)_{L^2} \quad 2\,\mathrm{Im}(\rho_e \bullet f(\rho_\varepsilon \bullet u_\varepsilon), u_\varepsilon)_{L^2} \\ &= \frac{d}{dt} \backslash u_\varepsilon(t) \backslash^2_{L^2} \quad 2\,\mathrm{Im}(f(\rho_\varepsilon \bullet u_\varepsilon), \rho_\varepsilon \bullet u_\varepsilon)_{L^2} \\ &= \frac{d}{dt} \backslash u_\varepsilon(t) \backslash^2_{L^2}, \end{aligned}$$

which yields

$$\backslash u_\varepsilon(t) \backslash_{L^2} = \backslash u_\varepsilon(t_0) \backslash_{L^2} = \backslash \rho_\varepsilon \bullet u(t_0) \backslash_{L^2}$$

for all $t \ni I_{\widetilde{T}}$. Combining (2.4.4) with $\rho_\varepsilon \bullet u(t_0) \uparrow \ u(t_0)$ as $\varepsilon \propto 0$, when $\varepsilon \propto 0$, we deduce

$$\backslash u(t) \backslash_{L^2} = \backslash u(t_0) \backslash_{L^2} \tag{2.4.7}$$

for all $t \ni I_{\widetilde{T}}$.

Moreover, multiplying the equation (2.4.1) by $\overline{\partial_t u_\varepsilon}$, integrating over $\mathbb{R}^n$, and taking the real part, we compute

$$\begin{aligned} 0 &= \quad 2\,\mathrm{Re}(i\partial_t u_\varepsilon + \Delta u_\varepsilon \quad \rho_\varepsilon \bullet f(\rho_\varepsilon \bullet u_\varepsilon), \partial_t u_\varepsilon)_{L^2} \\ &= 2\,\mathrm{Re}(\quad u_\varepsilon, \partial_t \quad u_\varepsilon)_{L^2} + 2\,\mathrm{Re}(f(\rho_\varepsilon \bullet u_\varepsilon), \partial_t(\rho_\varepsilon \bullet u_\varepsilon))_{L^2} \\ &= \frac{d}{dt} \Big) \backslash \quad u_\varepsilon \backslash^2_{L^2} + \frac{2\lambda}{p+1} \backslash \rho_\varepsilon \bullet u_\varepsilon \backslash^{p+1}_{L^{p+1}} \Big[, \end{aligned}$$

which implies

$$
\begin{aligned}
&\backslash \quad u_\varepsilon \backslash^2_{L^2} + \frac{2\lambda}{p+1} \backslash \rho_\varepsilon \bullet u_\varepsilon \backslash^{p+1}_{L^{p+1}} \\
&\quad = \backslash \quad (\widetilde{\rho_\varepsilon} \bullet u(t_0)) \backslash^2_{L^2} + \frac{2\lambda}{p+1} \backslash \widetilde{\rho_\varepsilon} \bullet u(t_0) \backslash^{p+1}_{L^{p+1}} \qquad (2.4.8)
\end{aligned}
$$

for all $t \ni I_{\widetilde{T}}$, where $\widetilde{\rho_\varepsilon} = \rho_\varepsilon \bullet \rho_\varepsilon$. By (2.4.3), (2.4.4) and Sobolev embedding, we obtain that

$$
\begin{aligned}
&\Big\| \backslash \rho_\varepsilon \bullet u_\varepsilon \backslash^{p+1}_{L^{p+1}} \quad \backslash u \backslash^{p+1}_{L^{p+1}} \Big\| \\
&\geq \bigcap_{\mathbb{R}^n} \Big\| |\rho_\varepsilon \bullet u_\varepsilon|^{p+1} \quad |\rho_\varepsilon \bullet u|^{p+1} \Big\| dx \\
&\quad + \bigcap_{\mathbb{R}^n} \Big\| |\rho_\varepsilon \bullet u|^{p+1} \quad |u|^{p+1} \Big\| dx \\
&\geq C \quad \backslash \rho_\varepsilon \bullet u_\varepsilon \backslash^p_{L^{p+1}} + \backslash \rho_\varepsilon \bullet u \backslash^p_{L^{p+1}} \langle \backslash \rho_\varepsilon \bullet u_\varepsilon \quad \rho_\varepsilon \bullet u \backslash_{L^{p+1}} \\
&\quad + C \quad \backslash \rho_\varepsilon \bullet u \backslash^p_{L^{p+1}} + \backslash u \backslash^p_{L^{p+1}} \langle \backslash \rho_\varepsilon \bullet u \quad u \backslash_{L^{p+1}} \\
&\geq C \quad \backslash u_\varepsilon \backslash^p_{L^{p+1}} + \backslash u \backslash^p_{L^{p+1}} \langle \backslash u_\varepsilon \quad u \backslash_{L^{p+1}} \\
&\quad + C \backslash u \backslash^p_{L^{p+1}} \backslash \rho_\varepsilon \bullet u \quad u \backslash_{L^{p+1}} \\
&\geq C \Big) \backslash u_\varepsilon \backslash^p_{L^| \;(I_{\widetilde{T}};H^1)} + \backslash u \backslash^p_{L^| \;(I_{\widetilde{T}};H^1)} \Big(\backslash u_\varepsilon \quad u \backslash_{L^{p+1}} \\
&\quad + C \backslash u \backslash^p_{L^| \;(I_{\widetilde{T}};H^1)} \backslash \rho_\varepsilon \bullet u \quad u \backslash_{L^{p+1}} \\
&\geq C \Big) \backslash u(t_0) \backslash^p_{H^1} + \backslash u \backslash^p_{L^| \;(I_{\widetilde{T}};H^1)} \Big(\backslash u_\varepsilon \quad u \backslash_{L^{p+1}} \\
&\quad + C \backslash u \backslash^p_{L^| \;(I_{\widetilde{T}};H^1)} \backslash \rho_\varepsilon \bullet u \quad u \backslash_{L^{p+1}} \\
&\uparrow \;\; 0
\end{aligned}
$$

as $\varepsilon \propto 0$. Similarly, we get

$$
\Big\| \backslash \widetilde{\rho_\varepsilon} \bullet u(t_0) \backslash^{p+1}_{L^{p+1}} \quad \backslash u(t_0) \backslash^{p+1}_{L^{p+1}} \Big\| \uparrow \;\; 0
$$

as $\varepsilon \propto 0$. Furthermore, we deduce from (2.4.3) and (2.4.4) that

$$
\begin{aligned}
&u_\varepsilon(t) \uparrow \;\; u(t) \quad \textit{weakly in } H^1 \quad \textit{as } \varepsilon \propto 0, \\
&\backslash \quad u(t) \backslash_{L^2} \geq \liminf_{\varepsilon \prime 0} \backslash \quad u_\varepsilon(t) \backslash_{L^2}, \\
&\backslash \quad (\widetilde{\rho_\varepsilon} \bullet u(t_0)) \backslash_{L^2} \geq \backslash \quad u(t_0) \backslash_{L^2}
\end{aligned}
$$

for any $t \ni I_{\widetilde{T}}$. We show that $u_\varepsilon(t) \uparrow \;\; u(t)$ weakly in H^1. From (2.4.3) and Theorem 6.2.2, it follows that there exist $\}\varepsilon_n\langle^{\in}_{n=1} \to (0, \in\,)$ with $\varepsilon_n \propto 0$ and $v \ni H^1$ such that $u_{\varepsilon_n} \uparrow \;\; v$ weakly in H^1 as $n \uparrow \;\; \in$. By (2.4.4), $u_\varepsilon \uparrow \;\; u$ in L^2 as $\varepsilon \propto 0$. Therefore, we deduce that

$$
\begin{aligned}
\| \rangle u \quad v, \varphi|_{\mathscr{S} \infty \mathscr{S}} \| \geq \| \rangle u \quad u_{\varepsilon_n}, \varphi|_{L^2 * L^2} \| + \| \rangle u_{\varepsilon_n} \quad v, \varphi|_{H^1 * H \;\; 1} \| \\
\uparrow \;\; 0
\end{aligned}
$$

as $\varepsilon \propto 0$, which implies $u(x) = v(x)\ a.e. x \ni \mathbb{R}^n$, where $\mathscr{S}^{\infty} = \mathscr{S}^{\infty}(\mathbb{R}^n)$ denotes the space of tempered distributions on $\mathbb{R}^n$. Hence, we obtain that $u_{\varepsilon_n} \uparrow \;\; u$ weakly in H^1 as $n \uparrow \;\; \in$. Putting $\}\varphi_m\langle^{\in}_{m=1} \to \mathscr{S}$ such that $\varphi_m \uparrow \;\; \varphi$ in $H \;\; ^1$, we have

$$
\| \rangle u_\varepsilon \quad u, \varphi|_{H^1 * H \;\; 1} \| \geq \| \rangle u_\varepsilon \quad u_{\varepsilon_n}, \varphi \quad \varphi_m|_{H^1 * H \;\; 1} \|
$$

$$
\begin{aligned}
&+ \|\rangle u_\varepsilon \quad u_{\varepsilon_n}, \varphi_m|_{L^2 * L^2}\| \\
&+ \|\rangle u_{\varepsilon_n} \quad u, \varphi|_{H^1 * H \ 1}\| \\
\geq\ & (\backslash u_\varepsilon \backslash_{H^1} + \backslash u_{\varepsilon_n} \backslash_{H^1}) \backslash \varphi \quad \varphi_m \backslash_{H \ 1} \\
&+ (\backslash u_\varepsilon \quad u \backslash_{L^2} + \backslash u \quad u_{\varepsilon_n} \backslash_{L^2}) \backslash \varphi_m \backslash_{L^2} \\
&+ \|\rangle u_{\varepsilon_n} \quad u, \varphi|_{H^1 * H \ 1}\| \\
\uparrow\ & 0
\end{aligned}
$$

as $\varepsilon \propto 0$ and n, $m \uparrow \ \in$. Note that ε, n and m is independent of each other. Thus, this implies $u_\varepsilon(t) \uparrow \ u(t)$ weakly in H^1 for all $t \ni I_{\widetilde{T}}$. Furthermore, By Thereom 6.2.1, we can give $\backslash \ u(t)\backslash_{L^2} \geq \liminf_{\varepsilon \prime 0} \backslash \ u_\varepsilon(t)\backslash_{L^2}$. In conclusion, it follows from (2.4.8) that if $\varepsilon \propto 0$, then

$$E(u(t)) \geq E(u(t_0)) \tag{2.4.9}$$

for any $t \ni I_{\widetilde{T}}$. Thus, taking $t_1 \ni I_{\widetilde{T}}$, when we consider the Cauchy problem for (2.4.1) replacing u_0 to $u(t_1)$, if T is sufficiently small (by using $\backslash u \backslash_{L^{\downarrow}_T H^1}$, we set $\widetilde{T}$ again), then we can take the same $\widetilde{T}$ as the existence time of the solution. Hence, in the same way as above, we have

$$E(u(t_0)) \geq E(u(t_1)). \tag{2.4.10}$$

By $E(u(t)) \geq E(u(t_0))$ for any $t \ni I_{\widetilde{T}}$, this contradicts (2.4.10). Therefore, (2.4.9) yields

$$E(u(t)) = E(u(t_0))$$

for all $t \ni I_{\widetilde{T}}$. In conclusion, for any $t_0 \ni [0, T)$, there exists $\widetilde{T} > 0$ such that

$$\backslash u(t)\backslash_{L^2} = \backslash u(t_0)\backslash_{L^2}, \quad E(u(t)) = E(u(t_0))$$

for all $t \ni I_{\widetilde{T}}$. Using the proof by contradiction, we deduce that

$$\backslash u(t)\backslash_{L^2} = \backslash u_0\backslash_{L^2}, \quad E(u(t)) = E(u_0)$$

for any $t \ni [0, T]$. This completes the proof. □

2.4.2 Justification of the derivation of conservation laws Applying the continuous dependence of solutions on the initial data

We present the other method to justify the calculation to derive the conservation law of the mass and the energy for H^1 solution of (2.2.1) as in Section 2.3.

Let $u_0 \ni H^1$. Let u be a corresponding solution $u \ni C([0, T], H^1)$ of (2.2.1) with $u(0) = u_0$ as in Theorem 2.2.2. We remark that $u \ni C([0, T], H^1)$ satisfies the continuous dependence on the initial data. Then, there exists $\}u_{0,n}\langle_{n=1}^{\in} \rightarrow H^2$ such that $u_{0,n} \uparrow \ u_0$ in H^1 as $n \uparrow \ \in$. Moreover, combining Remark 2.2.3 with the continuous dependence, there exist $u_n \ni C([0, T], H^2)$ such that a corresponding solution of (2.2.1) with $u_n = u_{0,n}$ if n is sufficiently large.

For each u_n, we can execute the calculation to derive the conservation of the mass and the energy as in Section 2.3. That is, we obtain

$$\backslash u_n(t)\backslash_{L^2} = \backslash u_{0,n}\backslash_{L^2}, \quad E(u_n(t)) = E(u_{0.n})$$

for all $t \ni [0, T]$ and each $n \ni \mathbb{N}$. Using the continuous dependence and Sobolev embedding, we deduce that for any $t \ni [0, T]$,

$$\begin{aligned} \| \backslash u_n(t) \backslash_{L^2} \quad \backslash u(t) \backslash_{L^2} \| &\geq \backslash u_n(t) \quad u(t) \backslash_{L^2} \\ &\geq \backslash u_n \quad u \backslash_{L^{\mid}_T L^2} \uparrow \ 0 \end{aligned}$$

as $n \uparrow \ \in$, and

$$\begin{aligned} \| E(u_n(t)) \quad E(u(t)) \| &\geq \left\| \backslash \ \ u_n(t) \backslash_{L^2}^2 \quad \backslash \ \ u(t) \backslash_{L^2}^2 \right\| \\ &\geq C \left\| \backslash u_n(t) \backslash_{L^{p+1}}^{p+1} \quad \backslash u(\ t) \backslash_{L^{p+1}}^{p+1} \right\| \\ &\geq (\backslash u_n \backslash_{L^{\mid}_T H^1} + \backslash u \backslash_{L^{\mid}_T H^1}) \backslash u_n \quad u \backslash_{L^{\mid}_T H^1} \\ &\quad + (\backslash u_n \backslash^{p \ \ 1}_{L^{\mid}_T H^1} + \backslash u \backslash^{p \ \ 1}_{L^{\mid}_T H^1}) \backslash u_n \quad u \backslash_{L^{\mid}_T H^1} \\ &\uparrow \ 0 \end{aligned}$$

as $n \uparrow \ \in$. Thus, for any $t \ni [0, T]$,

$$\backslash u(t) \backslash_{L^2} = \backslash u_0 \backslash_{L^2}, \quad E(u(t)) = E(u_0)$$

hold.

Remark 2.4.1. *Actually, to obtain the mass consevation law for H^1 solutions of (2.2.1), we don't need the approximating arguments as in Section 2.4.1- 2.4.2. However, for L^2 solutions, we need the similar argument.*

2.5 Global behavior of nonlinear Schrödinger equations with power type nonlinearity

In this section, we consider a global behavior of nonlinear Schrödinger equation (2.2.1).

2.5.1 Globall wellposedness results

First, we present results for global wellposedness of (2.2.1).

Theorem 2.5.1 ([24])**.** *Let $1 < p < 1 + 4/n$. Let $u_0 \ni L^2$. Assume that (q, r) is some admissible pair. Let $u \ni C([0, T_{\max}), L^2) \ \{ \ L^q_{loc}((0, T_{\max}), L^r)$ be the corresponding maximal solution of (2.2.1) in Theorem 2.2.1. Then $T_{\max} = \in$. Moreover $u \ni L^{\in}([0, \in), L^2)$.*

Proof. Obviously, the consrvation law of the mass implies the desired assertion. □

Theorem 2.5.2 ([16], cf. [23])**.** *Let $1 < p < \alpha(n)$. Let $u_0 \ni H^1$. Assume that (q, r) is some admissible pair. Let $u \ni C([0, T_{\max}), H^1) \ \{ \ L^q_{loc}((0, T_{\max}), W^{1,r})$ be the corresponding maximal solution of (2.2.1) in Theorem 2.2.2. If $\lambda > 0$, or $\lambda < 0$ and $p < 1 + 4/n$, then $T_{\max} = \in$. Moreover $u \ni L^{\in}([0, \in), H^1)$.*

Proof. Assume that $T_{\max} < \infty$. Combining the conservation of energy and mass with Gagliardo-Nirenberg inequality, we see that

$$
\begin{aligned}
\|\nabla u(t)\|_{L^2}^2 &\le E(u_0) + C\|u(t)\|_{L^{p+1}}^{p+1} \\
&\le E(u_0) + C\|\nabla u(t)\|_{L^2}^{\theta(p+1)}\|u(t)\|_{L^2}^{(1-\theta)(p+1)} \\
&\le E(u_0) + C\|\nabla u(t)\|_{L^2}^{\theta(p+1)}\|u_0\|_{L^2}^{(1-\theta)(p+1)}, \qquad (2.5.1)
\end{aligned}
$$

where $\theta = n(p-1)/2(p+1)$. Noting that $\theta(p+1) = \frac{n(p-1)}{2} < 2$, there exists $M > 0$ depending only on $\|u_0\|_{H^1}$ such that $\|\nabla u(t)\|_{L^2} \le M$ for all $t \in [0, T_{\max})$. From the conservation law of mass, this implies that there exists $M > 0$ depending only on $\|u_0\|_{H^1}$ such that

$$\|\nabla u(t)\|_{H^1} \le M$$

for all $t \in [0, T_{\max})$. We choose $t_0 \in [0, T_{\max})$. we can construct the corresponding solution u_1 with $u_1(0) = u(t_0)$ as in Theorem 2.2.2. Indeed, we can show that there exist $T > 0$ such that there exists a solution $u_1 \in X_T$ of (2.2.1) replacing u_0 with $u(t_0)$, where

$$
\begin{aligned}
X_T = \{&v \in C([t_0, t_0+T], H^1) \cap L^\sigma((t_0, t_0+T), W^{1,p+1}); \\
&\|v\|_{L^\infty([t_0,t_0+T];L^2)} + \|v\|_{L^\sigma([t_0,t_0+T];W^{1,p+1})} \le 4C_0\|u(t_0)\|_{H^1} \le 4C_0 M\}, \\
d(v_1, v_2) &= \|v_1 - v_2\|_{L^\sigma_T L^{p+1}}
\end{aligned}
$$

with $\sigma = 4(p+1)/n(p-1)$. Note that T depends only on M. Furthermore, we denote the solution $u^{(1)}$ on $[0, T]$ of (2.2.1) by

$$
u^{(1)}(t) = \begin{cases} u(t) & \text{if } t \in [0, t_0], \\ u_1(t - t_0) & \text{if } t \in [t_0, T]. \end{cases}
$$

Again, Combining the conservation of energy and mass with Gagliardo-Nirenberg inequality, by estimating $\|\nabla u^{(1)}(t)\|_{L^2}^2$ in the same way as (2.5.1), we implies that

$$\|\nabla u^{(1)}(t)\|_{H^1} \le M$$

for all $t \in [0, T]$. Note that repeating the procedure similar to the above, we can continue to take the same $T > 0$ as the time of existence of the solution. This follows a contradiction of $T_{\max} < \infty$. Namely, $T_{\max} = \infty$. Also, by the above argument, it is clear that $u \in L^\infty([0, \infty), H^1)$. □

2.5.2 Blow-up results

We use the virial identity as follows to prove the finite time blow-up of (2.2.1).

Proposition 2.5.1 (virial identity, [11])**.** *Let $u_0 \in \Sigma$. Let $u \in C([0,T], \Sigma)$ be a corresponding solution of (2.2.1) with $u(0) = u_0$. Then it holds that*

$$
\begin{aligned}
\|xu(t)\|_{L^2}^2 = \|xu_0\|_{L^2}^2 &+ 4t\,\mathrm{Im}\int_{\mathbb{R}^n} u_0\, x\cdot\nabla\overline{u_0}\,dx \\
&+ 4\int_0^t\int_0^s \left\{2\|\nabla u(s)\|_{L^2}^2 + \frac{\lambda n(p-1)}{p+1}\|u(s)\|_{L^{p+1}}^{p+1}\right\} d\tau ds
\end{aligned}
$$

for any $t \in [0, T)$.

Proof. First, integrating (2.3.2) from 0 to t, we get

$$\|xu(t)\|_{L^2}^2 = \|xu_0\|_{L^2}^2 + 4\int_0^t \operatorname{Im}\int_{\mathbb{R}^n} u(s)\, x\cdot\nabla\overline{u(s)}\,dx\,ds. \tag{2.5.2}$$

Moreover, integrating (2.3.4) from 0 to s, we deduce that

$$\operatorname{Im}\int_{\mathbb{R}^n} u\, x\cdot\nabla\bar{u}\,dx = \operatorname{Im}\int_{\mathbb{R}^n} u_0\, x\cdot\nabla\overline{u_0}\,dx + \int_0^s \left\{2\|\nabla u(\tau)\|_{L^2}^2 + \frac{\lambda n(p-1)}{p+1}\|u(\tau)\|_{p+1}^{p+1}\right\} d\tau. \tag{2.5.3}$$

Combining (2.5.2) with (2.5.3), we obtain the virial identity. □

Remark 2.5.1. *To justify the above proof, we need a regularization argument (see Secion 6.5 of [4]).*

Applying the virial identity, we can show the finite blow-up result of (2.2.1) as follows:

Theorem 2.5.3 ([11], cf. [23])**.** *Assume that $\lambda < 0$ and $1 + 4/n \ge p < \alpha(n)$. Let $u_0 \in \Sigma$. Let $u \in C([0,T],\Sigma)$ be a corresponding solution of (2.2.1) with $u(0) = u_0$. If $E(u_0) < 0$, then $T_{\max} < \infty$. Moreover, the solution u of (2.2.1) blows up in finite time. Namely, it holds that*

$$\lim_{t\downarrow T_{\max}} \|\nabla u(t)\|_{L^2} = \infty.$$

Proof. Assume that $T_{\max} = \infty$. Using the virial identity and the energy conservation law, since $\lambda < 0$ and $p \le 1 + 4/n$, we estimate

$$\begin{aligned}
\|xu(t)\|_{L^2}^2 &= \|xu_0\|_{L^2}^2 + 4t\operatorname{Im}\int_{\mathbb{R}^n} u_0\, x\cdot\nabla\overline{u_0}\,dx \\
&\quad + 4\int_0^t\int_0^s \left\{2E(u_0) + \frac{\lambda(n(p-1)-4)}{p+1}\|u\|_{p+1}^{p+1}\right\} d\tau\,ds \\
&= \|xu_0\|_{L^2}^2 + 4t\operatorname{Im}\int_{\mathbb{R}^n} u_0\, x\cdot\nabla\overline{u_0}\,dx \\
&\quad + 4t^2E(u_0) + \int_0^t\int_0^s \frac{4\lambda(n(p-1)-4)}{p+1}\|u\|_{p+1}^{p+1}\,d\tau\,ds \\
&< \|xu_0\|_{L^2}^2 + 4t\operatorname{Im}\int_{\mathbb{R}^n} u_0\, x\cdot\nabla\overline{u_0}\,dx + 4t^2E(u_0) \\
&=: Q(t)
\end{aligned}$$

for all $t > 0$. Here, $Q(t)$ is a quadratic function of t. In addition, because of $E(u_0) < 0$, the cofficient of t^2 is nagative. Hence, there exists $T_0 > 0$ such that $\|xu(t)\|_{L^2} < 0$ for any $t > T_0$. This is a contradiction. Therefore, we have $T_{\max} < \infty$.

Next, we assume $\lim_{t\downarrow T_{\max}} \|\nabla u(t)\|_{L^2} < \infty$. In particular, this implies

$$\liminf_{t\downarrow T_{\max}} \|\nabla u(t)\|_{L^2} < \infty.$$

Using the conservation law of mass, since $\inf_{t>t_0} \backslash \ \ u(t)\backslash_{L^2}$ is a mononical increasing, there exist $M > 0$ and a sequence $\}t_k\langle \to [0, T_{\max})$ such that $t_j \downarrow T_{\max}$ and $\backslash u(t_j)\backslash_{H^1} \geq M$ for all $j \ni \mathbb{N}$. Therefore, by Theorem 2.2.2, there exists $T_M > 0$ such that (2.2.1) replacing u_0 to $u(t_k)$ has a solution on $[t_k, t_k + T_M]$. Note that we can take T_M uniformly for k. Hence, putting $k_0 \ni \mathbb{N}$ such that $T_{\max} \quad t_{k_0} < T_M$, applying Theorem 2.2.2 again, (2.2.1) replacing u_0 to $u(t_{k_0})$ has a solution on $[t_{k_0}, t_{k_0} + T_M]$, which contradicts the definition of $T_{\max}$. This completes the proof. □

2.5.3 Application of the pseudo conformal conservation law

We state an application of the pseudo conformal conservation law.

Theorem 2.5.4 ([3])**.** *Let $\lambda > 0$. If $u_0 \ni \Sigma$ and if $u \ni C([0, \in), H^1)$ is a coresponding solution of (2.2.1), then the following properties hold:*

(i) If $p \approx 1 + \frac{4}{n}$, then for any $r \ni [2, \alpha(n) + 1]$ (but, $r \ni [2, \in]$ if $n = 1$, $r \ni [2, \in)$ if $n = 2$), it holds that

$$\backslash u(t)\backslash_{L^r} \geq C\|t\| \ ^{n\left(\frac{1}{2} \ \ \frac{1}{r}\right)} \tag{2.5.4}$$

for all $t \ni [0, \in)$.

(ii) If $p < 1 + \frac{4}{n}$, then for any $r \ni [2, \alpha(n) + 1]$ (but, $r \ni [2, \in]$ if $n = 1$, $r \ni [2, \in)$ if $n = 2$), it holds that

$$\backslash u(t)\backslash_{L^r} \geq C\|t\| \ ^{n\left(\frac{1}{2} \ \ \frac{1}{r}\right)(1 \ \ \theta(r))} \tag{2.5.5}$$

for all $t \ni [0, \in)$, where

$$\theta(r) = \begin{cases} 0 & \text{if } r \ni [2, p+1], \\ \frac{(r \ \ (p+1))(4 \ \ n(p \ \ 1))}{(r \ \ 2)(2(p+1) \ \ n(p \ \ 1))} & \text{if } r > p+1. \end{cases}$$

Proof. We show (2.5.4). If $p \approx 1 + 4/n$, then putting $v(t) = M(\ \ t)u$, using

$$(x + 2it \ \ \ )u = M(t)(2it \ \ \ )M(\ \ t)u, \tag{2.5.6}$$

we transform the psudo conformal conservation law (2.3.1) to

$$\begin{aligned} &4t^2\backslash \ \ v(t)\backslash_{L^2}^2 + \frac{8t^2\lambda}{p+1}\backslash v(t)\backslash_{p+1}^{p+1} \\ &= \backslash xu_0\backslash_{L^2}^2 \quad \frac{4\lambda(n(p \ \ \ 1) \ \ \ 4)}{p+1}\bigcap_0^t s\backslash u(s)\backslash_{p+1}^{p+1} ds. \end{aligned} \tag{2.5.7}$$

This implies that

$$4t^2\backslash \ \ v(t)\backslash_{L^2}^2 \geq \backslash xu_0\backslash_{L^2}^2. \tag{2.5.8}$$

Combining (2.5.8) with Gagliardo-Nirenberg's inequality, the mass conservation law $\backslash u(t)\backslash_{L^2} = \backslash u_0\backslash_{L^2}$, we calculate

$$\backslash u(t)\backslash_{L^r} = \backslash v(t)\backslash_{L^r}$$

$$
\begin{aligned}
&\geq C\|\nabla v(t)\|_{L^2}^{n(\frac{1}{2}-\frac{1}{r})}\|v(t)\|_{L^2}^{1-n(\frac{1}{2}-\frac{1}{r})}\\
&\geq Ct^{-n(\frac{1}{2}-\frac{1}{r})}\|xu_0\|_{L^2}^{n(\frac{1}{2}-\frac{1}{r})}\|u_0\|_{L^2}^{1-n(\frac{1}{2}-\frac{1}{r})}\\
&\geq Ct^{-n(\frac{1}{2}-\frac{1}{r})}.
\end{aligned}
$$

This completes the proof of (2.5.4).

Next, we assume $p < 1 + 4/n$ and $t \approx 1$. Concatenating the identity (2.3.6) and (2.5.6), we obtain

$$4t^2E(v(t)) = E(v(1)) - \frac{4\lambda(n(p-1)-4)}{p+1}\int_1^t s\|u(s)\|_{p+1}^{p+1}ds.$$

Thus, putting $h(t) = t^2\|u(t)\|_{p+1}^{p+1}$, this implies

$$h(t) \geq C + \frac{4-n(p-1)}{2}\int_1^t \frac{1}{s}h(s)ds.$$

Using Gronwall's Lemma, indeed, putting

$$H(t) = C + \frac{4-n(p-1)}{2}\int_1^t \frac{1}{s}h(s)ds,$$

we deduce that

$$h(t) \geq Ct^{\frac{4-n(p-1)}{2}}.$$

Hence, it follows that

$$\|v(t)\|_{L^{p+1}} \geq Ct^{-n(\frac{1}{2}-\frac{1}{p+1})}. \tag{2.5.9}$$

Combining (2.5.7) with (2.5.9), since $p < 1 + 4/n$, we compute

$$
\begin{aligned}
4t^2\|\nabla v(t)\|_{L^2}^2 &\geq C + C\int_0^t s\|v(s)\|_{L^{p+1}}^{p+1}ds\\
&\geq C + Ct^{2-\frac{n(p-1)}{2}},
\end{aligned}
$$

which yields

$$\|\nabla v(t)\|_{L^2} \geq Ct^{-\frac{n(p-1)}{4}}. \tag{2.5.10}$$

From (2.5.9), Holder's inequality and the mass conservation law $\|u(t)\|_{L^2} = \|u_0\|_{L^2}$, it follows that for $r \in [2, p+1]$,

$$
\begin{aligned}
\|u(t)\|_{L^r} &= \|v(t)\|_{L^r}\\
&\geq C\|v(t)\|_{L^{p+1}}^{\frac{2(p+1)}{p-1}(\frac{1}{2}-\frac{1}{r})}\|v(t)\|_{L^2}^{1-\frac{2(p+1)}{p-1}(\frac{1}{2}-\frac{1}{r})}\\
&\geq Ct^{-n(\frac{1}{2}-\frac{1}{r})}.
\end{aligned}
$$

This complete the proof of (2.5.5) if $r \in [2, p+1]$.

For $r \in (p+1, \frac{2n}{n-2}]$, Applying (2.5.9), (2.5.10) and Gagliardo-Nirenberg's inequality, we get

$$
\begin{aligned}
\|u(t)\|_{L^r} &= \|v(t)\|_{L^r}\\
&\geq C\|\nabla v(t)\|_{L^2}^{\frac{2n(r-p-1)}{r(2(p-1)+4-n(p-1))}}\|v(t)\|_{L^{p+1}}^{1-\frac{2n(r-p-1)}{r(2(p-1)+4-n(p-1))}}\\
&\geq Ct^{-n(\frac{1}{2}-\frac{1}{r})(1-\theta(r))}.
\end{aligned}
$$

This complete the proof. □

2.5.4 Application of the momentum conservation law

Under the assumption as in Theorem 2.5.3, we don't know whether $\backslash xu(t)\backslash_{L^2} \uparrow 0$ as $t \downarrow T_{\max}$. Applying the momentum conservation law, by the invariance of the equation (2.2.1) under the spatial translation, we can construct a solution of (2.2.1) such that $\backslash xu(t)\backslash_{L^2} \nrightarrow 0$ as $t \downarrow T_{\max}$. In detail, we have the following result:

Proposition 2.5.2 ([3]). *Suppose that* $u_0 \ni \Sigma$. *Let* u *be a corresponding maximal solution* $u \ni C([0, T_{\max}), \Sigma)$ *of (2.1.1) with the initial data* u_0. *Then there exists* $x_0 \ni \mathbb{R}^n$ *such that* $\backslash xu(t, \times + x_0)\backslash_{L^2} \nrightarrow 0$ *as* $t \downarrow T_{\max}$.

