広島大学学位請求論文

Bernstein type theorems for some types of parabolic *k*-Hessian equations

（ある種の放物型 k-Hessian 方程式に対する Bernstein 型定理）

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広島大学大学院理学研究科

数学専攻

中森　さおり

（広島大学 学術・社会産学連携室研究企画室）

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(1) Uniqueness of boundary blowup solutions to k-curvature equation.
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Journal of Mathematical Analysis and Applications,
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(2) A Bernstein type theorem for parabolic k-Hessian equations.
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117 (2015), 211-220.

主論文

Bernstein type theorems for some types of parabolic k-Hessian equations

Saori Nakamori[1]

Abstract

We are concerned with the characterization of entire solutions to the parabolic k-Hessian equation of the form $-u_t F_k(D^2u) = 1$ in $\mathbb{R}^n \times (-\infty, 0]$. We prove that for $1 \le k \le n$, any strictly convex-monotone solution $u = u(x,t) \in C^{4,2}(\mathbb{R}^n \times (-\infty, 0])$ to $-u_t F_k(D^2u) = 1$ in $\mathbb{R}^n \times (-\infty, 0]$ must be a linear function of t plus a quadratic polynomial of x, under some growth assumptions on u.

1 Introduction

In the early 20th century, Bernstein [3] proved the following theorem.

Theorem 1.1. *If $f \in C^2(\mathbb{R}^2)$ and the graph of $z = f(x,y)$ is a minimal surface in $\mathbb{R}^3$, that is, f satisfies*

$$(1+f_y^2)f_{xx} + 2f_x f_y f_{xy} + (1+f_x^2)f_{yy} = 0 \quad \text{in } \mathbb{R}^2, \tag{1.1}$$

then f is necessarily an affine function of x and y.

This theorem gives the characterization of entire solutions to the minimal surface equation defined in the whole plane $\mathbb{R}^2$.

Many problems on the classification of entire solutions to PDEs have been extensively studied. We list some results concerning Bernstein type theorems for *fully nonlinear* equations. First, for Monge-Ampère equation, the following theorem is known.

[1]Department of Mathematics, Graduate School of Science, Hiroshima University,
1-3-1 Kagamiyama, Higashi-Hiroshima city, Hiroshima 739-8526, Japan
E-mail : `d113989@hiroshima-u.ac.jp`

Theorem 1.2. *Let $u \in C^4(\mathbb{R}^n)$ be a convex solution to*

$$\det D^2 u = 1 \quad \text{in } \mathbb{R}^n. \tag{1.2}$$

Then u is a quadratic polynomial.

This theorem was proved by Jörgens [24] for $n = 2$, by Calabi [10] for $n \leq 5$, and by Pogorelov [36] for arbitrary $n \geq 2$ (see also [11] for a simpler proof). Caffarelli [5] proved that the result holds for viscosity solutions (see also [7]). Moreover, Jian and Wang [23] obtained Bernstein type result for a certain Monge-Ampère equation in the half space $\mathbb{R}^n_+$.

Here we note that the convexity assumption in Theorem 1.2 is quite natural, since Monge-Ampère operator $\det D^2 u$ is degenerate elliptic for convex functions so that we usually seek solutions in the class of convex functions when we deal with Monge-Ampère equation.

Later, Bao, Chen, Guan and Ji [2] extended this result to the so-called *k-Hessian equation* of the form

$$F_k(D^2 u) = 1 \quad \text{in } \mathbb{R}^n, \tag{1.3}$$

for $1 \leq k \leq n$. Here $F_k(D^2 u)$ is defined by

$$F_k(D^2 u) = S_k(\lambda_1, \ldots, \lambda_n), \tag{1.4}$$

where, for a C^2 function u, $\lambda_1, \ldots, \lambda_n$ denote the eigenvalues of the Hessian matrix $D^2 u$, and S_k denotes the k-th elementary symmetric function, that is

$$S_k(\lambda_1, \ldots, \lambda_n) = \sum \lambda_{i_1} \cdots \lambda_{i_k}, \tag{1.5}$$

where the sum is taken over all increasing k-tuples, $1 \leq i_1 < \cdots < i_k \leq n$.

Laplace operator Δu and Monge-Ampère operator $\det D^2 u$ correspond respectively to the special cases $k = 1$ and $k = n$ in (1.4). Hence, the class of k-Hessian equations includes important PDEs which arise in physics and geometry. Here we remark that (1.4) is a linear operator for $k = 1$ while it is a fully nonlinear operator for $k \geq 2$. It is much harder to study the intermediate case $2 \leq k \leq n-1$. Though, there are a number of papers concerning the analysis of k-Hessian equation, such as the solvability of the Dirichlet problem, see [8, 13, 20, 41, 42, 43, 44, 45, 46] for example.

Bao, Chen, Guan and Ji [2] proved the following Bernstein type theorem for k-Hessian equation (1.3).

Theorem 1.3. *Let $1 \leq k \leq n$ and $u \in C^4(\mathbb{R}^n)$ be a strictly convex solution to* (1.3)*. Suppose that there exist constants $A, B > 0$ such that for all $x \in \mathbb{R}^n$,*

$$u(x) \geq A|x|^2 - B. \tag{1.6}$$

Then u is a quadratic polynomial.

In this theorem, for the case $k = n$ which corresponds to Monge-Ampère equation, the assumption (1.6) can be removed, due to Theorem 1.2. Furthermore, for the case $k = 1$ which corresponds to Poisson equation $\Delta u = 1$, the assumption (1.6) can also be removed. It is because the classical convex solution to $\Delta u = 1$ in $\mathbb{R}^n$ must be quadratic, as it follows almost straightforward from Liouville's theorem for harmonic functions. The proof is given in [2], but we state the proof here for the reader's convenience. Assume $u \in C^2(\mathbb{R}^n)$ is a convex solution to $\Delta u = 1$ in $\mathbb{R}^n$ and let $\{e_1, \ldots, e_n\}$ be the canonical basis of $\mathbb{R}^n$. Then we see that $0 \leq D_{\xi\xi}u \leq 1$ and $D_{\xi\xi}u$ is harmonic for any unit vector ξ. It follows from Liouville's theorem for harmonic functions that $D_{\xi\xi}u$ is a constant. Therefore, $D_{ij}u$ is a constant for $i = j$. For $i \neq j$, by the fact $D_{ij}u = D_{\xi\xi}u - (D_{ii}u + D_{jj}u)/2$ for $\xi = (e_i + e_j)/\sqrt{2}$ we obtain that $D_{ij}u$ is also a constant. This ends the proof.

Next, Gutiérrez and Huang [19] extended Theorem 1.2 to the parabolic analogue of Monge-Ampère equation

$$-u_t \det D^2 u = 1 \quad \text{in } \mathbb{R}^n \times (-\infty, 0]. \tag{1.7}$$

Here D^2u means the matrix of second partial derivatives with respect to x. This type of equation was firstly proposed by Krylov [27].

The function $u = u(x,t) : \mathbb{R}^n \times (-\infty, 0] \to \mathbb{R}$ is said to be *convex-monotone* if it is convex in x and non-increasing in t. We state Bernstein type theorem for (1.7) which Gutiérrez and Huang [19] proved.

Theorem 1.4. *Let $u \in C^{4,2}(\mathbb{R}^n \times (-\infty, 0])$ be a convex-monotone solution to (1.7). Suppose that there exist constants $m_1 \geq m_2 > 0$ such that for all $(x,t) \in \mathbb{R}^n \times (-\infty, 0]$,*

$$-m_1 \leq u_t(x,t) \leq -m_2. \tag{1.8}$$

Then u has the form $u(x,t) = -mt + p(x)$ where $m > 0$ is a constant and p is a quadratic polynomial.

We note that Xiong and Bao [49] have recently obtained Bernstein type theorems for more general parabolic Monge-Ampère equations, such as $u_t = (\det D^2u)^{1/n}$ and $u_t = \log\det D^2u$. However, as far as we know, Bernstein type theorems for parabolic fully nonlinear equations are known only for the parabolic Monge-Ampère equations.

In this paper, we are concerned with the parabolic analogue of k-Hessian equation of the following form

$$-u_t F_k(D^2u) = 1 \quad \text{in } \mathbb{R}^n \times (-\infty, 0], \tag{1.9}$$

for $1 \leq k \leq n$. Here $F_k(D^2u)$ is the k-Hessian operator defined in (1.4). We call (1.9) "parabolic k-Hessian equation" in this paper. For the special case $k = n$, (1.9) reduces to the parabolic Monge-Ampère equation (1.7). We shall obtain Bernstein type theorem for (1.9). Moreover, we deal with other forms of parabolic k-Hessian equation $u_t = \rho(F_k(D^2u)^{1/k})$ (see Section 5).

This paper is divided as follows. In Section 2, we state our main result and give the strategy for the proof. In Section 3, we prove Pogorelov type lemma, which is used later. Section 4 is devoted to the proof of the main result. In Section 5, we consider more generalized parabolic k-Hessian equations and present Bernstein type theorem for them. In Section 6, we state some remarks and open problems. Finally, in Section 7, we prove some lemmas which are used in the proof of main theorem.

2 Main result

The function $u = u(x,t) : \mathbb{R}^n \times (-\infty, 0] \to \mathbb{R}$ is said to be *strictly convex-monotone* if u is strictly convex in x and decreasing in t. Here is our main result of this paper.

Theorem 2.1. *Let* $1 \leq k \leq n$ *and* $u \in C^{4,2}(\mathbb{R}^n \times (-\infty, 0])$ *be a strictly convex-monotone solution to* (1.9)*. Suppose that there exist constants* $m_1 \geq m_2 > 0$ *such that for all* $(x,t) \in \mathbb{R}^n \times (-\infty, 0]$,

$$-m_1 \leq u_t(x,t) \leq -m_2, \tag{2.1}$$

and that there exist constants $A, B > 0$ *such that for all* $x \in \mathbb{R}^n$,

$$u(x,0) \geq A|x|^2 - B. \tag{2.2}$$

Then u *has the form* $u(x,t) = -mt + p(x)$ *where* $m > 0$ *is a constant and* p *is a quadratic polynomial.*

Remark 2.1. For the case $k = n$ which corresponds to the parabolic Monge-Ampère equation (1.7), the assumption (2.2) can be removed, due to Theorem 1.4.

The proof of this theorem will be given in subsequent sections. Here we give the strategy for the proof:

Step 1. Derivation of a local gradient estimate of u.
Step 2. Pogorelov type lemma.
Step 3. Combining these results and Evans-Krylov type theorem, we obtain local α-Hölder estimates of D^2u and u_t.

3 Pogorelov type lemma

We introduce some notation. First, if $D \subset \mathbb{R}^n \times (-\infty, 0]$ and $t \leq 0$, $D(t)$ is denoted by

$$D(t) = \{x \in \mathbb{R}^n \mid (x,t) \in D\}.$$

Let $D \subset \mathbb{R}^n \times (-\infty, 0]$ be a bounded set and $t_0 = \inf\{t \leq 0 \mid D(t) \neq \emptyset\}$. The parabolic boundary $\partial_p D$ of D is defined by

$$\partial_p D = \left(\overline{D(t_0)} \times \{t_0\}\right) \cup \bigcup_{t \leq 0} \left(\partial D(t) \times \{t\}\right),$$

where $\overline{D(t_0)}$ denotes the closure of $D(t_0)$ and $\partial D(t)$ denotes the boundary of $D(t)$. We say that the domain $D \subset \mathbb{R}^n \times (-\infty, 0]$ is a *bowl-shaped* domain if $D(t)$ is convex for each $t \in (-\infty, 0]$ and $D(t_1) \subset D(t_2)$ for $t_1 \leq t_2 \leq 0$.

Next, for $\lambda = (\lambda_1, \ldots, \lambda_n)$ and $1 \leq m \leq n$, we define

$$S_{m;i_1 i_2 \ldots i_j}(\lambda) = \begin{cases} S_m(\lambda)\big|_{\lambda_{i_1}=\lambda_{i_2}=\cdots=\lambda_{i_j}=0} & \text{if } i_p \neq i_q \text{ for any } 1 \leq p < q \leq j, \\ 0 & \text{otherwise.} \end{cases}$$

In this section, we prove Pogorelov type lemma. This is an analogue of the result of Pogorelov [35], who derived interior C^2-estimates of a solution from C^1-estimates for Monge-Ampère equation. The idea of the proof of the following proposition is adapted from that of [12].

Proposition 3.1. *Let D be a bounded bowl-shaped domain in $\mathbb{R}^n \times (-\infty, 0]$ and $u \in C^{4,2}(\overline{D})$ a strictly convex-monotone solution to $-u_t F_k(D^2 u) = 1$ in D with $u = 0$ on $\partial_p D$, which satisfies (2.1) in D. Then there exists a constant $C = C(n, k, m_2, \|u\|_{C^1(D)})$ such that*

$$\sup_{(x,t) \in D} |u(x,t)|^4 |D^2 u(x,t)| \leq C. \tag{3.1}$$

Proof. We consider the auxiliary function

$$\Psi(x, t; \xi) = (-u(x,t))^4 \varphi\left(\frac{|Du(x,t)|^2}{2}\right) D_{\xi\xi} u(x,t), \quad (x,t) \in \overline{D}, \ |\xi| = 1,$$

where $\varphi(s) = (1 - s/M)^{-1/8}$ and $M = 2 \sup_{(x,t) \in D} |Du(x,t)|^2$.

Then we can take a point $(x_0, t_0) \in \overline{D}$ and a unit vector $\xi_0 \in \mathbb{R}^n$ which satisfy

$$\Psi(x_0, t_0; \xi_0) = \max\{\Psi(x, t; \xi) \mid (x,t) \in \overline{D}, \ |\xi| = 1\}.$$

The point (x_0, t_0) can be taken in $\overline{D} \setminus \partial_p D$ due to the boundary condition $u = 0$ on $\partial_p D$. Without loss of generality, we may assume $\xi_0 = e_1$ and $D^2 u(x_0, t_0)$ is diagonal with $D_{11} u(x_0, t_0) \geq D_{22} u(x_0, t_0) \geq \cdots \geq D_{nn} u(x_0, t_0) > 0$. Then $\Psi = \Psi(x, t; e_1) = (-u(x,t))^4 \varphi(|Du(x,t)|^2/2) D_{11} u(x,t)$ attains its maximum at (x_0, t_0) and the eigenvalues of $D^2 u(x_0, t_0)$ are $\lambda = (\lambda_1, \ldots, \lambda_n) = (u_{11}(x_0, t_0), \ldots, u_{nn}(x_0, t_0))$. It is enough to consider the case $\lambda_1 = u_{11}(x_0, t_0) \geq 1$. Here and throughout the paper, we denoted $D_i u$ by u_i, $D_{ij} u$ by u_{ij}, and so on.

Since Ψ attains its maximum at (x_0, t_0), direct calculation gives

$$(\log \Psi)_i = \frac{4u_i}{u} + \frac{\varphi_i}{\varphi} + \frac{u_{11i}}{u_{11}} = 0, \tag{3.2}$$

$$(\log \Psi)_{ii} = 4\left(\frac{u_{ii}}{u} - \frac{u_i^2}{u^2}\right) + \frac{\varphi_{ii}}{\varphi} - \frac{\varphi_i^2}{\varphi^2} + \frac{u_{11ii}}{u_{11}} - \frac{u_{11i}^2}{u_{11}^2} \leq 0, \tag{3.3}$$

$$(\log \Psi)_t = \frac{4u_t}{u} + \frac{\varphi_t}{\varphi} + \frac{u_{11t}}{u_{11}} \geq 0, \tag{3.4}$$

$$\varphi_i = \varphi'\left(\frac{|Du|^2}{2}\right)\sum_{j=1}^{n} u_j u_{ij} = \varphi'\left(\frac{|Du|^2}{2}\right) u_i u_{ii}, \tag{3.5}$$

$$\begin{aligned}\varphi_{ii} &= \varphi''\left(\frac{|Du|^2}{2}\right)\left(\sum_{j=1}^{n} u_j u_{ij}\right)^2 + \varphi'\left(\frac{|Du|^2}{2}\right)\left\{\sum_{j=1}^{n} u_{ij}^2 + \sum_{j=1}^{n} u_j u_{iij}\right\} \\ &= \varphi''\left(\frac{|Du|^2}{2}\right) u_i^2 u_{ii}^2 + \varphi'\left(\frac{|Du|^2}{2}\right)\left(u_{ii}^2 + \sum_{j=1}^{n} u_j u_{iij}\right),\end{aligned} \tag{3.6}$$

$$\varphi_t = \varphi'\left(\frac{|Du|^2}{2}\right)\sum_{j=1}^{n} u_j u_{jt} \tag{3.7}$$

at (x_0, t_0), for $i = 1, \ldots, n$. We set $f(D^2u) = F_k(D^2u)^{1/k}$, then u satisfies

$$(-u_t)^{\frac{1}{k}} f(D^2u) = 1 \quad \text{in } \overline{D}. \tag{3.8}$$

Differentiating (3.8) with respect to x_γ (and using (3.8) itself) yields

$$-\frac{1}{k}(-u_t)^{-1} u_{\gamma t} + (-u_t)^{\frac{1}{k}} f_{ij} u_{ij\gamma} = 0. \tag{3.9}$$

Here, for $f = f(M)$ where $M = (m_{ij})_{1\leq i,j\leq n}$, we write $f_{ij} = \partial f/\partial m_{ij}$. Multiplying (3.9) by $(-u_t)^{-1/k}$, differentiating once more with respect to x_γ and multiplying $(-u_t)^{1/k}$, we obtain

$$-\left(\frac{1}{k}+1\right)\frac{u_{\gamma t}^2}{k u_t^2} + \frac{u_{\gamma\gamma t}}{k u_t} + (-u_t)^{\frac{1}{k}} f_{ii} u_{ii\gamma\gamma} + (-u_t)^{\frac{1}{k}} f_{ij,rs} u_{ij\gamma} u_{rs\gamma} = 0, \tag{3.10}$$

where $f_{ij,rs} = \partial^2 f/\partial m_{ij}\partial m_{rs}$.