Proof. For all $x_0 \ni \mathbb{R}^n$, we compute

$$
\begin{aligned}
&\backslash xu(t, \times + x_0)\backslash_{L^2}^2 \\
&= \bigcap_{\mathbb{R}^n} |x|^2 |u(t, x + x_0)|^2 dx \\
&= \bigcap_{\mathbb{R}^n} |\widetilde{x} \quad x_0|^2 |u(t, \widetilde{x})|^2 d\widetilde{x} \\
&= \bigcap_{\mathbb{R}^n} |\widetilde{x}|^2 |u(t, \widetilde{x})|^2 d\widetilde{x} + |x_0|^2 \bigcap_{\mathbb{R}^n} |u(t, \widetilde{x})|^2 d\widetilde{x} \quad 2\bigcap_{\mathbb{R}^n} \widetilde{x} \times x_0 |u(t, \widetilde{x})|^2 d\widetilde{x} \\
&= \backslash xu(t)\backslash_{L^2}^2 + |x_0|^2 \backslash u_0 \backslash_{L^2}^2 \quad 2\bigcap_{\mathbb{R}^n} x \times x_0 |u(t)|^2 dx.
\end{aligned}
$$

Hence, it follows formally from the conservation law of the momentum that

$$
\begin{aligned}
\frac{d}{dt} \bigcap_{\mathbb{R}^n} x \times x_0 |u(t)|^2 dx &= 2\bigcap_{\mathbb{R}^n} x \times x_0 \operatorname{Re}(\bar{u}\partial_t u) dx \\
&= 2\bigcap_{\mathbb{R}^n} x \times x_0 \operatorname{Im}\}\bar{u}(\lambda |u|^{p \quad 1} u \quad \Delta u)\langle dx \\
&= \quad 2 \operatorname{Im} \bigcap_{\mathbb{R}^n} x \times x_0 \bar{u} \Delta u dx \\
&= 2 \operatorname{Im} \bigcap_{\mathbb{R}^n} \quad (x \times x_0 \bar{u}) \times \quad u dx \\
&= 2 \operatorname{Im} \bigcap_{\mathbb{R}^n} \bar{u}(t) x_0 \times \quad u(t) dx \\
&= 2 \operatorname{Im} \bigcap_{\mathbb{R}^n} \bar{u}_0 x_0 \times \quad u_0 dx.
\end{aligned}
$$

This yields

$$
\bigcap_{\mathbb{R}^n} x \times x_0 |u(t)|^2 dx = \bigcap_{\mathbb{R}^n} x \times x_0 |u_0|^2 dx + 2t \operatorname{Im} \bigcap_{\mathbb{R}^n} \bar{u_0} x_0 \times \quad u_0 dx.
$$

Note that we can justify the above computation by replacing $x \times x_0 |u(t)|^2$ to $e^{\quad \varepsilon\, x^{\,2}} x \times x_0 |u(t)|^2$ for $\varepsilon > 0$ and converging $\varepsilon \propto 0$. Therefore, we get

$$
\begin{aligned}
\backslash xu(t, \times + x_0)\backslash_{L^2}^2 &= \backslash xu(t)\backslash_{L^2}^2 + |x_0|^2 \backslash u_0 \backslash_{L^2}^2 \\
&\quad 2\bigcap_{\mathbb{R}^n} x \times x_0 |u_0|^2 dx \quad 4t \operatorname{Im} \bigcap_{\mathbb{R}^n} \bar{u_0} x_0 \times \quad u_0 dx
\end{aligned}
$$

for all $t \ni [0, T_{\max})$. Observe that for any $t \ni [0, T_{\max})$,

$$|x_0|^2 \backslash u_0 \backslash_{L^2}^2 \quad 2 \bigcap_{\mathbb{R}^n} x \times x_0 |u_0|^2 dx \quad 4t \operatorname{Im} \bigcap_{\mathbb{R}^n} \bar{u_0} x_0 \times \ u_0 dx = O(|x_0|^2)$$

as $|x_0| \uparrow \ \in$ and the cofficient of $|x_0|^2$ is positive. Thus, we implies that $\backslash x u(t, \times + x_0) \backslash_{L^2} \nrightarrow 0$ as $t \downarrow T_{\max}$ if $|x_0|$ is sufficiently large. $\square$

Chapter 3

Nonlinear Schrödinger equations with non-vanishing boundary conditions at spatial infinity

3.1 Previous Works

In this chapter, we consider defocusing nonlinear Schrödinger equations in dimension $n \geq 4$.

$$\left.\begin{array}{l} i\partial_t u + \Delta u = f(\|u|^2)u, \quad t \ni [0, \in\ ),\ x \ni \mathbb{R}^n, \\ u(0,x) = u_0(x), \quad x \ni \mathbb{R}^n, \end{array}\right\} \tag{3.1.1}$$

where $u(t,x) : [0, \in\ ) * \mathbb{R}^n \uparrow\ \mathbb{C}$. The unknown function u has the following boundary condition:

$$\|u(x)|^2 \uparrow\ \rho_0 \quad \text{as} \quad \|x\| \uparrow\ \in\ ,$$

where $\rho_0 > 0$ denotes the light intensity of the background. The nonlinear term f is assumed to be defocusing. Namely the real-valued function f satisfies the following assumption:

$$f(\rho_0) = 0, \quad f'(\rho_0) > 0. \tag{$\mathbf{H}_f$}$$

The aim of this chapter is to state previous works for the global wellposedness of Cauchy problem (3.1.1) in energy space

$$E_{\rho_0} = \}u \ni H^1_{loc}(\mathbb{R}^n); \quad u \ni L^2(\mathbb{R}^n), \|u|^2 \quad \rho_0 \ni L^2(\mathbb{R}^n)\langle.$$

First, we consider the Cauchy problem for the Gross-Pitaevskii equation

$$\left.\begin{array}{l} i\partial_t u + \Delta u = (\|u|^2 \quad 1)u, \quad t \ni [0, \in\ ),\ x \ni \mathbb{R}^n, \\ u(0,x) = u_0(x), \quad x \ni \mathbb{R}^n \end{array}\right\} \tag{3.1.2}$$

(that is (3.1.1) with $f(r) = r \quad 1$ and $\rho_0 = 1$). Béthnel-Saut [2] prove that the Caucy problem (3.1.2) is globally wellposed in $1 + H^1$ for $n = 2,\ 3$. We state the result. A first strategy of the proof is that (3.1.2) is transformed as follows to look for a solution of (3.1.2) under the form $1+v$:

$$\begin{cases} i\partial_t v + \Delta v = N(v), \quad t \in (0,T),\ x \in \mathbb{R}^n, \\ v(0,x) = v_0(x), \quad x \in \mathbb{R}^n, \end{cases} \tag{3.1.3}$$

where

$$N(v) = |v|^2 v + |v|^2 + 2\operatorname{Re}(v)v + 2\operatorname{Re}(v).$$

In a next strategy, he prove that (3.1.3) is locally well-posed in H^1 by using Strichartz's estimates and a contraction argument for the map

$$\Phi(v) = U(t)v_0 - i\int_0^t U(t-s)N(v(s))ds. \tag{3.1.4}$$

For a locally well-posedness, Béthnel-Saut [2] prove the following Theorem:

Theorem 3.1.1 ([2])**.** *Let $n = 2, 3$. For any $v_0 \in H^1(\mathbb{R}^n)$, there exists $T > 0$ such that (3.1.3) has a unique solution $v \in C([0,T], H^1) \cap L^{8/3}([0,T], W^{1,4})$. Moreover, the energy*

$$\mathcal{E}_1(1+v) = \|\nabla v\|_{L^2}^2 + \frac{1}{2}\||v|^2 + 2\operatorname{Re}(v)\|_{L^2}^2 \tag{3.1.5}$$

is conserved.

Sketch of proof of Theorem 3.1.1. We show that the map (3.1.4) is contraction mapping in the complete metric space

$$\begin{aligned} X_T = \{ & v \in C([0,T], H^1) \cap L^{8/3}((0,T), W^{1,4}); \\ & \|v\|_{L^\infty_T H^1} + \|v\|_{L^{8/3}_T W^{1,4}} \le 8C_0\|v_0\|_{H^1} =: M\}, \\ d(v_1, v_2) = {} & \|v_1 - v_2\|_{L^\infty_T H^1} + \|v_1 - v_2\|_{L^{8/3}_T W^{1,4}}. \end{aligned}$$

First, combining

$$|\nabla N(v)| \le C(|\nabla v||v|^2 + |\nabla v||v| + |\nabla v|)$$

with Strichartz's estimate, Hölder's inequality and Sobolev embedding, we deduce that for all $v \in X_T$,

$$\begin{aligned} & \|\nabla\Phi(u)\|_{L^\infty_T L^2} + \|\nabla\Phi(u)\|_{L^{8/3}_T L^4} \\ \le {} & \|U(t)\nabla v_0\|_{L^\infty_T L^2} + \left\|\int_0^t U(t-s)\nabla N(v(s))ds\right\|_{L^\infty_T L^2} \\ & + \|U(t)\nabla v_0\|_{L^{8/3}_T L^4} + \left\|\int_0^t U(t-s)\nabla N(v(s))ds\right\|_{L^{8/3}_T L^4} \\ \le {} & 2C_0\|\nabla v_0\|_{L^2} + C\left\||\nabla v||v|^2\right\|_{L^{8/5}_T L^{4/3}} + C\||\nabla v||v|\|_{L^{8/5}_T L^{4/3}} + C\|\nabla v\|_{L^1_T L^2} \\ \le {} & 2C_0\|\nabla v_0\|_{L^2} + CT^{1/4}\|\nabla v\|_{L^{8/3}_T L^4}\|v\|^2_{L^\infty_T H^1} \\ & + CT^{1/4}\|\nabla v\|_{L^{8/3}_T L^4}\|v\|_{L^\infty_T H^1} + CT\|\nabla v\|_{L^\infty_T L^2} \\ \le {} & 2C_0\|\nabla v_0\|_{L^2} + CT^{1/4}(M^3 + M^2 + M) \end{aligned}$$

$\geq M/2$

if T is sufficiently small depending only $\backslash v_0 \backslash_{H^1}$. Also, in the same way as above, we get

$$\backslash \Phi(v) \backslash_{L^{\rfloor}_T L^2} + \backslash \Phi(v) \backslash_{L^{8/3}_T L^4} \geq M/2$$

if T is sufficiently small depending only $\backslash v_0 \backslash_{H^1}$. Therefore, we see that

$$\backslash \Phi(v) \backslash_{L^{\rfloor}_T H^1} + \backslash \Phi(v) \backslash_{L^{8/3}_T W^{1,4}} \geq M,$$

which yields $\Phi(v) \ni X_T$. Moreover, by the same estimation in the above, it follows that the map Φ is a contraction mapping in X_T. Thus, we obtain the local solution of (3.1.3) in $C([0,T], H^1)$.

Furthermore, we can obtain formally the conservation law of the energy $\mathcal{F}_1(1+v)$ by multiplying the equation (3.1.3) by $\overline{\partial_t v}$, integrating over $\mathbb{R}^n$, and taking the real part as follows:

$$\begin{aligned} 0 = &\; 2\,\mathrm{Re}(i\partial_t v + \Delta v \quad (|v|^2 + \mathrm{Re}(v))(1+v), \partial_t v)_{L^2} \\ = &\; 2\,\mathrm{Re}(\Delta v, \partial_t v)_{L^2} + 2\,\mathrm{Re}((|v|^2 + \mathrm{Re}(v))(1+v), \partial_t v)_{L^2} \\ = &\; 2\,\mathrm{Re}(\quad v, \partial_t \quad v)_{L^2} + 2\,\mathrm{Re}((|v|^2 + \mathrm{Re}(v))(1+v), \partial_t v)_{L^2} \\ = &\; \frac{d}{dt}\mathcal{F}(1+v(t)). \end{aligned}$$

Note that the above procedure is justified as in Section 2.4.2. This complete the proof. $\square$

Theorem 3.1.2 ([2])**.** *Let $n = 2, 3$. For any $v_0 \ni H^1(\mathbb{R}^n)$, the local solution v of (3.1.3) as in Theorem 3.1.1 extends globally with*

$$v \ni C([0, \in), H^1) \{ L^{8/3}_{loc}((0, \in), W^{1,4}).$$

Proof. Using the energy conservation law

$$\mathcal{F}_1(1+v) = \backslash \quad v \backslash^2_{L^2} + \frac{1}{2} \backslash |v|^2 + 2\,\mathrm{Re}(v) \backslash^2_{L^2} = \mathcal{F}_1(1+v_0),$$

and Sobolev embedding, we get

$$\backslash \quad v \backslash_{L^2} \geq C(\backslash v_0 \backslash_{H^1}).$$

Moreover, by multiplying the equation (3.1.3) by $\overline{u}$, integrating over $\mathbb{R}^n$, and taking the imaginary part, we estimate

$$\begin{aligned} \frac{1}{2}\frac{d}{dt} \bigcap_{\mathbb{R}^n} |v(t)|^2 dx &= \bigcap_{\mathbb{R}^n} (|v(t)|^2 + 2\,\mathrm{Re}(v(t)))\,\mathrm{Im}(v(t)) dx \\ &\geq \Big) \bigcap_{\mathbb{R}^n} (|v(t)|^2 + 2\,\mathrm{Re}(v(t)))^2 dx \Big[^{1/2} \Big) \bigcap_{\mathbb{R}^n} |v(t)|^2 dx \Big[^{1/2} \\ &\geq \sqrt{2\mathcal{F}_1(1+v_0)} \backslash v(t) \backslash_{L^2}, \end{aligned}$$

which yields

$$\backslash v(t) \backslash^2_{L^2} \geq \backslash u_0 \backslash^2_{L^2} + \bigcap_0^t 2\sqrt{2\mathcal{F}_1(1+v_0)} \backslash v(s) \backslash_{L^2} ds.$$

Hence, by using Gronwall type Lemma, this implies

$$\backslash v(t)\backslash_{L^2} \geq \backslash v_0\backslash_{L^2} + \sqrt{2\mathcal{F}_1(1+v_0)}t$$

for any $t \ni [0,T]$. Since we obtain the local uniformaly estimate of $\backslash v(t)\backslash_{H^1}$, we can extend the local solution v to a global solution. □

Since equation (3.1.2) has the energy $\mathcal{F}_1(u) = \backslash \quad u\backslash^2_{L^2} + \frac{1}{2}\backslash\|u\|^2 \quad 1\backslash^2_{L^2}$, solution space $1 + H^1$ is not enough. In fact, Gravejat [13] shows the existence of a traveling wave not even to belong to $1 + L^2$. Hence, it was expected to prove the existence of solution of (3.1.2) in

$$E_1 = \}u \ni H^1_{loc}(\mathbb{R}^n); \quad u \ni L^2(\mathbb{R}^n), \|u\|^2 \quad 1 \ni L^2(\mathbb{R}^n)\langle,$$
$$d_E(u,v) = \backslash \quad u \qquad v\,\backslash_{L^2} + \backslash\|u\|^2 \quad \|v\,\|^2\backslash_{L^2}.$$

After that, Gallo [7] prove a local solution of (3.1.2) in Zhidkov space

$$X^k(\mathbb{R}^n) = \}u \ni L^{\in}(\mathbb{R}^n);\ \partial^\alpha_x u \ni L^2(\mathbb{R}^n),\ 1 \geq \|\alpha\| \geq k\langle$$

with $k \ni \mathbb{N}$ and $k > n/2$. Next, Goubet [12] shows the existence of a global solution of (3.1.2) in $X^2(\mathbb{R}^2)$ if the energy $\mathcal{F}(u)$ is finite. Moreover, for $n = 2$, 3, Gérard [9] prove that The Cauchy problem (3.1.2) is the globally wellposed in the energy space E_1 as follows:

Theorem 3.1.3 ([9])**.** *Let $n = 2$, 3. For any $u_0 \ni E_1$, there exists a global solution $u \ni C([0,\in), E_1)$ of (3.1.2).*

To obtain a local solution of (3.1.2) in E_1, Gérard [9] use the contraction argument and Strichartz's estimate in a similar way. The key Lemmas for the proof of Theorem 3.1.3 are as follows:

Lemma 3.1.1 ([9], cf. [8])**.** $E_{\rho_0} \to X^1(\mathbb{R}^n) + H^1(\mathbb{R}^n)$.

Proof. Using a cutoff function $\chi \ni C_0(\mathbb{C})$ with $\chi(z) = 1$ ($\|z\| \geq \nabla\overline{2\rho_0}$) and $\chi(z) = 0$ ($\|z\| \approx \nabla\overline{3\rho_0}$), we decompose $u \ni E_{\rho_0}$ as

$$E_{\rho_0} \,/\, u = \underbrace{\chi(u)u}_{=:u_1} + \underbrace{(1 \quad \chi(u))u}_{=:u_2}.$$

Since $\|\chi(u)\| = 1$ if $\|u\| \geq \nabla\overline{2\rho_0}$ and $\|\chi(u)\| = 0$ if $\|u\| \approx \nabla\overline{3\rho_0}$, we deduce that

$$\backslash u_1\backslash_{L^{\in}} \geq \sqrt{3\rho_0}.$$

Moreover, $\|u\| \approx \nabla\overline{2\rho_0}$ in supp(u_2) implies

$$\|u_2\| \geq \|(1 \quad \chi(u))u\| \geq \sqrt{\frac{2}{\rho_0}}\|\|u\|^2 \quad p_0\|,$$

which yields

$$\backslash u_2\backslash_{L^2} \geq C\backslash\|u\|^2 \quad \rho_0\backslash_{L^2}.$$

On the other hand, we see that

$$u_1 = (\partial_z\chi(u) \quad u + \partial_{\bar{z}}\chi(u) \quad \bar{u})u + \chi(u) \quad u$$

$$= (\chi(u) + u\partial_z\chi(u)) \quad u + u\partial_{\bar z}\chi(u) \quad \bar u,$$
$$u_2 = (\quad \partial_z\chi(u) \quad u \quad \partial_{\bar z}\chi(u) \quad \bar u)u + (1 \quad \chi(u)) \quad u$$
$$= (1 \quad \chi(u) \quad u\partial_z\chi(u)) \quad u \quad u\partial_{\bar z}\chi(u) \quad \bar u.$$

Therefore, putting $A(\chi) := 1 + \sup_{u\mathcal{D}\mathbb{C}} \|u\partial_{\bar z}\chi(u)\| + \sup_{u\mathcal{D}\mathbb{C}} \|u\partial_z\chi(u)\|$, it follows that

$$\backslash \quad u_1\backslash_{L^2} + \backslash \quad u_2\backslash_{L^2} \geq A(\chi)\backslash \quad u\backslash_{L^2}.$$

This complete the proof. □

Lemma 3.1.2 ([9]). *For $n = 2, 3, 4$. $E_{\rho_0} + H^1 \to E_{\rho_0}$.*

Proof. For any $v \ni E_{\rho_0}$ and $w \ni H^1$, $\quad (v + w) \ni L^2$ is trivial. Next,

$$\|v + w\|^2 \quad \rho_0 = \|v\|^2 \quad \rho_0 + 2\,\mathrm{Re}(\bar v w) + \|w\|^2$$

implies

$$\backslash\|v + w\|^2 \quad \rho_0\backslash_{L^2} \geq \backslash\|v\|^2 \quad \rho_0\backslash_{L^2} + 2\backslash vw\backslash_{L^2} + \backslash w\backslash^2_{L^2}. \tag{3.1.6}$$

Hence, from Lemma 3.1.1, (3.1.6) and Gagliardo-Nirenberg inequality, it follows that

$$\backslash\|v + w\|^2 \quad \rho_0\backslash_{L^2} \geq \backslash\|v\|^2 \quad \rho_0\backslash_{L^2} + C(1 + \sqrt{\mathcal{F}(1 + v)})(\backslash w\backslash_{L^2} + \backslash w\backslash_{L^4}) + \backslash w\backslash_{L^2}.$$

In conclusion, it holds that $E_{\rho_0} + H^1 \to E_{\rho_0}$. □

Lemma 3.1.3 ([9]). *$U(t)$: $X^1 + H^1 \uparrow \quad X^1 + H^1$. Moreover, $U(t)(E_{\rho_0}) \to E_{\rho_0}$.*

Proof. By the unitarity of $U(t)$ in H^1, we may assume $f \ni X^1$ without loss of generality. First, we decompose $U(t)f$ as

$$U(t)f = f + U(t)f \quad f.$$

Let $\chi \ni C_0^\in(\mathbb{R})$ be a cutoff function satisfying $0 \geq \chi \geq 1$, $\chi(s) = 1$ ($\|s\| \geq 1$ and $\chi(s) = 0$ ($\|s\| \approx 2$). Next, We factorize $e^{\quad it\ \xi\ ^2}$ as

$$e^{\quad it\ \xi\ ^2} \quad 1 = \int_{j=1}^{d} g_j(t,\xi)\xi_j,$$

where

$$g_j(t,\xi) := \quad it\chi(t\|\xi\|^2)\xi_j \bigcap_0^1 e^{\quad ist\ \xi\ ^2} ds + \frac{1 \quad \chi(t\|\xi\|^2)}{\|\xi\|^2}\xi_j(e^{\quad it\ \xi\ ^2} \quad 1) = O(\|t\|^{1/2})$$

uniformly in $\xi \ni \mathbb{R}^n$. Thus, using the equality in the above, we deduce that

$$\begin{aligned} &\backslash U(t)f \quad f\backslash_{L^2} \\ &= \backslash\mathcal{S} \quad ^1[(e^{\quad it\ \xi\ ^2} \quad 1)\hat f]\backslash_{L^2} \\ &= \backslash\mathcal{S} \quad ^1[\int_{j=1}^{d} g_j(t,\xi)\xi_j\hat f]\backslash_{L^2} \\ &\geq C\|t\|^{1/2}\backslash \quad f\backslash_{L^2}. \end{aligned}$$

Since we have $(U(t)f) \ni L^2$, we obtain $U(t)f \ni X^1 + H^1$.

Furthermore, combining Lemma 3.1.1 with Lemma 3.1.2, $U(t)f \ni X^1 + H^1$ for all $f \ni X^1 + H^1$, we obtain that for all $u \ni E_{\rho_0}$,

$$U(t)u = \underbrace{u}_{\mathcal{D}E_{\rho_0}} + \underbrace{(U(t)u \quad u)}_{\mathcal{D}H^1} \to E_{\rho_0}.$$

This complete the proof. □

Finally, Gallo [8] has considered the Cauchy problem for (3.1.1). He proved the following Theorem:

Theorem 3.1.4 (Theorem 1.1 in [8])**.** *Let $n \geq 4$ and $\rho_0 > 0$. Assume that $f \ni C^k(\mathbb{R}_+)$ ($k = 3$ if $n = 2, 3$, $k = 4$ if $n = 4$) satisfying ($\mathbf{H}_f$), and there exist $\alpha_1 \approx 1$, with a supplementary condition $\alpha_1 < \alpha_1^{\leq}$ if $n = 3, 4$ ($\alpha_1^{\leq} = 3$ if $n = 3$, $\alpha_1^{\leq} = 2$ if $n = 4$), and $\alpha_2 \ni \mathbb{R}$ with $\alpha_1 \quad \alpha_2 \geq 1/2$ such that*

$$\mathcal{D}C_0 > 0,\ \mathcal{D}A > \rho_0\ s.t. \begin{cases} \exists r \approx 1, \begin{cases} \|f^{\infty\infty}(r)\| \geq C_0 r^{\alpha_1 \quad 3} \text{ if } n = 1, 2, 3, \\ \|f^{\infty\infty\infty}(r)\| \geq C_0 r^{\alpha_1 \quad 4} \text{ if } n = 4, \end{cases} & (\mathbf{H}_{\alpha_1}) \\ \begin{cases} \text{if } \alpha_1 \geq 3/2,\ V \text{ is bounded from below}, \\ \text{if } \alpha_1 > 3/2,\ \exists r \approx A,\ r^{\alpha_2} \geq C_0 V(r), \end{cases} & (\mathbf{H}_{\alpha_2}) \end{cases}$$

where $V(r) := \sum_{\rho_0}^{r} f(s)ds$. Then for any function ϕ satisfying

$$\phi \ni C_b^{k+1}(\mathbb{R}^n), \qquad \phi \ni H^{k+1}(\mathbb{R}^n)^n, \quad \|\phi\|^2 \quad \rho_0 \ni L^2(\mathbb{R}^n), \qquad (\mathbf{H}_\phi)$$

(3.1.1) is globally well-posed in $\phi + H^1(\mathbb{R}^n)$. Namely, for any $w_0 \ni H^1(\mathbb{R}^n)$, there exists an unique $w \ni C([0, \in), H^1(\mathbb{R}^n))$ such that $\phi + w$ is the solution to (3.1.1) with the initial data $w(0) = w_0$. Moreover, The solution depends continuously on the initial data $w_0 \ni H^1$. Furthermore, $\mathcal{F}(\phi + w(t)) = \mathcal{F}(\phi + w_0)$ for all $t \ni [0, \in)$, where

$$\mathcal{F}(\phi + w) = \bigcap_{\mathbb{R}^n} \| \quad (\phi + w)\|^2 dx + \bigcap_{\mathbb{R}^n} V(\|\phi + w\|^2) dx.$$

Gallo [8] decompose the element of E_{ρ_0} as follows:

Lemma 3.1.4 (Proposition 1.1 in [8])**.** *For any $u_0 \ni E_{\rho_0}$, there exist $\phi \ni E$ satisfying the following condtion (3.1.7) and $w_0 \ni H^1$ such that $u_0 = \phi + w_0$:*

$$\phi \ni C_b^{\in}(\mathbb{R}^n), \qquad \phi \ni H^{\in}(\mathbb{R}^n)^n, \quad \|\phi\|^2 \quad \rho_0 \ni L^2(\mathbb{R}^n) \qquad (3.1.7)$$

Proof of Lemma 3.1.4. Using the cutoff function $\chi \ni C_0(\mathbb{C})$ with $\chi(z) = 1$ ($\|z\| \geq 1$) and $\chi(z) = 0$ ($\|z\| \approx 2$) and $\rho \ni C_0(\mathbb{R}^n)(\sum_{\mathbb{R}^n} \rho = 1)$, we decompose u_0 as

$$\begin{aligned} u_0 &= \chi(u_0)u_0 + (1 \quad \chi(u_0))u_0 \\ &= \underbrace{\rho \bullet (\chi(u_0)u_0)}_{E_{\rho_0} \text{ satisfying (3.1.7)}} + \underbrace{(\chi(u_0)u_0 \quad \rho \bullet (\chi(u_0)u_0)) + (1 \quad \chi(u_0))u_0}_{\mathcal{D}H^1} \end{aligned}$$

This complete the proof. □

For $n \geq 4$, Gallo [8] prove the globally well-posedness of (3.1.1). We state the result for $n = 2, 3, 4$. A first strategy of the proof is that (3.1.1) is transformed as follows to look for a solution of (3.1.1) under the form $\phi + w$.

$$\begin{cases} i\partial_t w + \Delta w = F(w(t)), \quad t \ni [0, \in\ ), \ x \ni \mathbb{R}^n, \\ w(0,x) = w_0(x), \quad x \ni \mathbb{R}^n, \end{cases} \tag{3.1.8}$$

where

$$F(w) := \quad \Delta\phi + f(|\phi + w|^2)(\phi + w).$$

In a next strategy, he proves that (3.1.8) is locally well-posed in H^1 by using Strichartz's estimates and a contraction argument for the map

$$\Phi(w) = U(t)w_0 \quad i \bigcap_0^t U(t \quad s)F(w(s))ds,$$

in the space

$$X_T := L_T^{\in} H^1 \ \{ \ L_T^p W^{1,q}$$

equipped with its natural norm

$$\backslash w \backslash_{X_T} := \backslash w \backslash_{L_T^{|} H^1} + \backslash w \backslash_{L_T^p W^{1,q}},$$

where a pair (p, q) is a admissible pair defined as $(p, q) := (6/n, 6)$ for $n = 2$, 3, $(p, q) := (2, 4)$ for $n = 4$. We remark that Gallo [8] takes $(p, q) := (4, 4)$ for $n = 2$. Note that our choice also works for getting local existence of solution to (3.1.1).

To prove that Φ is a contraction mapping in X_T, we need some of estimates for nonlinearity $F(w)$.

Lemma 3.1.5 ([8])**.** *Let $T > 0$. For any $w \ni X_T$, there exist*

$$F_1(w) \ni L_T^{\in} L^2, \qquad F_2(w) \ni L_T^{\in} L^{q^\infty}$$

such that

$$F(w) = F_1(w) + F_2(w).$$

Moreover it follows that

$$\begin{aligned} &\backslash F_1(w) \backslash_{L_T^1 L^2} + \backslash F_2(w) \backslash_{L_T^{p^\infty} L^{q^\infty}} \\ &\quad \geq CT(1 + \backslash w \backslash_{L_T^{|} L^2}) + CT^{1/p^\infty}(\backslash w \backslash^2_{L_T^{|} H^1} + \backslash w \backslash^{\max(2, 2\alpha_1 \ \ 1)}_{L_T^{|} H^1}), \end{aligned}$$

where C is a positive constant depending on T.

Lemma 3.1.6 ([8])**.** *Let $T > 0$. For any $w \ni X_T$, there exist*

$$G_1(w) \ni L_T^1 L^2, \qquad G_2(w) \ni L_T^{p^\infty} L^{q^\infty}.$$

such that

$$F(w) = G_1(w) + G_2(w).$$

Moreover it follows that there exists $\theta > 0$ such that

$$\begin{aligned} &\backslash G_1(w) \backslash_{L_T^1 L^2} + \backslash G_2(w) \backslash_{L_T^{p^\infty} L^{q^\infty}} \\ &\geq CT(1 + \backslash \quad w \backslash_{L_T^{|} L^2}) \\ &\quad + C(1 + \backslash \quad w \backslash_{L_T^{|} L^2})(T^{1/p^\infty} \backslash w \backslash_{L_T^{|} H^1} + T^\theta \backslash w \backslash^{\max(1, 2\alpha_1 \ \ 2)}_{X_T}), \end{aligned}$$

where C is a positive constant depending on T.

Lemma 3.1.7 ([8])**.** *Let $T > 0$. For any w_1, $w_2 \in X_T$, decomposing $f(|\phi + w|^2)(\phi + w)$ as Lemma 5.2.1, it follows that there exist $\theta_0 > 0$ and $\theta_1 > 0$ such that*

$$\begin{aligned}
&\|F_1(w_1) - F_1(w_2)\|_{L^1_T L^2} + \|F_2(w_1) - F_2(w_2)\|_{L^{p'}_T L^{q'}} \\
&\leq CT\|w_1 - w_2\|_{L^\infty_T H^1} + C\|w_1 - w_2\|_{L^\infty_T H^1} \\
&\quad \times (T^{\theta_0}(\|w_1\|_{L^\infty_T H^1} + \|w_2\|_{L^\infty_T H^1}) + T^{\theta_1}(\|w_1\|_{L^\infty_T H^1} + \|w_2\|_{L^\infty_T H^1})^{\max(1,2\alpha_1-2)}),
\end{aligned}$$

where C is a positive constant depending on T.