By the concavity of $S_k(\lambda)^{\frac{1}{k}}$, we obtain

$$\begin{aligned}\sum_{i,j=1}^{n} \frac{\partial^2}{\partial\lambda_i\partial\lambda_j}(S_k(\lambda))^{\frac{1}{k}} u_{ii\gamma} u_{jj\gamma} = \sum_{i,j=1}^{n} &\Big[\frac{1}{k}\left(\frac{1}{k}-1\right) S_k(\lambda)^{\frac{1}{k}-2} S_{k-1;i}(\lambda) S_{k-1;j}(\lambda) \\ &+ \frac{1}{k} S_k(\lambda)^{\frac{1}{k}-1} S_{k-2;ij}(\lambda)\Big] u_{ii\gamma} u_{jj\gamma} \\ \leq 0.&\end{aligned} \tag{3.11}$$

When the matrix D^2u is diagonal at (x_0, t_0), direct calculation gives

$$f_{ij,rs} = \begin{cases} \frac{1}{k}\left(\frac{1}{k}-1\right) S_k(\lambda)^{\frac{1}{k}-2} S_{k-1;i}(\lambda) S_{k-1;r}(\lambda) \\ \qquad\qquad + \frac{1}{k} S_k(\lambda)^{\frac{1}{k}-1} S_{k-2;ir}(\lambda) \quad \text{if } i=j,\ r=s, \\ -\frac{1}{k} S_k(\lambda)^{\frac{1}{k}-1} S_{k-2;ij}(\lambda) \quad \text{if } i\neq j,\ r=j, \text{ and } s=i, \\ 0 \quad \text{otherwise.} \end{cases} \tag{3.12}$$

By (3.11) and (3.12), we obtain

$$\begin{aligned} f_{ij,rs} u_{ij\gamma} u_{rs\gamma} = & \sum_{i,j=1}^{n} \Big[\frac{1}{k}\left(\frac{1}{k}-1\right) S_k(\lambda)^{\frac{1}{k}-2} S_{k-1;i}(\lambda) S_{k-1;j}(\lambda) \\ & \qquad + \frac{1}{k} S_k(\lambda)^{\frac{1}{k}-1} S_{k-2;ij}(\lambda) \Big] u_{ii\gamma} u_{jj\gamma} \\ & - \frac{1}{k} \sum_{i,j=1}^{n} S_k(\lambda)^{\frac{1}{k}-1} S_{k-2;ij}(\lambda) u_{ij\gamma}^2 \\ \le & -\frac{1}{k} \sum_{i,j=1}^{n} S_k(\lambda)^{\frac{1}{k}-1} S_{k-2;ij}(\lambda) u_{ij\gamma}^2 \end{aligned} \tag{3.13}$$

at (x_0, t_0). By using (3.10) and (3.13), we get the inequality

$$\frac{u_{\gamma\gamma t}}{k u_t} + (-u_t)^{\frac{1}{k}} f_{ii} u_{ii\gamma\gamma} \ge (-u_t)^{\frac{1}{k}} \frac{1}{k} \sum_{i,j=1}^{n} S_k(\lambda)^{\frac{1}{k}-1} S_{k-2;ij}(\lambda) u_{ij\gamma}^2$$

at (x_0, t_0). Letting $\gamma = 1$ and multiplying $1/u_{11}$, we get at (x_0, t_0)

$$\frac{u_{11t}}{k u_t u_{11}} + (-u_t)^{\frac{1}{k}} f_{ii} \frac{u_{11ii}}{u_{11}} \ge (-u_t)^{\frac{1}{k}} \frac{1}{k} \sum_{i,j=1}^{n} S_k(\lambda)^{\frac{1}{k}-1} S_{k-2;ij}(\lambda) \frac{u_{1ij}^2}{u_{11}}. \tag{3.14}$$

Let L be the linearized operator of (3.8) at (x_0, t_0). Then one can write

$$L = \frac{1}{k u_t(x_0, t_0)} D_t + (-u_t(x_0, t_0))^{\frac{1}{k}} f_{ij}(D^2 u(x_0, t_0)) D_{ij}.$$

By (3.3) and (3.4), we obtain

$$\begin{aligned} L(\log \Psi) = & (-u_t)^{\frac{1}{k}} f_{ii} \left(4\left(\frac{u_{ii}}{u} - \frac{u_i^2}{u^2}\right) + \frac{\varphi_{ii}}{\varphi} - \frac{\varphi_i^2}{\varphi^2} + \frac{u_{11ii}}{u_{11}} - \frac{u_{11i}^2}{u_{11}^2} \right) \\ & + \frac{1}{k u_t} \left(\frac{4 u_t}{u} + \frac{\varphi_t}{\varphi} + \frac{u_{11t}}{u_{11}} \right) \le 0. \end{aligned} \tag{3.15}$$

at (x_0, t_0). By substituting (3.5), (3.6), (3.7), (3.9) and (3.14) into (3.15), we obtain

$$
\begin{aligned}
&(-u_t)^{\frac{1}{k}} f_{ii}\left(4\left(\frac{u_{ii}}{u}-\frac{u_i^2}{u^2}\right)+\frac{\varphi''}{\varphi}u_i^2u_{ii}^2+\frac{\varphi'}{\varphi}u_{ii}^2-\frac{\varphi'^2}{\varphi^2}u_i^2u_{ii}^2-\frac{u_{11i}^2}{u_{11}^2}\right)\\
&\qquad+(-u_t)^{\frac{1}{k}}\frac{1}{k}\sum_{i,j=1}^{n}S_k(\lambda)^{\frac{1}{k}-1}S_{k-2;ij}(\lambda)\frac{u_{1ij}^2}{u_{11}}+\frac{4}{ku}\le 0 \qquad (3.16)
\end{aligned}
$$

at (x_0, t_0).

Now we split into two cases.

(i) $u_{kk} \ge K u_{11}$, where $K > 0$ is a small constant to be determined later.

By (3.2) and (3.5), we have

$$
\frac{u_{11i}^2}{u_{11}^2}=\left(\frac{4u_i}{u}+\frac{\varphi_i}{\varphi}\right)^2\le 2\left(\frac{16u_i^2}{u^2}+\frac{\varphi'^2u_i^2u_{ii}^2}{\varphi^2}\right) \qquad (3.17)
$$

at (x_0, t_0). Therefore (3.17) and the fact that the second term of the left hand side of (3.16) is non-negative yield

$$
(-u_t)^{\frac{1}{k}} f_{ii}\left(4\left(\frac{u_{ii}}{u}-\frac{9u_i^2}{u^2}\right)+\left(\frac{\varphi''}{\varphi}-\frac{3\varphi'^2}{\varphi^2}\right)u_i^2u_{ii}^2+\frac{\varphi'}{\varphi}u_{ii}^2\right)+\frac{4}{ku}\le 0 \qquad (3.18)
$$

at (x_0, t_0).

We prove the following inequality:

$$
\sum_{i=1}^{n} f_{ii}u_{ii}^2 > f_{kk}u_{kk}^2 \ge \theta_1\sum_{i=1}^{n} f_{ii}u_{11}^2 \qquad (3.19)
$$

for some constant $\theta_1 > 0$.

The left inequality of (3.19) follows from the fact that

$$
f_{ii}=S_k(\lambda)^{\frac{1}{k}-1}S_{k-1;i}(\lambda)>0
$$

due to $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n > 0$.

To prove the right inequality of (3.19), we need the following lemma.

Lemma 3.2. *It holds that there exists a constant $\theta > 0$ such that*

$$
\theta S_{k-1}(\mu) \le \mu_1 \cdots \mu_{k-1} \qquad (3.20)
$$

for all $\mu = (\mu_1, \ldots, \mu_n)$ with $\mu_1 \ge \mu_2 \ge \cdots \ge \mu_n > 0$.

Proof.

$$S_{k-1}(\mu) = \sum_{1\le i_1<\cdots<i_{k-1}\le n} \mu_1\cdots\mu_{i_{k-1}} \le \binom{n}{k-1}\mu_1\cdots\mu_{k-1}.$$

Hence (3.20) holds. $\square$

Therefore there exists some $\theta > 0$ such that

$$\theta S_{k-1}(\lambda) \le \lambda_1\cdots\lambda_{k-1}.$$

On the other hand,

$$f_{kk}u_{kk}^2 = \frac{1}{k}S_k(\lambda)^{\frac{1}{k}-1}S_{k-1;k}(\lambda)u_{kk}^2. \tag{3.21}$$

By using Lemma 7.2, we obtain

$$f_{kk}u_{kk}^2 \ge \frac{1}{k}S_k(\lambda)^{\frac{1}{k}-1}\tilde{\theta}\lambda_1\lambda_2\cdots\lambda_{k-1}u_{kk}^2 \tag{3.22}$$

for some constant $\tilde{\theta} > 0$.

By using the assumption $u_{kk} \ge Ku_{11}$ and (3.20) we obtain

$$\begin{aligned}
f_{kk}u_{kk}^2 &\ge \frac{1}{k}S_k(\lambda)^{\frac{1}{k}-1}\tilde{\theta}\lambda_1\lambda_2\cdots\lambda_{k-1}K^2u_{11}^2\\
&\ge \frac{1}{k}S_k(\lambda)^{\frac{1}{k}-1}\theta\tilde{\theta}S_{k-1}(\lambda)K^2u_{11}^2\\
&= \frac{\theta\tilde{\theta}K^2}{k(n-k+1)}S_k(\lambda)^{\frac{1}{k}-1}\sum_{i=1}^n S_{k-1;i}(\lambda)u_{11}^2\\
&= \theta_1\sum_{i=1}^n f_{ii}u_{11}^2
\end{aligned}$$

for some constant $\theta_1 > 0$. Here we used the equality (7.2).

By the inequality (3.19) and $\varphi''/\varphi - 3\varphi'^2/\varphi^2 \ge 0$, it can be derived by (3.18) that at (x_0, t_0)

$$(-u_t)^{\frac{1}{k}}\theta_2\sum_{i=1}^n f_{ii}u_{11}^2 - C(-u_t)^{\frac{1}{k}}\frac{1}{u^2}\sum_{i=1}^n f_{ii} + \frac{4}{u}\left(1+\frac{1}{k}\right) \le 0,$$

for some constant $\theta_2 > 0$. Here we used the fact that $\sum_{i=1}^n f_{ii}(D^2u)u_{ii} = f(D^2u) = (-u_t)^{-1/k}$ at (x_0, t_0), due to the homogeneity of f and (3.8). By multiplying $(-u)^8\varphi^2$, we obtain

$$(-u_t)^{\frac{1}{k}}\theta_2 \sum_{i=1}^n f_{ii}u_{11}^2(-u)^8\varphi^2 - C(-u_t)^{\frac{1}{k}}(-u)^6\varphi^2 \sum_{i=1}^n f_{ii} - 4\left(1+\frac{1}{k}\right)(-u)^7\varphi^2 \leq 0. \tag{3.23}$$

On the other hand, it holds that at (x_0, t_0)

$$\begin{aligned} \sum_{i=1}^n f_{ii}(D^2u) &= \sum_{i=1}^n \frac{1}{k} F_k(D^2u)^{\frac{1}{k}-1} \frac{\partial F_k}{\partial m_{ii}}(D^2u) \\ &= \frac{1}{k}(-u_t)^{1-\frac{1}{k}} \sum_{i=1}^n S_{k-1;i}(\lambda) \leq C(-u_t)^{1-\frac{1}{k}} u_{11}^{k-1}, \end{aligned} \tag{3.24}$$

and that

$$\begin{aligned} \sum_{i=1}^n f_{ii}(D^2u) &\geq f_{nn}(D^2u) \\ &= \frac{1}{k} F_k(D^2u)^{\frac{1}{k}-1} \frac{\partial F_k}{\partial m_{nn}}(D^2u) \\ &\geq \frac{1}{k} F_k(D^2u)^{\frac{1}{k}-1} \theta_3 u_{11} \cdots u_{k-1,k-1} \geq C(-u_t)^{1-\frac{1}{k}} u_{11}^{k-1}, \end{aligned} \tag{3.25}$$

for some constant $\theta_3 > 0$ (see Lemma 7.2), by the hypothesis $u_{kk} \geq Ku_{11}$. Substituting (3.24) and (3.25) into (3.23), we obtain

$$\Psi^2 \leq C(-u)^6\varphi^2 + \frac{(-u)^7\varphi^2}{(-u_t)u_{11}^{k-1}} \leq C(n, k, m_2, \|u\|_{C^1(D)}),$$

at (x_0, t_0). Therefore, for all $(x, t) \in D$ and $\xi \in \mathbb{R}^n$ with $|\xi| = 1$, $(-u)^4 u_{\xi\xi} \leq C$ holds, so that $(-u)^4|D^2u|$ can be estimated from above by some constant C.

(ii) $u_{kk} \leq Ku_{11}$, that is, $u_{jj} \leq Ku_{11}$ for $j = k, k+1, \ldots, n$.

By (3.2),

$$\frac{u_{111}}{u_{11}} = -\left(\frac{\varphi_1}{\varphi} + \frac{4u_1}{u}\right), \quad \frac{u_i}{u} = -\frac{1}{4}\left(\frac{\varphi_i}{\varphi} + \frac{u_{11i}}{u_{11}}\right), \quad i = 2, \ldots, n \tag{3.26}$$

at (x_0,t_0). Substituting (3.26) into (3.16), we obtain

$$
\begin{aligned}
0 \geq\ & (-u_t)^{\frac{1}{k}} f_{11}\left(4\left(\frac{u_{11}}{u}-\frac{u_1^2}{u^2}\right)+\frac{\varphi''}{\varphi}u_1^2u_{11}^2+\frac{\varphi'}{\varphi}u_{11}^2-\frac{\varphi'^2}{\varphi^2}u_1^2u_{11}^2-\left(\frac{\varphi_1}{\varphi}+\frac{4u_1}{u}\right)^2\right)\\
&+(-u_t)^{\frac{1}{k}}\sum_{i=2}^n f_{ii}\left(\frac{4u_{ii}}{u}-\frac{1}{4}\left(\frac{\varphi_i}{\varphi}+\frac{u_{11i}}{u_{11}}\right)^2+\frac{\varphi''}{\varphi}u_i^2u_{ii}^2+\frac{\varphi'}{\varphi}u_{ii}^2-\frac{\varphi'^2}{\varphi^2}u_i^2u_{ii}^2-\frac{u_{11i}^2}{u_{11}^2}\right)\\
&+(-u_t)^{\frac{1}{k}}\frac{1}{k}\sum_{i,j=1}^n S_k(\lambda)^{\frac{1}{k}-1}S_{k-2;ij}(\lambda)\frac{u_{1ij}^2}{u_{11}}+\frac{4}{ku}\\
\geq\ & \left[(-u_t)^{\frac{1}{k}}\sum_{i=1}^n f_{ii}\left(\frac{4u_{ii}}{u}+\left(\frac{\varphi''}{\varphi}-\frac{3\varphi'^2}{\varphi^2}\right)u_i^2u_{ii}^2+\frac{\varphi'}{\varphi}u_{ii}^2\right)-36(-u_t)^{\frac{1}{k}}f_{11}\frac{u_1^2}{u^2}\right]\\
&+\left[-\frac{3}{2}(-u_t)^{\frac{1}{k}}\sum_{i=2}^n f_{ii}\frac{u_{11i}^2}{u_{11}^2}+(-u_t)^{\frac{1}{k}}\frac{1}{k}\sum_{i,j=1}^n S_k(\lambda)^{\frac{1}{k}-1}S_{k-2;ij}(\lambda)\frac{u_{1ij}^2}{u_{11}}\right]+\frac{4}{ku}\\
=:\ & I_1+I_2+\frac{4}{ku}, \qquad (3.27)
\end{aligned}
$$

at (x_0,t_0). First, I_1 can be estimated from below as

$$
\begin{aligned}
I_1 &\geq (-u_t)^{\frac{1}{k}}\theta_1 f_{11}u_{11}^2+\frac{4}{u}-C(-u_t)^{\frac{1}{k}}\frac{f_{11}}{u^2}\\
&\geq (-u_t)^{\frac{1}{k}}\frac{1}{2}\theta_1 f_{11}u_{11}^2+\frac{4}{u}, \qquad (3.28)
\end{aligned}
$$

provided $u(x_0,t_0)^2u_{11}(x_0,t_0)^2 \geq 2C/\theta_1$. If $u(x_0,t_0)^2u_{11}(x_0,t_0)^2 < 2C/\theta_1$, then (3.1) is obvious. Hence we may assume $u(x_0,t_0)^2u_{11}(x_0,t_0)^2 \geq 2C/\theta_1$ hereafter. Second, I_2 can be also estimated from below as

$$
\begin{aligned}
I_2 &\geq -\frac{3}{2}(-u_t)^{\frac{1}{k}}\frac{1}{k}S_k(\lambda)^{\frac{1}{k}-1}\sum_{i=2}^n S_{k-1;i}(\lambda)\frac{u_{11i}^2}{u_{11}^2}+2(-u_t)^{\frac{1}{k}}\frac{1}{k}S_k(\lambda)^{\frac{1}{k}-1}\sum_{i=2}^n S_{k-2;1i}(\lambda)\frac{u_{11i}^2}{u_{11}}\\
&= 2(-u_t)^{\frac{1}{k}}\frac{1}{k}S_k(\lambda)^{\frac{1}{k}-1}\left(\sum_{i=2}^n\left(S_{k-2;1i}(\lambda)-\frac{3}{4}\frac{S_{k-1;i}(\lambda)}{\lambda_1}\right)\frac{u_{11i}^2}{\lambda_1}\right)\geq 0, \qquad (3.29)
\end{aligned}
$$

by using $\lambda_1 S_{k-2;1i}(\lambda) \geq 3S_{k-1;i}(\lambda)/4$ provided $K>0$ is sufficiently small (see Lemma 7.3 (ii)).

Substituting (3.28) and (3.29) into (3.27), we obtain

$$
0 \geq (-u_t)^{\frac{1}{k}}\frac{1}{2}\theta_1 f_{11}u_{11}^2+\frac{4}{u}\left(1+\frac{1}{k}\right). \qquad (3.30)
$$

By multiplying $(-u)^4\varphi$, we get

$$
0 \geq \frac{1}{2}\theta_1(-u_t)^{\frac{1}{k}}f_{11}u_{11}^2(-u)^4\varphi-C(-u)^3\varphi.
$$

It follows from Lemma 7.3 (i) that $\lambda_1 S_{k-1;1}(\lambda) \geq \theta_4 S_k(\lambda)$ for some constant $\theta_4 > 0$, which implies that

$$f_{11}u_{11}^2 = \frac{1}{k}S_k(\lambda)^{\frac{1}{k}-1}S_{k-1;1}(\lambda)\lambda_1^2 \geq \frac{\theta_4}{k}S_k(\lambda)^{\frac{1}{k}}\lambda_1 = \frac{\theta_4}{k}(-u_t)^{-\frac{1}{k}}u_{11}. \tag{3.31}$$

Hence the inequality

$$0 \geq \frac{\theta_1\theta_4}{2k}(-u)^4\varphi u_{11} - C(-u)^3\varphi$$

holds at (x_0, t_0). Then we have

$$\Psi \leq C(-u)^3\varphi \leq C(n, k, m_2, \|u\|_{C^1(D)}),$$

at (x_0, t_0). Therefore, $(-u)^4|D^2u|$ can be estimated from above by some constant C. □

4 Proof of Theorem 2.1

Before giving a proof of Theorem 2.1, we introduce some notation. For a subset $D \subset \mathbb{R}^n \times (-\infty, 0]$, a function v defined on D and $\alpha \in (0,1)$, α-Hölder seminorm of v over D is denoted by

$$[v]_{\alpha,D} = \sup_{\substack{(x,t),(y,s)\in D,\\ (x,t)\neq(y,s)}} \frac{|v(x,t) - v(y,s)|}{(|x-y|^2 + |t-s|)^{\frac{\alpha}{2}}}. \tag{4.1}$$

Moreover, $\mathbb{S}^{n\times n}$ is defined to be the set of all symmetric $n \times n$ matrices, and $\mathbb{S}_+^{n\times n}$ is the set of all non-negative definite symmetric $n \times n$ matrices.

Let $u \in C^{4,2}(\mathbb{R}^n \times (-\infty, 0])$ be a strictly convex-monotone solution to (1.9), which satisfies the growth conditions (2.1) and (2.2). We may assume without loss of generality that $u(0,0) = 0$, $Du(0,0) = 0$, by considering $u(x,t) - u(0,0) - Du(0,0) \cdot x$ instead of $u(x,t)$. Then it can be seen by (2.2) that there exists a constant $\tilde{A} > 0$ such that $u(x,0) \geq \tilde{A}|x|^2$ for all $x \in \mathbb{R}^n$,

Let $R > 0$ be fixed. We define $v(x,t) = v_R(x,t) = u(Rx, R^2t)/R^2$. Then v is also a strictly convex-monotone classical solution to (1.9), and satisfies $v_t(x,t) = u_t(Rx, R^2t)$ and $v_{ij}(x,t) = u_{ij}(Rx, R^2t)$. Moreover, it holds that for all $(x,t) \in \mathbb{R}^n \times (-\infty, 0]$,

$$-m_1 \leq v_t(x,t) \leq -m_2, \tag{4.2}$$

and that for all $x \in \mathbb{R}^n$,

$$v(x,0) \geq \tilde{A}|x|^2. \tag{4.3}$$

First, we shall obtain the local gradient estimate of the solution v. For $q > 0$, we set

$$\Omega_q = \{(x,t) \in \mathbb{R}^n \times (-\infty, 0] \mid v(x,t) < \tilde{A}q\}. \tag{4.4}$$

Then we can find that Ω_q is a bounded bowl-shaped domain and

$$\Omega_q(t) \subset \Omega_q(0) \subset B(0, \sqrt{q}), \tag{4.5}$$

due to (4.2), (4.3) and the strict parabolic-monotonicity of v. Now we establish the following estimate.

Lemma 4.1. *Let v and Ω_q be defined as above. Then there exists a constant $C > 0$, independent of q and R, such that for all $(x,t) \in \Omega_q$,*

$$|Dv(x,t)| \leq C\sqrt{q}. \tag{4.6}$$

Proof. We note that $v(x,t)$ is strictly convex in x, and that $v(x,t) - \tilde{A}q = 0$ on $\partial_p \Omega_q$. From Newton-Maclaurin inequality it follows that $(F_k(M)/\binom{n}{k})^{1/k} \geq F_n(M)^{1/n}$ for all $M \in \mathbb{S}^{n\times n}_+$.

By Aleksandrov's maximum principle (see Theorem 7.7), we obtain that at $(x_0, t) \in \Omega_q$,

$$\begin{aligned}
|v(x_0,t) - \tilde{A}q|^n &\leq C\left(\operatorname{diam}\Omega_q(t)\right)^{n-1} \operatorname{dist}(x_0, \partial\Omega_q(t)) |\partial v(\Omega_q(t))| \\
&\leq C(2\sqrt{q})^{n-1} \operatorname{dist}(x_0, \partial\Omega_q(t)) \int_{\Omega_q(t)} \det D^2 v(x,t)\, dx \\
&\leq Cq^{\frac{n-1}{2}} \operatorname{dist}(x_0, \partial\Omega_q(t)) \int_{\Omega_q(t)} F_k(D^2 v(x,t))^{\frac{n}{k}}\, dx \\
&= Cq^{\frac{n-1}{2}} \operatorname{dist}(x_0, \partial\Omega_q(t)) \int_{\Omega_q(t)} (-v_t)^{-\frac{n}{k}}\, dx \\
&\leq Cq^{\frac{n-1}{2}} \operatorname{dist}(x_0, \partial\Omega_q(t)) \cdot m_2^{-\frac{n}{k}} |B(0,\sqrt{q})| \\
&= Cq^{n-\frac{1}{2}} \operatorname{dist}(x_0, \partial\Omega_q(t)),
\end{aligned} \tag{4.7}$$

so that

$$|v(x_0,t) - \tilde{A}q| \leq Cq^{1-\frac{1}{2n}} \operatorname{dist}(x_0, \partial\Omega_q(t))^{\frac{1}{n}}. \tag{4.8}$$

Therefore for all $x_0 \in \Omega_{q/2}(t)$,

$$\tilde{A}q - \frac{1}{2}\tilde{A}q \leq \tilde{A}q - v(x_0, t) \leq Cq^{1-\frac{1}{2n}} \mathrm{dist}(x_0, \partial\Omega_q(t))^{\frac{1}{n}},$$

which implies the inequality

$$\mathrm{dist}(\Omega_{\frac{q}{2}}(t), \partial\Omega_q(t)) \geq Cq^{\frac{1}{2}}. \tag{4.9}$$

Therefore we can see that $|Dv(x,t)| \leq Cq^{1/2}$ for all $(x,t) \in \Omega_{q/2}$ by (4.9) and the convexity of v with respect to x. This ends the proof. $\square$

Especially, $|Dv(x,t)| \leq C$ for all $(x,t) \in \Omega_1$, in which C is independent of R. By applying (3.1) to the function $\tilde{A} - v(x,t)$, one obtains

$$\left(\tilde{A} - v(x,t)\right)^4 |D^2 v(x,t)| \leq C$$

in Ω_1. This implies that

$$|D^2 v(x,t)| \leq C \quad \text{in } \Omega_{1/2}. \tag{4.10}$$

The following Evans-Krylov type theorem is needed for the proof of Theorem 2.1. For the proof, see Section 7.

Theorem 4.2. *(Evans-Krylov type theorem)*

Let D and D' be bounded bowl-shaped domains which satisfy $D' \subset D$ and $\mathrm{dist}(D', \partial_p D) > 0$, and u be a $C^{4,2}(D)$ solution to the equation

$$G(u_t, D^2 u) = 0 \tag{4.11}$$

in D, where $G = G(q, M)$ is defined for all $(q, M) \in \mathbb{R} \times \mathbb{S}^{n\times n}$ with $G(\cdot, M) \in C^1(\mathbb{R})$ for each $M \in \mathbb{S}^{n\times n}$, and $G \in C^2(\mathbb{R} \times X)$ for some $X \subset \mathbb{S}^{n\times n}$ which is a neighborhood of $D^2u(D)$. Suppose that:

(i) G is uniformly parabolic, i.e., there exist positive constants λ and Λ such that

$$-\Lambda \leq G_q(q, M) \leq -\lambda, \tag{4.12}$$

$$\lambda\|N\| \leq G(q, M+N) - G(q, M) \leq \Lambda\|N\|, \tag{4.13}$$

for all $q \in \mathbb{R}$ and M, $N \in \mathbb{S}^{n\times n}$ with $N \geq O$.

(ii) G is concave with respect to M.

If $\|u\|_{C^{2,1}(D)} \leq K$, then there exist positive constants C depending on λ, Λ, n, K, D, D' and $G(0,0)$, and $\alpha \in (0,1)$ depending on λ, Λ and n such that

$$\|u\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(D')} \leq C.$$

Then we prove the next lemma in order to use Theorem 4.2.

Lemma 4.3. *There exists a constant $C > 0$, independent of R, such that*

$$\operatorname{dist}(\Omega_{\frac{1}{8}}, \partial_p \Omega_{\frac{1}{2}}) \geq C. \tag{4.14}$$

Proof. Take $(x,t) \in \Omega_{1/8}$ arbitrarily. Then, putting $q = 1/4$ in (4.9), we obtain

$$\operatorname{dist}(\Omega_{\frac{1}{8}}(t), \partial\Omega_{\frac{1}{4}}(t)) \geq C', \tag{4.15}$$

where C' is a positive constant independent of R. We set $\delta = \min\{\tilde{A}/(4m_1), C'\}$. If $\operatorname{dist}((x,t),(x',t')) < \delta$, then $|x - x'| < C'$ and $|t - t'| < \tilde{A}/(4m_1)$, which imply that

$$\begin{aligned} v(x',t') &= v(x',t) + \int_t^{t'} v_t(x',s)ds \\ &\leq v(x',t) + m_1|t-t'| \\ &\leq \frac{1}{4}\tilde{A} + m_1 \cdot \frac{\tilde{A}}{4m_1} = \frac{1}{2}\tilde{A}, \end{aligned}$$

due to (4.15). Therefore $(x',t') \in \overline{\Omega_{1/2}}$ and this completes the proof. $\square$

We set $G(q,M) = (-q)^{1/k}F_k(M)^{1/k} - 1 = (-q)^{1/k}f(M) - 1$ for $(q,M) \in [-m_1, -m_2] \times X$, where

$$X = \Big\{ M = (m_{ij}) \in \mathbb{S}_+^{n\times n} \,\Big|\, \frac{1}{m_1} \leq F_k(M) \leq \frac{1}{m_2}, \\ |m_{ij}| \leq C \text{ for } i,j = 1,\dots,n \Big\},$$

in which C is a constant appeared in (4.10).

Since $G_q(q,M) = (-q)^{1/k-1}F_k(M)^{1/k}/k$, we see that there exist constants $\lambda, \Lambda > 0$ such that (4.12) holds in $[-m_1,-m_2] \times X$. Moreover, we can also see that (4.13) and (ii) in Theorem 4.2 hold in $[-m_1,-m_2] \times X$, due to [8].

Next we can extend G in $\mathbb{R} \times \mathbb{S}^{n\times n}$ so that G satisfies (i) and (ii) in Theorem 4.2 for different constants $\lambda, \Lambda > 0$ if necessary. Then we apply Theorem 4.2 to $G(v_t, D^2 v) = 0$ in $\Omega_{1/2}$ and obtain that

$$\|v\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\Omega_{\frac{1}{8}})} \leq C.$$

Therefore it follows that $[D_{ij}v]_{\alpha,\Omega_{1/8}} \leq C$ for $i, j = 1, \dots, n$ and $[v_t]_{\alpha,\Omega_{1/8}} \leq C$.

By substituting $v(x,t) = u(Rx, R^2 t)/R^2$, we have

$$[D_{ij}u]_{\alpha,\{u(x,t)<\frac{\tilde{A}}{8}R^2\}} \leq CR^{-\alpha}, \tag{4.16}$$

$$[u_t]_{\alpha,\{u(x,t)<\frac{\tilde{A}}{8}R^2\}} \leq CR^{-\alpha}, \tag{4.17}$$

for any $R > 0$. This implies that for any bounded subset Ω of $\mathbb{R}^n \times (-\infty, 0]$, $[D_{ij}u]_{\alpha,\Omega} = 0$, and $[u_t]_{\alpha,\Omega} = 0$. Hence $D_{ij}u$ and u_t are constants in $\mathbb{R}^n \times (-\infty, 0]$ and this completes the proof of Theorem 2.1.

5 Recent progress

Up to this point, we considered the parabolic k-Hessian equation of the form $-u_t F_k(D^2u) = 1$, and obtained Bernstein type theorem for this equation. In this section, we deal with other forms of parabolic k-Hessian equations. There are different parabolic analogues of k-Hessian equation which have been studied in the literature.

Ivochkina and Ladyzhenskaya [22] have studied the solvability of the first initial boundary value problem for

$$-u_t + F_k(D^2u)^{\frac{1}{k}} = \psi. \tag{5.1}$$

X.J. Wang [48] considered a following version of parabolic equation,

$$-u_t + \log F_k(D^2u) = \psi. \tag{5.2}$$

For the case $k = n$, (5.2) reduces to

$$-u_t + \log \det D^2 u = \psi, \tag{5.3}$$

which was studied by G. Wang and W. Wang [47]. Moreover,

$$S_k(-u_t, \lambda_1, \ldots, \lambda_n) = \psi, \tag{5.4}$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of D^2u, i.e., $-u_t F_{k-1}(D^2u) + F_k(D^2u) = \psi$, was considered in [29].

Therefore it seems natural to study whether Bernstein type theorems for more general parabolic k-Hessian equations hold. We obtain the following theorem.