Lemma 3.1.8 ([8])**.** *Let $T > 0$. For any w_1, $w_2 \in X_T$, decomposing $f(|\phi + w|^2)(\phi + w)$ as Lemma 5.2.2, it follows that there exist $\theta_2 > 0$ and $\theta_3 > 0$ such that*

$$\begin{aligned}
&\|G_1(w_1) - G_1(w_2)\|_{L^1_T L^2} + \|G_2(w_1) - G_2(w_2)\|_{L^{p'}_T L^{q'}} \\
&\leq CT\|\nabla(w_1 - w_2)\|_{L^\infty_T L^2} \\
&\quad + CT^{1/p'}(1 + \|w_1\|_{L^\infty_T H^1} + \|w_2\|_{L^\infty_T H^1})^{\max(1,2\alpha_1-2)}\|w_1 - w_2\|_{L^\infty_T H^1} \\
&\quad + CT^{\theta_2}\|w_1 - w_2\|_{X_T}(\|w_1\|_{X_T}^{\max(1,2\alpha_1-2)} + \|w_2\|_{X_T}^{\max(1,2\alpha_1-2)}) \\
&\quad + CT^{\theta_3}\|w_1 - w_2\|_{X_T}(1 + \|w_1\|_{L^\infty_T H^1} + \|w_2\|_{L^\infty_T H^1}) \\
&\quad \times (\|w_1\|_{X_T}^{\max(0,2\alpha_1-3)} + \|w_2\|_{X_T}^{\max(0,2\alpha_1-3)}),
\end{aligned}$$

where C is a positive constant depending on T.

For locally well-posedness, Gallo [8] proves the following Theorem:

Theorem 3.1.5 ([8])**.** *Let $n = 2, 3, 4$. Let $\rho_0 > 0$, and $f \in C^2(\mathbb{R}_+)$ satisfying $(\mathbf{H}_f)$. Moreover, we assume that there exist $\alpha_1 \geq 1$, with a supplementary condition $\alpha_1 < \alpha_1^*$ if $n = 3, 4$ ($\alpha_1^* = 3$ if $n = 3$, $\alpha_1^* = 2$ if $n = 4$), and $\alpha_2 \in \mathbb{R}$ with $\alpha_1 - \alpha_2 \leq 1/2$ such that $(\mathbf{H}_{\alpha_1})$ and $(\mathbf{H}_{\alpha_2})$. Let ϕ be a function satisfying $(\mathbf{H}'_\phi)$.*

Then for any $R > 0$, there exists $T(R) > 0$ such that for any $w_0 \in H^1$ with $\|w_0\|_{H^1} \leq R$, there exists an unique solution $w \in X_{T(R)}$ of the integral equation

$$w(t) = U(t)w_0 - i\int_0^t U(t - t')F(w(t'))dt'. \tag{3.1.9}$$

Moreover $w \in C([0, T(R)], H^1)$.

If $\widetilde{w} \in C([0, T], H^1)$ solves (3.1.9) for some $T > 0$, then $\widetilde{w} \in X_T$, and $\widetilde{w} \in X_T$ is the unique solution to (3.1.9) in $C([0, T], H^1)$.

Also the flow map is locally Lipschitz continuous on the bounded sets of H^1, indeed for any $R > 0$, there exists $T(R) > 0$ such that for any $T' \in (0, T(R)]$ and w_0, $\widetilde{w_0} \in H^1$ with $\|w_0\|_{H^1} \leq R$ and $\|\widetilde{w_0}\|_{H^1} \leq R$, corresponding solutions w, $\widetilde{w} \in X_{T'}$ of (3.1.9) satisfy the following locally Lipschitz continuity:

$$\|w - \widetilde{w}\|_{X_{T'}} \leq C\|w_0 - \widetilde{w_0}\|_{H^1}, \tag{3.1.10}$$

where C is a positive constant depending on $\|w\|_{X_{T'}}$ and $\|\widetilde{w}\|_{X_{T'}}$. Especially, for the same constant C,

$$\|w - \widetilde{w}\|_{L^\infty_{T'} H^1} \leq C\|w_0 - \widetilde{w_0}\|_{H^1}.$$

Proof of Theorem 3.1.5. For any $R > 0$, we take $w_0 \ni X_T$ with $\backslash w_0 \backslash_{H^1} \geq R$. We show that the map

$$\Phi(w) = U(t)w_0 \quad i \bigcap_0^t U(t \quad s)F(w(s))ds$$

is a contraction mapping in

$$\widetilde{X_T} := \} w \ni X_T \ ; \ \backslash w \backslash_{X_T} \geq R+1 \langle \ ,$$
$$\backslash w \backslash_{\widetilde{X}_T} := \backslash w \backslash_{X_T}.$$

Combining Lemma 3.1.5 with Lemma 3.1.7, Strichartz's estimate and Holder's inequality, we deduce that for any $u \ni X_T$,

$$\begin{aligned}
\backslash \Phi(w) \backslash_{\widetilde{X}_T} &= \backslash \Phi(w) \backslash_{L_T^{\dagger} H^1} + \backslash \Phi(w) \backslash_{L_T^p W^{1,q}} \\
&= \backslash \Phi(w) \backslash_{L_T^{\dagger} L^2} + \backslash \Phi(w) \backslash_{L_T^p L^q} + \backslash \quad \Phi(w) \backslash_{L_T^{\dagger} L^2} + \backslash \quad \Phi(w) \backslash_{L_T^p L^q} \\
&\geq \backslash w_0 \backslash_{L^2} + CT(1 + \backslash w \backslash_{L_T^{\dagger} L^2}) \\
&\quad + CT^{1/p^\infty}(\backslash w \backslash^2_{L_T^{\dagger} H^1} + \backslash w \backslash^{\max(2,2\alpha_1 \quad 1)}_{L_T^{\dagger} H^1}) \\
&\quad + \backslash \quad w_0 \backslash_{L^2} + CT(1 + \backslash \quad w \backslash_{L_T^{\dagger} L^2}) \\
&\quad + C(1 + \backslash \quad w \backslash_{L_T^{\dagger} L^2})(T^{1/p^\infty} \backslash w \backslash_{L_T^{\dagger} H^1} + T^{\theta} \backslash w \backslash^{\max(1,2\alpha_1 \quad 2)}_{X_T}) \\
&\geq \backslash w_0 \backslash_{H^1} + CT(1 + \backslash \quad w \backslash_{L_T^{\dagger} H^1}) \\
&\quad + CT^{1/p^\infty}(\backslash w \backslash^2_{L_T^{\dagger} H^1} + \backslash w \backslash^{\max(2,2\alpha_1 \quad 1)}_{L_T^{\dagger} H^1}) \\
&\quad + C(1 + \backslash \quad w \backslash_{L_T^{\dagger} L^2})(T^{1/p^\infty} \backslash w \backslash_{L_T^{\dagger} H^1} + T^{\theta} \backslash w \backslash^{\max(1,2\alpha_1 \quad 2)}_{X_T}) \\
&\geq R + CT(1 + R + 1) + CT^{1/p^\infty}((R+1)^2 + (R+1)^{\max(2,2\alpha_1 \quad 1)}) \\
&\quad + C(1 + R + 1)(T^{1/p^\infty}(R+1) + T^{\theta}(R+1)^{\max(2,2\alpha_1 \quad 1)}) \\
&\geq R + C(R)T^{\gamma_1} \quad (\gamma_1 := \min(1, 1/p^\infty, \theta)) \\
&\geq R + 1
\end{aligned}$$

if we take $T = \min(1, (1/C(R))^{1/\gamma_1})$, which implies $\Phi(w) \ni \widetilde{X_T}$. Similarly, for all w_1, $w_2 \ni \widetilde{X_T}$, we estimate that

$$\begin{aligned}
&\backslash \Phi(w_1) \quad \Phi(w_2) \backslash_{X_T} \\
&= \backslash \Phi(w_1) \quad \Phi(w_2) \backslash_{L_T^{\dagger} L^2} + \backslash \Phi(w_1) \quad \Phi(w_2) \backslash_{L_T^p L^q} \\
&\quad + \backslash \quad \Phi(w_1) \qquad \Phi(w_2) \backslash_{L_T^{\dagger} L^2} + \backslash \quad \Phi(w_1) \qquad \Phi(w_2) \backslash_{L_T^p L^q} \\
&\geq CT \backslash w_1 \quad w_2 \backslash_{L_T^{\dagger} L^2} + C(\backslash w_1 \quad w_2 \backslash_{L_T^{\dagger} L^2} + \backslash w_1 \quad w_2 \backslash_{L_T^p L^q}) \\
&\quad * \ (T^{\theta_0}(\backslash w_1 \backslash_{L_T^{\dagger} H^1} + \backslash w_2 \backslash_{L_T^{\dagger} H^1}) + T^{\theta_1}(\backslash w_1 \backslash_{L_T^{\dagger} H^1} + \backslash w_2 \backslash_{L_T^{\dagger} H^1})^{\max(1,2\alpha_1 \quad 2)}) \\
&\quad + CT \backslash \quad (w_1 \quad w_2) \backslash_{L_T^{\dagger} L^2} + T^{1/p^\infty}(1 + \backslash w_1 \backslash_{L_T^{\dagger} H^1} + \backslash w_2 \backslash_{L_T^{\dagger} H^1})^{\max(1,2\alpha_1 \quad 2)} \\
&\quad * \ \backslash w_1 \quad w_2 \backslash_{L_T^{\dagger} H^1} \\
&\quad + CT^{\theta_2} \backslash w_1 \quad w_2 \backslash_{X_T}(\backslash w_1 \backslash^{\max(1,2\alpha_1 \quad 2)}_{X_T} + \backslash w_2 \backslash^{\max(1,2\alpha_1 \quad 2)}_{X_T}) \\
&\quad + CT^{\theta_3} \backslash w_1 \quad w_2 \backslash_{X_T}(1 + \backslash w_1 \backslash_{L_T^{\dagger} H^1} + \backslash w_2 \backslash_{L_T^{\dagger} H^1}) \\
&\quad * \ (\backslash w_1 \backslash^{\max(0,2\alpha_1 \quad 3)}_{X_T} + \backslash w_2 \backslash^{\max(0,2\alpha_1 \quad 3)}_{X_T})
\end{aligned}$$

$$
\begin{aligned}
&\geq CT\backslash w_1 \quad w_2\backslash_{L^|_T H^1} + C\backslash w_1 \quad w_2\backslash_{X_T} \\
&\quad * \; (T^{\theta_0}(2(R+1)) + T^{\theta_1}(2(R+1))^{\max(1,2\alpha_1 \;\; 2)}) \\
&\quad + CT^{1/p^\infty}(1+2(R+1))^{\max(1,2\alpha_1 \;\; 2)}\backslash w_1 \quad w_2\backslash_{X_T} \\
&\quad + CT^{\theta_2}\backslash w_1 \quad w_2\backslash_{X_T}(2(R+1)^{\max(1,2\alpha_1 \;\; 2)}) \\
&\quad + CT^{\theta_3}\backslash w_1 \quad w_2\backslash_{X_T}(1+2(R+1))(2(R+1)^{\max(0,2\alpha_1 \;\; 3)}) \\
&\geq C(R)T^{\gamma_2}\backslash w_1 \quad w_2\backslash_{X_T}. \quad (\gamma_2 := \min(1,\theta_0,\theta_1,\theta_2,\theta_2,1/p^\infty)) \\
&\geq \frac{1}{2}\backslash w_1 \quad w_2\backslash_{\widetilde{X}_T}
\end{aligned}
$$

If we take $T := \min(1,(1/2C(R))^{1/\gamma_2})$. Thus, we complete the proof of the local existence of a solution u of (3.1.9). To get remaining assertions, we wish referring to Gallo [8]. □

Proof of Theorem 3.1.4. We show the existence of the global solution. The local solution $\phi + w$ of (3.1.8) as in Theorem 3.1.5 has the conservation law of the energy

$$
\mathcal{F}(\phi + w) = \bigcap_{\mathbb{R}^n} \| \;\; (\phi + w)\|^2 dx + \bigcap_{\mathbb{R}^n} V(\|\phi + w\|^2)dx.
$$

In fact, when $(n,k) = (1,1)$, $(2,2)$, $(3,2)$ or $(4,3)$, $F : H^k(\mathbb{R}^n) \uparrow \;\; H^k(\mathbb{R}^n)$ is locally Lipshitz continuous. Hence, Gallo [8] proves the results as follows:

Theorem 3.1.6. *(Theorem 2.1 in [8], cf. Chapter 4 in [5]) Let* $(n,k) = (1,1)$, $(2,2)$, $(3,2)$ *or* $(4,3)$. *Let* $f \ni C^{k+1}(\mathbb{R}_+)$ *satisfying* $(\mathbf{H}_f)$ *and* ϕ *satisfying* $(\mathbf{H}_\phi)$. *For every* $w_0 \ni H^k(\mathbb{R}^n)$, *there exists* $T^{\leq}(w_0) > 0$ *such that there exists a unique solution* $w \ni C([0,T^{\leq}), H^k(\mathbb{R}^n))$ *of the intergral equation (3.1.9).*

If $T^{\leq} < \in$, *then* $\backslash w(t)\backslash_{H^k} \downarrow \in$ *as* $t \downarrow T^{\leq}$. *Moreover if* $w_0 \ni H^{k+2}(\mathbb{R}^n)$, $w \ni C([0,T^{\leq}), H^{k+2}(\mathbb{R}^n)) \{ \; C^1([0,T^{\leq}), H^k(\mathbb{R}^n))$.

Lemma 3.1.9. *(Lemma 3.1 in [8]) Let* $(n,k) = (1,1)$, $(2,2)$, $(3,2)$ *or* $(4,3)$. *Let* $f \ni C^{k+1}(\mathbb{R}_+)$ *satisfying* $(\mathbf{H}_f)$ *and* ϕ *satisfying* $(\mathbf{H}_\phi)$. $w \ni C([0,T^{\leq}), H^k(\mathbb{R}^n))$ *denotes the solution of intergral equation (3.1.9) with* $w_0 \ni H^k(\mathbb{R}^n)$ *as in Theorem 3.1.6. Then for any* $t \ni [0, T^{\leq}(w_0))$, $\mathcal{F}(\phi + w(t)) = \mathcal{F}(\phi + w_0)$.

Combining Theorem 3.1.6 with Lemma 3.1.9 and approximating the initial data $w_0 \ni H^1$ as a sequence $\}w_{0,n}\langle_{n=1}^{\in} \to H^k$ such that $w_{0,n} \uparrow \;\; w_0$ in H^1 as $n \uparrow \;\; \in$, we can prove the energy conservation law (see Chapter 5 in Gallo [8]).

Thus, Combining the condition $(\mathbf{H}_{\alpha_2})$ with Gronwall type Lemma, we obtain time local uniformaly estimate of $\backslash w(t)\backslash_{H^1}$. In conclusion, we can construct a time global soluton of (3.1.8) (see chapters 2 and 5 in Gallo [8]). □

Remark 3.1.1. *In Gallo [8], to prove Theorem 3.1.5, we need the assumption* $f \ni C^2(\mathbb{R}_+)$ *and* $(\mathbf{H}_{\alpha_1})$. *Moreover, to prove Theorem 3.1.4, we need the assumption* $f \ni C^k(\mathbb{R}_+)$ *(*$k = 3$ *if* $n = 2$, 3, $k = 4$ *if* $n = 4$*),* $(\mathbf{H}_{\alpha_1})$ *and* $(\mathbf{H}_{\alpha_2})$.

From Theorem 3.1.5, a local solution of (3.1.8) is constructed as the following Theorem:

Theorem 3.1.7. *Let $n = 2, 3, 4$. Let $w_0 \ni H^1$. Let $T > 0$ and let w be a mild solution of the integral equation (3.1.9) with $w \ni C([0,T], H^1)$. Then, for any $t_0 \ni [0,T]$, there exists $v(t_0) \ni H^{\ \ 1}$ such that*

$$\frac{w(t_0 + h) \quad w(t_0)}{h} \uparrow \ v(t_0) \quad \text{in } H^{\ \ 1} \text{ as } h \uparrow \ 0.$$

Moreover, denoting $v(t_0)$ by $\partial_t w(t_0)$, w is a solution of (3.1.8), indeed w satisfies

(i) $i\partial_t w(t) + \Delta w(t) = F(w(t))$ *in* $H^{\ \ 1}$ *for all* $t \ni [0,T]$,

(ii) $w(0) = w_0$.

Remark 3.1.2. *From $E_{\rho_0} + H^1 \to E_{\rho_0}$ and*

$$\phi + w = \phi + w_0 \quad w_0 + w = u_0 + (w \quad w_0) \ni E_{\rho_0} + H^1,$$

it follows that the solution $\phi + w$ given by Theorem 3.1.4 belongs to E_{ρ_0}.

Chapter 4

The new method to derive conservation laws for nonlinear Schrödinger equations with a power type nonlinearity

4.1 Ozawa's idea

In this chapter, we consider the Cauchy problem for the nonlinear Schrödinger equation

$$\begin{cases} i\partial_t u + \Delta u = \lambda |u|^{p\ \ 1} u, \quad t \ni [0, \in\ ),\ x \ni \mathbb{R}^n, \\ u(0,x) = u_0(x), \quad x \ni \mathbb{R}^n, \end{cases} \tag{4.1.1}$$

where, $\lambda \ni \mathbb{R}$, $p > 1$, $u(t,x) : [0, \in\ ) * \mathbb{R}^n \uparrow\ \mathbb{C}$ and the initial data u_0 is a complex valued function in $\mathbb{R}^n$. In what follows, $f(u)$ denotes $\lambda|u|^{p\ \ 1}u$.

We present a new method to derive the conservation laws of (4.1.1). To derive the conservation laws, we need to use the approximating argument as in Section 2.4. However, for (4.1.1), by using additional properties of solutions provided by Strichartz's estimates without using approximating argument, Ozawa [19] derives conservation laws of the charge and the energy as follows:

Proposition 4.1.1 ([19])**.** *Let (q,r) be some admissible pair. Let $u \ni L^q([0,T], L^r)$ be a mild solution of the intergral equation*

$$u(t) = U(t)u_0 \quad i\lambda \bigcap_0^t U(t \quad s)(|u|^{p\ \ 1}u)(s)ds \tag{4.1.2}$$

for some $u_0 \ni L^2$ and $T > 0$. Then $\backslash u(t)\backslash_{L^2} = \backslash u_0 \backslash_{L^2}$ for any $t \ni [\ \ T, T]$.

Proposition 4.1.2 ([19])**.** *Let (q,r) be some admissible pair. Let $u \ni L^q([0,T], L^r)$ be a mild solution of the intergral equation (4.1.2) for some $u_0 \ni H^1$ and $T > 0$.*

Then $E(u(t)) = E(u_0)$ *for any* $t \in [-T,T]$, *where*

$$E(u) = \|\nabla u\|_{L^2}^2 + \frac{2\lambda}{p+1}\|u\|_{L^{p+1}}^{p+1}.$$

Proof of Proposition 4.1.1. For all $t \in [0,T]$, we obtain

$$\begin{aligned}
&\|u(t)\|_{L^2}^2 \\
&= \|U(-t)u(t)\|_{L^2}^2 \\
&= \|u_0\|_{L^2}^2 - 2\,\mathrm{Im}\left(u_0, \int_0^t U(-s)f(u(s))ds \right)_{L^2} \\
&\quad + \left\| \int_0^t U(-s)f(u(s))ds \right\|_{L^2}^2 .
\end{aligned} \tag{4.1.3}$$

The second term on the RHS of (4.1.3) satisfies the following equality:

$$\begin{aligned}
&2\,\mathrm{Im}\left(u_0, \int_0^t U(-s)f(u(s))ds \right)_{L^2} \\
&\quad = -2\,\mathrm{Im}\int_0^t \left\langle U(s)u_0, \overline{f(u(s))} \right\rangle ds,
\end{aligned} \tag{4.1.4}$$

where combining Strichartz estimates with $f(u) \in L_T^{q'}L^{r'}$, the time integral of the scalar product is understood as the duality coupling on $(L_T^q L^r) \times (L_T^{q'} L^{r'})$ with $(q,r) = (4(p+1)/n(p-1), p+1)$. For the last term on the RHS of (4.1.3), using Fubini's theorem and (4.1.2), we get

$$\begin{aligned}
&\left\| \int_0^t U(-s)f(u(s))ds \right\|_{L^2}^2 \\
&= 2\,\mathrm{Re}\int_0^t \left\langle f(u(s)), \overline{\int_0^s U(s-s')f(u(s'))ds'} \right\rangle ds, \\
&= -2\,\mathrm{Im}\int_0^t \left\langle f(u(s)), \overline{u(s) + i\int_0^s U(s-s')f(u(s'))ds'} \right\rangle ds, \\
&= -2\,\mathrm{Im}\int_0^t \left\langle f(u(s)), \overline{U(s)u_0} \right\rangle ds, \\
&= 2\,\mathrm{Im}\int_0^t \left\langle U(s)u_0, \overline{f(u(s))} \right\rangle ds.
\end{aligned} \tag{4.1.5}$$

Combining (4.1.3) - (4.1.5), we complete the proof. □

Proof of Proposition 4.1.2. Acting ∇ on (4.1.2), for all $t \in [0,T]$, we obtain

$$\begin{aligned}
&\|\nabla u(t)\|_{L^2}^2 \\
&= \|\nabla U(-t)u(t)\|_{L^2}^2 \\
&= \|\nabla u_0\|_{L^2}^2 - 2\,\mathrm{Im}\left(\nabla u_0, \int_0^t U(-s)\nabla f(u(s))ds \right)_{L^2} \\
&\quad + \left\| \int_0^t U(-s)\nabla f(u(s))ds \right\|_{L^2}^2 .
\end{aligned} \tag{4.1.6}$$

The second term on the RHS of (4.1.6) satisfies the following equality:

$$
\begin{aligned}
&2\,\mathrm{Im}\left(u_0, \int_0^t U(-s)\, f(u(s))ds \right)_{L^2} \\
&= -2\,\mathrm{Im}\int_0^t \left\langle U(s)\, u_0, \overline{f(u(s))} \right\rangle ds,
\end{aligned}
\tag{4.1.7}
$$

where combining Strichartz estimates with $f(u) \in L_T^{q'}L^{r'}$, the time integral of the scalar product is understood as the duality coupling on $(L_T^q L^r) \times (L_T^{q'} L^{r'})$ with $(q,r) = (4(p+1)/n(p-1), p+1)$. For the last term on the RHS of (4.1.6), Fubini's theorem implies

$$
\begin{aligned}
&\left\| \int_0^t U(-s)\, f(u(s))ds \right\|_{L^2}^2 \\
&= 2\,\mathrm{Re}\int_0^t \left\langle f(u(s)), \overline{\int_0^s U(s-s')\, f(u(s'))ds'} \right\rangle ds,
\end{aligned}
\tag{4.1.8}
$$

where the time integral of the scalar product is understood as the duality coupling on $(L_T^{q'} L^{r'}) \times (L_T^q L^r)$. Concatenating (4.1.6) - (4.1.8), we compute

$$
\begin{aligned}
&\| u(t)\|_{L^2}^2 \\
&= \| u_0\|_{L^2}^2 - 2\,\mathrm{Im}\int_0^t \left\langle U(s)\, u_0, \overline{f(u(s))} \right\rangle ds \\
&\quad + 2\,\mathrm{Re}\int_0^t \left\langle f(u(s)), \overline{\int_0^s U(s-s')\, f(u(s'))ds'} \right\rangle ds \\
&= \| u_0\|_{L^2}^2 + 2\,\mathrm{Im}\int_0^t \left\langle f(u(s)), \overline{U(s)\, u_0} \right\rangle ds \\
&\quad + 2\,\mathrm{Im}\int_0^t \left\langle f(u(s)), \overline{-i\int_0^s U(s-s')\, f(u(s'))ds'} \right\rangle ds \\
&= \| u_0\|_{L^2}^2 + \lim_{\varepsilon \to 0} 2\,\mathrm{Im}\int_0^t \left\langle (1-\varepsilon\Delta)^{-1} f(u(s)), \overline{u(s)} \right\rangle ds,
\end{aligned}
$$

where the last equality in the above holds by using (4.1.2). Taking the duality coupling between the equation (4.1.1) and $(1-\varepsilon\Delta)^{-1}f(u)$ on $H^{-1} \times H^1$ and using $\mathrm{Im}\{\langle(1-\varepsilon\Delta)^{-1}f(u), \overline{f(u)}\rangle\} = 0$, we obtain

$$
\mathrm{Im}\left\langle(1-\varepsilon\Delta)^{-1} f(u), \overline{u}\right\rangle = \mathrm{Im}\left\{ i\left\langle(1-\varepsilon\Delta)^{-1}f(u), \overline{\partial_t u}\right\rangle\right\}.
$$

From these equalities, we can show

$$
\begin{aligned}
&\| u(t)\|_{L^2}^2 \\
&= \| u_0\|_{L^2}^2 - \lim_{\varepsilon \to 0} 2\,\mathrm{Re}\int_0^t \left\langle(1-\varepsilon\Delta)^{-1}f(u(s)), \overline{\partial_t u(s)}\right\rangle ds \\
&= \| u_0\|_{L^2}^2 - 2\,\mathrm{Re}\int_0^t \left\langle f(u(s)), \overline{\partial_t u(s)}\right\rangle ds.
\end{aligned}
\tag{4.1.9}
$$

Note that in the above, the time integral of the scalar product in the last line is understood as the duality coupling on $(L^{q'}_T W^{1,r'}) \times (L^q_T W^{-1,r})$. From (4.1.9), we can continue as follows:

$$
\begin{aligned}
\| v(t)\|_{L^2}^2 &= \|u_0\|_{L^2}^2 - \int_0^t \frac{d}{ds}\left(\frac{2\lambda}{p+1}\|u(s)\|_{L^{p+1}}^{p+1}\right) ds \\
&= \|u_0\|_{L^2}^2 - \frac{2\lambda}{p+1}\|u(t)\|_{L^{p+1}}^{p+1} + \frac{2\lambda}{p+1}\|u_0\|_{L^{p+1}}^{p+1}
\end{aligned}
$$

since we can show that

$$
\frac{d}{ds}\left(\frac{2\lambda}{p+1}\|u(s)\|_{L^{p+1}}^{p+1}\right) = 2\,\mathrm{Re}\langle f(u(s)), \overline{\partial_t u(s)}\rangle \ \ in\ L^1(0,T). \tag{4.1.10}
$$

We can justify the equality (4.1.10) above by combining the way to the proof of Lemma 5.1 of [18] with Lemma 6.2.3. This completes the proof.

□

Remark 4.1.1. *By using the method [19], we can derive the mass and energy conservation law not to use the approximating agrument as in Chapter 2.4.*

Moreover, we remark that applying the method [19], for the equation (4.1.1), we can derive the conservation law of the momentum and the pseudo conformal conservation law for time local solutions without approximating procedure (see Fujiwara-Miyazaki [6]).

Chapter 5

The energy conservation law for nonlinear Schrödinger equations with non-vanishing boundary conditions at spatial infinity

5.1 Introduction in this chapter

In this chapter, we consider defocusing nonlinear Schrödinger equations in dimension $n \geq 4$.

$$\begin{cases} i\partial_t u + \Delta u = f(|u|^2)u, & t \in [0, \infty),\ x \in \mathbb{R}^n, \\ u(0, x) = u_0(x), & x \in \mathbb{R}^n, \end{cases} \tag{5.1.1}$$

where $u(t, x) : [0, \infty) \times \mathbb{R}^n \to \mathbb{C}$. The unknown function u has the following boundary condition:

$$|u(x)|^2 \to \rho_0 \quad \text{as} \quad |x| \to \infty,$$

where $\rho_0 > 0$ denotes the light intensity of the background. The nonlinear term f is assumed to be defocusing. Namely the real-valued function f satisfies the following assumption:

$$f(\rho_0) = 0, \quad f'(\rho_0) > 0. \tag{$\mathbf{H}_f$}$$

In this chapter, for the equation (5.1.1) in $n = 2$, 3, 4, we derive the conservation law for time local solutions without approximating procedure. Instead of that, we use Ozawa's idea [19]. Note that when $n = 1$, because $H^1 \hookrightarrow L^\infty$, Gallo [8] derived it without approximating procedure, and that for $n \geq 2$, Gallo [8] derive it using the approximate argument (see a proof of Theorem 3.1.4). We follow Ozawa's idea, however, we can not derive the conservation law only by Ozawa's idea, due to the nonlinear term and the space of a solution. We derive the conservation law to combine Ozawa's idea with decomposing the nonlinear term by applying the method for the decomposition of Schrödinger operator in Gérard [9] (see Lemma 3.1.3). Moreover, we remove some of technical assumptions of the nonlinearity necessary to derive the conservation law. Our main result in this thesis is as follows:

Theorem 5.1.1. *Let $n = 2, 3, 4$. Let $\rho_0 > 0$, and $f \ni C^2(\mathbb{R}_+)$ satisfying $(\mathbf{H}_f)$. Moreover, we assume that there exist $\alpha_1 \approx 1$, with a supplementary condition $\alpha_1 < \alpha_1^{\leq}$ if $n = 3, 4$ ($\alpha_1^{\leq} = 3$ if $n = 3$, $\alpha_1^{\leq} = 2$ if $n = 4$) such that*

$$\mathcal{D}C_0 > 0,\ s.t.\ \exists r \approx 1,\ \|f^{(k)}(r)\| \geq C_0 r^{\alpha_1 \quad 1 \quad k}\ (k = 1, 2). \qquad (\mathbf{H}^{\infty}_{\alpha_1})$$

Let ϕ be a function satisfying

$$\phi \ni C_b^2(\mathbb{R}^n), \qquad \phi \ni H^2(\mathbb{R}^n)^n, \quad \|\phi\|^2 \quad \rho_0 \ni L^2(\mathbb{R}^n). \qquad (\mathbf{H}^{\infty}_{\phi})$$

(Note that such function ϕ is called as a regular function of finite energy.) Let $w \ni C([0,T], H^1(\mathbb{R}^n))$ be a mild solution of the integral equation

$$w(t) = U(t)w_0 \quad i\bigcap_0^t U(t \quad t^{\infty})F(w(t^{\infty}))dt^{\infty} \qquad (5.1.2)$$

for some $w_0 \ni H^1$ and $T > 0$, where $F(w) := \quad \Delta\phi + f(\|\phi + w\|^2)(\phi + w)$.

Then $\mathcal{F}(w(t)) = \mathcal{F}(w_0)$ for all $t \ni [0,T]$, where

$$\mathcal{F}(w) := \bigcap_{\mathbb{R}^n} \| \quad (\phi + w)\|^2 dx + \bigcap_{\mathbb{R}^n} V(\|\phi + w\|^2)dx,$$

and

$$V(r) := \bigcap_{\rho_0}^r f(s)ds.$$

Remark 5.1.1. *Gallo [8] proves the energy conservation law under $f \ni C^k(\mathbb{R}_+)$ ($k = 3$ if $n = 2, 3$, $k = 4$ if $n = 4$) satisfying $(\mathbf{H}_f)$, $(\mathbf{H}_{\alpha_1})$ and $(\mathbf{H}_{\alpha_2})$ for some $\alpha_1 \approx 1$ and $\alpha_2 \ni \mathbb{R}$ with $\alpha_1 \quad \alpha_2 \geq 1/2$, and ϕ satisfying $(\mathbf{H}_\phi)$, but we can prove it under $f \ni C^2(\mathbb{R}_+)$ with $(\mathbf{H}_f)$ and $(\mathbf{H}^{\infty}_{\alpha_1})$ for some $\alpha_1 \approx 1$, and ϕ with $(\mathbf{H}^{\infty}_{\phi})$.*

Remark 5.1.2. *For proofs of the a priori estimate of $f(\|\phi + w\|^2)(\phi + w)$ (that is Lemmas 5.2.1 - 5.2.4, and Lemmas 4.1 - 4.4 in Gallo [8]) and boundedness of H^1 norm of w on bounded intervals (that is Lemma 3.3 in Gallo [8]), we need that there exists $C_{\alpha_1} > 0$ such that for any $r \approx 0$,*

$$r^{1/2}\|f^{(k)}(r)\| \geq C_\alpha(1 + r^{\max(0, \alpha_1 \quad (2k+1)/2)}) \quad (k = 1, 2), \qquad (5.1.3)$$

where $1 \geq \alpha_1$ with the same supplementary condition in Theorem 3.1.4. If $3/2 < \alpha_1 \geq 2$, then we can not obtain (5.1.3) from $(\mathbf{H}_{\alpha_1})$ or $(\mathbf{H}_{\alpha_2})$. Therefore by replacing $(\mathbf{H}_{\alpha_1})$ with $(\mathbf{H}^{\infty}_{\alpha_1})$, we deduce (5.1.3) from $(\mathbf{H}^{\infty}_{\alpha_1})$ only. To show (5.1.3), we do not need $(\mathbf{H}_{\alpha_2})$. That is, Theorem 3.1.5 can be shown only assuming $(\mathbf{H}^{\infty}_{\alpha_1})$. To show only the local existence Theorem, we do not need $(\mathbf{H}_{\alpha_2})$.