Theorem 5.1. *Let $\rho \in C^2(0,\infty)$, $1 \le k \le n$ and $u \in C^{4,2}(\mathbb{R}^n \times (-\infty, 0])$ be a strictly convex-monotone solution to*

$$u_t = \rho\left(F_k(D^2u)^{\frac{1}{k}}\right) \quad \text{in } \mathbb{R}^n \times (-\infty, 0]. \tag{5.5}$$

Suppose that there exist constants $m_1 \ge m_2 > 0$ such that for all $(x,t) \in \mathbb{R}^n \times (-\infty, 0]$,

$$-m_1 \le u_t(x,t) \le -m_2, \tag{5.6}$$

and that there exist constants $A, B > 0$ such that for all $x \in \mathbb{R}^n$,

$$u(x,0) \ge A|x|^2 - B. \tag{5.7}$$

Moreover, suppose that for all $s \in (0,\infty)$,

$$\rho'(s) > 0, \quad \rho''(s) \le 0, \tag{5.8}$$

and that

$$\rho^{-1}([-m_1, -m_2]) = [r_1, r_2] \tag{5.9}$$

for some positive constants r_1, r_2, where m_1 and m_2 are constants appeared in (5.6).

Then, u has the form $u(x,t) = -mt + p(x)$ where $m > 0$ is a constant and p is a quadratic polynomial.

Remark 5.1. Set $\widetilde{F}(M) = \rho(F_k(M)^{1/k}) = \rho(f(M))$. Then the condition (5.8) guarantees that $\widetilde{F}$ is concave in $\mathbb{S}_+^{n\times n}$. Indeed, easy calculation shows that

$$\widetilde{F}_{ij,rs} = \rho'' f_{ij} f_{rs} + \rho' f_{ij,rs},$$

which yields that for all $\xi = (\xi_{ij})_{\substack{1\le i\le n\\ 1\le j\le n}} \in \mathbb{R}^{n\times n}$,

$$\widetilde{F}_{ij,rs}\xi_{ij}\xi_{rs} = \rho'' \left(\sum_{i,j=1}^{n} f_{ij}\xi_{ij} \right)^2 + \rho' f_{ij,rs}\xi_{ij}\xi_{rs} \le 0,$$

due to the concavity of f in $\mathbb{S}_{+}^{n\times n}$.

Proof. We shall prove Pogolerov type lemma for (5.5):

> *Let D be a bounded bowl-shaped domain in $\mathbb{R}^n \times (-\infty, 0]$ and $u \in C^{4,2}(\overline{D})$ a strictly convex-monotone solution to $u_t = \rho(F_k(D^2u)^{1/k})$ in D with $u = 0$ on $\partial_p D$, which satisfies* (5.6) *in D. Suppose that ρ satisfies* (5.8) *for all $s \in (0,\infty)$ and* (5.9) *for some positive constants r_1, r_2. Then there exists a constant $C = C(n, k, m_1, m_2, \rho, \|u\|_{C^1(D)})$ such that*
>
> $$\sup_{(x,t)\in D} |u(x,t)|^4 |D^2u(x,t)| \le C. \tag{5.10}$$

The function u satisfies

$$-u_t + \rho(f(D^2u)) = 0 \quad \text{in } \overline{D}, \tag{5.11}$$

where $f(M) = F_k(M)^{1/k}$. Differentiating (5.11) with respect to x_γ yields

$$-u_{\gamma t} + \rho'(f(D^2u)) f_{ij} u_{ij\gamma} = 0. \tag{5.12}$$

Differentiating once more with respect to x_γ, we obtain

$$\begin{aligned} -u_{\gamma\gamma t} + \rho''(f(D^2u))(f_{ij}u_{ij\gamma})^2 + \rho'(f(D^2u)) f_{ij} u_{ij\gamma\gamma} \\ + \rho'(f(D^2u)) f_{ij,rs} u_{ij\gamma} u_{rs\gamma} = 0. \end{aligned} \tag{5.13}$$

As before, we consider the auxiliary function

$$\Psi(x,t;\xi) = (-u(x,t))^4 \varphi\left(\frac{|Du(x,t)|^2}{2} \right) D_{\xi\xi}u(x,t), \quad (x,t) \in \overline{D},\ |\xi| = 1,$$

where $\varphi(s) = (1 - s/M)^{-1/8}$ and $M = 2\sup_{(x,t)\in D} |Du(x,t)|^2$. Then we can take a point $(x_0, t_0) \in \overline{D} \setminus \partial_p D$ and a unit vector $\xi_0 \in \mathbb{R}^n$ which satisfy

$$\Psi(x_0, t_0; \xi_0) = \max\{\Psi(x,t;\xi) \mid (x,t) \in \overline{D},\ |\xi| = 1\}.$$

Rotating the coordinates appropriately, we may take $\xi_0 = e_1$ and $D^2u(x_0, t_0)$ is diagonal with $u_{11}(x_0, t_0) \geq u_{22}(x_0, t_0) \geq \cdots \geq u_{nn}(x_0, t_0) > 0$. Then $\Psi = \Psi(x, t; e_1) = (-u(x, t))^4\varphi(|Du(x, t)|^2/2)u_{11}(x, t)$ attains its maximum at (x_0, t_0). It is enough to consider the case $\lambda_1 = u_{11}(x_0, t_0) \geq 1$.

Letting $\gamma = 1$ in (5.13) and using (3.13) and (5.8), we get at (x_0, t_0)

$$-u_{11t} + \rho' f_{ii}u_{11ii} \geq \frac{1}{k}\rho' \sum_{i,j=1}^{n} S_k(\lambda)^{\frac{1}{k}-1}S_{k-2;ij}(\lambda)u_{1ij}^2. \tag{5.14}$$

Let L be the linearized operator of (5.11) at (x_0, t_0):

$$L = -D_t + \rho'(f(D^2u(x_0, t_0)))f_{ij}(D^2u(x_0, t_0))D_{ij}.$$

By (3.3) and (3.4), we obtain

$$\begin{aligned} L(\log\Psi) = &-\left(\frac{4u_t}{u} + \frac{\varphi_t}{\varphi} + \frac{u_{11t}}{u_{11}}\right) \\ &+ \rho' f_{ii}\left(4\left(\frac{u_{ii}}{u} - \frac{u_i^2}{u^2}\right) + \frac{\varphi_{ii}}{\varphi} - \frac{\varphi_i^2}{\varphi^2} + \frac{u_{11ii}}{u_{11}} - \frac{u_{11i}^2}{u_{11}^2}\right) \leq 0 \end{aligned} \tag{5.15}$$

at (x_0, t_0). By substituting (3.5), (3.6), (3.7), (5.12) and (5.14) into (5.15), we obtain

$$\begin{aligned} &-\frac{4u_t}{u} + \frac{1}{k}\rho' \sum_{i,j=1}^{n} S_k(\lambda)^{\frac{1}{k}-1}S_{k-2;ij}(\lambda)\frac{u_{1ij}^2}{u_{11}} \\ &+ \rho' f_{ii}\left(4\left(\frac{u_{ii}}{u} - \frac{u_i^2}{u^2}\right) + \frac{\varphi''}{\varphi}u_i^2u_{ii}^2 + \frac{\varphi'}{\varphi}u_{ii}^2 - \frac{\varphi'^2}{\varphi^2}u_i^2u_{ii}^2 - \frac{u_{11i}^2}{u_{11}^2}\right) \leq 0 \end{aligned} \tag{5.16}$$

at (x_0, t_0).

Now we split into two cases.

(i) $u_{kk} \geq Ku_{11}$, where $K > 0$ is a small constant to be determined later.

From (3.17) and the fact that the second term of the left hand side of (5.16) is non-negative, it follows that

$$-\frac{4u_t}{u} + \rho' f_{ii}\left(4\left(\frac{u_{ii}}{u} - \frac{9u_i^2}{u^2}\right) + \left(\frac{\varphi''}{\varphi} - \frac{3\varphi'^2}{\varphi^2}\right)u_i^2u_{ii}^2 + \frac{\varphi'}{\varphi}u_{ii}^2\right) \leq 0 \tag{5.17}$$

at (x_0, t_0). Since (3.19) holds for some constant $\theta_1 > 0$, $\varphi''/\varphi - 3\varphi'^2/\varphi^2 \geq 0$ and $\sum_{i=1}^{n} f_{ii}(D^2u)u_{ii} = f(D^2u)$ at (x_0, t_0), it can be derived by (5.17) that at (x_0, t_0)

$$-\frac{4u_t}{u} + \rho'\left(\theta_2 \sum_{i=1}^{n} f_{ii}u_{11}^2 + \frac{4}{u}f(D^2u) - \frac{C\rho'(r_1)}{u^2}\sum_{i=1}^{n} f_{ii}\right) \leq 0,$$

for some constant $\theta_2 > 0$. We see that $f(D^2u) = \rho^{-1}(u_t) \in [r_1, r_2]$ in D which implies that $\rho'(f(D^2u)) \in [\rho'(r_2), \rho'(r_1)]$ (we note that ρ' is non-increasing). Therefore we obtain at (x_0, t_0)

$$\frac{4(m_1 + r_2\rho'(r_1))}{u} + \rho'(r_2)\theta_2 \sum_{i=1}^{n} f_{ii}u_{11}^2 - \frac{C\rho'(r_1)}{u^2} \sum_{i=1}^{n} f_{ii} \leq 0. \tag{5.18}$$

It holds that at (x_0, t_0)

$$\begin{aligned}\sum_{i=1}^{n} f_{ii}(D^2u) &= \sum_{i=1}^{n} \frac{1}{k} F_k(D^2u)^{\frac{1}{k}-1} \frac{\partial F_k}{\partial m_{ii}}(D^2u) \\ &= \frac{1}{k}\left(\rho^{-1}(u_t)\right)^{1-k} \sum_{i=1}^{n} S_{k-1;i}(\lambda) \leq Cu_{11}^{k-1},\end{aligned} \tag{5.19}$$

and that

$$\begin{aligned}\sum_{i=1}^{n} f_{ii}(D^2u) &\geq f_{nn}(D^2u) \\ &= \frac{1}{k} F_k(D^2u)^{\frac{1}{k}-1} \frac{\partial F_k}{\partial m_{nn}}(D^2u) \\ &\geq \frac{1}{k}\left(\rho^{-1}(u_t)\right)^{1-k} \theta_3 u_{11} \cdots u_{k-1,k-1} \geq cu_{11}^{k-1},\end{aligned} \tag{5.20}$$

for some constants $\theta_3, c > 0$, by Lemma 7.2 and the hypothesis $u_{kk} \geq Ku_{11}$. Substituting (5.19) and (5.20) into (5.18), we obtain

$$\theta u_{11}^{k+1} + \frac{C}{u} - \frac{C}{u^2} u_{11}^{k-1} \leq 0, \tag{5.21}$$

for some constant $\theta > 0$. Multiplying $(-u)^8\varphi^2 u_{11}^{-(k-1)}/\theta$ by (5.21), we obtain

$$(-u)^8\varphi^2 u_{11}^2 \leq C\frac{(-u)^7\varphi^2}{u_{11}^{k-1}} + C(-u)^6\varphi^2,$$

from which follows that $\Psi^2 \leq C(n, k, m_1, m_2, \rho, \|u\|_{C^1(D)})$ at (x_0, t_0).

(ii) $u_{kk} \leq Ku_{11}$, that is, $u_{jj} \leq Ku_{11}$ for $j = k, k+1, \ldots, n$.

Substituting (3.26) into (5.16), we obtain

$$
\begin{aligned}
0 \geq{}& \rho' f_{11}\left(4\left(\frac{u_{11}}{u}-\frac{u_1^2}{u^2}\right)+\frac{\varphi''}{\varphi}u_1^2u_{11}^2+\frac{\varphi'}{\varphi}u_{11}^2-\frac{\varphi'^2}{\varphi^2}u_1^2u_{11}^2-\left(\frac{\varphi_1}{\varphi}+\frac{4u_1}{u}\right)^2\right)\\
&+\rho'\sum_{i=2}^n f_{ii}\left(\frac{4u_{ii}}{u}-\frac{1}{4}\left(\frac{\varphi_i}{\varphi}+\frac{u_{11i}}{u_{11}}\right)^2+\frac{\varphi''}{\varphi}u_i^2u_{ii}^2+\frac{\varphi'}{\varphi}u_{ii}^2-\frac{\varphi'^2}{\varphi^2}u_i^2u_{ii}^2-\frac{u_{11i}^2}{u_{11}^2}\right)\\
&+\rho'\frac{1}{k}\sum_{i,j=1}^n S_k(\lambda)^{\frac{1}{k}-1}S_{k-2;ij}(\lambda)\frac{u_{1ij}^2}{u_{11}}-\frac{4u_t}{u}\\
\geq{}& \rho'\left[\sum_{i=1}^n f_{ii}\left(\frac{4u_{ii}}{u}+\left(\frac{\varphi''}{\varphi}-\frac{3\varphi'^2}{\varphi^2}\right)u_i^2u_{ii}^2+\frac{\varphi'}{\varphi}u_{ii}^2\right)-36f_{11}\frac{u_1^2}{u^2}\right]\\
&+\rho'\left[-\frac{3}{2}\sum_{i=2}^n f_{ii}\frac{u_{11i}^2}{u_{11}^2}+\frac{1}{k}\sum_{i,j=1}^n S_k(\lambda)^{\frac{1}{k}-1}S_{k-2;ij}(\lambda)\frac{u_{1ij}^2}{u_{11}}\right]-\frac{4u_t}{u}\\
=:{}& I_1+I_2-\frac{4u_t}{u}
\end{aligned}
\tag{5.22}
$$

at (x_0,t_0). I_1 can be estimated from below as

$$
\begin{aligned}
I_1 &\geq \rho'\left[\frac{4\rho^{-1}(u_t)}{u}+\theta_1 f_{11}u_{11}^2-\frac{C}{u^2}f_{11}\right]\\
&\geq \rho'\left[\frac{1}{2}\theta_1 f_{11}u_{11}^2-\frac{4\rho^{-1}(u_t)}{u}\right],
\end{aligned}
\tag{5.23}
$$

provided $u(x_0,t_0)^2u_{11}(x_0,t_0)^2 \geq 2C/\theta_1$. If $u(x_0,t_0)^2u_{11}(x_0,t_0)^2 < 2C/\theta_1$, then (5.10) is obvious. I_2 can be estimated by 0 from below, provided $K>0$ is sufficiently small, as in Section 3. Therefore (5.22) yields

$$
f_{11}u_{11}^2 \leq -\frac{C}{u} \tag{5.24}
$$

at (x_0,t_0). On the other hand, it holds that

$$
f_{11}u_{11}^2=\frac{1}{k}S_k(\lambda)^{\frac{1}{k}-1}S_{k-1;1}(\lambda)\lambda_1^2 \geq \frac{\theta_4}{k}S_k(\lambda)^{\frac{1}{k}}\lambda_1=\frac{\theta_4}{k}\rho^{-1}(u_t)u_{11}, \tag{5.25}
$$

for some constant θ_4, by Lemma 7.3 (i). Combining (5.24) and (5.25), we obtain

$$
u_{11} \leq -\frac{C}{u} \tag{5.26}
$$

at (x_0,t_0). Multiplying $(-u)^4\varphi$ by (5.26), we get

$$
(-u)^4\varphi u_{11} \leq C(-u)^3\varphi,
$$

from which follows that $\Psi \le C(n, k, m_1, m_2, \rho, \|u\|_{C^1(D)})$ at (x_0, t_0).

Hence (5.10) is proved. The rest of the proof of Theorem 5.1 is similar to that of Theorem 2.1, so we omit it. $\square$

Example 5.1. (1) If we set $\rho(s) = -s^{-k}$, then the equation (5.5) reduces to (1.9). Therefore we can obtain Theorem 2.1 again.

(2) If we set $\rho(s) = -1/s$, then we can obtain Bernstein type theorem for

$$-u_t F_k(D^2u)^{\frac{1}{k}} = 1 \quad \text{in } \mathbb{R}^n \times (-\infty, 0].$$

(3) If we set $\rho(s) = k \log s$, then we can obtain Bernstein type theorem for the following equation

$$u_t = \log F_k(D^2u),$$

which has been studied by X.J. Wang [48]. It should be noted that in this case the condition (5.6) can be replaced by the boundedness of u_t in $\mathbb{R}^n \times (-\infty, 0]$. Therefore, u needs not to be decreasing in t, while u must be strictly convex in t. Indeed, if we consider $v(x,t) = u(x,t) - ct$ for sufficiently large $c > 0$ and set $\rho(s) = k \log s - c$, then we get the desired result.

(4) For the following version of parabolic k-Hessian equation

$$u_t = F_k(D^2u)^{\frac{1}{k}} \quad \text{in } \mathbb{R}^n \times (-\infty, 0], \tag{5.27}$$

which has been studied by Ivochkina and Ladyzhenskaya [22], we can also obtain Bernstein type theorem. We remark that for $k = 1$, (5.27) reduces to the heat equation which is well-known.