Moreover, as a corollary to the main result, we can deduce a globally well-posedness of (5.1.1). Due to Theorem 5.1.1, we can remove a technical assumption of the nonlinear term. We have the following result:

Corolary 5.1.1. *Let $n = 2,\ 3,\ 4$. We assume that f and ϕ satisfy the same assumptions as in Theorem 5.1.1, with a supplementary assumption as f satisfying $(\mathbf{H}_{\alpha_2})$ for some $\alpha_2 \ni \mathbb{R}$ with $\alpha_1 \quad \alpha_2 \geq 1/2$. Then (5.1.1) is globally well-posed in $\phi + H^1(\mathbb{R}^n)$. That is, for any $w_0 \ni H^1(\mathbb{R}^n)$, there exist a unique $w \ni C(\mathbb{R}, H^1(\mathbb{R}^n))$ such that $\phi + w$ solves (5.1.1) with the initial data $w(0) = w_0$. Moreover, for any $T > 0$, the flow map $w_0 \forall\ w\ (H^1 \uparrow\ C([0,T], H^1))$ is Lipschitz continuous on the bounded sets of $H^1(\mathbb{R}^n)$. The energy $\mathcal{F}(w)$ is conserved by the flow.*

5.2 The estimates of nonlinear terms

In what follows, we put $\widetilde{F(w)} = f(|\phi + w|^2)(\phi + w)$. Applying directly the decomposition of $F(w)$ that Gallo [8] gave, we can deduce the following decompositions for $\widetilde{F(w)}$. Note that we can show Lemmas 5.2.1 - 5.2.4 by applying the same method to $\widetilde{F(w)}$ as corresponding Lemmas for $F(w)$ in Gallo [8]. The statements of Lemma 5.2.1 and Lemma 5.2.3 is slightly different from these Lemmas in Gallo [8]. Therefore we only prove them.

Lemma 5.2.1. *Let $T > 0$. For any $w \ni X_T$, there exist*

$$\widetilde{F_1}(w) \ni L^{\in}_T L^2, \qquad \widetilde{F_2}(w) \ni L^{\in}_T L^{q^\infty}$$

such that

$$\widetilde{F(w)} = \widetilde{F_1}(w) + \widetilde{F_2}(w).$$

Moreover it follows that

$$\begin{aligned} &\backslash \widetilde{F_1}(w) \backslash_{L^{|}_T L^2} + \backslash \widetilde{F_2}(w) \backslash_{L^{|}_T L^{q^\infty}} \\ &\quad \geq C(1 + \backslash w \backslash_{L^{|}_T L^2}) + C(\backslash w \backslash^2_{L^{|}_T H^1} + \backslash w \backslash^{\max(2, 2\alpha_1 \ \ 1)}_{L^{|}_T H^1}), \end{aligned}$$

where C is a positive constant depending on T. Also for the same decomposition of $\widetilde{F(w)}$ in the above, we have $\widetilde{F_2}(w) \ni L^p_T L^2$ and

$$\backslash \widetilde{F_2}(w) \backslash_{L^p_T L^2} \geq C(\backslash w \backslash^2_{L^{|}_T H^1} + \backslash w \backslash^{\max(2, 2\alpha_1 \ \ 1)}_{X_T}),$$

where C is a positive constant depending on T. Thus $\widetilde{F_2}(w) \ni L^p_T L^2$.

Lemma 5.2.2 ([8])**.** *Let $T > 0$. For any $w \ni X_T$, there exist*

$$\widetilde{G_1}(w) \ni L^{\in}_T L^2, \qquad \widetilde{G_2}(w) \ni L^{p^\infty}_T L^{q^\infty}.$$

such that

$$\widetilde{F(w)} = \widetilde{G_1}(w) + \widetilde{G_2}(w).$$

Moreover it follows that

$$\begin{aligned} &\backslash \widetilde{G_1}(w) \backslash_{L^{|}_T L^2} + \backslash \widetilde{G_2}(w) \backslash_{L^{p^\infty}_T L^{q^\infty}} \\ &\geq C(1 + \backslash \ \ w \backslash_{L^{|}_T L^2}) \\ &\quad + C(1 + \backslash \ \ w \backslash_{L^{|}_T L^2})(\backslash w \backslash_{L^{|}_T H^1} + \backslash w \backslash^{\max(1, 2\alpha_1 \ \ 2)}_{X_T}), \end{aligned}$$

where C is a positive constant depending on T.

Lemma 5.2.3. *Let $T > 0$. For any w_1, $w_2 \in X_T$, decomposing $f(|\phi+w|^2)(\phi+w)$ as Lemma 5.2.1, it follows that*

$$
\begin{aligned}
&\|\widetilde{F_1}(w_1) - \widetilde{F_1}(w_2)\|_{L^1_T L^2} + \|\widetilde{F_2}(w_1) - \widetilde{F_2}(w_2)\|_{L^1_T L^{q'}} \\
&\leq C\|w_1 - w_2\|_{L^1_T H^1} + C\|w_1 - w_2\|_{L^1_T H^1} \\
&\quad \times \left((\|w_1\|_{L^1_T H^1} + \|w_2\|_{L^1_T H^1}) + (\|w_1\|_{L^1_T H^1} + \|w_2\|_{L^1_T H^1})^{\max(1,2\alpha_1-2)}\right),
\end{aligned}
$$

where C is a positive constant depending on T.

Lemma 5.2.4 ([8])**.** *Let $T > 0$. For any w_1, $w_2 \in X_T$, decomposing $f(|\phi+w|^2)(\phi+w)$ as Lemma 5.2.2, it follows that*

$$
\begin{aligned}
&\|\widetilde{G_1}(w_1) - \widetilde{G_1}(w_2)\|_{L^1_T L^2} + \|\widetilde{G_2}(w_1) - \widetilde{G_2}(w_2)\|_{L^{p'}_T L^{q'}} \\
&\leq C\|\nabla(w_1 - w_2)\|_{L^1_T L^2} \\
&\quad + C(1 + \|w_1\|_{L^1_T H^1} + \|w_2\|_{L^1_T H^1})^{\max(1,2\alpha_1-2)}\|w_1 - w_2\|_{L^1_T H^1} \\
&\quad + C\|w_1 - w_2\|_{X_T}(\|w_1\|_{X_T}^{\max(1,2\alpha_1-2)} + \|w_2\|_{X_T}^{\max(1,2\alpha_1-2)}) \\
&\quad + C\|w_1 - w_2\|_{X_T}(1 + \|w_1\|_{L^1_T H^1} + \|w_2\|_{L^1_T H^1}) \\
&\quad \times (\|w_1\|_{X_T}^{\max(0,2\alpha_1-3)} + \|w_2\|_{X_T}^{\max(0,2\alpha_1-3)}),
\end{aligned}
$$

where C is a positive constant depending on T.

Proof of Lemma 5.2.1. We decompose $f(|\phi+w|^2)(\phi+w)$ as

$$
f(|\phi+w|^2)(\phi+w) = \widetilde{F_1}(w) + \widetilde{F_2}(w), \tag{5.2.1}
$$

where

$$
\begin{aligned}
&\widetilde{F_1}(w) := f(|\phi|^2)(\phi+w) + 2\,\mathrm{Re}[\bar{\phi}w]f'(|\phi|^2)\phi, \\
&\widetilde{F_2}(w) := \{f(|\phi+w|^2)(\phi+w) - f(|\phi|^2)(\phi+w)\} - 2\,\mathrm{Re}[\bar{\phi}w]f'(|\phi|^2)\phi.
\end{aligned}
$$

According to Lemma 4.1 in Gallo [8], by the assumption $(\mathbf{H}^\infty_\phi)$ and $f(|\phi|^2) \in L^2$, we deduce that

$$
\|\widetilde{F_1}(w)\|_{L^1_T L^2} \leq C(1 + \|w\|_{L^1_T L^2}), \quad |\widetilde{F_2}(w)| \leq C|w|^2(1+|w|)^{\max(0,2\alpha_1-3)},
$$

Therefore for all $t \in [0,T]$, we estimate that

$$
\begin{aligned}
\|\widetilde{F_2}(w(t))\|_{L^{q'}} &\leq C\||w(t)|^2(1+|w(t)|)^{\max(0,2\alpha_1-3)}\|_{L^{q'}} \\
&\leq C\|w(t)\|^2_{L^{2q'}} + C\|w(t)\|^{\max(2,2\alpha_1-1)}_{L^{q'\max(2,2\alpha_1-1)}} \\
&\leq C\|w(t)\|^2_{H^1} + C\|w(t)\|^{\max(2,2\alpha_1-1)}_{H^1}.
\end{aligned}
$$

Hence, we deduce that

$$
\|\widetilde{F_2}(w)\|_{L^1_T L^{q'}} \leq C\|w\|^2_{L^1_T H^1} + C\|w\|^{\max(2,2\alpha_1-1)}_{L^1_T H^1}.
$$

In conclusion, we get

$$
\|\widetilde{F_1}(w)\|_{L^1_T L^2} + \|\widetilde{F_2}(w)\|_{L^1_T L^{q'}}
$$

$$\geq C(1 + \backslash w\backslash_{L^{\downharpoonleft}_T L^2}) + C(\backslash w\backslash^2_{L^{\downharpoonleft}_T H^1} + \backslash w\backslash^{\max(2,2\alpha_1 \;\; 1)}_{L^{\downharpoonleft}_T H^1}).$$

Next, we show $\widetilde{F(w)} \ni L^p_T L^2$. We apply an interpolation method (see Lemma 4.2 in Gallo [8]). Thanks to Hölder's inequality and Gagliardo-Nirenberg's inequality, we estimate

$$\begin{aligned}\backslash\widetilde{F_2}(w)\backslash_{L^p_T L^2} &\geq C\backslash w\backslash^2_{L^{2p}_T L^4} + \backslash w\backslash^{\max(2,2\alpha_1 \;\; 1)}_{L^{p\max(2,2\alpha_1 \;\; 1)}_T L^{2\max(2,2\alpha_1 \;\; 1)}} \\ &\geq C\backslash w\backslash^2_{L^{\downharpoonleft}_T H^1} + \backslash w\backslash^{\max(2,2\alpha_1 \;\; 1)}_{L^s_T W^{1,r}},\end{aligned} \tag{5.2.2}$$

where we choose the pair (s, r) such that

≤ If $\frac{1}{2} \;\; \frac{1}{n} \geq \frac{1}{p\max(2,2\alpha_1 \;\; 1)}$ (which means that $H^1 \uparrow \;\; L^{p\max(2,2\alpha_1 \;\; 1)}$), then $(s, r) = (\in, 2)$.

≤ If $\frac{1}{2} \;\; \frac{1}{n} > \frac{1}{p\max(2,2\alpha_1 \;\; 1)}$, then $r > 2$ and

(i) $\frac{2}{s} + \frac{n}{r} = \frac{n}{2}$ (which means that (s, r) is an admissible pair),

(ii) $0 \geq \frac{1}{r} \;\; \frac{1}{n} \geq \frac{1}{p\max(2,2\alpha_1 \;\; 1)}$ (which gives the Sobolev embedding $W^{1,r} \uparrow \;\; L^{p\max(2,2\alpha_1 \;\; 1)}$),

(iii) $\frac{1}{p\max(2,2\alpha_1 \;\; 1)} \approx \frac{1}{s}$ (which gives $L^s_T \uparrow \;\; L^{p\max(2,2\alpha_1 \;\; 1)}_T$).

Such the choice of s and r is possible if and only if s and r satisfy the following inequality:

$$\frac{n}{2} \;\; 1 \geq \frac{2+n}{p\max(2, 2\alpha_1 \;\; 1)}. \tag{5.2.3}$$

Indeed, if (5.2.3) is true, then it is sufficient to choose

$$\frac{n}{r} \ni \left]\frac{n}{2} \;\; \frac{2}{p\max(2,2\alpha_1 \;\; 1)},\ 1 + \frac{n}{p\max(2,2\alpha_1 \;\; 1)}\right\{.$$

Moreover, since $H^1 \uparrow \;\; L^{p\max(2,2\alpha_1 \;\; 1)}$ if $n = 2$ or if $n = 3$ and $1 \geq \alpha_1 \geq 2$ or if $n = 4$ and $1 \geq \alpha_1 \geq 3/2$, we consider that $n = 3$ and $2 < \alpha_1 < 3$ or $n = 4$ and $3/2 < \alpha_1 < 2$. Since $2 < r < 3$ and (s, r) is an admissible pair, we can choose $\widetilde{\theta} \ni (0, 1)$ satisfying

$$\frac{1 \;\; \widetilde{\theta}}{2} + \frac{\widetilde{\theta}}{q} = \frac{1}{r}, \quad \frac{1 \;\; \widetilde{\theta}}{\in} + \frac{\widetilde{\theta}}{p} = \frac{1}{s}.$$

Thus, using interpolation method,

$$\begin{aligned}\backslash w\backslash_{L^s_T W^{1,r}} &\geq C\backslash w\backslash^{1 \;\; \widetilde{\theta}}_{L^{\downharpoonleft}_T H^1}\backslash w\backslash^{\widetilde{\theta}}_{L^p_T W^{1,q}} \\ &\geq C(\backslash w\backslash_{L^{\downharpoonleft}_T H^1} + \backslash w\backslash_{L^p_T W^{1,q}}) \\ &= C\backslash w\backslash_{X_T}.\end{aligned} \tag{5.2.4}$$

From (5.2.2) and (5.2.4), we deduce that

$$\backslash F_2(w)\backslash_{L^p_T L^2} \geq C(\backslash w\backslash^2_{L^{\downharpoonleft}_T H^1} + \backslash w\backslash^{\max(2,2\alpha_1 \;\; 1)}_{X_T}).$$

Thus, we get $F(w) \ni L^p_T L^2$. □

Proof of Lemma 5.2.3. we use the decomposition (5.2.1) again. As is in Gallo [8], we also have

$$|\widetilde{F_1}(w_1) - \widetilde{F_1}(w_2)| \leq C|w_1 - w_2|,$$
$$|\widetilde{F_2}(w_2) - \widetilde{F_2}(w_2)| \leq C|w_1 - w_2|(|w_1| + |w_2|)(1 + |w_1| + |w_2|)^{\max(0,2\alpha_1 - 3)}.$$

Therefore we deduce that

$$\|\widetilde{F_1}(w_1) - \widetilde{F_1}(w_2)\|_{L^1_T L^2} \leq C\|w_1 - w_2\|_{L^1_T L^2}.$$

Moreover let

$$(q_1, q_2) := \begin{cases} (2,3) & \text{if } n = 2, \text{ or } n = 3 \text{ and } \alpha_1 \leq 2, \\ \left(\dfrac{q}{q - 1 - \max(1, 2\alpha_1 - 2)}, \dfrac{q}{\max(1, 2\alpha_1 - 2)}\right) & \\ & \text{if } n = 3 \text{ and } 2 < \alpha_1 < 3 \text{ or } n = 4, \end{cases}$$

with $\frac{1}{q'} = \frac{1}{q_1} + \frac{1}{q_2}$. Since if $n = 3$ and $2 < \alpha_1 < 3$ or $n = 4$, then $H^1 \hookrightarrow L^{q_1}$, for all $t \in [0, T]$, we estimate

$$\begin{aligned}
&\|\widetilde{F_2}(w_2(t)) - \widetilde{F_2}(w_2(t))\|_{L^{q'}} \\
&\leq C\|w_1(t) - w_2(t)\|_{L^{2q'}}(\|w_1(t)\|_{L^{2q'}} + \|w_2(t)\|_{L^{2q'}}) \\
&\quad + C\|w_1(t) - w_2(t)\|_{L^{q_1}}\||w_1(t)| + |w_2(t)|\|^{\max(1,2\alpha_1 - 2)}_{L^{q_2 \max(1,2\alpha_1 - 2)}} \\
&\leq C\|w_1(t) - w_2(t)\|_{L^{2q'}}(\|w_1(t)\|_{L^{2q'}} + \|w_2(t)\|_{L^{2q'}}) \\
&\quad + C\|w_1(t) - w_2(t)\|_{L^{q_1}}\||w_1(t)| + |w_2(t)|\|^{\max(1,2\alpha_1 - 2)}_{L^{q}} \\
&\leq C\|w_1(t) - w_2(t)\|_{H^1}(\|w_1(t)\|_{H^1} + \|w_2(t)\|_{H^1}) \\
&\quad + C\|w_1(t) - w_2(t)\|_{H^1}(\|w_1(t)\|_{H^1} + \|w_2(t)\|_{H^1})^{\max(1,2\alpha_1 - 2)}.
\end{aligned}$$

In conclusion, we get

$$\begin{aligned}
&\|\widetilde{F}(w_1) - \widetilde{F}(w_2)\|_{L^1_T L^2 + L^1_T L^{q'}} \\
&\leq CT\|w_1 - w_2\|_{L^1_T L^2} + C\|w_1 - w_2\|_{L^1_T H^1} \\
&\quad \times ((\|w_1\|_{L^1_T H^1} + \|w_2\|_{L^1_T H^1}) + (\|w_1\|_{L^1_T H^1} + \|w_2\|_{L^1_T H^1})^{\max(1,2\alpha_1 - 2)}).
\end{aligned}$$

□

Remark 5.2.1. *For Lemma 5.2.2 and Lemma 5.2.4, decomposing* $\widetilde{F}(w)$ *as*

$$\widetilde{F}(w) = \widetilde{G_1}(w) + \widetilde{G_2}(w), \tag{5.2.5}$$

where

$$\widetilde{G_1}(w) = f(|\phi|^2)\nabla(\phi + w) + 2\,\mathrm{Re}[\bar{\phi}\nabla(\phi + w)]f'(|\phi|^2)\phi,$$
$$\begin{aligned}
\widetilde{G_2}(w) = {} & 2\,\mathrm{Re}[(\overline{\phi + w})\nabla(\phi + w)]f'(|\phi + w|^2)(\phi + w) - 2\,\mathrm{Re}[\bar{\phi}\nabla(\phi + w)]f'(|\phi|^2)\phi \\
& + \{f(|\phi + w|^2)\nabla(\phi + w) - f(|\phi|^2)\nabla(\phi + w)\},
\end{aligned}$$

we can prove the assertions in Lemma 5.2.2 and Lemma 5.2.4 from way similar to proof of Lemma 3.1.6 and Lemma 3.1.8 in Chapter 3.

Remark 5.2.2. *Let $T > 0$. Lemma 5.2.1 and Sobolev embedding $H^1 \hookrightarrow L^q$, imply that for any $w \in C([0,T], H^1)$ and $t \in [0,T]$, $F(w(t)) \in H^{-1}$. Furthermore, for any $t_0 \in [0,T]$, Lemma 5.2.3 yields*

$$\begin{aligned} \|F(w(t)) - F(w(t_0))\|_{H^{-1}} &\leq C\|w(t) - w(t_0)\|_{H^1} \\ &\to 0 \quad as \quad t \to t_0, \end{aligned}$$

where C is a positive constant depending on $\|w\|_{L^\infty_T H^1}$. To show it, for $w \in C([0,T], H^1)$, it suffices to put $w_1(s) = w(t)$ and $w_2(s) = w(t_0)$ in Lemma 5.2.3 $(0 \leq s \leq T)$. Thus we also obtain $F(w) \in C([0,T], H^{-1})$.

In the proof of the main result, we use the following Lemma:

Lemma 5.2.5. *For any $\eta \in L^2 + L^{q^\infty}$, it follows that*

$$\|\chi(D_x)\eta\|_{H^1} \leq C\|\eta\|_{L^2+L^{q^\infty}} \tag{5.2.6}$$

Moreover for any $\eta \in \mathscr{S}'(\mathbb{R}^n)$ with $\nabla\eta \in L^2 + L^{q^\infty}$, we obtain

$$\|(1 - \chi(D_x))\eta\|_{H^1} \leq C\|\nabla\eta\|_{L^2+L^{q^\infty}} \tag{5.2.7}$$

Note that if X and Y are Banach spaces, then $X + Y$ is a Banach space equipped with the norm

$$\|v\|_{X+Y} := \inf\{\|v_1\|_X + \|v_2\|_Y \ : \ v = v_1 + v_2,\ v_1 \in X,\ v_2 \in Y\}.$$

Proof of Lemma 5.2.5. For any $\eta \in L^2 + L^{q^\infty}$, There exist $\eta_1 \in L^2$ and $\eta_2 \in L^{q^\infty}$ such that $\eta = \eta_1 + \eta_2$. $Q(\xi)$ denotes $(1 + |\xi|^2)\chi(\xi)$. Also, $Q(\xi)$ satisfies (6.2.1) since $\chi \in C_0^\infty(\mathbb{R}^n)$. Therefore, using Fourier multiplier Theorem, we obtain

$$\begin{aligned} \|\chi(D_x)\eta\|_{H^1} &= \|Q(D_x)\eta\|_{H^{-1}} \\ &\leq \|Q(D_x)\eta_1\|_{L^2} + \|Q(D_x)\eta_2\|_{L^{q^\infty}} \\ &\leq C\left(\|\eta_1\|_{L^2} + \|\eta_2\|_{L^{q^\infty}}\right). \end{aligned}$$

Therefore, we deduce that

$$\|\chi(D_x)\eta\|_{H^1} \leq C\|\eta\|_{L^2+L^{q^\infty}}$$

Next, for any $\eta \in \mathscr{S}'(\mathbb{R}^n)$ with $\nabla\eta \in L^2 + L^{q^\infty}$, there exist $(\zeta_1^j(w))_{j=1,\cdots,n} \in L^2$ and $(\zeta_2^j(w))_{j=1,\cdots,n} \in L^{q^\infty}$ such that $\nabla\eta = \zeta_1 + \zeta_2$. Using $P_j(\xi) := -i\xi_j/|\xi|^2$ $(\xi := (\xi_j)_{j=1,\cdots,n} \in \mathbb{R}^n)$, we have

$$\begin{aligned} (1 - \chi(D_x))\eta &= (1 - \chi(D_x))\sum_{j=1}^n P_j(D_x)\partial_j\eta \\ &= \sum_{j=1}^n (1 - \chi(D_x))P_j(D_x)\zeta_1^j + \sum_{j=1}^n (1 - \chi(D_x))P_j(D_x)\zeta_2^j. \end{aligned}$$

Fourier multiplier Theorem implies

$$\|(1 - \chi(D_x))\eta\|_{H^1+W^{1,q^\infty}}$$

$$\geq \int_{j=1}^{n} \|(1 \quad \chi(D_x))P_j(D_x)\zeta_1^j\|_{H^1} + \int_{j=1}^{n} \|(1 \quad \chi(D_x))P_j(D_x)\zeta_2^j\|_{W^{1,q^\infty}}$$
$$\geq C\|\zeta_1\|_{L^2} + C\|\zeta_2\|_{L^{q^\infty}}$$

Thus we get

$$\|(1 \quad \chi(D_x))\eta\|_{H^1+W^{1,q^\infty}} \geq C\| \quad \eta\|_{L^2+L^{q^\infty}}$$

□

5.3 Regularities of time-derivative term

In this section, we shall show properties of the time-derivative term $\partial_t u$.

Lemma 5.3.1. *Let $n = 2,\ 3,\ 4$, and let $(p,q) := (6/n, 6)$ for $n = 2,\ 3$, $(p,q) := (2,4)$ for $n = 4$. Let w be a solution of equation (3.1.8) belonging to $C([0,T], H^1)$ for some $T > 0$ with the initial data $w(0) = w_0 \ni H^1$. Then for any $0 < \varepsilon < T^\infty < T$,*

(i) $$\left\| \frac{w(\cdot + h) \quad w(\cdot)}{h} \quad \partial_t w(\cdot) \right\|_{C([\varepsilon,T^\infty],H^{\ 1})} \uparrow\ 0 \quad as \quad h \uparrow\ 0,$$

and

(ii) $$\left\| \frac{w(\cdot + h) \quad w(\cdot)}{h} \quad \partial_t w(\cdot) \right\|_{L^p((\varepsilon,T^\infty),W^{\ 1,q})} \uparrow\ 0 \quad as \quad h \uparrow\ 0.$$

Proof. Note that equation (3.1.8) implies

$$\partial_t w = i(\Delta w \quad F(w)). \tag{5.3.1}$$

We show (i) and (ii) using (5.3.1).
Proof of (i). Note that from Theorem 3.1.7, for any $0 \geq t \geq T$, $\partial_t w(t) \ni H^{\ 1}$ exists in strong sense. Hence, it suffices to show continuity of $\partial_t w(t)$ on $[0,T]$. Clearly,

$$\|\Delta w\|_{H^{\ 1}} \geq \| \quad w\|_{L^2}, \tag{5.3.2}$$

which yields $\Delta w \ni C([0,T], H^{\ 1})$. Using (5.3.1), (5.3.2) and Remark 5.2.2, we obtain

$$\partial_t w \ni C([0,T], H^{\ 1}).$$

Hence, it follows that for all t_0, $t \ni [0,T]$,

$$w(t) \quad w(t_0) = \bigcap_{t_0}^{t} \partial_t w(s) ds \quad in\ H^{\ 1}. \tag{5.3.3}$$

We take $0 < \varepsilon < T^\infty < T$. For all $t_0 \ni [\varepsilon, T^\infty]$ and sufficiently small $h \ni \mathbb{R}$,

$$\left\| \frac{w(t_0 + h) \quad w(t_0)}{h} \quad \partial_t w(t_0) \right\|_{H^{\ 1}} \geq \frac{1}{|h|} \left| \bigcap_{t_0}^{t_0+h} \|\partial_t w(s) \quad \partial_t w(t_0)\|_{H^{\ 1}} ds \right|$$

$$\leq \sup_{|s-t_0|\leq h} \|\partial_t w(s) - \partial_t w(t_0)\|_{H^{-1}}.$$

Since $t \mapsto \partial_t u(t) \in H^{-1}(\mathbb{R}^n)$ is uniformly continuous on $[0,T]$, we obtain (i).
Proof of (ii). Since $W^{1,q'}(\mathbb{R}^n) \hookrightarrow L^2(\mathbb{R}^n)$ and $w \in C([0,T], H^{-1})$ and ϕ satisfies $(\mathbf{H}_\phi)$, we clearly get

$$\Delta w \in L^p([0,T], W^{-1,q}) \quad \text{and} \quad \Delta\phi \in L^p([0,T], W^{-1,q}). \tag{5.3.4}$$

Moreover, using Sobolev embedding and duality argument, we conclude $L^2 \hookrightarrow W^{-1,q}$. Thus Lemma 5.2.1 yields

$$F(w) \in L^p([0,T], W^{-1,q}). \tag{5.3.5}$$

Therefore, concatenating (5.3.1), (5.3.4) and (5.3.5), we obtain

$$\partial_t w \in L^p([0,T], W^{-1,q}).$$

Let $t_0 \in [0,T]$. By (5.3.3), for any $t \in [0,T]$,

$$w(t) - w(t_0) = \int_{t_0}^{t} \partial_t w(s)ds \quad in\ \mathcal{S}'(\mathbb{R}^n),$$

where $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ denote Schwartz space on $\mathbb{R}^n$ and the space of tempered distributions on $\mathbb{R}^n$, respectively. Using Hölder's inequality, we get

$$\begin{aligned}
\left\|\int_{t_0}^{\cdot} \partial_t w(s)ds\right\|_{L^p([0,T],W^{-1,q})} &\leq \left\|\int_{t_0}^{\cdot} \|\partial_t w(s)\|_{W^{-1,q}}ds\right\|_{L^p([0,T])} \\
&\leq \left[\int_0^T (t-t_0)^{p/p'}\left(\int_{t_0}^{t} \|\partial_t w(s)\|^p_{W^{-1,q}}ds\right) dt\right]^{1/p} \\
&\leq T^{1/p'}\left[\int_0^T \left(\int_{t_0}^{t} \|\partial_t w(s)\|^p_{W^{-1,q}}ds\right) dt\right]^{1/p} \\
&\leq T^{1/p'}\left(T \cdot \int_0^T \|\partial_t w(s)\|^p_{W^{-1,q}}ds\right)^{1/p} \\
&\leq T\|\partial_t w\|_{L^p_T W^{-1,q}}.
\end{aligned}$$

Therefore, for all $t_0 \in [0,T]$,

$$w(\cdot) - w(t_0) = \int_{t_0}^{\cdot} \partial_t w(s)ds \quad in \quad L^p([0,T], W^{-1,q}). \tag{5.3.6}$$

Combining (5.3.6) with Strichartz's estimate, in a way similar to the preceding argument, for all $0 < \varepsilon < T' < T$, we obtain

$$\begin{aligned}
&\left\|\frac{w(\cdot+h) - w(\cdot)}{h} - \partial_t w(\cdot)\right\|_{L^p([\varepsilon,T'],W^{-1,q})} \\
&\leq \left\|\frac{1}{h}\int_{\cdot}^{\cdot+h} \|\partial_t w(s) - \partial_t w(\cdot)\|_{W^{-1,q}}ds\right\|_{L^p([\varepsilon,T'])}
\end{aligned}$$

$$
\begin{aligned}
&= \left(\int_{\varepsilon}^{T'} \left\| \frac{1}{h} \int_{t_0}^{t_0+h} \| \partial_t w(s) - \partial_t w(t_0) \|_{W^{1,q}} ds \right\|^p dt_0 \right)^{1/p} \\
&\le h^{1/p'-1} \left(\int_{\varepsilon}^{T'} \left(\int_{t_0}^{t_0+h} \| \partial_t w(s) - \partial_t w(t_0) \|_{W^{1,q}}^p ds \right) dt_0 \right)^{1/p} \\
&= h^{1/p'-1} \left(\int_{0}^{h} \left(\int_{\varepsilon}^{T'} \| \partial_t w(t_0+s) - \partial_t w(t_0) \|_{W^{1,q}}^p dt_0 \right) ds \right)^{1/p} \\
&\le \sup_{0\le s\le h} \left(\int_{\varepsilon}^{T'} \| \partial_t w(t_0+s) - \partial_t w(t_0) \|_{W^{1,q}}^p dt_0 \right)^{1/p} \\
&\to 0 \quad as \quad h \to 0.
\end{aligned}
$$

This completes the proof of Lemma 5.3.1. □

5.4 The proof of the main result

Since Schrödinger operator $U(t)$ becomes bounded operator from $\phi + H^1$ to itself (see Lemma 3.1.3), we can obtain

$$
\phi = U(t)\phi - i \int_0^t U(t - t')\Delta\phi dt'.
$$

Combining the above equality with (5.1.2), we get

$$
\phi + w(t) = U(t)(\phi + w_0) - i \int_0^t U(t - t')\widetilde{F}(w(t'))dt', \tag{5.4.1}
$$

where $\widetilde{F}(w) := -f(|\phi + w|^2)(\phi + w)$. From now on, we deduce the proof in a way similar to Ozawa [19]. Acting ∇ on (5.4.1), we obtain

$$
\begin{aligned}
&\|\nabla(\phi + w(t))\|_{L^2}^2 \\
&= \|\nabla U(-t)(\phi + w(t))\|_{L^2}^2 \\
&= \|\nabla(\phi + w_0)\|_{L^2}^2 - 2\,\mathrm{Im} \left(\nabla(\phi + w_0), \int_0^t U(-t')\nabla\widetilde{F}(w(t'))dt' \right)_{L^2} \\
&\quad + \left\| \int_0^t U(-t')\nabla\widetilde{F}(w(t'))dt' \right\|_{L^2}^2 .
\end{aligned} \tag{5.4.2}
$$

The second term on the RHS of (5.4.2) satisfies the following equality:

$$
\begin{aligned}
&-2\,\mathrm{Im} \left(\nabla(\phi + w_0), \int_0^t e^{i(-t')\Delta}\nabla\widetilde{F}(u(t'))dt' \right)_{L^2} \\
&= -2\,\mathrm{Im} \int_0^t \langle U(t')\nabla(\phi + w_0), \overline{\nabla(\widetilde{F}(w(t')))} \rangle dt',
\end{aligned} \tag{5.4.3}
$$

where the time integral of the scalar product is understood as the duality coupling on $(L_T^1 L^2 \cap L_t^p L^q) \times (L_T^\infty L^2 + L_T^{p'} L^{q'})$ with $(p, q) = (6/n, 6)$ if $n = 2$, 3, $(p,q)=(2,4)$ if $n=4$. For the last term on the RHS of (5.4.2), Fubini's Theorem implies

$$\begin{aligned}
&\left\| \int_0^t U(-t')\,(\widetilde{F}(w(t')))\,dt' \right\|_{L^2}^2 \\
&\quad = 2\,\mathrm{Re} \int_0^t \left\langle (\widetilde{F}(w(t'))), \overline{\int_0^{t'} e^{i(t'-t'')\Delta}\,(\widetilde{F}(w(t'')))\,dt''} \right\rangle dt', \qquad (5.4.4)
\end{aligned}$$

where the time integral of the scalar product is understood as the duality coupling on $(L_T^\infty L^2 + L_T^{p'} L^{q'}) \times (L_T^1 L^2 \cap L_T^p L^q)$. Concatenating (5.4.2) - (5.4.4), we compute

$$\begin{aligned}
&\|(\phi + w(t))\|_{L^2}^2 \\
&= \|(\phi + w_0)\|_{L^2}^2 - 2\,\mathrm{Im} \int_0^t \langle U(t')(\phi + w_0), \overline{(\widetilde{F}(w(t')))}\rangle\, dt' \\
&\quad + 2\,\mathrm{Re} \int_0^t \left\langle (\widetilde{F}(w(t'))), \overline{\int_0^{t'} U(t'-t'')\,(\widetilde{F}(w(t'')))\,dt''} \right\rangle dt' \\
&= \|(\phi + w_0)\|_{L^2}^2 + 2\,\mathrm{Im} \int_0^t \langle (\widetilde{F}(w(t')), \overline{U(t')(\phi + w_0)}\rangle\, dt' \\
&\quad + 2\,\mathrm{Im} \int_0^t \left\langle (\widetilde{F}(w(t'))), \overline{-i\int_0^{t'} U(t'-t'')\,(\widetilde{F}(w(t'')))\,dt''} \right\rangle dt' \\
&= \|(\phi + w_0)\|_{L^2}^2 + \lim_{\varepsilon\to 0} 2\,\mathrm{Im} \int_0^t \langle (1-\varepsilon\Delta)^{-1}(\widetilde{F}(w(t'))), \overline{w(t')}\rangle\, dt',
\end{aligned}$$

where the last equality in the above holds by using (5.4.1). Taking the duality coupling between the equation (5.1.1) and $(1-\varepsilon\Delta)^{-1}(\widetilde{F}(w))$ on $H^{-1}\times H^1$ and using $\mathrm{Im}\{\langle(1-\varepsilon\Delta)^{-1}\widetilde{F}(w), \overline{\widetilde{F}(w)}\rangle\}=0$, we obtain

$$\mathrm{Im}\langle(1-\varepsilon\Delta)^{-1}(\widetilde{F}(w)), \overline{w}\rangle = \mathrm{Im}\{-i\langle(1-\varepsilon\Delta)^{-1}\widetilde{F}(w), \overline{\partial_t w}\rangle\}.$$

From these equalities, we can show

$$\begin{aligned}
&\|(\phi + w(t))\|_{L^2}^2 \\
&= \|(\phi + w_0)\|_{L^2}^2 - \lim_{\varepsilon\to 0} 2\,\mathrm{Re} \int_0^t \langle(1-\varepsilon\Delta)^{-1}\widetilde{F}(w(t')), \overline{\partial_t w(t')}\rangle\, dt' \\
&= \|(\phi + w_0)\|_{L^2}^2 - 2\,\mathrm{Re} \int_0^t \langle\widetilde{F}(w(t')), \overline{\partial_t w(t')}\rangle\, dt'. \qquad (5.4.5)
\end{aligned}$$

Note that in the above time integral of the scalar product in the last line is understood as the duality coupling on $(L_T^\infty H^1 + (L_T^\infty H^1 + L_T^{p'} W^{1,q'})) \times ((L_T^1 H^{-1}) \cap (L_T^1 H^{-1} \cap L_T^p W^{-1,q}))$ by applying the idea used Lemma 3 in Gérard [9] (see Lemma 3.1.3), that is, we decompose $\widetilde{F}(w)$ as

$$\widetilde{F}(w) = \chi(D_x)\widetilde{F}(w) + \sum_{j=1}^n (1-\chi(D_x))P_j(D_x)\partial_{x_j}\widetilde{F}(w),$$

where $\chi \in C_0^\infty(\mathbb{R}^n)$ is a cutoff function such that $0 \le \chi \le 1$, $\chi(\xi) = 1$ for $|\xi| \le 1$ and $\chi(\xi) = 0$ for $|\xi| \ge 2$, and $P_j(\xi) = i\xi_j/|\xi|^2$.

We show (5.4.5). It follows from Theorem 3.1.7, Lemma 5.2.1, 5.2.2, 5.2.5 and 5.3.1 that

$$
\begin{aligned}
&\left|\int_0^t \langle \widetilde{F}(w(t')), \overline{\partial_t w(t')}\rangle \, dt'\right| \\
&\le \int_0^t |\langle \chi(D_x)\widetilde{F}(w(t')), \overline{\partial_t w(t')}\rangle| \, dt' + \int_0^t |\langle (1-\chi(D_x))\widetilde{F}(w(t')), \overline{\partial_t w(t')}\rangle| \, dt' \\
&\le \|\chi(D_x)\widetilde{F}(w)\|_{L^\infty_T H^1}\|\partial_t w\|_{L^1_T H^{-1}} \\
&\qquad + \|(1-\chi(D_x))\widetilde{F}(w)\|_{L^{p'}_T(H^1+W^{1,q'})}\|\partial_t w\|_{L^p_T(H^{-1}\cap W^{-1,q})}. \\
&\le C(\|\widetilde{F_1}(w)\|_{L^\infty_T L^2} + \|\widetilde{F_2}(w)\|_{L^\infty_T L^{q'}})\|\partial_t w\|_{L^1_T H^{-1}} \\
&\quad + C(\|\widetilde{G_1}(w)\|_{L^{p'}_T L^2} + \|\widetilde{G_2}(w)\|_{L^{p'}_T L^{q'}})\|\partial_t w\|_{L^p_T(H^{-1}\cap W^{-1,q})}. \qquad (5.4.6)
\end{aligned}
$$

Furthermore, by using a similar argument to the above and Lebesgue convergence Theorem, we deduce that

$$\lim_{\varepsilon\to 0}\int_0^t \langle (1-\varepsilon\Delta)^{-1}\widetilde{F}(w(t')), \overline{\partial_t w(t')}\rangle \, dt' = \int_0^t \langle \widetilde{F}(w(t')), \overline{\partial_t w(t')}\rangle \, dt',$$

which yields (5.4.5).

From (5.4.5), formally, we can continue as follows:

$$
\begin{aligned}
\|\nabla(\phi + w(t))\|_{L^2}^2 &= \|\nabla(\phi + w_0)\|_{L^2}^2 - \int_0^t \frac{\partial}{\partial t}\left(\int_{\mathbb{R}^n} V(|\phi + w(t')|^2)dx\right) dt' \\
&= \|\nabla(\phi + w_0)\|_{L^2}^2 - \int_{\mathbb{R}^n} V(|\phi + w(t)|^2)dx + \int_{\mathbb{R}^n} V(|\phi|^2)dx,
\end{aligned}
$$

since a formal argument implies

$$\frac{\partial}{\partial t}\left(\int_{\mathbb{R}^n} V(|\phi + w(t)|^2)dx\right) = 2\,\mathrm{Re}\langle \widetilde{F}(w(t)), \overline{\partial_t w(t)}\rangle .$$

Hence, to justify the argument above, we need to show the following Lemma.

Lemma 5.4.1. $\displaystyle\int_{\mathbb{R}^n} V(|\phi + w(\cdot)|^2)dx \in W^{1,1}((0,T))$
and

$$\frac{\partial}{\partial t}\left(\int_{\mathbb{R}^n} V(|\phi + w(t)|^2)dx\right) = 2\,\mathrm{Re}\langle \widetilde{F}(w(t)), \overline{\partial_t w(t)}\rangle_{A^*, B} \quad in \quad L^1((0,T)),$$

where $A := (H^1 + (H^1 + W^{1,q'}))$, $B := (H^{-1} \cap (H^{-1} \cap W^{-1,q}))$.

Proof. Put $I = (0,T)$ for simplicity. Moreover, $\mathcal{D}(I)$ and $\mathcal{D}'(I)$ denote the Fréchet space of C^∞ functions $I \to \mathbb{C}$ compactly supported in I and the space of distributions on I, respectively. Note that as is in Gallo [8], from $(\mathbf{H}_f)$, the mapping $w \mapsto V(|\phi + w|^2)$ become a bounded operator from $H^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$. Thus, for any $\varphi \in C_0^\infty(0,T)$, we have

$$\left\langle \frac{\partial}{\partial t}\int_{\mathbb{R}^n} V(|\phi + w|^2)dx, \varphi \right\rangle_{\mathcal{D}'(I), \mathcal{D}(I)}$$

$$= \Big\rangle \bigcap_{\mathbb{R}^n} V(|\phi + w|^2)dx, \partial_t \varphi \Big(_{\cap \propto I) * \cap (I)}$$

$$= \bigcap_I \Big) \bigcap_{\mathbb{R}^n} V(|\phi + w(t)|^2)dx \Big[\partial_t \varphi(t)dt.$$

Take $0 < \varepsilon < T^{\infty} < T$ such that $\mathrm{supp}(\varphi) \to [\varepsilon, T^{\propto}]$. Using Lebesgue convergence Theorem, we compute

$$\bigcap_I \Big) \bigcap_{\mathbb{R}^n} V(|\phi + w(t)|^2)dx \Big[\partial_t \varphi(t)dt$$

$$= \lim_{h \uparrow 0} \Big\} \bigcap_{\varepsilon}^{T^{\infty}} \Big) \bigcap_{\mathbb{R}^n} V(|\phi + w(t)|^2)dx \Big[\frac{\varphi(t+h) \quad \varphi(t)}{h} dt \Big\langle$$

$$= \lim_{h \propto 0} \Big\} \bigcap_{\varepsilon}^{T^{\infty}} \Big) \bigcap_{\mathbb{R}^n} \frac{V(|\phi + w(t + h^{\propto})|^2) \quad V(|\phi + w(t)|^2)}{h^{\infty}} dx \Big[\varphi(t)dt \Big\langle$$

$$= \bigcap_I 2\,\mathrm{Re} \rangle F(u(t)), \overline{\partial_t w(t)} |_{A* \, B} \varphi(t)dt.$$

We need to justify the limiting procedure of the last line in the above. Since $(\partial/\partial \bar{z})(V(|z|^2)) = \widetilde{F}(|z|)$ for any $z \ni \mathbb{C}$, it follows that

$$\Big\| \bigcap_{\mathbb{R}^n} \frac{V(|\phi + w(t+h)|^2) \quad V(|\phi + w(t)|^2)}{h} dx \quad 2\,\mathrm{Re} \rangle \widetilde{F}(w(t)), \overline{\partial_t w(t)} |_{A* \, B} \Big\|$$

$$\geq \Big\| \bigcap_{\mathbb{R}^n} 2\,\mathrm{Re} \Big) \bigcap_0^1 \frac{\partial V}{\partial \bar{z}} (|\phi + w(t) + \theta(w(t+h) \quad w(t))|^2) d\theta$$

$$* \; \frac{\overline{(w(t+h) \quad w(t))}}{h} \Big[dx \quad 2\,\mathrm{Re} \rangle \widetilde{F}(w(t)), \overline{\partial_t w(t)} |_{A* \, B} \Big\|$$

$$\geq 2 \Big\| \bigcap_{\mathbb{R}^n} \Big) \bigcap_0^1 \Big) \frac{\partial V}{\partial \bar{z}} (|\phi + w(t) + \theta(w(t+h) \quad w(t))|^2) \quad \widetilde{F}(w(t)) \Big[d\theta$$

$$* \; \frac{\overline{(w(t+h) \quad w(t))}}{h} \Big[dx \Big\|$$

$$+ 2 \Big\| \bigcap_{\mathbb{R}^n} \widetilde{F}(w(t)) \frac{\overline{(w(t+h) \quad w(t))}}{h} dx \quad \rangle \widetilde{F}(w(t)), \overline{\partial_t w(t)} |_{A* \, B} \Big\|$$

$$\geq 2 \Big\| \bigcap_{\mathbb{R}^n} \Big) \bigcap_0^1 \Big) \widetilde{F}(w(t) + \theta(w(t+h) \quad w(t))) \quad \widetilde{F}(w(t)) \Big(d\theta$$

$$* \; \frac{\overline{(w(t+h) \quad w(t))}}{h} \Big[dx \Big\|$$

$$+ 2 \Big\| \Big\rangle \widetilde{F}(w(t)), \frac{\overline{(w(t+h) \quad w(t))}}{h} \Big(_{H^{\;\;1} * \, H^1} \quad \rangle \widetilde{F}(w(t)), \overline{\partial_t w(t)} |_{A* \, B} \Big\|$$

$$=: 2L_1 + 2L_2. \tag{5.4.7}$$

<u>The estimation of L_1.</u> Choose the cutoff function $\chi \ni C_0^{\in}(\mathbb{R}^n)$ such that $0 \geq \chi \geq 1$, $\chi(\xi) = 1$ for $|\xi| \geq 1$ and $\chi(\xi) = 0$ for $|\xi| \approx 2$. Using $\chi(D_x)$, we decompose

L_1 as follows:

$$L_1 \geq \left\| \bigcap_{\mathbb{R}^n} \right) \bigcap_0^1 \chi(D_x) \Big\} \widetilde{F}(w(t) + \theta(w(t+h) \quad w(t))) \quad \widetilde{F}(w(t)) \surd d\theta$$
$$* \, \overline{\frac{(w(t+h) \quad w(t))}{h}} \Big[\, dx \Big\|$$
$$+ \left\| \bigcap_{\mathbb{R}^n} \right) \bigcap_0^1 (1 \quad \chi(D_x)) \Big\} \widetilde{F}(w(t) + \theta(w(t+h) \quad w(t))) \quad \widetilde{F}(w(t)) \surd d\theta$$
$$* \, \overline{\frac{(w(t+h) \quad w(t))}{h}} \Big[\, dx \Big\|$$
$$=: K_1 + K_2.$$

From now on, $L^p_{[\varepsilon,T]}X$ denotes the Banach space $L^p([\varepsilon, T]; X)$ for $p \ni [1, \in]$ and a Banach space X.

<u>The estimation of K_1.</u> By Lemma 5.2.3, we get

$$\backslash \widetilde{F}(w(\cdot) + \theta(w(\cdot + h) \quad w(\cdot))) \quad \widetilde{F}(w(\cdot)) \backslash_{L^{|}_{[\varepsilon,T]}L^2 + L^{|}_{[\varepsilon,T]}L^{q^\infty}}$$
$$\geq \backslash \widetilde{F_1}(w(\cdot) + \theta(w(\cdot + h) \quad w(\cdot))) \quad \widetilde{F_1}(w(\cdot)) \backslash_{L^{|}_{[\varepsilon,T]}L^2}$$
$$+ \backslash \widetilde{F_2}(w(\cdot) + \theta(w(\cdot + h) \quad w(\cdot))) \quad \widetilde{F_2}(w(\cdot)) \backslash_{L^{|}_{[\varepsilon,T]}L^{q^\infty}}$$
$$\geq C \backslash w(\cdot + h) \quad w(\cdot) \backslash_{L^{|}_{[\varepsilon,T]}L^2} + C \backslash w(\cdot + h) \quad w(\cdot) \backslash_{L^{|}_{[\varepsilon,T]}H^1}$$
$$* \, \Big\}(\backslash w(\cdot + h) \backslash_{L^{|}_{[\varepsilon,T]}H^1} + \backslash w(\cdot) \backslash_{L^{|}_{[\varepsilon,T]}H^1})$$
$$+ (\backslash w(\cdot + h) \backslash_{L^{|}_{[\varepsilon,T]}H^1} + \backslash w(\cdot) \backslash_{L^{|}_{[\varepsilon,T]}H^1})^{\max(1, 2\alpha_1 \quad 2)} \langle$$
$$\geq C \backslash w(\cdot + h) \quad w(\cdot) \backslash_{L^{|}_{[\varepsilon,T]}H^1} + C \backslash w(\cdot + h) \quad w(\cdot) \backslash_{L^{|}_{[\varepsilon,T]}H^1}$$
$$* \, (\backslash w \backslash_{L^{|}_T H^1} + \backslash w \backslash^{\max(1, 2\alpha_1 \quad 2)}_{L^{|}_T H^1})$$
$$\geq C \backslash w(\cdot + h) \quad w(\cdot) \backslash_{L^{|}_{[\varepsilon,T]}H^1} (1 + \backslash w \backslash_{L^{|}_T H^1} + \backslash w \backslash^{\max(1, 2\alpha_1 \quad 2)}_{L^{|}_T H^1})$$
$$\geq C \backslash w(\cdot + h) \quad w(\cdot) \backslash_{L^{|}_{[\varepsilon,T]}H^1}, \tag{5.4.8}$$

where C depends on the norm $\backslash w \backslash_{X_T}$ of the space X_T.
$X_{[\varepsilon,T]}$ denotes $L^{\in}_{[\varepsilon,T]}H^1 \{ L^p_{[\varepsilon,T]}W^{1,q}$. Using the estimate similar to (5.2.6) and (5.4.8), we obtain

$$\bigcap_\varepsilon^{T^\infty} K_1 dt$$
$$\geq \bigcap_\varepsilon^{T^\infty} \Big) \bigcap_0^1 \Big(\chi(D_x) \Big\} \widetilde{F}(w(t) + \theta(w(t+h) \quad w(t))) \quad \widetilde{F}(w(t)) \surd \Big(_{H^1} \, d\theta$$
$$* \, \Big(\overline{\frac{w(t+h) \quad w(t)}{h}} \Big(_{H^{\ 1}} \Big[\, dt$$
$$\geq C \bigcap_0^1 \Big) \backslash \widetilde{F_1}(w(\cdot) + \theta(w(\cdot + h) \quad w(\cdot))) \quad \widetilde{F_1}(w(\cdot)) \backslash_{L^{|}_{[\varepsilon,T]}L^2}$$
$$+ \backslash \widetilde{F_2}(w(\cdot) + \theta(w(\cdot + h) \quad w(\cdot))) \quad \widetilde{F_2}(w(\cdot)) \backslash_{L^{|}_{[\varepsilon,T]}L^{q^\infty}} \Big(d\theta$$

$$
\begin{aligned}
&\quad \times \left\| \frac{w(\cdot+h) - w(\cdot)}{h} \right\|_{L^1_{[\varepsilon,T)}H^{-1}} \\
&\geq C\int_0^1 \|w(\cdot+h) - w(\cdot)\|_{X_{[\varepsilon,T)}} d\theta \left\| \frac{w(\cdot+h) - w(\cdot)}{h} \right\|_{L^1_{[\varepsilon,T)}H^{-1}} \\
&\geq C\|w(\cdot+h) - w(\cdot)\|_{X_{[\varepsilon,T)}} \left\| \frac{w(\cdot+h) - w(\cdot)}{h} \right\|_{L^1_{[\varepsilon,T)}H^{-1}} .
\end{aligned}
$$

By Lemma 5.2.4, we have

$$
\begin{aligned}
&\| \widetilde{F}(w(\cdot) + \theta(w(\cdot+h) - w(\cdot))) - \widetilde{F}(w(\cdot))\|_{L^1_{[\varepsilon,T)}L^2 + L^{p_\infty}_{[\varepsilon,T)}L^{q_\infty}} \\
&\geq \|\widetilde{G_1}(w(\cdot) + \theta(w(\cdot+h) - w(\cdot))) - \widetilde{G_1}(w(\cdot))\|_{L^1_{[\varepsilon,T)}L^2} \\
&\quad + \|\widetilde{G_2}(w(\cdot) + \theta(w(\cdot+h) - w(\cdot))) - \widetilde{G_2}(w(\cdot))\|_{L^{p_\infty}_{[\varepsilon,T)}L^{q_\infty}} \\
&\geq C\|w(\cdot+h) - w(\cdot)\|_{L^1_{[\varepsilon,T)}H^1} \\
&\quad + C(1 + \|w(\cdot+h)\|_{L^1_{[\varepsilon,T)}H^1} + \|w(\cdot)\|_{L^1_{[\varepsilon,T)}H^1})^{\max(1,2\alpha_1-2)} \\
&\qquad \times \|w(\cdot+h) - w(\cdot)\|_{L^1_{[\varepsilon,T)}H^1} \\
&\quad + C\|w(\cdot+h) - w(\cdot)\|_{X_{[\varepsilon,T)}}(\|w(\cdot+h)\|^{\max(1,2\alpha_1-2)}_{X_{[\varepsilon,T)}} + \|w(\cdot)\|^{\max(1,2\alpha_1-2)}_{X_{[\varepsilon,T)}}) \\
&\quad + C\|w(\cdot+h) - w(\cdot)\|_{X_{[\varepsilon,T)}}(1 + \|w(\cdot+h)\|_{L^1_{[\varepsilon,T)}H^1} + \|w(\cdot)\|_{L^1_{[\varepsilon,T)}H^1}) \\
&\qquad \times (\|w(\cdot+h)\|^{\max(0,2\alpha_1-3)}_{X_{[\varepsilon,T)}} + \|w(\cdot)\|^{\max(0,2\alpha_1-3)}_{X_{[\varepsilon,T)}}) \\
&\geq C\|w(\cdot+h) - w(\cdot)\|_{X_{[\varepsilon,T)}}, \qquad (5.4.9)
\end{aligned}
$$

where C depends on $\|w\|_{X_T}$. Using the estimate similar to (5.2.7) and (5.4.9), we have

$$
\begin{aligned}
K_2 &\geq \left\| \int_0^1 (1 - \chi(D_x)) \left\{ \widetilde{F}(w(t) + \theta(w(t+h) - w(t))) - \widetilde{F}(w(t)) \right\} d\theta \right\|_{H^1 + W^{1,q_\infty}} \\
&\quad \times \left\| \frac{\overline{(w(t+h) - w(t))}}{h} \right\|_{H^{-1} \cap W^{-1,q}} \\
&\geq \int_0^1 \left\| (1 - \chi(D_x)) \left\{ \widetilde{F}(w(t) + \theta(w(t+h) - w(t))) - \widetilde{F}(w(t)) \right\} \right\|_{H^1 + W^{1,q_\infty}} d\theta \\
&\quad \times \left\| \frac{\overline{(w(t+h) - w(t))}}{h} \right\|_{H^{-1} \cap W^{-1,q}} \\
&\geq C\int_0^1 \Big(\|\widetilde{G_1}(w(t) + \theta(w(t+h) - w(t))) - \widetilde{G_1}(w(t))\|_{L^2} \\
&\qquad\qquad + \|\widetilde{G_2}(w(t) + \theta(w(t+h) - w(t))) - \widetilde{G_2}(w(t))\|_{L^{q_\infty}} \Big) d\theta \\
&\quad \times \left\| \frac{w(t+h) - w(t)}{h} \right\|_{H^{-1} \cap W^{-1,q}} .
\end{aligned}
$$

Hence, we get

$$
\int_\varepsilon^{T_\infty} K_2 dt
$$

$$
\begin{aligned}
&\ge C\Big\|\int_0^1 \Big(\|\widetilde{G_1}(w(\cdot)+\theta(w(\cdot+h)-w(\cdot)))-\widetilde{G_1}(w(\cdot))\|_{L^2} \\
&\qquad\qquad +\|\widetilde{G_2}(w(\cdot)+\theta(w(\cdot+h)-w(\cdot)))-\widetilde{G_2}(w(\cdot))\|_{L^{q'}}\,d\theta\Big)\Big\|_{L^{p'}_{[\varepsilon,T']}} \\
&\qquad \times \Big\|\frac{w(\cdot+h)-w(\cdot)}{h}\Big\|_{L^p_{[\varepsilon,T']}(H^{-1}\cap W^{-1,q})} \\
&\ge C\int_0^1 \Big(\|\widetilde{G_1}(w(\cdot)+\theta(w(\cdot+h)-w(\cdot)))-\widetilde{G_1}(w(\cdot))\|_{L^2_{[\varepsilon,T']}L^2} \\
&\qquad\qquad +\|\widetilde{G_2}(w(\cdot)+\theta(w(\cdot+h)-w(\cdot)))-\widetilde{G_2}(w(\cdot))\|_{L^{p'}_{[\varepsilon,T']}L^{q'}}\Big]\,d\theta \\
&\qquad \times \Big\|\frac{w(\cdot+h)-w(\cdot)}{h}\Big\|_{L^2_{[\varepsilon,T']}H^{-1}\cap L^p_{[\varepsilon,T']}W^{-1,q}} \\
&\ge C\int_0^1 \|w(\cdot+h)-w(\cdot)\|_{X_{[\varepsilon,T']}}\,d\theta\,\Big\|\frac{w(\cdot+h)-w(\cdot)}{h}\Big\|_{L^2_{[\varepsilon,T']}H^{-1}\cap L^p_{[\varepsilon,T']}W^{-1,q}} \\
&\ge C\|w(\cdot+h)-w(\cdot)\|_{X_{[\varepsilon,T']}}\Big\|\frac{w(\cdot+h)-w(\cdot)}{h}\Big\|_{L^2_{[\varepsilon,T']}H^{-1}\cap L^p_{[\varepsilon,T']}W^{-1,q}}.
\end{aligned}
$$

Thus, we obtain

$$
\begin{aligned}
\int_\varepsilon^{T'} L_1\,dt &= \int_\varepsilon^{T'} K_1\,dt+\int_\varepsilon^{T'} K_2\,dt \\
&\ge C\|w(\cdot+h)-w(\cdot)\|_{X_{[\varepsilon,T']}} \\
&\quad \times \Big(\Big\|\frac{w(\cdot+h)-w(\cdot)}{h}\Big\|_{L^2_{[\varepsilon,T']}H^{-1}}+\Big\|\frac{w(\cdot+h)-w(\cdot)}{h}\Big\|_{L^p_{[\varepsilon,T']}W^{-1,q}}\Big).
\end{aligned}
\tag{5.4.10}
$$

<u>The estimation of L_2.</u> It follows from Lemma 5.2.1, Lemma 5.2.3 and Lemma 5.2.5 that for almost all $t\in[\varepsilon,T']$,

$$
\begin{aligned}
L_2 &\ge \Big|\Big\langle \chi(D_x)\widetilde{F}(w(t)),\,\overline{\frac{w(t+h)-w(t)}{h}}-\overline{\partial_t w(t)}\Big\rangle_{H^1\times H^{-1}}\Big| \\
&\quad +\Big|\Big\langle (1-\chi(D_x))\widetilde{F}(w(t)),\,\overline{\frac{w(t+h)-w(t)}{h}}-\overline{\partial_t w(t)}\Big\rangle_{(H^1+W^{1,q'})\times(H^{-1}\cap W^{-1,q})}\Big| \\
&\ge \|\chi(D_x)\widetilde{F}(w(t))\|_{H^1}\Big\|\frac{w(t+h)-w(t)}{h}-\partial_t w(t)\Big\|_{H^{-1}} \\
&\quad +\|(1-\chi(D_x))\widetilde{F}(w(t))\|_{H^1+W^{q'}}\Big\|\frac{(w(t+h)-w(t))}{h}-\partial_t w(t)\Big\|_{H^{-1}\cap W^{-1,q}} \\
&\ge C\big(\|\widetilde{F_1}(w(t))\|_{L^2}+\|\widetilde{F_2}(w(t))\|_{L^{q'}}\big)\Big\|\frac{w(t+h)-w(t)}{h}-\partial_t w(t)\Big\|_{H^{-1}} \\
&\quad + C\big(\|\widetilde{G_1}(w(t))\|_{L^2}+\|\widetilde{G_2}(w(t))\|_{L^{q'}}\big)
\end{aligned}
$$

$$
\times \left(\left\| \frac{w(t+h) - w(t)}{h} - \partial_t w(t) \right\|_{H^{-1}} + \left\| \frac{w(t+h) - w(t)}{h} - \partial_t w(t) \right\|_{W^{-1,q}} \right).
$$

Hence, we deduce that

$$
\begin{aligned}
&\int_{\varepsilon}^{T'} L_2 dt \\
&\leq C \left(\|\widetilde{F_1}(w)\|_{L_T^1 L^2} + \|\widetilde{F_2}(w)\|_{L_T^1 L^{q'}} \right) \left\| \frac{w(\cdot+h) - w(\cdot)}{h} - \partial_t w(\cdot) \right\|_{L^\infty_{[\varepsilon,T']} H^{-1}} \\
&\quad + C \left(\|\widetilde{G_1}(w)\|_{L_T^{p'} L^2} + \|\widetilde{G_2}(w)\|_{L_T^{p'} L^{q'}} \right) \\
&\qquad \times \left(\left\| \frac{w(\cdot+h) - w(\cdot)}{h} - \partial_t w(\cdot) \right\|_{L^\infty_{[\varepsilon,T']} H^{-1}} \right. \\
&\qquad\quad \left. + \left\| \frac{w(\cdot+h) - w(\cdot)}{h} - \partial_t w(\cdot) \right\|_{L^p_{[\varepsilon,T']} W^{-1,q}} \right).
\end{aligned}
\tag{5.4.11}
$$

In conclusion, concatenating (5.4.7), (5.4.10) and (5.4.11),

$$
\begin{aligned}
&\left| \int_{\varepsilon}^{T'} \int_{\mathbb{R}^n} \left(\frac{V(|\phi + w(t+h')|^2) - V(|\phi + w(t)|^2)}{h'} \right) \varphi(t) dt dx \right. \\
&\qquad \left. - \int_{\varepsilon}^{T'} (\mathrm{Re}\langle \widetilde{F}(w(t)), \overline{\partial_t w(t)} \rangle) \varphi(t) dt \right| \\
&\leq 2 \int_{\varepsilon}^{T'} L_1 |\varphi(t)| dt + 2 \int_{\varepsilon}^{T'} L_2 |\varphi(t)| dt \\
&\leq C \|w(\cdot+h) - w(\cdot)\|_{X_{[\varepsilon,T']}} \\
&\qquad \times \left(\left\| \frac{w(\cdot+h) - w(\cdot)}{h} \right\|_{L^\infty_{[\varepsilon,T']} H^{-1}} + \left\| \frac{w(\cdot+h) - w(\cdot)}{h} \right\|_{L^p_{[\varepsilon,T']} W^{-1,q}} \right) \\
&\quad + C \left(\|\widetilde{F_1}(w)\|_{L_T^1 L^2} + \|\widetilde{F_2}(w)\|_{L_T^1 L^{q'}} \right) \left\| \frac{w(\cdot+h) - w(\cdot)}{h} - \partial_t w(\cdot) \right\|_{L^\infty_{[\varepsilon,T']} H^{-1}} \\
&\quad + C \left(\|\widetilde{G_1}(w)\|_{L_T^{p'} L^2} + \|\widetilde{G_2}(w)\|_{L_T^{p'} L^{q'}} \right) \\
&\qquad \times \left(\left\| \frac{w(\cdot+h) - w(\cdot)}{h} - \partial_t w(\cdot) \right\|_{L^\infty_{[\varepsilon,T']} H^{-1}} \right. \\
&\qquad\quad \left. + \left\| \frac{w(\cdot+h) - w(\cdot)}{h} - \partial_t w(\cdot) \right\|_{L^p_{[\varepsilon,T']} W^{-1,q}} \right).
\end{aligned}
$$

Noting Lemma 5.3.1 and the fact that a local Lipschitz continuity (3.1.10) in Theorem 3.1.5 yields

$$
\|w(\cdot+h) - w(\cdot)\|_{X_{[\varepsilon,T']}} \leq C \|w(|h|) - w_0\|_{H^1}
$$

$$\uparrow\ 0 \quad as \quad h \uparrow\ 0,$$

we obtain

$$\Big\rangle \frac{\partial}{\partial t} \bigcap_{\mathbb{R}^n} V(u)dx, \varphi \Big(_{\cap \infty \cap (I)} = 2 \bigcap_I \mathrm{Re}\rangle \widetilde{F}(w(t)), \overline{\partial_t w(t)} | \varphi(t) dt.$$

Since the estimation (5.4.6) means $\mathrm{Re}\rangle \widetilde{F}(w(t)), \overline{\partial_t w(t)}| \ni L^1(I)$, we complete the proof of Lemma 5.4.1. □

In conclusion, by Lemma 5.4.1, we complete the proof of the main result. □

Chapter 6

Appendix

6.1 Notation

We present the notations used throughout this thesis.