Corollary 5.2. *Let $1 \le k \le n$ and $u \in C^{4,2}(\mathbb{R}^n \times (-\infty, 0])$ be a solution to (5.27) which is strictly convex in x. Suppose that there exist constants $c_2 \ge c_1 > 0$ such that for all $(x,t) \in \mathbb{R}^n \times (-\infty, 0]$,*

$$c_1 \le u_t(x,t) \le c_2, \tag{5.28}$$

and that there exist constants $A, B > 0$ such that (5.7) holds for all $x \in \mathbb{R}^n$.

Then, u has the form $u(x,t) = Ct + p(x)$ where $C > 0$ is a constant and p is a quadratic polynomial.

Proof. We set $v(x,t) = u(x,t) - (c_2+1)t$. Then $u \in C^{4,2}(\mathbb{R}^n \times (-\infty, 0])$ is strictly convex-monotone solution to

$$v_t = F_k(D^2 v)^{\frac{1}{k}} - (c_2 + 1) \quad \text{in } \mathbb{R}^n \times (-\infty, 0]$$

and satisfies $-(c_2 - c_1 + 1) \le v_t \le -1$ in $\mathbb{R}^n \times (-\infty, 0]$ and $v(x,0) \ge A|x|^2 - B$ for all $x \in \mathbb{R}^n$. Applying Theorem 5.1 for $\rho(s) = s - (c_2+1)$, we are done. $\square$

6 Final remarks

(i) Viscosity solutions

We define the notion of viscosity solutions of the parabolic k-Hessian equation

$$-u_t F_k(D^2 u) = \psi(x,t) \quad \text{in } \Omega \tag{6.1}$$

where Ω is an arbitrary bowl-shaped domain in $\mathbb{R}^n \times (-\infty, 0]$ and $\psi \in C(\mathbb{R}^n \times (-\infty, 0])$ is a non-negative function. The theory of viscosity solutions to the first order equations and the second order ones was developed in the 1980's by Crandall, Evans, Ishii, Koike, Lions and so on. See, for example, [14, 15, 16, 25, 31]. But the equation (6.1) is not parabolic on all smooth functions, so that the definitions need to be modified slightly. A definition of the viscosity solutions for k-Hessian equations can be seen in, for example, [46].

Let D be a domain in $\mathbb{R}^n$. First, we define the admissible set of elementary symmetric function S_k by

$$\begin{aligned}\Gamma_k &= \{\lambda \in \mathbb{R}^n \mid S_k(\lambda + \eta) > S_k(\lambda) \quad \text{for all } \ \eta_i \ge 0\} \\ &= \{\lambda \in \mathbb{R}^n \mid S_j(\lambda) > 0, \quad \text{for all } \ j = 1, \dots, k\}.\end{aligned}$$

We say that a function $v \in C^2(D)$ is *k-convex* for the operator F_k if $\lambda = (\lambda_1, \dots, \lambda_n)$ belongs to $\overline{\Gamma_k}$ for every point $x \in \Omega$, where $\lambda_1, \dots, \lambda_n$ denote the eigenvalues of $D^2 v$ (at x). Except for the case $k=1$, the operator $F_k(D^2 v)$ is not elliptic on all functions $v \in C^2(D)$, but Caffarelli, Nirenberg and Spruck [8] have shown that F_k is degenerate elliptic for k-convex functions. Obviously,

$$\Gamma_1 \supset \Gamma_2 \supset \cdots \supset \Gamma_n = \Gamma_+ = \{\lambda \in \mathbb{R}^n \mid \lambda_i > 0, \quad i = 1, \dots, n\},$$

which implies that $v \in C^2(D)$ is convex if and only if v is n-convex, and that if $v \in C^2(D)$ is convex, then it is k-convex for any $k = 1, \dots, n$. Alternative characterizations of Γ_k are also known (see [26]).

Let Ω be a bowl-shaped domain in $\mathbb{R}^n \times (-\infty, 0]$. We define a viscosity solution to (6.1). We say that $\varphi \in C^{2,1}(\Omega)$ is said to be *parabolically k-convex* if φ is k-convex in x and non-increasing in t. Therefore, if $\varphi \in C^{2,1}(\Omega)$ is convex-monotone, then it is parabolically k-convex for $k = 1, \dots, n$.

Definition 6.1. Let Ω be a bowl-shaped domain in $\mathbb{R}^n \times (-\infty, 0]$.

(i) A function $u \in C(\Omega)$ is said to be a *viscosity subsolution* to (6.1) in Ω if for any parabolically k-convex function $\varphi \in C^{2,1}(\Omega)$ and any point $(x_0, t_0) \in \Omega$ which is a maximum point of $u - \varphi$, we have

$$-\varphi_t(x_0, t_0) F_k(D^2\varphi(x_0, t_0)) \geq \psi(x_0, t_0). \tag{6.2}$$

(ii) A function $u \in C(\Omega)$ is said to be a *viscosity supersolution* to (6.1) in Ω if for any parabolically k-convex function $\varphi \in C^{2,1}(\Omega)$ and any point $(x_0, t_0) \in \Omega$ which is a minimum point of $u - \varphi$, we have

$$-\varphi_t(x_0, t_0) F_k(D^2\varphi(x_0, t_0)) \leq \psi(x_0, t_0). \tag{6.3}$$

(iii) A function $u \in C(\Omega)$ is said to be a *viscosity solution* to (6.1) in Ω if it is both a viscosity subsolution and supersolution to (6.1) in Ω.

We note that the notion of viscosity subsolution does not change if all $C^{2,1}(\Omega)$ functions which are non-increasing in t are allowed as comparison functions φ. One can prove that a function $u \in C^{2,1}(\Omega)$ is a viscosity solution to (6.1) if and only if it is a parabolically k-convex classical solution.

Here we consider whether Theorem 2.1 also holds for viscosity solutions to the parabolic k-Hessian equation (1.9). We can show the following proposition.

Proposition 6.2. *Let $1 \leq k \leq n$. Then there exists a convex-monotone viscosity solution $u \in C(\mathbb{R}^n \times (-\infty, 0])$ to (1.9), which does not have the form $u(x,t) = -mt + p(x)$ where $m \geq 0$ and p is a quadratic polynomial.*

Proof. Let $t_0 \geq 0$ be an arbitrary number. We set u by

$$u(x,t) = C(-t+t_0)^{\alpha}|x|^{\beta} \quad \text{in } \mathbb{R}^n \times (-\infty, 0], \tag{6.4}$$

where $\alpha = 1/(k+1)$, $\beta = 2k/(k+1)$ and

$$C = \left\{ \alpha\beta^k \left[(\beta - 1)\binom{n-1}{k-1} + \binom{n-1}{k} \right] \right\}^{-\frac{1}{k+1}}. \tag{6.5}$$

Now we define $\binom{n-1}{n} = 0$. Then it can be easily seen that u is convex-monotone in $\mathbb{R}^n \times (-\infty, 0]$ and that u is a classical solution to (1.9) in $(\mathbb{R}^n \setminus \{0\}) \times (-\infty, 0]$.

For $t \leq 0$, there exists no $C^{2,1}$ function φ which touches u at $(0,t)$ from above, because $\beta < 2$. While, for any parabolically k-convex $C^{2,1}$ function φ which touches u from below at $(0,t)$, $\varphi_t(0,t)$ must be 0, because $u(0,\cdot) \equiv 0$. This implies that $-\varphi_t(0,t)F_k(D^2\varphi(0,t)) = 0 \leq 1$. Therefore u is a viscosity solution to (1.9) in $\mathbb{R}^n \times (-\infty, 0]$. □

For $k = n$ which corresponds to the parabolic Monge-Ampère equation's case, the function u constructed above is almost the same as the one in [19]. We remark that this function u satisfies neither (2.1) nor (2.2), for arbitrary $t_0 \geq 0$. Also, it is not *strictly* convex-monotone. We would like to know whether Theorem 2.1 holds for viscosity solutions under the assumptions (2.1) and (2.2).

(ii) Relaxing the assumptions : Growth conditions and convexity

We would like to remove growth conditions (1.6), (1.8), (2.1) and (2.2) in Theorems 1.3, 1.4 and 2.1 (or, to prove growth conditions are necessary). As we have stated in Section 1 and Remark 2.1 before, Theorem 1.3 remains valid without the growth condition (1.6) when $k = 1$ (the case of Poisson equation) and $k = n$ (the case of Monge-Ampère equation), and Theorem 2.1 is true without (2.2) when $k = n$. However, we do not know any more for other cases.

As we have said in (i), k-Hessian operator F_k is degenerate elliptic for k-convex functions (see [8] for the proof). Therefore, when we study k-Hessian equation, it is natural to seek solutions in the class of k-convex functions, rather than in the class of convex functions. It seems an interesting open problem whether Theorems 1.3 and 2.1 remain true if one replaces "strictly convex" by "strictly k-convex."

(iii) k-curvature equation

Here we deal with the so-called curvature equations of the form

$$H_k[u] = S_k(\kappa_1, \ldots, \kappa_n) = \psi \quad \text{in } \mathbb{R}^n, \tag{6.6}$$

for $1 \le k \le n$, where ψ is a function defined in $\mathbb{R}^n$ and for a function $u \in C^2(\mathbb{R}^n)$, $\kappa = (\kappa_1, \ldots, \kappa_n)$ denotes the *principal curvatures* of the graph of the function u, namely, the eigenvalues of the matrix

$$\mathcal{C} = D\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = \frac{1}{\sqrt{1+|Du|^2}}\left(I - \frac{Du \otimes Du}{1+|Du|^2}\right) D^2u. \tag{6.7}$$

Also S_k denotes the k-th elementary symmetric function which is defined in (1.5). The mean, scalar and Gauss curvature equations correspond respectively to the special cases $k = 1, 2, n$ in (6.6). We call the equation (6.6) "k-curvature equation."

We remark that (6.6) is a quasilinear equation for $k = 1$ while it is a *fully non-linear* equation for $k \ge 2$. In the particular case that $k = n$, it is an equation of Monge-Ampère type. The cases $k = 1$ and $k = n$, corresponding to the mean and Gauss curvature equations respectively, are well understood. Although it is much harder to analyze the intermediate cases $2 \le k \le n-1$, some progress have been made in the last three decades, such as the study of the classical Dirichlet problem. See for instance [9, 21, 40]. Recently, the author and Takimoto [34] considered the boundary blowup problem for k-curvature equations and obtained the uniqueness of a boundary blowup solution under some hypotheses (see also [38]). It was the first result for the uniqueness of boundary blowup solutions for k-curvature equations, even for the mean curvature equation which corresponds to the case of $k = 1$ for (6.6).

When $n = 2$, $k = 1$ and $\psi \equiv 0$, the graph of a solution $u = u(x_1, x_2)$ to (6.6) is a minimal surface in $\mathbb{R}^3$, so that u must be an affine function due to Theorem 1.1, which is the classical Bernstein's theorem for the minimal surface equation. We remark that Theorem 1.1 can also be derived from Bernstein type theorem for Monge-Ampère equation (Theorem 1.2). For the detail, see [24, 33].

It is quite natural to study whether Bernstein type theorem holds for k-curvature equation (6.6). First we consider the case $\psi \equiv 0$ and give the following problem.

Problem. Let $u = u(x_1, \ldots, x_n)$ be a solution to the homogeneous k-curvature equation

$$H_k[u] = 0 \quad \text{in } \mathbb{R}^n. \tag{6.8}$$

Then, can we say that u must be an affine function of $x_1, \ldots, x_n$, that is, the graph of u must be a hyperplane in $\mathbb{R}^{n+1}$?

For the case $k = 1$ which corresponds to the minimal surface equation, Bernstein conjectured that it is true. Many mathematicians have attacked to this problem. It was solved affirmatively by de Giorgi [17] for $n = 3$, by Almgren [1] for $n = 4$, and by Simons [37] for $n \leq 7$. However, Bombieri, de Giorgi and Giusti [4] proved that for $n \geq 8$, there exists a solution to the minimal surface equation in $\mathbb{R}^n$ which is *not* an affine function.

While, for the case $k \geq 2$, our problem can be solved negatively, even if we add additional hypotheses such as the convexity of u. In fact, $u(x) = \varphi(x_1)$ where φ is any C^2 function defined in $\mathbb{R}$ solves (6.8), because $n-1$ principal curvatures of u are 0.

Next, we consider the case $\psi \equiv \text{const.} > 0$. In this case, however, there exist no (k-admissible) solutions to $H_k[u] = \psi \equiv \text{const.} > 0$ in $\mathbb{R}^n$, due to [39]. Indeed, the condition

$$k \int_{B(0,R)} \psi \, dx \leq (1-\chi) \int_{\partial B(0,R)} H_{k-1}[\partial B(0,R)] \, ds, \tag{6.9}$$

for some positive constant χ does not hold for sufficiently large R. Here, for a C^2 domain $\Omega \subset \mathbb{R}^n$, $H_{k-1}[\partial\Omega] = S_{k-1}(\kappa'_1, \ldots, \kappa'_{n-1})$ where $\kappa'_1, \ldots, \kappa'_{n-1}$ are the principal curvatures of $\partial\Omega$. It is because the left-hand side of (6.9) is $\text{const.} \times R^n$ while the right-hand side is $\text{const.} \times R^{n-k}$.

Our next task is to consider the appropriate formulation of Bernstein type problems for k-curvature equations and parabolic k-curvature equations.

7 Appendix

We begin with some notation. The k-th elementary symmetric function of n variables S_k is considered in the corresponding cone Γ_k in $\mathbb{R}^n$, given by

$$\Gamma_k = \{\lambda \in \mathbb{R}^n \mid S_j(\lambda) > 0, \quad \text{for all } \ j = 1, \dots, k\}.$$

The following properties of the functions S_k are used in Section 3 and in the proof of Lemma 7.1.

$$S_k(\lambda) = S_{k;i}(\lambda) + \lambda_i S_{k-1;i}(\lambda), \tag{7.1}$$

$$\sum_{i=1}^{n} S_{k;i}(\lambda) = (n-k)S_k(\lambda) \tag{7.2}$$

for all $\lambda \in \mathbb{R}^n$. Furthermore, if $\lambda \in \Gamma_k$, then at least k of the numbers $\lambda_1, \dots, \lambda_n$ are positive and moreover

$$S_{l;i_1 i_2 \dots i_s}(\lambda) > 0$$

for all $\{i_1, i_2, \dots, i_s\} \subseteq \{1, 2, \dots, n\}$, $l + s \leq k$ (see [26]).

It is known that the Newton inequalities

$$S_k(\lambda)S_{k-2}(\lambda) \leq \frac{(k-1)(n-k+1)}{k(n-k+2)}[S_{k-1}(\lambda)]^2 \tag{7.3}$$

for $\lambda \in \mathbb{R}^n$, $k \geq 2$ and the Maclaurin inequalities

$$\left[\frac{S_k(\lambda)}{\binom{n}{k}}\right]^{\frac{1}{k}} \leq \left[\frac{S_l(\lambda)}{\binom{n}{l}}\right]^{\frac{1}{l}} \tag{7.4}$$

for $\lambda \in \Gamma_k$, $k \geq l \geq 1$ hold (see [32]).

The following lemma is used in Section 3 ([30]).

Lemma 7.1. *There exists a constant $\theta > 0$, depending only on n and k, such that*

$$S_{k-1;k}(\lambda) \geq \theta S_{k-1}(\lambda), \tag{7.5}$$

for all $\lambda \in \Gamma_k$.

Proof. First, we prove

$$|S_{k-1;1k}(\lambda)| \leq C_k S_{k-1;k}(\lambda), \qquad C_k = \sqrt{\frac{k(n-k)}{n-1}}. \tag{7.6}$$

By using (7.1), we have

$$\begin{aligned} S_{k;1k}(\lambda) + \lambda_1 S_{k-1;1k}(\lambda) = S_{k;k}(\lambda) &= S_k(\lambda) - \lambda_k S_{k-1;k}(\lambda) \\ &\geq -\lambda_k S_{k-1;k}(\lambda), \end{aligned} \tag{7.7}$$

$$S_{k-1;1k}(\lambda) + \lambda_1 S_{k-2;1k}(\lambda) = S_{k-1;k}(\lambda). \tag{7.8}$$

Eliminating λ_1 from (7.7) and (7.8) gives

$$\begin{aligned} (S_{k-1;1k}(\lambda))^2 - S_{k;1k}(\lambda) S_{k-2;1k}(\lambda) &\leq S_{k-1;k}(\lambda)(S_{k-1;1k}(\lambda) + \lambda_k S_{k-2;1k}(\lambda)) \\ &= S_{k-1;k}(\lambda) S_{k-1;1}(\lambda) \end{aligned}$$

so that by Newton's inequality (7.3) we obtain

$$\left(1 - \frac{(k-1)(n-k-1)}{k(n-k)}\right)(S_{k-1;1k}(\lambda))^2 \leq (S_{k-1;k}(\lambda))^2. \tag{7.9}$$

Therefore (7.6) follows.