- $\leq$ Let $\mathbb{R}_+ = [0, \in\ )$.
- $\leq$ for a function f in $\mathbb{R}^n$, we defined the Fourier transform $\mathcal{S}[f] = \hat{f}$ of f by

$$\hat{f}(\xi) = \frac{1}{(2\pi)^n} \bigcap_{\mathbb{R}^n} e^{\quad ix \cdot \xi} f(x)dx.$$

  Moreover, we denote the inverse Fourier Transform $\mathcal{S}^{\quad 1}[f] = \check{f}$ of a function f in $\mathbb{R}^n$ by

$$\check{f}(x) = \frac{1}{(2\pi)^n} \bigcap_{\mathbb{R}^n} e^{ix \cdot \xi} f(\xi)d\xi.$$

- $\leq$ For $p \ni [1, \in\ ]$, let $L^p(\mathbb{R}^n) = L^p$ be the Banach space in $\mathbb{R}^n$ defined by

$$L^p = \}f:\ \mathbb{R}^n \uparrow\ \mathbb{C}\ \text{(measurable)};\ \backslash f \backslash_{L^p} < \in \langle$$

  equipped with the norm

$$\backslash f \backslash_{L^p} = \begin{cases} \left(\bigcap_{\mathbb{R}^n} \|f(x)\|^p dx \right[^{1/p} & \text{if } p < \in \\ \operatorname{esssup}_{x \mathcal{D} \mathbb{R}^n} \|f(x)\| & \text{if } p = \in\ . \end{cases}$$

- $\leq$ For $m \ni \mathbb{N} \cap \}0\langle$ and $p \ni [1, \in\ ]$, let $W^{m,p}(\mathbb{R}^n) = W^{m,p}$ be the Banach space in $\mathbb{R}^n$ defined by

$$W^{m,p} = \}f \ni L^p;\ \backslash f \backslash_{W^{m,p}} < \in \langle$$

  equipped with the norm

$$\backslash f \backslash_{W^{m,p}} = \int_{\alpha \geq m} \backslash \partial_x^\alpha f \backslash_{L^p}.$$

- ≤ For a Banach space X, $T > 0$ and $p \ni [1, \in]$, $L^p_T X$ denotes the Banach space $L^p([0,T], X)$ equipped with its natural norm.
- ≤ Let $U(t)$ be the Schrödinger operator $e^{it\Delta}$.
- ≤ We denote by $f(u)$ the nonlinearity $\lambda|u|^{p\ \ 1}u$.
- ≤ We put $\alpha(n) = 1 + \frac{4}{n\ \ 2}$ if $n \approx 3$ and $\alpha(n) = \in$ if $n = 1, 2$.
- ≤ we define $\Sigma = \}f \ni H^1;\ xf \ni L^2\langle$.
- ≤ For the interval $I \rightarrow \mathbb{R}$, $\mathcal{E}(I)$ and $\mathcal{E}^{\propto}(I)$ denote the Fréchet space of $C^{\in}$ functions $I \uparrow \mathbb{C}$ compactly supported in I and the space of distributions on I, respectively.
- ≤ $\mathscr{S}(\mathbb{R}^n) = \mathscr{S}$ denotes a Schwartz space on $\mathbb{R}^n$.
- ≤ $\mathscr{S}^{\propto}(\mathbb{R}^n) = \mathscr{S}^{\propto}$ is the space of tempered distributions on $\mathbb{R}^n$.
- ≤ For $s \ni \mathbb{R}$ and $p \ni [1, \in]$, Let $H^{s,p}(\mathbb{R}^n) = H^{s,p}$ be the generalized Sobolev space in $\mathbb{R}^n$ defined by

$$H^{s,p} = \}f \ni \mathscr{S}^{\propto};\ \mathcal{S}^{\ \ 1}[(1 + |\xi|^2)^{\frac{s}{2}} \hat{f}] \ni L^p\langle$$

equipped with the norm

$$\backslash f \backslash_{H^{s,p}} = \backslash \mathcal{S}^{\ \ 1}[(1 + |\xi|^2)^{\frac{s}{2}} \hat{f}] \backslash_{L^p}.$$

- ≤ For $s \ni \mathbb{R}$, let $H^s(\mathbb{R}^n) = H^s$ be the generalized Sobolev space in $\mathbb{R}^n$ defined by

$$H^s = \}f \ni \mathscr{S}^{\propto};\ (1 + |\xi|^2)^{\frac{s}{2}} \hat{f} \ni L^2\langle$$

equipped with the norm

$$\backslash f \backslash_{H^s} = \backslash (1 + |\xi|^2)^{\frac{s}{2}} \hat{f} \backslash_{L^2}.$$

- ≤ For $s \ni \mathbb{R}$ and $p, q \ni [1, \in]$, Let $B^s_{p,q}(\mathbb{R}^n) = B^s_{p,q}$ be the inhomogeneous Besov space in $\mathbb{R}^n$ defined by

$$B^s_{p,q} = \}f \ni \mathscr{S};\ \backslash f \backslash_{B^s_{p,q}} < \in \langle$$

equipped with the norm

$$\backslash f \backslash_{B^s_{p,q}} = \backslash \psi \bullet f \backslash_{L^p} + \begin{cases} \left(\int_{j=1}^{\in} 2^{jsq} \backslash \phi_j \bullet f \backslash^q_{L^p} \right)^{1/q} & \text{if } q < \in \\ \sup_{j>0} 2^{sj} \backslash \phi_j \bullet f \backslash_{L^p} & \text{if } q = \in . \end{cases}$$

where $\}\phi_j\langle_{j\mathcal{D}\mathbb{Z}}$ is the Littlewood-Paley decomposition, and $\hat{\psi}(\xi) = 1\ \int_{j>0} \hat{\phi}_j(\xi)$.

$\leq$ For $s \ni \mathbb{R}$ and $p, q \ni [1, \in]$, Let $\dot{B}^s_{p,q}(\mathbb{R}^n) = \dot{B}^s_{p,q}$ be the homogeneous Besov space in $\mathbb{R}^n$ defined by

$$\dot{B}^s_{p,q} = \}f \ni \mathscr{S};\ \backslash f \backslash_{\dot{B}^s_{p,q}} < \in \langle$$

equipped with the norm

$$\backslash f \backslash_{\dot{B}^s_{p,q}} = \begin{cases} \Big(\int_{j \neq \in}^{\in} 2^{jsq} \backslash \phi_j \bullet f \backslash^q_{L^p} \Big\langle^{1/q} & \text{if } q < \in \\ \sup_{j \mathcal{D} \mathbb{Z}} 2^{sj} \backslash \phi_j \bullet f \backslash_{L^p} & \text{if } q = \in . \end{cases}$$

6.2 The results used in this thesis

We explain the results used in this thesis.

Functional Analysis

Definition 6.2.1. *Let (X, d) be a complete metric space. We say that the map $\Phi : X \uparrow X$ is a contraction mapping in X if there exists $\alpha \ni (0, 1)$ such that*

$$d(\Phi(f), \Phi(g)) \geq \alpha d(f, g)$$

for any f, $g \ni X$.

Theorem 6.2.1 (Banach's fixed point Theorem)**.** *(X, d) be a complete metric space. Φ denotes a contraction mapping in X. Then there exists a unique fix point $u \ni X$ of Φ. That is, there exists a unique $u \ni X$ such that $F(u) = u$.*

Lemma 6.2.1. *Let X be a Banach space. Let $x \ni X$ and a sequence $\}x_n\langle_{n=1} \rightarrow X$. If $x_n \uparrow x$ weakly in X as $n \uparrow \in$, then the sequence $\}x_n\langle_{n=1}$ is bounded in X, and it holds that*

$$\backslash x \backslash_X \geq \liminf_{n \uparrow \in} \backslash x_n \backslash_X .$$

Lemma 6.2.2. *Let X be a Banach space. $\}x_n\langle^{\in}_{n=1}$ denotes a sequence in X. If X is reflexive, then there exists a subsequence $\}x_{n_k}\langle^{\in}_{k=1}$ such that converges weakly in X.*

Lemma 6.2.3. *Let X and Y be Banach spaces such that $Y \uparrow X$ and $X^{\leq} \uparrow Y^{\leq}$ with dense embedding, where $X^{\leq}$ and $Y^{\leq}$ denote the dual spaces of X and Y, respectively. Then if a bounded sequence $\}\varphi_n\langle^{\in}_{n=1} \rightarrow Y$ satisfies $\varphi_n \uparrow 0$ in X as $n \uparrow \in$, then for any $f \ni Y^{\leq}$, $\rangle \varphi_n, f|_{Y * Y'} \uparrow 0$ as $n \uparrow \in$.*

Real analysis

Theorem 6.2.2 (Fourier multiplier Theorem, e.g. [21])**.** *Let $1 < p < \in$. For some integer $s > n/2$, suppose that $m(\xi) \ni C^s(\mathbb{R}^n \setminus \}0\langle) \{ L^{\in}(\mathbb{R}^n)$. Assume also that for all multi-index α with $\|\alpha\| \geq s$, there exists a positive constant C_α such that*

$$\|\partial^\alpha_\xi m(\xi)\| \geq C_\alpha \|\xi\|^{\alpha} . \quad (\xi \ni \mathbb{R}^n \setminus \}0\langle) \tag{6.2.1}$$

Then, there exists a positive constant C depending on p, C_α, d, s such that

$$\backslash m(D_x) f \backslash_{L^p} \geq C \backslash f \backslash_{L^p} .$$

Besov space

Lemma 6.2.4. *Let* $1 < p < \in$ *and* $s \ni \mathbb{R}$. *Then*

$$B^s_{p,\min(p,2)} \uparrow \;\; H^{s,p} \uparrow \;\; B^s_{p,\max(p,2)}.$$

Especially, $B^s_{2,2} = H^s$ *and* $\dot{B}^s_{2,2} = \dot{H}^s$.

Lemma 6.2.5. *(i) If* $s > 0$, *then* $\backslash u \backslash_{B^s_{p,q}} \subset \backslash u \backslash_{L^p} + \backslash u \backslash_{\dot{B}^s_{p,q}}$.

(ii) If $0 < s < 1$, *then*

$$\backslash u \backslash_{\dot{B}^s_{p,q}} \subset \begin{cases} \left(\bigcap_0^{\in} \right) t \quad {}^s \sup_{y \geq t} \backslash u(\times \quad y) \quad u(\times) \backslash_{L^p(\mathbb{R}^n)} \left[{}^q \frac{dt}{t} \right[^{1/q} & \text{if } q < \in, \\ \sup_{t>0} \sup_{y \geq t} \backslash u(\times \quad y) \quad u(\times) \backslash_{L^p(\mathbb{R}^n)} & \text{if } q = \in . \end{cases}$$

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The derivation of the conservation law for defocusing nonlinear Schrödinger equations with non-vanishing initial data at infinity

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ABSTRACT

For nonlinear Schrödinger equations in less than or equal to four dimension, with non-vanishing initial data at infinity, a new approach to derive the conservation law is obtained. Since this approach does not contain approximating procedure, the argument is simplified and some of technical assumption of the nonlinearity to derive the conservation law and time global solutions, is removed.

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1. Introduction

In this paper, we consider defocusing nonlinear Schrödinger equations in dimension $n \leqslant 4$

$$\begin{cases} i\dfrac{\partial u}{\partial t} + \Delta u + f\left(|u|^2\right)u = 0, \quad t \in (0,T),\ x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), \quad x \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $u(t,x) : (0,T) \times \mathbb{R}^n \to \mathbb{C}$. The initial data u_0 has the following boundary condition:

$$\left|u_0(x)\right|^2 \to \rho_0 \quad \text{as } |x| \to \infty,$$

where $\rho_0 > 0$ denotes the light intensity of the background. The nonlinear term f is assumed to be defocusing. Namely the real-valued function f satisfies the following assumption:

$$f(\rho_0) = 0, \qquad f'(\rho_0) < 0. \tag{$\mathbf{H}_f$}$$

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Eq. (1.1) with non-vanishing initial data at infinity appears as a relevant model in various physical problems: for example, Bose–Einstein condensation and superfluidity (see [1,5,6]), and nonlinear topics (dark solitons, optical vortices) (see [9,7]). Two important model cases for (1.1) have been extensively studied both in the physical and mathematical literatures: the Gross–Pitaevskii equation (where $f(r) = 1 - r$) and the so-called "cubic-quintic" Schrödinger equation (where $f(r) = (r - \rho_0)(2a + \rho_0 - 3r)$, $0 < a < \rho_0$). Gallo [3] has considered the Cauchy problem for (1.1). He proved the following theorem:

Theorem 1.1. *(See Theorem 1.1 in Gallo [3].) Let $n \leqslant 4$ and $\rho_0 > 0$. Assume that $f \in C^k(\mathbb{R}_+)$ ($k = 3$ if $n = 2, 3$, $k = 4$ if $n = 4$) satisfying ($\mathbf{H}_f$), and there exist $\alpha_1 \geqslant 1$, with a supplementary condition $\alpha_1 < \alpha_1^*$ if $n = 3, 4$ ($\alpha_1^* = 3$ if $n = 3$, $\alpha_1^* = 2$ if $n = 4$), and $\alpha_2 \in \mathbb{R}$ with $\alpha_1 - \alpha_2 \leqslant 1/2$ such that*

$$\exists C_0 > 0,\ \exists A > \rho_0 \quad s.t. \quad \begin{cases} \forall r \geqslant 1, \quad \begin{cases} |f''(r)| \leqslant C_0 r^{\alpha_1 - 3} & \text{if } n = 1, 2, 3, \\ |f'''(r)| \leqslant C_0 r^{\alpha_1 - 4} & \text{if } n = 4, \end{cases} & (\mathbf{H}_{\alpha_1}) \\ \begin{cases} \text{if } \alpha_1 \leqslant 3/2,\ V \text{ is bounded from below}, \\ \text{if } \alpha_1 > 3/2,\ \forall r \geqslant A,\ r^{\alpha_2} \leqslant C_0 V(r), \end{cases} & (\mathbf{H}_{\alpha_2}) \end{cases}$$

where $V(r) := \int_r^{\rho_0} f(s)\,ds$. Then for any function ϕ satisfying

$$\phi \in C_b^{k+1}(\mathbb{R}^n), \qquad \nabla\phi \in H^{k+1}(\mathbb{R}^n)^n, \qquad |\phi|^2 - \rho_0 \in L^2(\mathbb{R}^n), \tag{$\mathbf{H}_\phi$}$$

(1.1) is globally well-posed in $\phi + H^1(\mathbb{R}^n)$. Namely, for any $w_0 \in H^1(\mathbb{R}^n)$, there exists a unique $w \in C(\mathbb{R}, H^1(\mathbb{R}^n))$ such that $\phi + w$ is the solution to (1.1) with the initial data $w(0) = w_0$. Moreover, the solution depends continuously on the initial data $w_0 \in H^1$.

Generally, we take two steps to construct a time global solution for the Cauchy problem of usual nonlinear Schrödinger equations ((NLS)s) (see [2]). The first step is to construct a time local solution to Duhamel's integral equation by using a contraction argument. The next step is to extend the solution to the time global solution by using conservation laws. For Cauchy problem (1.1), we follow the same steps stated above. Thus, to get time global solutions, it is important to obtain conservation laws. We obtain formally the conservation law of energy by multiplying Eq. (1.1) by $\bar{u}_t$, integrating over $\mathbb{R}^n$, and taking the real part. There are basically two methods to justify the procedure above. One is that solutions are approximated by a sequence of regular solutions, using the continuous dependence of solutions on the initial data. The other is to use a sequence of regularized equations of (1.1) whose solutions have enough regularities to perform the procedure above (see [8]). However, these two methods involve a limiting procedure on approximate solutions. Instead, for (NLS)s with a local interaction nonlinearity, Ozawa [8] derives conservation laws by using additional properties of solutions provided by Strichartz estimates. We need the following definitions to mention it:

Definition 1.1.

(i) A positive exponent p' is called the dual exponent of p if p and p' satisfy $1/p + 1/p' = 1$.
(ii) A pair of two exponents (p, q) is called an admissible pair if (p, q) satisfies

$$\frac{2}{p} + \frac{n}{q} = \frac{n}{2}, \qquad p \geqslant 2, \qquad (p, q) \neq (2, \infty).$$

Strichartz estimates are described as the following lemma:

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Lemma 1.1 *(Strichartz estimates). (See [2].) Let (p_1, q_1) and (p_2, q_2) be admissible pairs. Then*

(i) *for all $f \in L^2(\mathbb{R}^n)$,*

$$\left\|e^{it\Delta} f\right\|_{L^{p_1}(\mathbb{R}, L^{q_1}(\mathbb{R}^n))} \leqslant C\|f\|_{L^2(\mathbb{R}^n)},$$

(ii) *let $T > 0$, for all $f \in L^{p_1'}([0,T], L^{q_1'}(\mathbb{R}^n))$,*

$$\left\| \int_{-\infty}^{t} e^{i(t-\tau)\Delta} f(\tau)\, d\tau \right\|_{L^{p_2}([0,T], L^{q_2}(\mathbb{R}^n))} \leqslant C\|f\|_{L^{p_1'}([0,T], L^{q_1'}(\mathbb{R}^n))},$$

where p_1' and p_2' are the dual exponents of p_1 and p_2, respectively.

In this paper, for Eq. (1.1) in $n = 2, 3, 4$, we derive the conservation law for time local solutions without approximating procedure. Instead of that, we use Ozawa's idea [8]. Note that when $n = 1$, because $H^1 \hookrightarrow L^\infty$, Gallo [3] derived it without approximating procedure (see Sections 2, 3 in Gallo [3]), and that for $n \geqslant 2$, Gallo [3] derived it using the approximate argument (see Section 5 in Gallo [3]). We follow Ozawa's idea, however, we cannot derive the conservation law only by Ozawa's idea, due to the nonlinearity and the space of a solution. We derive the conservation law to combine Ozawa's idea with decomposing the nonlinear term by applying the method for the decomposition of Schrödinger operator in Gérard [4]. Moreover, we remove some of technical assumptions of the nonlinearity necessary to derive the conservation law. Our main result is as follows.

Theorem 1.2. *Let $n = 2, 3, 4$. Let $\rho_0 > 0$, and $f \in C^2(\mathbb{R}_+)$ satisfying $(\mathbf{H}_f)$. Moreover, we assume that there exists $\alpha_1 \geqslant 1$, with a supplementary condition $\alpha_1 < \alpha_1^*$ if $n = 3, 4$ ($\alpha_1^* = 3$ if $n = 3$, $\alpha_1^* = 2$ if $n = 4$) such that*

$$\exists C_0 > 0, \quad s.t. \quad \forall r \geqslant 1, \quad \left|f^{(k)}(r)\right| \leqslant C_0 r^{\alpha_1 - 1 - k} \quad (k = 1, 2). \qquad (\mathbf{H}_{\alpha_1}')$$

Let ϕ be a function satisfying

$$\phi \in C_b^2(\mathbb{R}^n), \qquad \nabla\phi \in H^2(\mathbb{R}^n)^n, \qquad |\phi|^2 - \rho_0 \in L^2(\mathbb{R}^n). \qquad (\mathbf{H}_\phi')$$

(Note that such function ϕ is called as a regular function of finite energy.) Let $w \in C([0,T], H^1(\mathbb{R}^n))$ be a mild solution of the integral equation

$$w(t) = e^{it\Delta} w_0 - i \int_0^t e^{i(t-t')\Delta} F\left(w(t')\right) dt' \qquad (1.2)$$

for some $w_0 \in H^1$ and $T > 0$, where $F(w) := -\Delta\phi - f(|\phi + w|^2)(\phi + w)$.

Then $\mathcal{E}(w(t)) = \mathcal{E}(w_0)$ for all $t \in [0,T]$, where

$$\mathcal{E}(w) := \int_{\mathbb{R}^n} \left|\nabla(\phi + w)\right|^2 dx + \int_{\mathbb{R}^n} V\left(|\phi + w|^2\right) dx,$$

and

$$V(r) := \int_r^{\rho_0} f(s)\, ds.$$

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Remark 1.1. Gallo [3] proved the energy conservation law under $f \in C^k(\mathbb{R}_+)$ ($k = 3$ if $n = 2, 3$, $k = 4$ if $n = 4$) satisfying $(\mathbf{H}_f)$, $(\mathbf{H}_{\alpha_1})$ and $(\mathbf{H}_{\alpha_2})$ for some $\alpha_1 \geqslant 1$ and $\alpha_2 \in \mathbb{R}$ with $\alpha_1 - \alpha_2 \leqslant 1/2$, and ϕ satisfying $(\mathbf{H}_\phi)$, but we can prove it under $f \in C^2(\mathbb{R}_+)$ with $(\mathbf{H}_f)$ and $(\mathbf{H}'_{\alpha_1})$ for some $\alpha_1 \geqslant 1$, and ϕ with $(\mathbf{H}'_\phi)$.

Remark 1.2. For proofs of the a priori estimate of $f(|\phi + w|^2)(\phi + w)$ (that is Lemmas 3.1–3.4, and Lemmas 4.1–4.4 in Gallo [3]) and boundedness of H^1 norm of w on bounded intervals (that is Lemma 3.3 in Gallo [3]), we need that there exists $C_{\alpha_1} > 0$ such that for any $r \geqslant 0$,

$$r^{1/2}\left|f^{(k)}(r)\right| \leqslant C_\alpha\left(1 + r^{\max(0,\alpha_1-(2k+1)/2)}\right) \quad (k = 1, 2), \tag{1.3}$$

where $1 \leqslant \alpha_1$ with the same supplementary condition in Theorem 1.1. If $3/2 < \alpha_1 \leqslant 2$, then we cannot obtain (1.3) from $(\mathbf{H}_{\alpha_1})$ or $(\mathbf{H}_{\alpha_2})$. Therefore by replacing $(\mathbf{H}_{\alpha_1})$ with $(\mathbf{H}'_{\alpha_1})$, we deduce (1.3) from $(\mathbf{H}'_{\alpha_1})$ only. To show (1.3), we do not need $(\mathbf{H}_{\alpha_2})$.

Moreover, as a corollary to the main result, we can deduce a globally well-posedness of (1.1). Due to Theorem 1.2, we can remove a technical assumption of the nonlinear term. We have the following result:

Corollary 1.1. *Let $n = 2, 3, 4$. We assume that f and ϕ satisfy the same assumptions as in Theorem 1.2, with a supplementary assumption as f satisfying $(\mathbf{H}_{\alpha_2})$ for some $\alpha_2 \in \mathbb{R}$ with $\alpha_1 - \alpha_2 \leqslant 1/2$. Then (1.1) is globally well-posed in $\phi + H^1(\mathbb{R}^n)$. That is, for any $w_0 \in H^1(\mathbb{R}^n)$, there exists a unique $w \in C(\mathbb{R}, H^1(\mathbb{R}^n))$ such that $\phi + w$ solves (1.1) with the initial data $w(0) = w_0$. Moreover, for any $T > 0$, the flow map $w_0 \mapsto w$ $(H^1 \to C([0, T], H^1))$ is Lipschitz continuous on the bounded sets of $H^1(\mathbb{R}^n)$. The energy $\mathcal{E}(w)$ is conserved by the flow.*

The structure of this paper is as follows. In Section 2, we introduce the previous results of Gallo [3] on the local existence of solutions of (1.1). In Sections 3 and 4, we give estimates of the nonlinear term and results of the time-derivative term needed for the proof of the main result, respectively. In Section 5, we prove the main result.

Notation. For a Banach space X, $T > 0$ and $p \in [1, \infty]$, $L^p_T X$ denotes the Banach space $L^p([0, T], X)$ equipped with its natural norm.

2. Previous results

For $n \leqslant 4$, Gallo [3] proved the globally well-posedness of (1.1). We state the result for $n = 2, 3, 4$. A first strategy of the proof is that (1.1) is transformed as follows to look for a solution of (1.1) under the form $\phi + w$

$$\begin{cases} i\dfrac{\partial w}{\partial t} + \Delta w = F\big(w(t)\big), & t \in (0, T),\ x \in \mathbb{R}^n, \\ w(0, x) = w_0(x), & x \in \mathbb{R}^n, \end{cases} \tag{2.1}$$

where

$$F(w) := -\Delta\phi - f\left(|\phi + w|^2\right)(\phi + w).$$

In a next strategy, he proves that (2.1) is locally well-posed in H^1 by using Strichartz estimates and a contraction argument for the map

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$$\Phi(w) = e^{it\Delta} w_0 - i \int_0^t e^{i(t-s)\Delta} F\big(w(s)\big)\, ds,$$

in the space

$$X_T := L_T^\infty H^1 \cap L_T^p W^{1,q}$$

equipped with its natural norm

$$\|w\|_{X_T} := \|w\|_{L_T^\infty H^1} + \|w\|_{L_T^p W^{1,q}},$$

where a pair (p,q) is an admissible pair defined as $(p,q) := (6/n, 6)$ for $n = 2,3$, $(p,q) := (2,4)$ for $n = 4$. We remark that Gallo [3] takes $(p,q) := (4,4)$ for $n = 2$. Note that our choice also works for getting local existence of solution to (1.1). For locally well-posedness, Gallo [3] proves the following theorem:

Theorem 2.1. *(See Gallo [3].) Let $n = 2,3,4$. Let $\rho_0 > 0$, and $f \in C^2(\mathbb{R}_+)$ satisfying $(\mathbf{H}_f)$. Moreover, we assume that there exist $\alpha_1 \geqslant 1$, with a supplementary condition $\alpha_1 < \alpha_1^*$ if $n = 3,4$ ($\alpha_1^* = 3$ if $n = 3$, $\alpha_1^* = 2$ if $n = 4$), and $\alpha_2 \in \mathbb{R}$ with $\alpha_1 - \alpha_2 \leqslant 1/2$ such that $(\mathbf{H}_{\alpha_1})$ and $(\mathbf{H}_{\alpha_2})$. Let ϕ be a function satisfying $(\mathbf{H}'_\phi)$.*

Then for any $R > 0$, there exists $T(R) > 0$ such that for any $w_0 \in H^1$ with $\|w_0\|_{H^1} \leqslant R$, there exists a unique solution $w \in X_{T(R)}$ of the integral equation (1.2). Moreover $w \in C([0,T(R)], H^1)$.

If $\widetilde{w} \in C([0,T], H^1)$ solves (1.2) for some $T > 0$, then $\widetilde{w} \in X_T$, and $\widetilde{w} \in X_T$ is the unique solution to (1.2) in $C([0,T], H^1)$.

Also the flow map is locally Lipschitz continuous on the bounded sets of H^1, indeed for any $R > 0$, there exists $T(R) > 0$ such that for any $T' \in (0, T(R)]$ and $w_0, \widetilde{w}_0 \in H^1$ with $\|w_0\|_{H^1} \leqslant R$ and $\|\widetilde{w}_0\|_{H^1} \leqslant R$, corresponding solutions $w, \widetilde{w} \in X_{T'}$ of (1.2) satisfy the following locally Lipschitz continuity:

$$\|w - \widetilde{w}\|_{X_{T'}} \leqslant C \|w_0 - \widetilde{w}_0\|_{H^1}, \tag{2.2}$$

where C is a positive constant depending on $\|w\|_{X_{T'}}$ and $\|\widetilde{w}\|_{X_{T'}}$. Especially, for the same constant C,

$$\|w - \widetilde{w}\|_{L_{T'}^\infty H^1} \leqslant C \|w_0 - \widetilde{w}_0\|_{H^1}.$$

Furthermore, the energy $\mathcal{E}(w(t))$ is conserved for all $t \in [0,T]$.

Remark 2.1. To obtain the local existence theorem above, it seems too much to assume both $(\mathbf{H}_{\alpha_1})$ and $(\mathbf{H}_{\alpha_2})$ for some $\alpha_1 \geqslant 1$ and $\alpha_2 \in \mathbb{R}$ with $\alpha_1 - \alpha_2 \leqslant 1/2$. Theorem 2.1 can be shown only assuming $(\mathbf{H}'_{\alpha_1})$. To show only the local existence theorem, we do not need $(\mathbf{H}_{\alpha_2})$.

From Theorem 2.1, a local solution of (2.1) is constructed as the following theorem:

Theorem 2.2. *Let $n = 2,3,4$. Let $w_0 \in H^1$. Let $T > 0$ and let w be a mild solution of the integral equation (1.2) with $w \in C([0,T], H^1)$. Then, for any $t_0 \in [0,T]$, there exists $v(t_0) \in H^{-1}$ such that*

$$\frac{w(t_0 + h) - w(t_0)}{h} \to v(t_0) \quad \text{in } H^{-1} \text{ as } h \to 0.$$

Moreover, denoting $v(t_0)$ by $\partial_t w(t_0)$, w is a solution of (2.1), indeed w satisfies

(i) *$i\partial_t w(t) + \Delta w(t) = F(w(t))$ in H^{-1} for all $t \in [0,T]$,*
(ii) *$w(0) = w_0$.*

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3. The estimates of nonlinear terms

In what follows, we put $\widetilde{F}(w) = -f(|\phi+w|^2)(\phi+w)$. Applying directly the decomposition of $F(w)$ that Gallo [3] used, we can deduce the following decompositions for $\widetilde{F}(w)$. Note that we can show Lemmas 3.1–3.4 by applying the same method to $\widetilde{F}(w)$ as corresponding lemmas for $F(w)$ in Gallo [3]. The statements of Lemma 3.1 and Lemma 3.3 are slightly different from these lemmas in Gallo [3]. Therefore we prove them in Appendix A.