By using (7.6) and (7.8), we obtain

$$C_k S_{k-1;k}(\lambda) \geq -S_{k-1;k}(\lambda) + \lambda_1 S_{k-2;1k}(\lambda) \tag{7.10}$$

so that

$$S_{k-1;k}(\lambda) \geq \frac{\lambda_1}{1 + C_k} S_{k-2;1k}(\lambda). \tag{7.11}$$

Let us now suppose that (7.5) is valid wherever k and n are replaced by $k-1$ and $n-1$, that is for some positive constant $\theta = \theta(k-1, n-1)$, we have

$$S_{k-2;1k}(\lambda) \geq \theta S_{k-2;1}(\lambda). \tag{7.12}$$

By using (7.11) and (7.12), we obtain

$$S_{k-1;k}(\lambda) \geq \frac{\lambda_1 \theta}{1 + C_k} S_{k-2;1}(\lambda) = \frac{\theta}{1 + C_k}(S_{k-1}(\lambda) - S_{k-1;1}(\lambda)).$$

Therefore we obtain

$$S_{k-1;k}(\lambda) \geq \frac{\theta}{\theta + 1 + C_k} S_{k-1}(\lambda). \tag{7.13}$$

Hence (7.5) holds. $\square$

Moreover, we prove some lemmas used in Section 3. These lemmas can be proved by using properties of the k-th elementary symmetric function S_k.

Lemma 7.2. *It holds that there exists some constant $\theta > 0$ such that*

$$S_{k-1;i}(\lambda) \geq \theta\lambda_1\lambda_2\cdots\lambda_{k-1} \tag{7.14}$$

for $i \geq k$ and all $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Gamma_k$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$.

Proof. By using the inequality (7.11), we obtain

$$\begin{aligned}
S_{k-1;i}(\lambda) &= S_{k-1}(\lambda_1, \ldots, \lambda_{k-1}, \lambda_k, \ldots, \lambda_{i-1}, 0, \lambda_{i+1}, \ldots, \lambda_n) \\
&\geq S_{k-1}(\lambda_1, \ldots, \lambda_{k-1}, \lambda_{k+1}, \ldots, \lambda_i, 0, \lambda_{i+1}, \ldots, \lambda_n) \\
&= S_{k-1;k}(\lambda) \\
&\geq \theta_1\lambda_1 S_{k-2;1k}(\lambda) \\
&\geq \theta_2\lambda_1\lambda_2 S_{k-3;12k}(\lambda) \\
&\geq \cdots \\
&\geq \theta\lambda_1\lambda_2\cdots\lambda_{k-1}S_{0;12\cdots k}(\lambda) \\
&= \theta\lambda_1\lambda_2\cdots\lambda_{k-1},
\end{aligned}$$

for some positive constants θ_1, $\theta_2, \ldots,$ and θ. ☐

Lemma 7.3. *Suppose $\lambda \in \overline{\Gamma_k}$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$.*

(i) There exists some $\theta > 0$, which depends only n and k, such that

$$\lambda_1 S_{k-1;1}(\lambda) \geq \theta S_k(\lambda). \tag{7.15}$$

(ii) For any $\delta \in (0,1)$ there exists $K > 0$ such that if

$$S_k(\lambda) \leq K\lambda_1^k \ \text{ or } \ |\lambda_i| \leq K\lambda_1 \ \text{ for } \ i = k+1, \ldots, n,$$

we have

$$\lambda_1 S_{k-1;1}(\lambda) \geq (1-\delta) S_k(\lambda). \tag{7.16}$$

Proof. (i) We have

$$S_k(\lambda) = \lambda_1 S_{k-1;1}(\lambda) + S_{k;1}(\lambda). \tag{7.17}$$

By

$$S_{k;1}(\lambda) \le C_{n,k} S_{k-1;1}^{\frac{k}{k-1}}(\lambda) \le C\lambda_1 S_{k-1;1}(\lambda),$$

(7.15) follows.

(ii) To prove (7.16), we first consider the case $S_k(\lambda) \le K\lambda_1^k$. We may assume $S_k(\lambda) = 1$. If (7.16) is not true, then

$$S_{k-1;1}(\lambda) < \frac{1-\delta}{\lambda_1} \le K^{\frac{1}{k}},$$

hence

$$S_{k;1}(\lambda) \le C S_{k-1;1}^{\frac{k}{k-1}}(\lambda) \le CK^{\frac{1}{k-1}}.$$

In view of (7.17), (7.16) follows.

Next, we consider the case $|\lambda_i| \le K\lambda_1$ for $i = k+1, \ldots, n$. Observing that if $\lambda_k \ll \lambda_1$, we have $S_k(\lambda) \ll \lambda_1^k$, and so (7.16) holds. Hence we may assume that $|\lambda_i| \ll \lambda_k$ for $i = k+1, \ldots, n$. In this case, both $S_k(\lambda)$ and $\lambda_1 S_{k-1;1}(\lambda)$ are equal to $\lambda_1\lambda_2 \cdots \lambda_k(1 + o(1))$ with $o(1) \to 0$ as $K \to 0$. Again (7.16) holds. □

Next, we prove a theorem called Alexsandrov's maximum principle. Before giving a proof, we introduce some notation.

Definition 7.4. *(supporting hyperplane)*

Let Ω be an open subset of $\mathbb{R}^n$ and $u : \Omega \to \mathbb{R}$. Given $x_0 \in \Omega$, a *supporting hyperplane* to the function u at the point $(x_0, u(x_0))$ is an affine function $l(x) = u(x_0) + p \cdot (x - x_0)$ such that $u(x) \ge l(x)$ for all $x \in \Omega$.

Definition 7.5. *(normal mapping)*

Let Ω be an open set in $\mathbb{R}^n$ and $u \in C(\Omega)$. The *normal mapping* of u, or *subdifferential* of u, is the set-valued function $\partial u : \Omega \to \mathcal{P}(\mathbb{R}^n)$ defined by

$$\partial u(x_0) = \{p \in \mathbb{R}^n \mid u(x) \ge u(x_0) + p \cdot (x - x_0), \quad \textit{for all}\, x \in \Omega\}.$$

Given $E \subset \Omega$, we define $\partial u(E) = \cup_{x \in E} \partial u(x)$.

Lemma 7.6. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, and $u, v \in C(\overline{\Omega})$. If $u = v$ on $\partial\Omega$ and $v \geq u$ in Ω, then*

$$\partial v(\Omega) \subset \partial u(\Omega).$$

Proof. Let $p \in \partial v(\Omega)$. There exists $x_0 \in \Omega$ such that

$$v(x) \geq v(x_0) + p \cdot (x - x_0), \qquad \forall x \in \Omega.$$

Let

$$a = \sup_{x \in \Omega} \{v(x_0) + p \cdot (x - x_0) - u(x)\}.$$

Since $v(x_0) \geq u(x_0)$, it follows that $a \geq 0$. If $a > 0$, we claim that $v(x_0) + p \cdot (x - x_0) - a$ is a supporting hyperplane to the function u at some point in Ω. Since Ω is bounded, there exists $x_1 \in \overline{\Omega}$ such that $a = v(x_0) + p \cdot (x_1 - x_0) - u(x_1)$. We have

$$v(x_1) \geq v(x_0) + p \cdot (x_1 - x_0) = u(x_1) + a.$$

Hence, since $a > 0$, then $x_1 \notin \partial\Omega$, so the claim holds in this case.

On the other hand, if $a = 0$, then

$$u(x) \geq v(x_0) + p \cdot (x - x_0) \geq u(x_0) + p \cdot (x - x_0), \quad \forall x \in \Omega$$

and consequently $u(x_0) + p \cdot (x - x_0)$ is a supporting hyperplane to u at x_0. Therefore, it holds that $p \in \partial u(\Omega)$. □

Now we prove Aleksandrov's maximum principle (see [18]). It is used in the proof of Lemma 4.1, in order to derive a local gradient estimate of a solution to (1.9).

Theorem 7.7. *(Aleksandrov's maximum principle)*

If $\Omega \subset \mathbb{R}^n$ is a bounded, open and convex set with diameter Δ, and $u \in C(\overline{\Omega})$ is convex with $u = 0$ on $\partial\Omega$, then

$$|u(x_0)|^n \leq C_n \Delta^{n-1} \operatorname{dist}(x_0, \partial\Omega) |\partial u(\Omega)|$$

for all $x_0 \in \Omega$, where C_n is a constant depending only on the dimension n.

Proof. Fix $x_0 \in \Omega$ and let v be the convex function whose graph is the upside down cone with vertex $(x_0, u(x_0))$ and base Ω, with $v = 0$ on $\partial\Omega$. Since u is convex, $v \geq u$ in Ω. By Lemma 7.6

$$\partial v(\Omega) \subset \partial u(\Omega). \tag{7.18}$$

To prove this theorem, we shall estimate $|\partial v(\Omega)|$ from below. We first notice that the set $\partial v(\Omega)$ is convex. This is true, because, if $p \in \partial v(\Omega)$, then there exists $x_1 \in \Omega$ such that $p \in \partial v(x_1)$. If $x_1 \neq x_0$, since the graph of v is a cone, then $v(x_1) + p \cdot (x - x_1)$ is a supporting hyperplane at x_0, that is $p \in \partial v(x_0)$. So $\partial v(\Omega) = \partial v(x_0)$ and since $\partial v(x_0)$ is convex, we are done.

Second, we notice that there exists $p_0 \in \partial v(\Omega)$ such that $|p_0| = \dfrac{-u(x_0)}{\operatorname{dist}(x_0, \partial\Omega)}$. This follows because Ω is convex. Indeed, we take $x_1 \in \partial\Omega$ such that $|x_1 - x_0| = \operatorname{dist}(x_0, \partial\Omega)$ and H is a supporting hyperplane to the set Ω at x_1. The hyperplane in $\mathbb{R}^{n+1}$ generated by H and the point $(x_0, u(x_0))$ is a supporting hyperplane to v that has the desired slope.

Now notice that the ball B with center at the origin and radius $\dfrac{-u(x_0)}{\Delta}$ is contained in $\partial v(\Omega)$, and $|p_0| \geq \dfrac{-u(x_0)}{\Delta}$. Hence the convex hull of B and p_0 is contained in $\partial v(\Omega)$ and it has measure

$$C_n \left(\frac{-u(x_0)}{\Delta} \right)^{n-1} |p_0|. \tag{7.19}$$

By the definition of $|p_0|$, (7.18) and (7.19), we obtain

$$|\partial u(\Omega)| \geq |\partial v(\Omega)| \geq \frac{C_n |u(x_0)|^n}{\Delta^{n-1} \operatorname{dist}(x_0, \partial\Omega)}.$$

Therefore the proof is completed. □

Finally, we prove Evans-Krylov type theorem (see [18]). By using this theorem, we can estimate local α-Hölder estimates of D^2u and u_t in the proof of Theorem 2.1.

Theorem 4.2 *(Evans-Krylov type theorem)*

Let D and D' be bounded bowl-shaped domains which satisfy $D' \subset D$ and $\operatorname{dist}(D', \partial_p D) > 0$, and u be a $C^{4,2}(D)$ solution to the equation

$$G(u_t, D^2u) = 0 \tag{4.11}$$

in D, where $G = G(q, M)$ is defined for all $(q, M) \in \mathbb{R} \times \mathbb{S}^{n\times n}$ with $G(\cdot, M) \in C^1(\mathbb{R})$ for each $M \in \mathbb{S}^{n\times n}$, and $G \in C^2(\mathbb{R} \times X)$ for some $X \subset \mathbb{S}^{n\times n}$ which is a neighborhood of $D^2u(D)$. Suppose that:

(i) G is uniformly parabolic, i.e., there exist positive constants λ and Λ such that

$$-\Lambda \le G_q(q, M) \le -\lambda, \tag{4.12}$$

$$\lambda\|N\| \le G(q, M + N) - G(q, M) \le \Lambda\|N\|, \tag{4.13}$$

for all $q \in \mathbb{R}$ and M, $N \in \mathbb{S}^{n\times n}$ with $N \ge O$.

(ii) G is concave with respect to M.

If $\|u\|_{C^{2,1}(D)} \le K$, then there exist positive constants C depending on λ, Λ, n, K, D, D' and $G(0, 0)$, and $\alpha \in (0, 1)$ depending on λ, Λ and n such that

$$\|u\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(D')} \le C.$$

Proof. By the smoothness of G on the range of D^2u and differentiating the equation (4.11) with respect to t, we obtain

$$G_q(u_t, D^2u)(u_t)_t + G_{ij}(u_t, D^2u)D_{ij}(u_t) = 0,$$

where

$$G_{ij} = \frac{\partial G}{\partial m_{ij}}.$$

Dividing the last equation by G_q, by (4.12), we obtain a uniformly parabolic equation, and by Harnack inequality [29], we obtain

$$[u_t]_{\gamma, D_{3/4}} \le C\|u_t\|_{L^\infty(D)},$$

where $D_{3/4} = B(0, 3/4) \times (-3/4, 0]$ and some $0 < \gamma < 1$. For the estimation of second x-derivatives, fix t. Then, $v(x) = u(x, t)$ satisfies

$$\widetilde{G}(x, D^2v(x)) = G(u_t(x, t), D^2u(x, t)) = 0.$$

By [6, Theorem 8.1], we have the estimate $\|D^2 v\|_{C^\beta(B(0,1/2))} \le C$ uniformly in t for some $0 < \beta < 1$. To show that $D^2 u$ is Hölder continuous in t, we note that by differentiating (4.11) with respect to x_k we get that $D_k u$ satisfies

$$G_q(u_t, D^2 u)(D_k u)_t + G_{ij}(u_t, D^2 u) D_{ij}(D_k u) = 0,$$

and as before, we get

$$[Du]_{\alpha, D_{1/2}} \le C \|Du\|_{L^\infty(D)},$$

for some $0 < \alpha < 1$. We have

$$|Du(x, t_1) - Du(x, t_2)| \le C_1 |x_1 - x_2|^\alpha \tag{7.20}$$

and

$$|D^2 u(x_1, t) - D^2 u(x_2, t)| \le C_2 |x_1 - x_2|^\beta. \tag{7.21}$$

This implies $|D^2 u(x, t_1) - D^2 u(x, t_2)| \le C |t_1 - t_2|^{\alpha\beta/(1+\beta)}$. Here we shall give its proof, which is found in [28, p.78].

We fix $x \in B(0, 1/2)$, $1 \le i, j \le n$ and $t_1, t_2 \in (-1/2, 0]$ which satisfies $|t_1 - t_2| \le \varepsilon$ where $\varepsilon > 0$ is a constant which will be determined later.

We set $h(s) = |D_i u(x + s e_j, t_1) - D_i u(x, t_1) - D_i u(x + s e_j, t_2) + D_i u(x, t_2)|$ for $s \in \mathbb{R}$ with $x + s e_j \in B(0, 1/2)$. First (7.20) yields

$$\begin{aligned} h(s) &\le |D_i u(x + s e_j, t_1) - D_i u(x + s e_j, t_2)| + |D_i u(x, t_1) - D_i u(x, t_2)| \\ &\le 2C_1 |t_1 - t_2|^\alpha. \end{aligned} \tag{7.22}$$

Next, $h(s)$ can be estimated as

$$\begin{aligned} h(s) &= \left| \int_0^s \big(D_{ij} u(x + \xi e_j, t_1) - D_{ij} u(x + \xi e_j, t_2) \big)\, d\xi \right| \\ &= \Big| \int_0^s \big(D_{ij} u(x + \xi e_j, t_1) - D_{ij} u(x, t_1) \big)\, d\xi \\ &\qquad - \int_0^s \big(D_{ij} u(x + \xi e_j, t_2) - D_{ij} u(x, t_1) \big)\, d\xi \Big| \\ &\ge \left| \int_0^s \big(D_{ij} u(x + \xi e_j, t_1) - D_{ij} u(x, t_1) \big)\, d\xi \right| \\ &\qquad - \left| \int_0^s \big(D_{ij} u(x + \xi e_j, t_2) - D_{ij} u(x, t_1) \big)\, d\xi \right| \\ &=: I_1 - I_2. \end{aligned} \tag{7.23}$$

It follows from (7.21) that

$$\begin{aligned} I_1 &\le \left| \int_0^s \left| D_{ij}u(x+\xi e_j,t_1) - D_{ij}u(x,t_1) \right| d\xi \right| \\ &\le \left| \int_0^s C_2|\xi|^\beta \, d\xi \right| \\ &= \frac{C_2}{1+\beta}|s|^{1+\beta} \le C_2|s|^{1+\beta}. \end{aligned} \tag{7.24}$$

And It follows from the mean value theorem that

$$I_2 = \left| s\big(D_{ij}u(x+\widetilde{\xi}e_j,t_2) - D_{ij}u(x,t_1)\big) \right|,$$

for some $\widetilde{\xi}$ which is between 0 and s. Therefore, by (7.21) we obtain that

$$\begin{aligned} I_2 &= |s| \left| \big(D_{ij}u(x+\widetilde{\xi}e_j,t_2) - D_{ij}u(x,t_2)\big) - \big(D_{ij}u(x,t_1) - D_{ij}u(x,t_2)\big) \right| \\ &\ge |s|\left|D_{ij}u(x,t_1) - D_{ij}u(x,t_2)\right| - C_2|s||\widetilde{\xi}|^\beta \\ &\ge |s|\left|D_{ij}u(x,t_1) - D_{ij}u(x,t_2)\right| - C_2|s|^{1+\beta}. \end{aligned} \tag{7.25}$$

Substituting (7.24) and (7.25) into (7.23), we have

$$h(s) \ge |s|\left|D_{ij}u(x,t_1) - D_{ij}u(x,t_2)\right| - C_2|s|^{1+\beta}. \tag{7.26}$$

Combining (7.22) and (7.26), we obtain that

$$\left|D_{ij}u(x,t_1) - D_{ij}u(x,t_2)\right| \le \frac{2C_1}{|s|}|t_1-t_2|^\alpha + 2C_2|s|^\beta.$$

If we choose $s = (C_1/(\beta C_2))^{1/(1+\beta)}|t_1-t_2|^{\alpha/(1+\beta)}$ or $s = -(C_1/(\beta C_2))^{1/(1+\beta)}|t_1 - t_2|^{\alpha/(1+\beta)}$, then we have the following inequality:

$$\left|D_{ij}u(x,t_1) - D_{ij}u(x,t_2)\right| \le C|t_1-t_2|^{\frac{\alpha\beta}{1+\beta}}. \tag{7.27}$$

Here C is some positive constant. We note that there exists some positive constant $\varepsilon > 0$ such that if $|t_1-t_2| < \varepsilon$ then the point $x+se_j$ for $s = (C_1/(\beta C_2))^{1/(1+\beta)}|t_1 - t_2|^{\alpha/(1+\beta)}$ or $s = -(C_1/(\beta C_2))^{1/(1+\beta)}|t_1-t_2|^{\alpha/(1+\beta)}$ is in $B(0,1/2)$. Therefore (7.27) holds for $x \in B(0,1/2)$ and $|t_1-t_2|$ is suffiently small, and the desired Hölder continuity follows. □

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(1) Uniqueness of boundary blowup solutions to k-curvature equation.
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UNIQUENESS OF BOUNDARY BLOWUP SOLUTIONS TO k-CURVATURE EQUATION

SAORI NAKAMORI AND KAZUHIRO TAKIMOTO

Abstract. We consider the boundary blowup problem for k-curvature equation, i.e., $H_k[u] = f(u)g(|Du|)$ in an n-dimensional domain Ω, with the boundary condition $u(x) \to \infty$ as $\operatorname{dist}(x, \partial\Omega) \to 0$. We prove the uniqueness result under some hypotheses.

1. Introduction

This paper deals with the so-called curvature equations of the form

$$H_k[u] = S_k(\kappa_1, \dots, \kappa_n) = f(u)g(|Du|) \quad \text{in } \Omega, \tag{1.1}$$

with the following boundary condition

$$u(x) \to \infty \quad \text{as } \operatorname{dist}(x, \partial\Omega) \to 0. \tag{1.2}$$

Here Ω is a bounded domain in $\mathbb{R}^n$ and for a function $u \in C^2(\Omega)$, $\kappa = (\kappa_1, \dots, \kappa_n)$ denotes the principal curvatures of the graph of the function u, namely, the eigenvalues of the matrix

$$\mathcal{C} = D\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = \frac{1}{\sqrt{1+|Du|^2}}\left(I - \frac{Du \otimes Du}{1+|Du|^2}\right) D^2 u, \tag{1.3}$$

and $S_k, k = 1, \dots, n$, denotes the k-th elementary symmetric function, i.e.,

$$S_k(\kappa) = \sum \kappa_{i_1} \cdots \kappa_{i_k}, \tag{1.4}$$

where the sum is taken over increasing k-tuples, $1 \le i_1 < i_2 < \cdots < i_k \le n$. The mean, scalar and Gauss curvatures correspond respectively to the special cases $k = 1, 2, n$ in (1.4). In this paper we call the equation (1.1) "k-curvature equation."

In [28] we have studied the existence and non-existence result of a solution to (1.1)-(1.2). In addition, we have obtained the result for the asymptotic behavior near $\partial\Omega$ of such solution. In this paper, we deal with the uniqueness of solutions to (1.1)-(1.2).

We remark that (1.1) is a quasilinear equation for $k = 1$ while it is a *fully nonlinear* equation for $k \ge 2$. In the particular case that $k = n$, it is an equation of Monge-Ampère type. It is much harder to analyze fully nonlinear equations, but the study of the classical Dirichlet problem for k-curvature equations in the case that $2 \le k \le n-1$ has been developed in the last two decades, see for instance [4, 14, 29].

The condition (1.2) is called the "boundary blowup condition," and a solution which satisfies (1.2) is called a "boundary blowup solution," a "large solution," or an "explosive solution." The boundary blowup problems arise from physics, geometry and many branches of mathematics, see for instance

[15, 22, 26]. The existence and the asymptotic behavior of solutions for such problems starts from the pioneering works of Bieberbach [3] and Rademacher [26] who considered $\Delta u = e^u$ in two and three dimensional domain respectively. For the case of semilinear equations, they have extensively been studied (see, for example, [16, 25] and [2, 5, 7, 17, 18, 19, 22, 23, 24]). The case of quasilinear equations of divergence type to which the mean curvature equation ($k = 1$ in (1.1)) belongs has been treated in [1, 11, 12]. However, there are only a few results concerning such problems for fully nonlinear PDEs, such as [6, 13, 20] for Monge-Ampère equation, [27] for k-Hessian equations, and [28] by the author for k-curvature equations.

In some works among them, the uniqueness of boundary blowup solutions has been also discussed, see [1, 19, 22, 24, 27] for example. But there were no results for the uniqueness of boundary blowup solutions for k-curvature equations, even for the mean curvature equation which corresponds to the case of $k = 1$ for (1.1). In this paper, we shall obtain the uniqueness result for (1.1)-(1.2), which is stated in Sections 3 and 4.

Throughout the paper, we assume the following conditions on f and g:

- Let $t_0 \in [-\infty, \infty)$. $f \in C^\infty(t_0, \infty)$ is a positive function and satisfies $f'(t) > 0$ for all $t \in (t_0, \infty)$.
- If $t_0 > -\infty$, then $f(t) \to 0$ as $t \to t_0+0$; otherwise (i.e., if $t_0 = -\infty$),

$$\int_{-\infty}^{t} f(s)\, ds < \infty \quad \text{for all } t \in \mathbb{R}. \tag{1.5}$$

- $g \in C^\infty[0, \infty)$ is a positive function.

The first condition assures us that the comparison principle for solutions to (1.1) holds. The typical examples of f are $f(t) = t^p$ ($p > 0$), $t_0 = 0$ and $f(t) = e^t$, $t_0 = -\infty$.

This paper is divided as follows. In the next section, we state our results for the existence and the estimate of the asymptotic behavior of a solution near the boundary to the boundary blowup problem (1.1)-(1.2), for the sake of completeness. This includes the improved results for the asymptotic behavior of boundary blowup solutions. In Section 3, we state our uniqueness result and prove it. However, the case $k = n$ is excluded from these theorems. We consider the particular case in Section 4.

2. Results for existence and asymptotic behavior of a solution

In this section we review the results for the existence and the asymptotic behavior of a solution to (1.1)-(1.2). The following existence result has been proved in [28].

Theorem 2.1. *Let $2 \le k \le n-1$. We assume that Ω, f and g satisfy the following conditions.*

(A1) *Ω is a bounded and uniformly k-convex domain with boundary $\partial\Omega \in C^\infty$.*

(A2) *There exists a constant $T > 0$ such that g is non-increasing in $[T, \infty)$, and $\lim_{t\to\infty} g(t) = 0$.*

(A3) *Set* $\tilde{g}(t) = g(t)/t$ *and* $F(t) = \int_{t_0}^{t} f(s)\,ds$. *Then*

$$\int^{\infty} \frac{dt}{\tilde{g}^{-1}\left(\frac{1}{F(t)}\right)} < \infty. \tag{2.1}$$

(A4) *Set*

$$H(t) = \int_0^t \frac{s^k}{g(s)\,(1+s^2)^{(k+2)/2}}\,ds. \tag{2.2}$$

Then $\lim_{t\to\infty} H(t) = \infty$.

(A5) *Set* $\varphi(t) = g(t)(1+t^2)^{k/2}$. *Then* $\varphi(t)$ *is a convex function in* $[0,\infty)$.

(A6) $\limsup_{t\to\infty} |g'(t)|t^2 < \infty$.

Then there exists a viscosity solution to (1.1)-(1.2).

We note that for $k = 1$ the existence has been already studied in [12], so that we focus here on the case $k \geq 2$. For the definition and the general theory of viscosity solutions to PDEs, we refer to, for example, [8, 9, 10, 21]. For the viscosity theory for curvature equations in particular, see [29].

Example 2.1. Let $2 \leq k \leq n-1$ and p, q be positive constants. Suppose Ω is a bounded and uniformly k-convex domain with boundary $\partial\Omega \in C^\infty$. We consider these two equations:

(i) $H_k[u] = u^p/\left(1+|Du|^2\right)^{q/2}$ in Ω.

From Theorem 2.1, it follows that there exists a boundary blowup solution provided $p > q$ and $1 \leq q \leq k-1$.

(ii) $H_k[u] = e^{pu}/\left(1+|Du|^2\right)^{q/2}$ in Ω.

There exists a boundary blowup solution provided $1 \leq q \leq k-1$.

Remark 2.1. In the preceding paper [28], we have also obtained a necessary condition for boundary blowup solutions to exist, so that we have given an example of f and g for which there does not exist any boundary blowup solution.

Next we establish the asymptotic behavior of a solution to (1.1)-(1.2) near $\partial\Omega$. We shall prove the following, which is slightly improved than the corresponding one in [28], so that we give its proof here.

Theorem 2.2. *Let* $1 \leq k \leq n-1$. *We assume that (A1), (A2) and (A3) in Theorem 2.1 and the conditions given below are satisfied.*

(B1) $t_0 = -\infty$, *or* $t_0 > -\infty$ *and* $f^{1/k}$ *is Lipschitz continuous at* t_0.

(B2) *There exists a constant* $T' > t_0$ *such that* f *is a convex function in* $[T', \infty)$.

(B3) *Set* $h(t) = \dfrac{t}{g(t)^{1/k}\sqrt{1+t^2}}$. *Then there exists a constant* $\alpha > 0$ *such that* $h(t)/t^\alpha$ *is non-decreasing in* $(0,\infty)$.

(B4) $\displaystyle\lim_{t\to\infty} \frac{g(t)}{(1+t^2)g'(t)} = 0$.

Then there exist positive constants C_1, C_2 *such that every solution* u *to (1.1)-(1.2) satisfies*

$$\psi^{-1}(C_1\,\mathrm{dist}(x,\partial\Omega)) - O(1) \leq u(x) \leq \psi^{-1}(C_2\,\mathrm{dist}(x,\partial\Omega)) + O(1) \tag{2.3}$$

near $\partial\Omega$*, where* ψ *is defined by*

$$\psi(t) = \int_t^\infty \frac{ds}{h^{-1}\left(f(s)^{1/k}\right)}. \tag{2.4}$$

Proof. Let u be a solution to (1.1)-(1.2). From now on, we use the following notation: $d(x) = \mathrm{dist}(x, \partial\Omega)$ and $\Omega_r = \{x \in \Omega \mid d(x) < r\}$ for $r > 0$.

It follows from (A1) that there exists a positive constant R such that the following conditions are satisfied:

(a) $d = d(x)$ is a C^∞ function in Ω_R;
(b) For each point $x \in \Omega_R$, there exists a unique point $z(x) \in \partial\Omega$ such that $d(x) = |x - z(x)|$;
(c) There exist positive constants m, M such that for every point $x \in \Omega_R$, it holds that

$$\tilde\kappa = (\tilde\kappa_1, \dots, \tilde\kappa_{n-1}) := \left(\frac{\kappa'_1}{1 - d(x)\kappa'_1}, \dots, \frac{\kappa'_{n-1}}{1 - d(x)\kappa'_{n-1}}\right) \in \Gamma_k(\mathbb{R}^{n-1}) \tag{2.5}$$

and that

$$m \le S_k(\tilde\kappa) \le M, \tag{2.6}$$

where $\kappa'_1, \dots, \kappa'_{n-1}$ denote the principal curvatures of $\partial\Omega$ at $z(x)$.

First, we prove the first inequality in (2.3). Let $\tilde v_1 = \tilde v_1(r)$ be a solution to the following problem

$$\begin{cases} \binom{n-1}{k-1} \frac{u''}{(1+u'^2)^{3/2}} \left(\frac{u'}{r\sqrt{1+u'^2}}\right)^{k-1} & \\ \qquad + \binom{n-1}{k} \left(\frac{u'}{r\sqrt{1+u'^2}}\right)^{k} = f(u)g(|u'|), & \text{in } (0, \mathrm{diam}\,\Omega), \\ u(0) = u_0 > t_0, \quad u'(0) = 0, & \\ u(r) \to \infty & \text{as } r \to \mathrm{diam}\,\Omega - 0. \end{cases} \tag{2.7}$$

The existence of the solution $\tilde v_1$ is guaranteed by the hypotheses (A2), (A3), (B1) and (B3); see [28, Theorem 3.6] for the proof. We set $v_1(x) = \tilde v_1(|x|)$, so that v_1 is a classical radially symmetric solution to (1.1) with the boundary blowup condition

$$v_1(x) \to \infty \quad \text{as } \mathrm{dist}(x, B_{\mathrm{diam}\Omega}(0)) \to 0. \tag{2.8}$$

For $y \in \Omega$ which satisfies $d(y) = 3R/4$, it follows from the comparison principle that

$$u(x) \ge v_1(x - y) \quad \text{in } \left\{ x \in \Omega \,\middle|\, |x - y| < \frac{R}{2} \right\}. \tag{2.9}$$

Therefore, setting $C := v_1(0)$, we obtain that for any solution u to (1.1) and any point $y \in \Omega$ which satisfies $d(y) = 3R/4$,

$$u(y) \ge C. \tag{2.10}$$

Next, we see that there exists a constant $w_1 > t_0$ such that a non-increasing, convex solution w on $(0, R]$ to the following problem

$$\begin{cases} f(w)g(|w'|) = \left(\frac{|w'|}{\sqrt{1+w'^2}}\right)^k \cdot m & \text{in } (0,R), \\ w(r) \to \infty & \text{as } r \to +0, \\ w(R) = w_1 \end{cases} \tag{2.11}$$

exists. Indeed, by the same argument as in [28, Section 3], one can prove the existence of such solution. We omit its proof. For $\varepsilon \in (0, R/4)$, we define

$$v_{1\varepsilon}(x) = w(d(x)+\varepsilon) + L, \qquad x \in \overline{\Omega_{3R/4}}, \tag{2.12}$$

where $L = \min\{C - w(3R/4), 0\}$. Then it follows that $u(x) \to \infty$ as $d(x) \to 0$ while $v_{1\varepsilon}(x)$ takes finite value on the set $\{d(x) = 0\} = \partial\Omega$. Moreover, for any x which satisfies $d(x) = 3R/4$ we have

$$v_{1\varepsilon}(x) = w\left(\frac{3}{4}R + \varepsilon\right) + L \le w\left(\frac{3}{4}R\right) + L \le C \le u(x) \tag{2.13}$$

due to (2.10). Finally, it holds that for $x \in \Omega_{3R/4}$

(2.14)

$$\begin{aligned} H_k[v_{1\varepsilon}](x) = &\left(\frac{|w'(d(x)+\varepsilon)|}{\sqrt{1+w'(d(x)+\varepsilon)^2}}\right)^k S_k(\tilde{\kappa}) \\ &+ \frac{w''(d(x)+\varepsilon)}{(1+w'(d(x)+\varepsilon)^2)^{3/2}} \left(\frac{|w'(d(x)+\varepsilon)|}{\sqrt{1+w'(d(x)+\varepsilon)^2}}\right)^{k-1} S_{k-1}(\tilde{\kappa}) \\ \ge &\left(\frac{|w'(d(x)+\varepsilon)|}{\sqrt{1+w'(d(x)+\varepsilon)^2}}\right)^k \cdot m \\ = &\, f(w(d(x)+\varepsilon))\, g(|w'(d(x)+\varepsilon)|) \\ = &\, f(v_{1\varepsilon}(x) - L)\, g(|Dv_{1\varepsilon}(x)|) \ge f(v_{1\varepsilon}(x))\, g(|Dv_{1\varepsilon}(x)|). \end{aligned}$$

Here we note that $L \le 0$. Therefore we can deduce by the comparison principle that

$$v_{1\varepsilon}(x) = w(d(x)+\varepsilon) + L \le u(x) \tag{2.15}$$

for $x \in \Omega_{3R/4}$. Taking the limit $\varepsilon \to +0$, we get that

$$w(d(x)) + L \le u(x) \tag{2.16}$$

for $x \in \Omega_{3R/4}$.