Lemma 3.1. *Let $T>0$. For any $w \in X_T$, there exist*

$$\widetilde{F}_1(w) \in L_T^\infty L^2, \qquad \widetilde{F}_2(w) \in L_T^\infty L^{q'}$$

such that

$$\widetilde{F}(w) = \widetilde{F}_1(w) + \widetilde{F}_2(w).$$

Moreover it follows that

$$\big\|\widetilde{F}_1(w)\big\|_{L_T^\infty L^2} + \big\|\widetilde{F}_2(w)\big\|_{L_T^\infty L^{q'}} \leqslant C\big(1+\|w\|_{L_T^\infty L^2}\big) + C\big(\|w\|_{L_T^\infty H^1}^2 + \|w\|_{L_T^\infty H^1}^{\max(2,2\alpha_1-1)}\big), \tag{3.1}$$

where C is a positive constant depending on T. Also for a same decomposition of $\widetilde{F}(w)$ in the above, we have $\widetilde{F}_2(w) \in L_T^p L^2$ and

$$\big\|\widetilde{F}_2(w)\big\|_{L_T^p L^2} \leqslant C\big(\|w\|_{L_T^\infty H^1}^2 + \|w\|_{X_T}^{\max(2,2\alpha_1-1)}\big),$$

where C is a positive constant depending on T. Thus $F_2(w) \in L_T^p L^2$.

Lemma 3.2. *Let $T>0$. For any $w \in X_T$, there exist*

$$\widetilde{G}_1(w) \in L_T^\infty L^2, \qquad \widetilde{G}_2(w) \in L_T^{p'} L^{q'},$$

such that

$$\nabla\widetilde{F}(w) = \widetilde{G}_1(w) + \widetilde{G}_2(w).$$

Moreover it follows that

$$\begin{aligned}\big\|\widetilde{G}_1(w)\big\|_{L_T^\infty L^2} + \big\|\widetilde{G}_2(w)\big\|_{L_T^{p'} L^{q'}} &\leqslant C\big(1+\|\nabla w\|_{L_T^\infty L^2}\big)\\ &\quad + C\big(1+\|\nabla w\|_{L_T^\infty L^2}\big)\big(\|w\|_{L_T^\infty H^1} + \|w\|_{X_T}^{\max(1,2\alpha_1-2)}\big),\end{aligned}$$

where C is a positive constant depending on T.

Lemma 3.3. *Let $T>0$. For any $w_1, w_2 \in X_T$, decomposing $f(|\phi+w|^2)(\phi+w)$ as Lemma 3.1, it follows that*

$$\begin{aligned}&\big\|\widetilde{F}_1(w_1)-\widetilde{F}_1(w_2)\big\|_{L_T^\infty L^2} + \big\|\widetilde{F}_2(w_1)-\widetilde{F}_2(w_2)\big\|_{L_T^\infty L^{q'}}\\ &\quad \leqslant C\|w_1-w_2\|_{L_T^\infty H^1} + C\|w_1-w_2\|_{L_T^\infty H^1}\Big(\big(\|w_1\|_{L_T^\infty H^1}+\|w_2\|_{L_T^\infty H^1}\big)\\ &\qquad + \big(\|w_1\|_{L_T^\infty H^1}+\|w_2\|_{L_T^\infty H^1}\big)^{\max(1,2\alpha_1-2)}\Big),\end{aligned}$$

where C is a positive constant depending on T.

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Lemma 3.4. *Let $T>0$. For any $w_1, w_2 \in X_T$, decomposing $f(|\phi+w|^2)(\phi+w)$ as Lemma 3.2, it follows that*

$$\begin{aligned}
&\big\|\widetilde{G}_1(w_1)-\widetilde{G}_1(w_2)\big\|_{L_T^\infty L^2}+\big\|\widetilde{G}_2(w_1)-\widetilde{G}_2(w_2)\big\|_{L_T^{p'} L^{q'}}\\
&\quad\leqslant C\big\|\nabla(w_1-w_2)\big\|_{L_T^\infty L^2}+C\big(1+\|w_1\|_{L_T^\infty H^1}+\|w_2\|_{L_T^\infty H^1}\big)^{\max(1,2\alpha_1-2)}\|w_1-w_2\|_{L_T^\infty H^1}\\
&\qquad+C\|w_1-w_2\|_{X_T}\big(\|w_1\|_{X_T}^{\max(1,2\alpha_1-2)}+\|w_2\|_{X_T}^{\max(1,2\alpha_1-2)}\big)\\
&\qquad+C\|w_1-w_2\|_{X_T}\big(1+\|w_1\|_{L_T^\infty H^1}+\|w_2\|_{L_T^\infty H^1}\big)\big(\|w_1\|_{X_T}^{\max(0,2\alpha_1-3)}+\|w_2\|_{X_T}^{\max(0,2\alpha_1-3)}\big),
\end{aligned}$$

where C is a positive constant depending on T.

Remark 3.1. Let $T>0$. Lemma 3.1 and Sobolev embedding $H^1 \hookrightarrow L^q$, imply that for any $w \in C([0,T],H^1)$ and $t \in [0,T]$, $F(w(t)) \in H^{-1}$. Furthermore, for any $t_0 \in [0,T]$, Lemma 3.3 yields

$$\begin{aligned}
\big\|F\big(w(t)\big)-F\big(w(t_0)\big)\big\|_{H^{-1}} &\leqslant C\big\|w(t)-w(t_0)\big\|_{H^1}\\
&\to 0 \quad \text{as } t\to t_0,
\end{aligned}$$

where C is a positive constant depending on $\|w\|_{L_T^\infty H^1}$. To show it, for $w \in C([0,T],H^1)$, it suffices to put $w_1(s)=w(t)$ and $w_2(s)=w(t_0)$ $(0 \leqslant s \leqslant T)$. Thus we also obtain $F(w) \in C([0,T],H^{-1})$.

In the proof of the main result, we use the following lemma:

Lemma 3.5. *For any $\eta \in L^2+L^{q'}$, it follows that*

$$\big\|\chi(D_x)\eta\big\|_{H^1} \leqslant C\|\eta\|_{L^2+L^{q'}}. \tag{3.2}$$

Moreover for any $\eta \in \mathcal{S}'(\mathbb{R}^n)$ with $\nabla\eta \in L^2+L^{q'}$, we obtain

$$\big\|\big(1-\chi(D_x)\big)\eta\big\|_{H^1} \leqslant C\|\nabla\eta\|_{L^2+L^{q'}}. \tag{3.3}$$

Note that if X and Y are Banach spaces, then $X+Y$ is a Banach space equipped with the norm

$$\|v\|_{X+Y} := \inf\big\{\|v_1\|_X+\|v_2\|_Y\colon\ v=v_1+v_2,\ v_1\in X,\ v_2\in Y\big\}.$$

We use the following theorem to prove Lemma 3.5.

Theorem 3.1 *(Fourier multiplier theorem). (See [10].) Let $1<p<\infty$. For some integer $s>n/2$, suppose that $m(\xi) \in C^s(\mathbb{R}^n\backslash\{0\}) \cap L^\infty(\mathbb{R}^n)$. Assume also that for all multi-index α with $|\alpha| \leqslant s$, there exists a positive constant C_α such that*

$$\big|\partial_\xi^\alpha m(\xi)\big| \leqslant C_\alpha|\xi|^{-|\alpha|} \quad \big(\xi \in \mathbb{R}^n \setminus \{0\}\big). \tag{3.4}$$

Then, there exists a positive constant C depending on p, C_α, d, s such that

$$\big\|m(D_x)f\big\|_{L^p} \leqslant C\|f\|_{L^p}.$$

Proof of Lemma 3.5. For any $\eta \in L^2+L^{q'}$, there exist $\eta_1 \in L^2$ and $\eta_2 \in L^{q'}$ such that $\eta=\eta_1+\eta_2$. $Q(\xi)$ denotes $(1+|\xi|^2)\chi(\xi)$. Also, $Q(\xi)$ satisfies (3.4) since $\chi \in C_0^\infty(\mathbb{R}^n)$. Therefore, using Fourier multiplier theorem, we obtain

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$$\begin{aligned}\|\chi(D_x)\eta\|_{H^1} &= \|Q(D_x)\eta\|_{H^{-1}} \\ &\leqslant \|Q(D_x)\eta_1\|_{L^2} + \|Q(D_x)\eta_2\|_{L^{q'}} \\ &\leqslant C\big(\|\eta_1\|_{L^2} + \|\eta_2\|_{L^{q'}}\big).\end{aligned}$$

Therefore, we deduce that

$$\|\chi(D_x)\eta\|_{H^1} \leqslant C\|\eta\|_{L^2+L^{q'}}.$$

Next, for any $\eta \in \mathcal{S}'(\mathbb{R}^n)$ with $\nabla\eta \in L^2 + L^{q'}$, there exist $(\zeta_1^j(w))_{j=1,\dots,n} \in L^2$ and $(\zeta_2^j(w))_{j=1,\dots,n} \in L^{q'}$ such that $\nabla\eta = \zeta_1 + \zeta_2$. Using $P_j(\xi) := -i\xi_j/|\xi|^2$ ($\xi := (\xi_j)_{j=1,\dots,n} \in \mathbb{R}^n$), we have

$$\begin{aligned}\big(1-\chi(D_x)\big)\eta &= \big(1-\chi(D_x)\big)\sum_{j=1}^n P_j(D_x)\partial_j\eta \\ &= \sum_{j=1}^n \big(1-\chi(D_x)\big)P_j(D_x)\zeta_1^j + \sum_{j=1}^n \big(1-\chi(D_x)\big)P_j(D_x)\zeta_2^j.\end{aligned}$$

Fourier multiplier theorem implies

$$\begin{aligned}\big\|\big(1-\chi(D_x)\big)\eta\big\|_{H^1+W^{1,q'}} &\leqslant \sum_{j=1}^n \big\|\big(1-\chi(D_x)\big)P_j(D_x)\zeta_1^j\big\|_{H^1} + \sum_{j=1}^n \big\|\big(1-\chi(D_x)\big)P_j(D_x)\zeta_2^j\big\|_{W^{1,q'}} \\ &\leqslant C\|\zeta_1\|_{L^2} + C\|\zeta_2\|_{L^{q'}}.\end{aligned}$$

Thus we get

$$\big\|\big(1-\chi(D_x)\big)\eta\big\|_{H^1+W^{1,q'}} \leqslant C\|\nabla\eta\|_{L^2+L^{q'}}. \qquad \square$$

4. Regularities of time-derivative term

In this section, we shall show properties of the time-derivative term $\partial_t u$.

Lemma 4.1. *Let $n = 2,3,4$, and let $(p,q) := (6/n, 6)$ for $n = 2,3$, $(p,q) := (2,4)$ for $n = 4$. Let w be a solution of Eq. (2.1) belonging to $C([0,T], H^1)$ for some $T > 0$ with the initial data $w(0) = w_0 \in H^1$. Then for any $0 < \varepsilon < T' < T$,*

$$\text{(i)} \quad \left\|\frac{w(\cdot+h)-w(\cdot)}{h} - \partial_t w(\cdot)\right\|_{C([\varepsilon,T'],H^{-1})} \to 0 \quad \textit{as } h \to 0,$$

and

$$\text{(ii)} \quad \left\|\frac{w(\cdot+h)-w(\cdot)}{h} - \partial_t w(\cdot)\right\|_{L^p([\varepsilon,T'],W^{-1,q})} \to 0 \quad \textit{as } h \to 0.$$

Proof. Note that Eq. (2.1) implies

$$\partial_t w = i\big(\Delta w - F(w)\big). \tag{4.1}$$

We show (i) and (ii) using (4.1).

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Proof of (i). Note that from Theorem 2.2, for any $0 \leqslant t \leqslant T$, $\partial_t w(t) \in H^{-1}$ exists in strong sense. Hence, it suffices to show continuity of $\partial_t w(t)$ on $[0,T]$. Clearly,

$$\|\Delta w\|_{H^{-1}} \leqslant \|\nabla w\|_{L^2}, \tag{4.2}$$

which yields $\Delta w \in C([0,T],H^{-1})$. Using (4.1), (4.2) and Remark 3.1, we obtain

$$\partial_t w \in C\big([0,T],H^{-1}\big).$$

Hence, it follows that for all t_0, $t \in [0,T]$,

$$w(t) - w(t_0) = \int_{t_0}^{t} \partial_t w(s)\,ds \quad \text{in } H^{-1}. \tag{4.3}$$

We take $0 < \varepsilon < T' < T$. For all $t_0 \in [\varepsilon, T']$ and sufficiently small $h \in \mathbb{R}$,

$$\begin{aligned}\left\|\frac{w(t_0+h)-w(t_0)}{h} - \partial_t w(t_0)\right\|_{H^{-1}} &\leqslant \frac{1}{|h|}\left|\int_{t_0}^{t_0+h} \big\|\partial_t w(s) - \partial_t w(t_0)\big\|_{H^{-1}}\,ds\right| \\ &\leqslant \sup_{|s-t_0|\leqslant|h|} \big\|\partial_t w(s) - \partial_t w(t_0)\big\|_{H^{-1}}.\end{aligned}$$

Since $t \mapsto \partial_t u(t) \in H^{-1}(\mathbb{R}^n)$ is uniformly continuous on $[0,T]$, we obtain (i).

Proof of (ii). Since $W^{1,q'}(\mathbb{R}^n) \hookrightarrow L^2(\mathbb{R}^n)$ and $w \in C([0,T],H^{-1})$ and ϕ satisfies $(\mathbf{H}_\phi)$, we clearly get

$$\Delta w \in L^p\big([0,T],W^{-1,q}\big) \quad \text{and} \quad \Delta\phi \in L^p\big([0,T],W^{-1,q}\big). \tag{4.4}$$

Moreover, using Sobolev embedding and duality argument, we conclude $L^2 \hookrightarrow W^{-1,q}$. Thus Lemma 3.1 yields

$$F(w) \in L^p\big([0,T],W^{-1,q}\big). \tag{4.5}$$

Therefore, concatenating (4.1), (4.4) and (4.5), we obtain

$$\partial_t w \in L^p\big([0,T],W^{-1,q}\big).$$

Let $t_0 \in [0,T]$. By (4.3), for any $t \in [0,T]$,

$$w(t) - w(t_0) = \int_{t_0}^{t} \partial_t w(s)\,ds \quad \text{in } \mathcal{S}'\big(\mathbb{R}^n\big),$$

where $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ denote Schwartz space on $\mathbb{R}^n$ and the space of tempered distributions on $\mathbb{R}^n$, respectively. Using Hölder's inequality, we get

$$\begin{aligned}\left\|\int_{t_0}^{\cdot} \partial_t w(s)\,ds\right\|_{L^p([0,T],W^{-1,q})} &\leqslant \left\|\int_{t_0}^{\cdot} \big\|\partial_t w(s)\big\|_{W^{-1,q}}\,ds\right\|_{L^p([0,T])} \\ &\leqslant \left[\int_0^T (t-t_0)^{p/p'}\left(\int_{t_0}^{t} \big\|\partial_t w(s)\big\|_{W^{-1,q}}^p\,ds\right)dt\right]^{1/p}\end{aligned}$$

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$$\leqslant T^{1/p'}\left[\int_0^T\left(\int_{t_0}^t \|\partial_t w(s)\|_{W^{-1,q}}^p\,ds\right)dt\right]^{1/p}$$

$$\leqslant T^{1/p'}\left(T\times\int_0^T \|\partial_t w(s)\|_{W^{-1,q}}^p\,ds\right)^{1/p}$$

$$\leqslant T\|\partial_t w\|_{L_T^p W^{-1,q}}.$$

Therefore, for all $t_0\in[0,T]$,

$$w(\cdot)-w(t_0)=\int_{t_0}^{\cdot}\partial_t w(s)\,ds\quad\text{in } L^p\big([0,T],W^{-1,q}\big). \tag{4.6}$$

Combining (4.6) with Strichartz's estimate, in a way similar to the preceding argument, for all $0<\varepsilon<T'<T$, we obtain

$$\left\|\frac{w(\cdot+h)-w(\cdot)}{h}-\partial_t w(\cdot)\right\|_{L^p([\varepsilon,T'],W^{-1,q})}$$

$$\leqslant\left\|\frac{1}{h}\int_{\cdot}^{\cdot+h}\|\partial_t w(s)-\partial_t w(\cdot)\|_{W^{-1,q}}\,ds\right\|_{L^p([\varepsilon,T'])}$$

$$=\left\{\int_\varepsilon^{T'}\left|\frac{1}{h}\int_{t_0}^{t_0+h}\|\partial_t w(s)-\partial_t w(t_0)\|_{W^{-1,q}}\,ds\right|^p dt_0\right\}^{1/p}$$

$$\leqslant h^{1/p'-1}\left\{\int_\varepsilon^{T'}\left(\int_{t_0}^{t_0+h}\|\partial_t w(s)-\partial_t w(t_0)\|_{W^{-1,q}}^p\,ds\right)dt_0\right\}^{1/p}$$

$$=h^{1/p'-1}\left\{\int_0^{h}\left(\int_{\varepsilon}^{T'}\|\partial_t w(t_0+s)-\partial_t w(t_0)\|_{W^{-1,q}}^p\,dt_0\right)ds\right\}^{1/p}$$

$$\leqslant\sup_{0\leqslant s\leqslant h}\left(\int_\varepsilon^{T'}\|\partial_t w(t_0+s)-\partial_t w(t_0)\|_{W^{-1,q}}^p\,dt_0\right)^{1/p}$$

$$\to 0\quad\text{as } h\to 0.$$

This completes the proof of Lemma 4.1. □

5. The proof of the main result

Since Schrödinger operator $e^{it\Delta}$ becomes bounded operator from $\phi+H^1$ to itself (see Lemma 3 in Gérard [4]), we can obtain

$$\phi=e^{it\Delta}\phi-i\int_0^t e^{i(t-t')\Delta}\Delta\phi\,dt'.$$

Combining the above equality with (1.2), we get

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$$\phi + w(t) = e^{it\Delta}(\phi + w_0) - i \int_0^t e^{i(t-t')\Delta} \widetilde{F}\big(w(t')\big)\, dt', \tag{5.1}$$

where $\widetilde{F}(w) := -f(|\phi + w|^2)(\phi + w)$. From now on, we deduce the proof in a way similar to Ozawa [8]. Acting ∇ on (5.1), we obtain

$$\begin{aligned} \big\|\nabla\big(\phi + w(t)\big)\big\|_{L^2}^2 &= \big\|\nabla e^{i(-t)\Delta}\big(\phi + w(t)\big)\big\|_{L^2}^2 \\ &= \big\|\nabla(\phi + w_0)\big\|_{L^2}^2 - 2\,\mathrm{Im}\left(\nabla(\phi + w_0), \int_0^t e^{i(-t')\Delta} \nabla \widetilde{F}\big(w(t')\big)\, dt'\right)_{L^2} \\ &\quad + \left\|\int_0^t e^{i(-t')\Delta} \nabla \widetilde{F}\big(w(t')\big)\, dt'\right\|_{L^2}^2. \end{aligned} \tag{5.2}$$

The second term on the RHS of (5.2) satisfies the following equality:

$$\begin{aligned} &-2\,\mathrm{Im}\left(\nabla(\phi + w_0), \int_0^t e^{i(-t')\Delta} \nabla \widetilde{F}\big(u(t')\big)\, dt'\right)_{L^2} \\ &\quad = -2\,\mathrm{Im} \int_0^t \big\langle e^{it'\Delta} \nabla(\phi + w_0), \overline{\nabla\big(\widetilde{F}\big(w(t')\big)\big)}\big\rangle\, dt', \end{aligned} \tag{5.3}$$

where the time integral of the scalar product is understood as the duality coupling on $(L_T^1 L^2 \cap L_t^p L^q) \times (L_T^\infty L^2 + L_T^{p'} L^{q'})$ with $(p, q) = (6/n, 6)$ if $n = 2, 3$, $(p, q) = (2, 4)$ if $n = 4$. For the last term on the RHS of (5.2), Fubini's theorem implies

$$\begin{aligned} &\left\|\int_0^t e^{i(-t')\Delta} \nabla\big(\widetilde{F}\big(w(t')\big)\big)\, dt'\right\|_{L^2}^2 \\ &\quad = 2\,\mathrm{Re} \int_0^t \left\langle \nabla\big(\widetilde{F}\big(w(t')\big)\big), \overline{\int_0^{t'} e^{i(t'-t'')\Delta} \nabla\big(\widetilde{F}\big(w(t'')\big)\big)\, dt''} \right\rangle dt', \end{aligned} \tag{5.4}$$

where the time integral of the scalar product is understood as the duality coupling on $(L_T^\infty L^2 + L_T^{p'} L^{q'}) \times (L_T^1 L^2 \cap L_T^p L^q)$. Concatenating (5.2)–(5.4), we compute

$$\begin{aligned} \big\|\nabla\big(\phi + w(t)\big)\big\|_{L^2}^2 &= \big\|\nabla(\phi + w_0)\big\|_{L^2}^2 - 2\,\mathrm{Im} \int_0^t \big\langle e^{it'\Delta} \nabla(\phi + w_0), \overline{\nabla\big(\widetilde{F}\big(w(t')\big)\big)}\big\rangle\, dt' \\ &\quad + 2\,\mathrm{Re} \int_0^t \left\langle \nabla\big(\widetilde{F}\big(w(t')\big)\big), \overline{\int_0^{t'} e^{i(t'-t'')\Delta} \nabla\big(\widetilde{F}\big(w(t'')\big)\big)\, dt''} \right\rangle dt' \\ &= \big\|\nabla(\phi + w_0)\big\|_{L^2}^2 + 2\,\mathrm{Im} \int_0^t \big\langle \nabla\big(\widetilde{F}\big(w(t')\big)\big), \overline{e^{it'\Delta} \nabla(\phi + w_0)}\big\rangle\, dt' \end{aligned}$$

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$$+ 2\,\mathrm{Im}\int_0^t \left\langle \nabla\big(\widetilde{F}\big(w(t')\big)\big), -i\overline{\int_0^{t'} e^{i(t'-t'')\Delta}\nabla\big(\widetilde{F}\big(w(t'')\big)\big)\,dt''} \right\rangle dt'$$

$$= \big\|\nabla(\phi + w_0)\big\|_{L^2}^2 + \lim_{\varepsilon\downarrow 0} 2\,\mathrm{Im}\int_0^t \big\langle (1-\varepsilon\Delta)^{-1}\nabla\big(\widetilde{F}\big(w(t')\big)\big), \overline{\nabla w(t')}\big\rangle\,dt',$$

where the last equality in the above holds by using (1.2). Taking the duality coupling between Eq. (2.1) and $(1-\varepsilon\Delta)^{-1}\nabla(\widetilde{F}(w))$ on $H^{-1}\times H^1$ and using $\mathrm{Im}\{\langle (1-\varepsilon\Delta)^{-1}\widetilde{F}(w), \overline{\widetilde{F}(w)}\rangle\} = 0$, we obtain

$$\mathrm{Im}\big\langle (1-\varepsilon\Delta)^{-1}\nabla\big(\widetilde{F}(w)\big), \overline{\nabla w}\big\rangle = \mathrm{Im}\big\{-i\big\langle (1-\varepsilon\Delta)^{-1}\widetilde{F}(w), \overline{\partial_t w}\big\rangle\big\}.$$

From these equalities, we can show

$$\big\|\nabla\big(\phi + w(t)\big)\big\|_{L^2}^2 = \big\|\nabla(\phi + w_0)\big\|_{L^2}^2 - \lim_{\varepsilon\downarrow 0} 2\,\mathrm{Re}\int_0^t \big\langle (1-\varepsilon\Delta)^{-1}F\big(u(t')\big), \overline{\partial_t w(t')}\big\rangle\,dt'$$

$$= \big\|\nabla(\phi + w_0)\big\|_{L^2}^2 - 2\,\mathrm{Re}\int_0^t \big\langle F\big(w(t')\big), \overline{\partial_t w(t')}\big\rangle\,dt'. \tag{5.5}$$

Note that in the above time integral of the scalar product in the last line is understood as the duality coupling on $(L_T^\infty H^1 + (L_T^\infty H^1 + L_T^{p'} W^{1,q'})) \times ((L_T^1 H^{-1}) \cap (L_T^1 H^{-1} \cap L_T^p W^{-1,q}))$ by applying the idea used in Lemma 3 in Gérard [4], that is, we decompose $F(w)$ as

$$F(w) = \chi(D_x)F(w) + \sum_{j=1}^n \big(1-\chi(D_x)\big)P_j(D_x)\partial_{x_j}F(w),$$

where $\chi \in C_0^\infty(\mathbb{R}^n)$ is a cutoff function such that $0 \leqslant \chi \leqslant 1$, $\chi(\xi) = 1$ for $|\xi| \leqslant 1$ and $\chi(\xi) = 0$ for $|\xi| \geqslant 2$, and $P_j(\xi) = i\xi_j/|\xi|^2$.

We show (5.5). It follows from Theorem 2.2, Lemmas 3.1, 3.2, 3.5 and 4.1 that

$$\begin{aligned}
\left|\int_0^t \big\langle \widetilde{F}\big(w(t')\big), \overline{\partial_t w(t')}\big\rangle\,dt'\right| &\leqslant \int_0^t \big|\big\langle \chi(D_x)\widetilde{F}\big(w(t')\big), \overline{\partial_t w(t')}\big\rangle\big|\,dt' + \int_0^t \big|\big\langle \big(1-\chi(D_x)\big)\widetilde{F}\big(w(t')\big), \overline{\partial_t w(t')}\big\rangle\big|\,dt' \\
&\leqslant \big\|\chi(D_x)\widetilde{F}(w)\big\|_{L_T^\infty H^1}\|\partial_t w\|_{L_T^1 H^{-1}} \\
&\quad + \big\|\big(1-\chi(D_x)\big)\widetilde{F}(w)\big\|_{L_T^{p'}(H^1+W^{1,q'})}\|\partial_t w\|_{L_T^p(H^{-1}\cap W^{-1,q})} \\
&\leqslant C\big(\big\|F_1(w)\big\|_{L_T^\infty L^2} + \big\|F_2(w)\big\|_{L_T^\infty L^{q'}}\big)\|\partial_t w\|_{L_T^\infty H^{-1}} \\
&\quad + C\big(\big\|G_1(w)\big\|_{L_T^{p'} L^2} + \big\|G_2(w)\big\|_{L_T^{p'} L^{q'}}\big)\|\partial_t w\|_{L_T^p(H^{-1}\cap W^{-1,q})}.
\end{aligned} \tag{5.6}$$

Furthermore, by using a similar argument to the above and Lebesgue convergence theorem, we deduce that

$$\lim_{\varepsilon\downarrow 0}\int_0^t \big\langle (1-\varepsilon\Delta)^{-1}\widetilde{F}\big(w(t')\big), \overline{\partial_t w(t')}\big\rangle\,dt' = \int_0^t \big\langle \widetilde{F}\big(w(t')\big), \overline{\partial_t w(t')}\big\rangle\,dt',$$

which yields (5.5).

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From (5.5), formally, we can continue as follows:

$$\begin{aligned}\left\|\nabla\left(\phi+w(t)\right)\right\|_{L^2}^2 &= \left\|\nabla(\phi+w_0)\right\|_{L^2}^2 - \int_0^t \frac{\partial}{\partial t}\left(\int_{\mathbb{R}^n} V\left(\left|\phi+w(t')\right|^2\right)dx\right)dt' \\ &= \left\|\nabla(\phi+w_0)\right\|_{L^2}^2 - \int_{\mathbb{R}^n} V\left(\left|\phi+w(t)\right|^2\right)dx + \int_{\mathbb{R}^n} V\left(|\phi|^2\right)dx,\end{aligned}$$

since a formal argument implies

$$\frac{\partial}{\partial t}\left(\int_{\mathbb{R}^n} V\left(\left|\phi+w(t)\right|^2\right)dx\right) = 2\,\mathrm{Re}\left\langle \widetilde{F}\left(w(t)\right), \overline{\partial_t w(t)}\right\rangle.$$

Hence, to justify the argument above, we need to show the following lemma.

Lemma 5.1. *$\int_{\mathbb{R}^n} V(|\phi+w(\cdot)|^2)\,dx \in W^{1,1}((0,T))$ and*

$$\frac{\partial}{\partial t}\left(\int_{\mathbb{R}^n} V\left(\left|\phi+w(t)\right|^2\right)dx\right) = 2\,\mathrm{Re}\left\langle \widetilde{F}\left(w(t)\right), \overline{\partial_t w(t)}\right\rangle_{A\times B} \quad \text{in } L^1\left((0,T)\right),$$

where $A := (H^1 + (H^1 + W^{1,q'}))$, $B := (H^{-1} \cap (H^{-1} \cap W^{-1,q}))$.

Proof. Put $I = (0,T)$ for simplicity. Moreover, $\mathcal{D}(I)$ and $\mathcal{D}'(I)$ denote the Fréchet space of C^∞ functions $I \to \mathbb{C}$ compactly supported in I and the space of distributions on I, respectively. Note that as is in Gallo [3], from $(\mathbf{H}_f)$, the mapping $w \mapsto V(|\phi+w|^2)$ becomes a bounded operator from $H^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$. Thus, for any $\varphi \in C_0^\infty(0,T)$, we have

$$\begin{aligned}\left\langle \frac{\partial}{\partial t}\int_{\mathbb{R}^n} V\left(|\phi+w|^2\right)dx, \varphi\right\rangle_{\mathcal{D}'(I)\times\mathcal{D}(I)} &= \left\langle -\int_{\mathbb{R}^n} V\left(|\phi+w|^2\right)dx, \partial_t\varphi\right\rangle_{\mathcal{D}'(I)\times\mathcal{D}(I)} \\ &= -\int_I \left(\int_{\mathbb{R}^n} V\left(\left|\phi+w(t)\right|^2\right)dx\right)\partial_t\varphi(t)\,dt.\end{aligned}$$

Take $0 < \varepsilon < T' < T$ such that $\mathrm{supp}(\varphi) \subset [\varepsilon, T']$. Using Lebesgue convergence theorem, we compute

$$\begin{aligned}&-\int_I \left(\int_{\mathbb{R}^n} V\left(\left|\phi+w(t)\right|^2\right)dx\right)\partial_t\varphi(t)\,dt \\ &= \lim_{h\to 0}\left\{-\int_\varepsilon^{T'} \left(\int_{\mathbb{R}^n} V\left(\left|\phi+w(t)\right|^2\right)dx\right)\frac{\varphi(t+h)-\varphi(t)}{h}\,dt\right\} \\ &= \lim_{h'\to 0}\left\{\int_\varepsilon^{T'} \left(\int_{\mathbb{R}^n} \frac{V(|\phi+w(t+h')|^2) - V(|\phi+w(t)|^2)}{h'}\,dx\right)\varphi(t)\,dt\right\} \\ &= \int_I 2\,\mathrm{Re}\left\langle F\left(u(t)\right), \overline{\partial_t w(t)}\right\rangle_{A\times B}\varphi(t)\,dt.\end{aligned}$$

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We need to justify the limiting procedure of the last line in the above. Since $(\partial/\partial\bar{z})(V(|z|^2)) = \widetilde{F}(|z|)$ for any $z \in \mathbb{C}$, it follows that

$$\begin{aligned}
&\left| \int_{\mathbb{R}^n} \frac{V(|\phi + w(t+h)|^2) - V(|\phi + w(t)|^2)}{h}\, dx - 2\operatorname{Re}\langle \widetilde{F}(w(t)), \overline{\partial_t w(t)} \rangle_{A\times B} \right| \\
&\leqslant \left| \int_{\mathbb{R}^n} 2\operatorname{Re}\left(\int_0^1 \frac{\partial V}{\partial \bar{z}} \left(\left|\phi + w(t) + \theta\big(w(t+h) - w(t)\big)\right|^2 \right) d\theta\, \frac{\overline{(w(t+h) - w(t))}}{h} \right) dx \right. \\
&\quad \left. - 2\operatorname{Re}\langle \widetilde{F}(w(t)), \overline{\partial_t w(t)} \rangle_{A\times B} \right| \\
&\leqslant 2\left| \int_{\mathbb{R}^n} \left(\int_0^1 \left(\frac{\partial V}{\partial \bar{z}} \left(\left|\phi + w(t) + \theta\big(w(t+h) - w(t)\big)\right|^2 \right) - \widetilde{F}(w(t)) \right) d\theta\, \frac{\overline{(w(t+h) - w(t))}}{h} \right) dx \right| \\
&\quad + 2\left| \int_{\mathbb{R}^n} \widetilde{F}(w(t)) \frac{\overline{(w(t+h) - w(t))}}{h}\, dx - \langle \widetilde{F}(w(t)), \overline{\partial_t w(t)} \rangle_{A\times B} \right| \\
&\leqslant 2\left| \int_{\mathbb{R}^n} \left(\int_0^1 \left(\widetilde{F}\big(w(t) + \theta\big(w(t+h) - w(t)\big)\big) - \widetilde{F}(w(t))\right) d\theta\, \frac{\overline{(w(t+h) - w(t))}}{h} \right) dx \right| \\
&\quad + 2\left| \left\langle \widetilde{F}(w(t)), \frac{\overline{(w(t+h) - w(t))}}{h} \right\rangle_{H^{-1}\times H^1} - \langle \widetilde{F}(w(t)), \overline{\partial_t w(t)} \rangle_{A\times B} \right| \\
&=: 2L_1 + 2L_2.
\end{aligned} \tag{5.7}$$

The estimation of L_1. Choose the cutoff function $\chi \in C_0^\infty(\mathbb{R}^n)$ such that $0 \leqslant \chi \leqslant 1$, $\chi(\xi) = 1$ for $|\xi| \leqslant 1$ and $\chi(\xi) = 0$ for $|\xi| \geqslant 2$. Using $\chi(D_x)$, we decompose L_1 as follows:

$$\begin{aligned}
L_1 &\leqslant \left| \int_{\mathbb{R}^n} \left(\int_0^1 \chi(D_x)\left\{\widetilde{F}\big(w(t) + \theta\big(w(t+h) - w(t)\big)\big) - \widetilde{F}(w(t))\right\} d\theta\, \frac{\overline{(w(t+h) - w(t))}}{h} \right) dx \right| \\
&\quad + \left| \int_{\mathbb{R}^n} \left(\int_0^1 \big(1 - \chi(D_x)\big)\left\{\widetilde{F}\big(w(t) + \theta\big(w(t+h) - w(t)\big)\big) - \widetilde{F}(w(t))\right\} d\theta\, \frac{\overline{(w(t+h) - w(t))}}{h} \right) dx \right| \\
&=: K_1 + K_2.
\end{aligned}$$

From now on, $L^p_{[\varepsilon,T']}X$ denotes the Banach space $L^p([\varepsilon, T'], X)$ for $p \in [1, \infty]$ and a Banach space X.

The estimation of K_1. By Lemma 3.3, we get

$$\begin{aligned}
&\left\| F\big(w(\cdot) + \theta\big(w(\cdot + h) - w(\cdot)\big)\big) - F(w(\cdot)) \right\|_{L^\infty_{[\varepsilon,T']}L^2 + L^\infty_{[\varepsilon,T']}L^{q'}} \\
&\quad \leqslant \left\| F_1\big(w(\cdot) + \theta\big(w(\cdot + h) - w(\cdot)\big)\big) - F_1(w(\cdot)) \right\|_{L^\infty_{[\varepsilon,T']}L^2} \\
&\qquad + \left\| F_2\big(w(\cdot) + \theta\big(w(\cdot + h) - w(\cdot)\big)\big) - F_2(w(\cdot)) \right\|_{L^\infty_{[\varepsilon,T']}L^{q'}} \\
&\quad \leqslant C\left\|w(\cdot + h) - w(\cdot)\right\|_{L^\infty_{[\varepsilon,T']}L^2} + C\left\|w(\cdot + h) - w(\cdot)\right\|_{L^\infty_{[\varepsilon,T']}H^1} \left\{ \left(\left\|w(\cdot + h)\right\|_{L^\infty_{[\varepsilon,T']}H^1} + \left\|w(\cdot)\right\|_{L^\infty_{[\varepsilon,T']}H^1}\right) \right. \\
&\qquad \left. + \left(\left\|w(\cdot + h)\right\|_{L^\infty_{[\varepsilon,T']}H^1} + \left\|w(\cdot)\right\|_{L^\infty_{[\varepsilon,T']}H^1}\right)^{\max(1, 2\alpha_1 - 2)} \right\}
\end{aligned}$$

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$$\begin{aligned}
&\leqslant C\|w(\cdot+h)-w(\cdot)\|_{L^\infty_{[\varepsilon,T']}H^1}+C\|w(\cdot+h)-w(\cdot)\|_{L^\infty_{[\varepsilon,T']}H^1}\big(\|w\|_{L^\infty_T H^1}+\|w\|^{\max(1,2\alpha_1-2)}_{L^\infty_T H^1}\big)\\
&\leqslant C\|w(\cdot+h)-w(\cdot)\|_{L^\infty_{[\varepsilon,T']}H^1}\big(1+\|w\|_{L^\infty_T H^1}+\|w\|^{\max(1,2\alpha_1-2)}_{L^\infty_T H^1}\big)\\
&\leqslant C\|w(\cdot+h)-w(\cdot)\|_{L^\infty_{[\varepsilon,T']}H^1}, \qquad (5.8)
\end{aligned}$$

where C depends on the norm $\|w\|_{X_T}$ of the space X_T.

$X_{[\varepsilon,T']}$ denotes $L^\infty_{[\varepsilon,T']}H^1\cap L^p_{[\varepsilon,T']}W^{1,q}$. Using the estimate similar to (3.2) and (5.8), we obtain

$$\begin{aligned}
\int_\varepsilon^{T'} K_1\,dt &\leqslant \int_\varepsilon^{T'}\bigg(\int_0^1\big\|\chi(D_x)\big\{\widetilde{F}\big(w(t)+\theta\big(w(t+h)-w(t)\big)\big)-\widetilde{F}\big(w(t)\big)\big\}\big\|_{H^1}\,d\theta\,\bigg\|\frac{\overline{w(t+h)-w(t)}}{h}\bigg\|_{H^{-1}}\bigg)\,dt\\
&\leqslant C\int_0^1\big(\big\|\widetilde{F}_1\big(w(\cdot)+\theta\big(w(\cdot+h)-w(\cdot)\big)\big)-\widetilde{F}_1\big(w(\cdot)\big)\big\|_{L^\infty_{[\varepsilon,T']}L^2}\\
&\quad+\big\|\widetilde{F}_2\big(w(\cdot)+\theta\big(w(\cdot+h)-w(\cdot)\big)\big)-\widetilde{F}_2\big(w(\cdot)\big)\big\|_{L^\infty_{[\varepsilon,T']}L^{q'}}\big)\,d\theta\,\bigg\|\frac{w(\cdot+h)-w(\cdot)}{h}\bigg\|_{L^\infty_{[\varepsilon,T']}H^{-1}}\\
&\leqslant C\int_0^1\big\|w(\cdot+h)-w(\cdot)\big\|_{X_{[\varepsilon,T']}}\,d\theta\,\bigg\|\frac{w(\cdot+h)-w(\cdot)}{h}\bigg\|_{L^\infty_{[\varepsilon,T']}H^{-1}}\\
&\leqslant C\big\|w(\cdot+h)-w(\cdot)\big\|_{X_{[\varepsilon,T']}}\bigg\|\frac{w(\cdot+h)-w(\cdot)}{h}\bigg\|_{L^\infty_{[\varepsilon,T']}H^{-1}}.
\end{aligned}$$

By Lemma 3.4, we have

$$\begin{aligned}
&\big\|\nabla F\big(w(\cdot)+\theta\big(w(\cdot+h)-w(\cdot)\big)\big)-\nabla F\big(w(\cdot)\big)\big\|_{L^\infty_{[\varepsilon,T']}L^2+L^{p'}_{[\varepsilon,T']}L^{q'}}\\
&\quad\leqslant\big\|\widetilde{G}_1\big(w(\cdot)+\theta\big(w(\cdot+h)-w(\cdot)\big)\big)-\widetilde{G}_1\big(w(\cdot)\big)\big\|_{L^\infty_{[\varepsilon,T']}L^2}\\
&\qquad+\big\|\widetilde{G}_2\big(w(\cdot)+\theta\big(w(\cdot+h)-w(\cdot)\big)\big)-\widetilde{G}_2\big(w(\cdot)\big)\big\|_{L^{p'}_{[\varepsilon,T']}L^{q'}}\\
&\quad\leqslant C\big\|w(\cdot+h)-w(\cdot)\big\|_{L^\infty_{[\varepsilon,T']}H^1}\\
&\qquad+C\big(1+\big\|w(\cdot+h)\big\|_{L^\infty_{[\varepsilon,T']}H^1}+\big\|w(\cdot)\big\|_{L^\infty_{[\varepsilon,T']}H^1}\big)^{\max(1,2\alpha_1-2)}\big\|w(\cdot+h)-w(\cdot)\big\|_{L^\infty_{[\varepsilon,T']}H^1}\\
&\qquad+C\big\|w(\cdot+h)-w(\cdot)\big\|_{X_{[\varepsilon,T']}}\big(\big\|w(\cdot+h)\big\|^{\max(1,2\alpha_1-2)}_{X_{[\varepsilon,T']}}+\big\|w(\cdot)\big\|^{\max(1,2\alpha_1-2)}_{X_{[\varepsilon,T']}}\big)\\
&\qquad+C\big\|w(\cdot+h)-w(\cdot)\big\|_{X_{[\varepsilon,T']}}\big(1+\big\|w(\cdot+h)\big\|_{L^\infty_{[\varepsilon,T']}H^1}+\big\|w(\cdot)\big\|_{L^\infty_{[\varepsilon,T']}H^1}\big)\\
&\qquad\times\big(\big\|w(\cdot+h)\big\|^{\max(0,2\alpha_1-3)}_{X_{[\varepsilon,T']}}+\big\|w(\cdot)\big\|^{\max(0,2\alpha_1-3)}_{X_{[\varepsilon,T']}}\big)\\
&\quad\leqslant C\big\|w(\cdot+h)-w(\cdot)\big\|_{X_{[\varepsilon,T']}}, \qquad (5.9)
\end{aligned}$$

where C depends on $\|w\|_{X_T}$. Using the estimate similar to (3.3) and (5.9), we have

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$$
\begin{aligned}
K_2 &\leqslant \left\| \int_0^1 \big(1-\chi(D_x)\big)\big\{\widetilde{F}\big(w(t)+\theta\big(w(t+h)-w(t)\big)\big)-\widetilde{F}\big(w(t)\big)\big\}\,d\theta \right\|_{H^1+W^{1,q'}} \\
&\quad \times \left\| \frac{\overline{(w(t+h)-w(t))}}{h} \right\|_{H^{-1}\cap W^{-1,q}} \\
&\leqslant \int_0^1 \big\| \big(1-\chi(D_x)\big)\big\{\widetilde{F}\big(w(t)+\theta\big(w(t+h)-w(t)\big)\big)-\widetilde{F}\big(w(t)\big)\big\}\big\|_{H^1+W^{1,q'}}\,d\theta \\
&\quad \times \left\| \frac{\overline{(w(t+h)-w(t))}}{h} \right\|_{H^{-1}\cap W^{-1,q}} \\
&\leqslant C \int_0^1 \Big(\big\| \widetilde{G}_1\big(w(t)+\theta\big(w(t+h)-w(t)\big)\big)-\nabla\widetilde{G}_1\big(w(t)\big)\big\|_{L^2} \\
&\quad + \big\| \widetilde{G}_2\big(w(t)+\theta\big(w(t+h)-w(t)\big)\big)-\nabla\widetilde{G}_2\big(w(t)\big)\big\|_{L^{q'}} \Big)\,d\theta \\
&\quad \times \left\| \frac{w(t+h)-w(t)}{h} \right\|_{H^{-1}\cap W^{-1,q}}.
\end{aligned}
$$

Hence, we get

$$
\begin{aligned}
\int_\varepsilon^{T'} K_2\,dt &\leqslant C \left\| \int_0^1 \Big(\big\| \widetilde{G}_1\big(w(\cdot)+\theta\big(w(\cdot+h)-w(\cdot)\big)\big)-\widetilde{G}_1\big(w(\cdot)\big)\big\|_{L^2} \right. \\
&\quad \left. + \big\| \widetilde{G}_2\big(w(\cdot)+\theta\big(w(\cdot+h)-w(\cdot)\big)\big)-\widetilde{G}_2\big(w(\cdot)\big)\big\|_{L^{q'}}\,d\theta \Big) \right\|_{L^{p'}_{[\varepsilon,T']}} \\
&\quad \times \left\| \frac{w(\cdot+h)-w(\cdot)}{h} \right\|_{L^p_{[\varepsilon,T']}(H^{-1}\cap W^{-1,q})} \\
&\leqslant C \int_0^1 \Big(\big\| \widetilde{G}_1\big(w(\cdot)+\theta\big(w(\cdot+h)-w(\cdot)\big)\big)-\widetilde{G}_1\big(w(\cdot)\big)\big\|_{L^\infty_{[\varepsilon,T']}L^2} \\
&\quad + \big\| \widetilde{G}_2\big(w(\cdot)+\theta\big(w(\cdot+h)-w(\cdot)\big)\big)-\widetilde{G}_2\big(w(\cdot)\big)\big\|_{L^{p'}_{[\varepsilon,T']}L^{q'}} \Big)\,d\theta \\
&\quad \times \left\| \frac{w(\cdot+h)-w(\cdot)}{h} \right\|_{L^\infty_{[\varepsilon,T']}H^{-1}\cap L^p_{[\varepsilon,T']}W^{-1,q}} \\
&\leqslant C \int_0^1 \big\| w(\cdot+h)-w(\cdot) \big\|_{X_{[\varepsilon,T']}}\,d\theta \left\| \frac{w(\cdot+h)-w(\cdot)}{h} \right\|_{L^\infty_{[\varepsilon,T']}H^{-1}\cap L^p_{[\varepsilon,T']}W^{-1,q}} \\
&\leqslant C \big\| w(\cdot+h)-w(\cdot) \big\|_{X_{[\varepsilon,T']}} \left\| \frac{w(\cdot+h)-w(\cdot)}{h} \right\|_{L^\infty_{[\varepsilon,T']}H^{-1}\cap L^p_{[\varepsilon,T']}W^{-1,q}}.
\end{aligned}
$$

Thus, we obtain

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$$\begin{aligned}\int_{\varepsilon}^{T'} L_1\,dt &= \int_{\varepsilon}^{T'} K_1\,dt + \int_{\varepsilon}^{T'} K_2\,dt \\ &\leqslant C\big\|w(\cdot+h)-w(\cdot)\big\|_{X_{[\varepsilon,T']}} \\ &\quad\times\left(\left\|\frac{w(\cdot+h)-w(\cdot)}{h}\right\|_{L^\infty_{[\varepsilon,T']}H^{-1}} + \left\|\frac{w(\cdot+h)-w(\cdot)}{h}\right\|_{L^p_{[\varepsilon,T']}W^{-1,q}}\right).\end{aligned} \tag{5.10}$$

The estimation of L_2. It follows from Lemma 3.1, Lemma 3.3 and Lemma 3.5 that for almost all $t\in[\varepsilon,T']$,

$$\begin{aligned}L_2 &\leqslant \left|\left\langle \chi(D_x)\widetilde{F}\big(w(t)\big), \overline{\frac{w(t+h)-w(t)}{h}} - \overline{\partial_t w(t)}\right\rangle_{H^1\times H^{-1}}\right| \\ &\quad+ \left|\left\langle \big(1-\chi(D_x)\big)\widetilde{F}\big(w(t)\big), \overline{\frac{w(t+h)-w(t)}{h}} - \overline{\partial_t w(t)}\right\rangle_{\substack{(H^1+W^{1,q'})\\ \times(H^{-1}\cap W^{-1,q})}}\right| \\ &\leqslant \big\|\chi(D_x)\widetilde{F}\big(w(t)\big)\big\|_{H^1}\left\|\frac{w(t+h)-w(t)}{h}-\partial_t w(t)\right\|_{H^{-1}} \\ &\quad+ \big\|\big(1-\chi(D_x)\big)\widetilde{F}\big(w(t)\big)\big\|_{H^1+W^{q'}}\left\|\frac{(w(t+h)-w(t))}{h}-\partial_t w(t)\right\|_{H^{-1}\cap W^{-1,q}} \\ &\leqslant C\big(\big\|\widetilde{F}_1\big(w(t)\big)\big\|_{L^2}+\big\|\widetilde{F}_2\big(w(t)\big)\big\|_{L^{q'}}\big)\left\|\frac{w(t+h)-w(t)}{h}-\partial_t w(t)\right\|_{H^{-1}} \\ &\quad+ C\big(\big\|\widetilde{G}_1\big(w(t)\big)\big\|_{L^2}+\big\|\widetilde{G}_2\big(w(t)\big)\big\|_{L^{q'}}\big) \\ &\quad\times\left(\left\|\frac{w(t+h)-w(t)}{h}-\partial_t w(t)\right\|_{H^{-1}}+\left\|\frac{w(t+h)-w(t)}{h}-\partial_t w(t)\right\|_{W^{-1,q}}\right).\end{aligned}$$

Hence, we deduce that

$$\begin{aligned}\int_{\varepsilon}^{T'} L_2\,dt &\leqslant C\big(\big\|\widetilde{F}_1(w)\big\|_{L^1_T L^2}+\big\|\widetilde{F}_2(w)\big\|_{L^1_T L^{q'}}\big)\left\|\frac{w(\cdot+h)-w(\cdot)}{h}-\partial_t w(\cdot)\right\|_{L^\infty_{[\varepsilon,T']}H^{-1}} \\ &\quad+ C\big(\big\|\widetilde{G}_1(w)\big\|_{L^{p'}_T L^2}+\big\|\widetilde{G}_2(w)\big\|_{L^{p'}_T L^{q'}}\big) \\ &\quad\times\left(\left\|\frac{w(\cdot+h)-w(\cdot)}{h}-\partial_t w(\cdot)\right\|_{L^\infty_{[\varepsilon,T']}H^{-1}}\right. \\ &\quad\left.+\left\|\frac{w(\cdot+h)-w(\cdot)}{h}-\partial_t w(\cdot)\right\|_{L^p_{[\varepsilon,T']}W^{-1,q}}\right).\end{aligned} \tag{5.11}$$

In conclusion, concatenating (5.7), (5.10) and (5.11),

$$\begin{aligned}&\left|\int_{\varepsilon}^{T'}\int_{\mathbb{R}^n}\left(\frac{V(|\phi+w(t+h')|^2)-V(|\phi+w(t)|^2)}{h'}\right)\varphi(t)\,dt\,dx - \int_{\varepsilon}^{T'}\big(\mathrm{Re}\big\langle\widetilde{F}\big(w(t)\big),\overline{\partial_t w(t)}\big\rangle\big)\varphi(t)\,dt\right| \\ &\leqslant 2\int_{\varepsilon}^{T'} L_1\big|\varphi(t)\big|\,dt + 2\int_{\varepsilon}^{T'} L_2\big|\varphi(t)\big|\,dt \\ &\leqslant C\big\|w(\cdot+h)-w(\cdot)\big\|_{X_{[\varepsilon,T']}}\end{aligned}$$

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$$\times \left(\left\| \frac{w(\cdot+h)-w(\cdot)}{h} \right\|_{L^\infty_{[\varepsilon,T']}H^{-1}} + \left\| \frac{w(\cdot+h)-w(\cdot)}{h} \right\|_{L^p_{[\varepsilon,T']}W^{-1,q}} \right)$$
$$+ C\big(\|\widetilde{F}_1(w)\|_{L^1_T L^2} + \|\widetilde{F}_2(w)\|_{L^1_T L^{q'}}\big) \left\| \frac{w(\cdot+h)-w(\cdot)}{h} - \partial_t w(\cdot) \right\|_{L^\infty_{[\varepsilon,T']}H^{-1}}$$
$$+ C\big(\|\widetilde{G}_1(w)\|_{L^{p'}_T L^2} + \|\widetilde{G}_2(w)\|_{L^{p'}_T L^{q'}}\big)$$
$$\times \left(\left\| \frac{w(\cdot+h)-w(\cdot)}{h} - \partial_t w(\cdot) \right\|_{L^\infty_{[\varepsilon,T']}H^{-1}} + \left\| \frac{w(\cdot+h)-w(\cdot)}{h} - \partial_t w(\cdot) \right\|_{L^p_{[\varepsilon,T']}W^{-1,q}} \right).$$

Noting Lemma 4.1 and the fact that a local Lipschitz continuity (2.2) in Theorem 2.1 yields

$$\big\|w(\cdot+h)-w(\cdot)\big\|_{X_{[\varepsilon,T']}} \leqslant C\big\|w(|h|)-w_0\big\|_{H^1} \to 0 \quad \text{as } h \to 0,$$

we obtain

$$\left\langle \frac{\partial}{\partial t}\int_{\mathbb{R}^n} V(u)\,dx, \varphi \right\rangle_{\mathcal{D}'\times\mathcal{D}(I)} = 2\int_I \operatorname{Re}\big\langle \widetilde{F}\big(w(t)\big), \overline{\partial_t w(t)}\big\rangle \varphi(t)\,dt.$$

Since the estimation (5.6) means $\operatorname{Re}\langle \widetilde{F}(w(t)), \overline{\partial_t w(t)}\rangle \in L^1(I)$, we complete the proof of Lemma 5.1. □

In conclusion, by Lemma 5.1, we complete the proof of the main result. □

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Appendix A

Proof of Lemma 3.1. We decompose $-f(|\phi+w|^2)(\phi+w)$ as

$$-f\big(|\phi+w|^2\big)(\phi+w) = \widetilde{F}_1(w) + \widetilde{F}_2(w), \tag{A.1}$$

where

$$\widetilde{F}_1(w) := -f\big(|\phi|^2\big)(\phi+w) - 2\operatorname{Re}[\bar{\phi}w]f'\big(|\phi|^2\big)\phi,$$
$$\widetilde{F}_2(w) := -\big\{f\big(|\phi+w|^2\big)(\phi+w) - f\big(|\phi|^2\big)(\phi+w)\big\} + 2\operatorname{Re}[\bar{\phi}w]f'\big(|\phi|^2\big)\phi.$$

According to Lemma 4.1 in Gallo [3], by the assumption $(\mathbf{H}'_\phi)$ and $f(|\phi|^2) \in L^2$, we deduce that

$$\big\|\widetilde{F}_1(w)\big\|_{L^\infty_T L^2} \leqslant C\big(1+\|w\|_{L^\infty_T L^2}\big), \qquad \big|\widetilde{F}_2(w)\big| \leqslant C|w|^2\big(1+|w|\big)^{\max(0,2\alpha_1-3)}.$$

Therefore for all $t \in [0,T]$, we estimate that

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$$\begin{aligned}\big\|\widetilde{F}_2\big(w(t)\big)\big\|_{L^{q'}} &\leqslant C\big\|\big|w(t)\big|^2\big(1+\big|w(t)\big|\big)^{\max(0,2\alpha_1-3)}\big\|_{L^{q'}}\\ &\leqslant C\big\|w(t)\big\|_{L^{2q'}}^2 + C\big\|w(t)\big\|_{L^{q'\max(2,2\alpha_1-1)}}^{\max(2,2\alpha_1-1)}\\ &\leqslant C\big\|w(t)\big\|_{H^1}^2 + C\big\|w(t)\big\|_{H^1}^{\max(2,2\alpha_1-1)}.\end{aligned}$$

Hence, we deduce that

$$\big\|\widetilde{F}_2(w)\big\|_{L_T^\infty L^{q'}} \leqslant C\|w\|_{L_T^\infty H^1}^2 + C\|w\|_{L_T^\infty H^1}^{\max(2,2\alpha_1-1)}.$$

In conclusion, we get

$$\big\|\widetilde{F}_1(w)\big\|_{L_T^\infty L^2} + \big\|\widetilde{F}_2(w)\big\|_{L_T^\infty L^{q'}} \leqslant C\big(1+\|w\|_{L_T^\infty L^2}\big) + C\big(\|w\|_{L_T^\infty H^1}^2 + \|w\|_{L_T^\infty H^1}^{\max(2,2\alpha_1-1)}\big).$$

Next, we show $\widetilde{F}(w) \in L_T^p L^2$. We apply an interpolation method (see Lemma 4.2 in Gallo [3]). Thanks to the Hölder inequality and Gagliardo–Nirenberg's inequality, we estimate

$$\begin{aligned}\big\|\widetilde{F}_2(w)\big\|_{L_T^p L^2} &\leqslant C\|w\|_{L_T^{2p} L^4}^2 + \|w\|_{L_T^{p\max(2,2\alpha_1-1)} L^{2\max(2,2\alpha_1-1)}}^{\max(2,2\alpha_1-1)}\\ &\leqslant C\|w\|_{L_T^\infty H^1}^2 + \|w\|_{L_T^s W^{1,r}}^{\max(2,2\alpha_1-1)},\end{aligned}\tag{A.2}$$

where we choose the pair (s,r) such that

- If $\frac{1}{2}-\frac{1}{n} \leqslant \frac{1}{p\max(2,2\alpha_1-1)}$ (which means that $H^1 \hookrightarrow L^{p\max(2,2\alpha_1-1)}$), then $(s,r)=(\infty,2)$.
- If $\frac{1}{2}-\frac{1}{n} > \frac{1}{p\max(2,2\alpha_1-1)}$, then $r>2$, and
  - (i) $\frac{2}{s}+\frac{n}{r}=\frac{n}{2}$ (which means that (s,r) is an admissible pair),
  - (ii) $0 \leqslant \frac{1}{r}-\frac{1}{n} \leqslant \frac{1}{p\max(2,2\alpha_1-1)}$ (which gives the Sobolev embedding $W^{1,r} \hookrightarrow L^{p\max(2,2\alpha_1-1)}$),
  - (iii) $\frac{1}{p\max(2,2\alpha_1-1)} \geqslant \frac{1}{s}$ (which gives $L_T^s \hookrightarrow L_T^{p\max(2,2\alpha_1-1)}$).

Such the choice of s and r is possible if and only if s and r satisfy the following inequality:

$$\frac{n}{2}-1 \leqslant \frac{2+n}{p\max(2,2\alpha_1-1)}.\tag{A.3}$$

Indeed, if (A.3) is true, then it is sufficient to choose

$$\frac{n}{r} \in \left[\frac{n}{2}-\frac{2}{p\max(2,2\alpha_1-1)}, 1+\frac{n}{p\max(2,2\alpha_1-1)}\right].$$

Moreover, since $H^1 \hookrightarrow L^{p\max(2,2\alpha_1-1)}$ if $n=2$ or if $n=3$ and $1 \leqslant \alpha_1 \leqslant 2$ or if $n=4$ and $1 \leqslant \alpha_1 \leqslant 3/2$, we consider that $n=3$ and $2<\alpha_1<3$ or $n=4$ and $3/2<\alpha_1<2$. Since $2<r<3$ and (s,r) is an admissible pair, we can choose $\widetilde{\theta} \in (0,1)$ satisfying

$$\frac{1-\widetilde{\theta}}{2}+\frac{\widetilde{\theta}}{q}=\frac{1}{r}, \qquad \frac{1-\widetilde{\theta}}{\infty}+\frac{\widetilde{\theta}}{p}=\frac{1}{s}.$$

Thus, using interpolation method,

$$\begin{aligned}\|w\|_{L_T^s W^{1,r}} &\leqslant C\|w\|_{L_T^\infty H^1}^{1-\widetilde{\theta}}\|w\|_{L_T^p W^{1,q}}^{\widetilde{\theta}}\\ &\leqslant C\big(\|w\|_{L_T^\infty H^1}+\|w\|_{L_T^p W^{1,q}}\big)\\ &= C\|w\|_{X_T}.\end{aligned}\tag{A.4}$$

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From (A.2) and (A.4), we deduce that

$$\left\|F_2(w)\right\|_{L_T^p L^2} \leqslant C\left(\|w\|_{L_T^\infty H^1}^2 + \|w\|_{X_T}^{\max(2,2\alpha_1-1)}\right).$$

Thus, we get $F(w) \in L_T^p L^2$. □

Proof of Lemma 3.3. we use the decomposition (A.1) again. As is in Gallo [3], we also have

$$\begin{aligned}
&\left|\widetilde{F}_1(w_1) - \widetilde{F}_1(w_2)\right| \leqslant C|w_1 - w_2|,\\
&\left|\widetilde{F}_2(w_2) - \widetilde{F}_2(w_2)\right| \leqslant C|w_1 - w_2|\left(|w_1| + |w_2|\right)\left(1 + |w_1| + |w_2|\right)^{\max(0,2\alpha_1-3)}.
\end{aligned}$$

Therefore we deduce that

$$\left\|\widetilde{F}_1(w_1) - \widetilde{F}_1(w_2)\right\|_{L_T^\infty L^2} \leqslant C\|w_1 - w_2\|_{L_T^\infty L^2}.$$

Moreover let

$$(q_1, q_2) := \begin{cases} (2,3) & \text{if } n = 2, \text{ or } n = 3 \text{ and } \alpha_1 \leqslant 2,\\ \left(\frac{q}{q-1-\max(1,2\alpha_1-2)}, \frac{q}{\max(1,2\alpha_1-2)}\right) & \text{if } n = 3 \text{ and } 2 < \alpha_1 < 3 \text{ or } n = 4, \end{cases}$$

with $\frac{1}{q'} = \frac{1}{q_1} + \frac{1}{q_2}$. Since if $n = 3$ and $2 < \alpha_1 < 3$ or $n = 4$, then $H^1 \hookrightarrow L^{q_1}$, for all $t \in [0, T]$, we estimate

$$\begin{aligned}
\left\|\widetilde{F}_2\big(w_2(t)\big) - \widetilde{F}_2\big(w_2(t)\big)\right\|_{L^{q'}} &\leqslant C\left\|w_1(t) - w_2(t)\right\|_{L^{2q'}}\left(\left\|w_1(t)\right\|_{L^{2q'}} + \left\|w_2(t)\right\|_{L^{2q'}}\right)\\
&\quad + C\left\|w_1(t) - w_2(t)\right\|_{L^{q_1}}\left\|\left|w_1(t)\right| + \left|w_2(t)\right|\right\|_{L^{q_2 \max(1,2\alpha_1-2)}}^{\max(1,2\alpha_1-2)}\\
&\leqslant C\left\|w_1(t) - w_2(t)\right\|_{L^{2q'}}\left(\left\|w_1(t)\right\|_{L^{2q'}} + \left\|w_2(t)\right\|_{L^{2q'}}\right)\\
&\quad + C\left\|w_1(t) - w_2(t)\right\|_{L^{q_1}}\left\|\left|w_1(t)\right| + \left|w_2(t)\right|\right\|_{L^q}^{\max(1,2\alpha_1-2)}\\
&\leqslant C\left\|w_1(t) - w_2(t)\right\|_{H^1}\left(\left\|w_1(t)\right\|_{H^1} + \left\|w_2(t)\right\|_{H^1}\right)\\
&\quad + C\left\|w_1(t) - w_2(t)\right\|_{H^1}\left(\left\|w_1(t)\right\|_{H^1} + \left\|w_2(t)\right\|_{H^1}\right)^{\max(1,2\alpha_1-2)}.
\end{aligned}$$

In conclusion, we get

$$\begin{aligned}
&\left\|\widetilde{F}(w_1) - \widetilde{F}(w_2)\right\|_{L_T^\infty L^2 + L_T^\infty L^{q'}}\\
&\quad \leqslant CT\|w_1 - w_2\|_{L_T^\infty L^2} + C\|w_1 - w_2\|_{L_T^\infty H^1}\\
&\quad \times \left(\left(\|w_1\|_{L_T^\infty H^1} + \|w_2\|_{L_T^\infty H^1}\right) + \left(\|w_1\|_{L_T^\infty H^1} + \|w_2\|_{L_T^\infty H^1}\right)^{\max(1,2\alpha_1-2)}\right). \qquad \square
\end{aligned}$$

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