Using (2.11) and the condition (B3), we obtain that

$$\begin{aligned} |w'| = -w' &= h^{-1}\left(m^{-1/k} f(w)^{1/k}\right) \\ &\le \max\left\{1, m^{-1/\alpha k}\right\} h^{-1}(f(w)^{1/k}). \end{aligned} \tag{2.17}$$

Integrating from 0 to r yields that

$$\psi(w(r)) = \int_{w(r)}^{\infty} \frac{ds}{h^{-1}(f(s)^{1/k})} \le \max\left\{1, m^{-1/\alpha k}\right\} r \tag{2.18}$$

for $r \in (0, R)$. Combining (2.16) and (2.18), we conclude that the first inequality in (2.3) holds.

Next we prove the second inequality in (2.3). As we have argued before, we see that there exists a constant $w_2 > t_0$ such that a non-increasing, convex solution $\tilde{w}$ on $(0, R]$ to the following problem

$$\begin{cases} f(\tilde{w})g(|\tilde{w}'|) = \left(\frac{|\tilde{w}'|}{\sqrt{1+(\tilde{w}')^2}}\right)^k \cdot M & \text{in } (0,R) \\ \tilde{w}(r) \to \infty & \text{as } r \to +0, \\ \tilde{w}(R) = w_2 \end{cases} \tag{2.19}$$

exists. We choose a constant $R' \in (0, R)$ such that $\tilde{w}(R') \geq T'$, where T' is a constant which appears in the condition (B2). For $\varepsilon \in (0, R'/4)$, we define

$$v_{2\varepsilon}(x) = \tilde{w}(d(x) - \varepsilon) + L', \quad x \in \overline{\Omega_{R'} \setminus \Omega_\varepsilon}, \tag{2.20}$$

where L' is a positive constant to be determined later.

Hereafter, we use the abbreviation: $v_{2\varepsilon} = v_{2\varepsilon}(x)$ and $\tilde{w} = \tilde{w}(d(x) - \varepsilon)$. Then it follows from (B2) that

$$f(\tilde{w}) = f(v_{2\varepsilon} - L') \leq f(v_{2\varepsilon}) - L' f'(\tilde{w}) \quad \text{in } \Omega_{R'}. \tag{2.21}$$

By differentiating the ODE in (2.19), we have

$$\tilde{w}'' = \frac{f'(\tilde{w})\tilde{w}' g(|\tilde{w}'|)^2 \left(1 + (\tilde{w}')^2\right)^{3/2}}{M|\tilde{w}'| \left(1 + (\tilde{w}')^2\right) g'(|\tilde{w}'|) - Mkg(|\tilde{w}'|)} \left(\frac{|\tilde{w}'|}{\sqrt{1+(\tilde{w}')^2}}\right)^{-(k-1)}, \tag{2.22}$$

which implies that

(2.23)

$$\begin{aligned} H_k[v_{2\varepsilon}] &= \left(\frac{|\tilde{w}'|}{\sqrt{1+(\tilde{w}')^2}}\right)^k S_k(\tilde{\kappa}) + \frac{\tilde{w}''}{(1+(\tilde{w}')^2)^{3/2}} \left(\frac{|\tilde{w}'|}{\sqrt{1+(\tilde{w}')^2}}\right)^{k-1} S_{k-1}(\tilde{\kappa}) \\ &\leq f(\tilde{w})g(|\tilde{w}'|) + \frac{f'(\tilde{w})\tilde{w}' g(|\tilde{w}'|)^2}{M|\tilde{w}'| \left(1 + (\tilde{w}')^2\right) g'(|\tilde{w}'|) - Mkg(|\tilde{w}'|)} S_{k-1}(\tilde{\kappa}) \\ &\leq g(|Dv_{2\varepsilon}|) \left(f(v_{2\varepsilon}) - f'(\tilde{w}) \left(L' + \frac{S_{k-1}(\tilde{\kappa})}{M\frac{(1+(\tilde{w}')^2)g'(|\tilde{w}'|)}{g(|\tilde{w}'|)} + \frac{Mk}{\tilde{w}'}} \right) \right). \end{aligned}$$

Here we used (2.21). By the boundedness of $S_{k-1}(\tilde{\kappa})$ in $\Omega_{R'}$ and the condition (B4), one sees that there exists $R'' \in (0, R')$ (which depends on L', but does not depend on ε) such that

$$H_k[v_{2\varepsilon}] \leq f(v_{2\varepsilon})\, g(|Dv_{2\varepsilon}|) \qquad \text{in } \Omega_{R''} \setminus \Omega_\varepsilon. \tag{2.24}$$

Now we choose L' sufficiently large so that $L' > \tilde{v}_2(0) - w_2$ where $\tilde{v}_2 = \tilde{v}_2(r)$ is a solution to

$$\begin{cases} \binom{n-1}{k-1} \frac{u''}{\left(1+u'^2\right)^{3/2}} \left(\frac{u'}{r\sqrt{1+u'^2}}\right)^{k-1} \\ \qquad + \binom{n-1}{k} \left(\frac{u'}{r\sqrt{1+u'^2}}\right)^k = f(u)g(|u'|), \quad r > 0 \\ u(0) = \tilde{v}_2(0) > t_0, \quad u'(0) = 0, \\ u(r) \to \infty & \text{as } r \to R''/2 - 0. \end{cases} \tag{2.25}$$

It is possible because as L' is larger and larger, we can choose R'' larger and larger so that $\tilde{v}_2(0)$ becomes smaller and smaller. We set $v_2(x) = \tilde{v}_2(|x|)$.

Then, it follows from the comparison principle that $u(y) \le v_2(0) = \tilde{v}_2(0)$ for any $y \in \Omega$ which satisfies $d(y) = R''$. Thus we have that

$$v_{2\varepsilon}(y) = \tilde{w}(R'' - \varepsilon) + L' \ge w_2 + L' > \tilde{v}_2(0) \ge u(y) \tag{2.26}$$

for any $y \in \Omega$ which satisfies $d(y) = R''$. Moreover, it holds that $v_{2\varepsilon}(x) \to \infty$ as $d(x) \to \varepsilon + 0$ while $u(x)$ takes finite value if $d(x) = \varepsilon$. Therefore, we can deduce by the comparison principle that

$$v_{2\varepsilon}(x) = \tilde{w}(d(x) - \varepsilon) + L' \ge u(x) \tag{2.27}$$

for $x \in \Omega_{R''} \setminus \Omega_\varepsilon$. Taking the limit $\varepsilon \to +0$, we get that

$$\tilde{w}(d(x)) + L' \ge u(x) \tag{2.28}$$

for $x \in \Omega_{R''}$.

Using (2.19) and the condition (B3), we obtain that

$$\begin{aligned} -\tilde{w}' &= h^{-1}\left(M^{-1/k} f(\tilde{w})^{1/k}\right) \\ &\ge \min\left\{1, M^{-1/\alpha k}\right\} h^{-1}(f(\tilde{w})^{1/k}). \end{aligned} \tag{2.29}$$

Integrating from 0 to r yields that

$$\psi(w(r)) = \int_{w(r)}^{\infty} \frac{ds}{h^{-1}(f(s)^{1/k})} \ge \min\left\{1, M^{-1/\alpha k}\right\} r \tag{2.30}$$

for $r \in (0, R'')$. Combining (2.28) and (2.30), we conclude that the second inequality in (2.3) holds. $\square$

Example 2.2. Let $1 \le k \le n-1$ and $p, q > 0$. Suppose Ω is a bounded and uniformly k-convex domain with boundary $\partial\Omega \in C^\infty$. We consider the same equations as in Example 2.1:

(i) $H_k[u] = u^p / \left(1 + |Du|^2\right)^{q/2}$ in Ω.

Theorem 2.2 implies that a boundary blowup solution u (if it exists) satisfies

$$C_1 \operatorname{dist}(x, \partial\Omega)^{-\frac{q}{p-q}} \le u(x) \le C_2 \operatorname{dist}(x, \partial\Omega)^{-\frac{q}{p-q}} \quad \text{near } \partial\Omega \tag{2.31}$$

for some constants $C_1, C_2 > 0$, provided $p \ge k$ and $p > q$.

(ii) $H_k[u] = e^{pu} / \left(1 + |Du|^2\right)^{q/2}$ in Ω.

We can also see that a boundary blowup solution u (if it exists) satisfies

$$u(x) = -\frac{q}{p} \log \operatorname{dist}(x, \partial\Omega) + O(1) \quad \text{near } \partial\Omega, \tag{2.32}$$

provided $q > 0$.

Remark 2.2. The case $k = n$, which corresponds to Gauss curvature equation, is excluded from Theorems 2.1 and 2.2. Alternative results for the case $k = n$ are given in Section 4.

3. Uniqueness results for boundary blowup problem

In this section, we give the uniqueness result for the boundary blowup problem (1.1)-(1.2) for $1 \le k \le n-1$.

Theorem 3.1. *Let $1 \le k \le n-1$. We assume that the conditions in Theorem 2.2 are satisfied. Also, we assume the following.*

(C1) *Ω is star-shaped (with respect to some point $x_0 \in \Omega$).*

(C2) *There exists constants $\beta > 0$ and $T'' > 0$ such that $f(t)/t^\beta$ is non-decreasing in $[T'', \infty)$.*

(C3) *$\lim_{s\to+0} s\psi^{-1}(s) = 0$, where ψ is defined in Theorem 2.2.*

Then the problem (1.1)-(1.2) has at most one viscosity solution.

Proof. In this proof, we denote the notation $d(x) = \operatorname{dist}(x, \partial\Omega)$ again. Without loss of generality, we may assume that $x_0 = 0$. Suppose that u_1 and u_2 be solutions to (1.1)-(1.2). In the following proof, we argue in the classical sense, but one can justify it in the viscosity sense.

For $\lambda \in (1,2)$, we define a function $\tilde{u}_{2,\lambda}$ in Ω by $\tilde{u}_{2,\lambda}(x) = \lambda u_2(x/\lambda) - \phi(\lambda)$, where $\phi(\lambda)$ is a positive constant to be determined later. It can be defined due to the condition (C1). Then it holds that

$$H_k[\tilde{u}_{2,\lambda}] = \frac{1}{\lambda^k} H_k[u_2]\left(\frac{x}{\lambda}\right). \tag{3.1}$$

Later, we will determine $\phi(\lambda)$ appropriately, in such a way as to satisfy that $\tilde{u}_{2,\lambda}$ is a subsolution to (1.1) and that $\phi(\lambda) \to 0$ as $\lambda \to 1+0$.

Meanwhile, we suppose that one can choose $\phi(\lambda)$ as above. Now $u_1(x) \to \infty$ as $d(x) \to 0$, while $\tilde{u}_{2,\lambda}$ has finite value on $\partial\Omega$. It follows from the comparison principle that

$$u_1(x) \ge \tilde{u}_{2,\lambda}\left(\frac{x}{\lambda}\right) - \phi(\lambda). \tag{3.2}$$

Letting $\lambda \to 1+0$, we get $u_1 \ge u_2$ in Ω.

Now we prove that $\phi(\lambda)$ can be chosen as desired. Noticing $D\tilde{u}_{2,\lambda}(x) = Du_2(x/\lambda)$, we have by (3.1) that

$$H_k[\tilde{u}_{2,\lambda}] = \frac{1}{\lambda^k} f\left(u_2\left(\frac{x}{\lambda}\right)\right) g(|D\tilde{u}_{2,\lambda}(x)|). \tag{3.3}$$

Therefore, $\tilde{u}_{2,\lambda}$ is a subsolution to (1.1) if and only if it holds that for any $x \in \Omega$,

$$\frac{1}{\lambda^k} f\left(u_2\left(\frac{x}{\lambda}\right)\right) \ge f(\tilde{u}_{2,\lambda}(x)) = f\left(\lambda u_2\left(\frac{x}{\lambda}\right) - \phi(\lambda)\right). \tag{3.4}$$

By Theorem 2.2, we obtain that there exist constants $c > t_0$ and $c_1, c_2 > 0$ such that for any $x \in \Omega$,

$$c \le u_2\left(\frac{x}{\lambda}\right) \le \psi^{-1}\left(\frac{\lambda-1}{\lambda}c_1\right) + c_2, \tag{3.5}$$

because $d(x/\lambda) \ge (1-1/\lambda)c_1$ for some $c_1 > 0$ which is independent of $x \in \Omega$. Therefore it is enough to prove that there exists $\phi(\lambda) > 0$ such that

$$f^{-1}\left(\frac{1}{\lambda^k} f(s)\right) \ge \lambda s - \phi(\lambda) \quad \text{for any } c \le s \le \psi^{-1}\left(\frac{\lambda-1}{\lambda}c_1\right) + c_2, \tag{3.6}$$

and that $\phi(\lambda) \to 0$ as $\lambda \to 1+0$.

We set

$$\psi(\lambda, s) = \lambda s - f^{-1}\left(\frac{1}{\lambda^k} f(s)\right), \quad \lambda \in [1,2],\ s \in (s_0, \infty), \tag{3.7}$$

and

$$\eta(\lambda) = \sup_{c \le s \le 2^{k/\beta} T''} |\psi(\lambda, s)|, \quad \lambda \in [1,2], \tag{3.8}$$

where β and T'' are constants which appear in the condition (C2). We note that $\psi(1,s) \equiv 0$ which implies $\eta(1) = 0$. Then it is easily seen that $\lim_{\lambda \to 1+0} \eta(\lambda) = 0$.

Now we define

$$\phi(\lambda) = \left(\lambda - \lambda^{-k/\beta}\right)\left(\psi^{-1}\left(\frac{\lambda - 1}{\lambda} c_1\right) + c_2\right) + \eta(\lambda). \tag{3.9}$$

We fix arbitrary s which satisfies $c \le s \le \psi^{-1}((\lambda-1)c_1/\lambda) + c_2$. First, if $c \le s \le 2^{k/\beta} T''$, then we get that

$$f^{-1}\left(\frac{1}{\lambda^k} f(s)\right) = \lambda s - \psi(\lambda, s) \ge \lambda s - \eta(\lambda) \ge \lambda s - \phi(\lambda). \tag{3.10}$$

Next, if $2^{k/\beta} T'' \le s \le \psi^{-1}((\lambda-1)c_1/\lambda) + c_2$, then it holds that

$$f^{-1}\left(\frac{1}{\lambda^k} f(s)\right) \ge \frac{s}{\lambda^{k/\beta}} = \lambda s - \left(\lambda - \lambda^{-k/\beta}\right) s \ge \lambda s - \phi(\lambda). \tag{3.11}$$

Here we used the condition (C2). Furthermore, it holds that

$$\phi(\lambda) = \frac{\lambda\left(\lambda - \lambda^{-k/\beta}\right)}{(\lambda-1)c_1}\left[\frac{(\lambda-1)c_1}{\lambda}\left(\psi^{-1}\left(\frac{\lambda-1}{\lambda} c_1\right) + c_2\right)\right] + \eta(\lambda) \to 0 \tag{3.12}$$

as $\lambda \to 1+0$, due to the condition (C3). This completes the proof.

By the similar argument, we see that $u_1 \le u_2$ in Ω and hence $u_1 = u_2$ in Ω. □

Example 3.1. Let $1 \le k \le n-1$ and $p, q > 0$. Suppose Ω is a bounded, star-shaped and uniformly k-convex domain with boundary $\partial\Omega \in C^\infty$. We consider again the same equations as in the last section:

(i) $H_k[u] = u^p / \left(1 + |Du|^2\right)^{q/2}$ in Ω.

Theorem 3.1 implies that there exists at most one boundary blowup solution, provided $p \ge k$ and $p > 2q$.

(ii) $H_k[u] = e^{pu} / \left(1 + |Du|^2\right)^{q/2}$ in Ω.

It follows that there exists at most one boundary blowup solution for any $p, q > 0$.

4. The case of Gauss curvature equation

The case $k = n$, which corresponds to Gauss curvature equation

$$\frac{\det D^2 u}{(1 + |Du|^2)^{(n+2)/2}} = f(u)g(|Du|) \tag{4.1}$$

is excluded from all theorems in Sections 2 and 3. In this section, we shall obtain the alternative results for the case $k = n$.

First, we state results for the existence and for the asymptotic behavior of a boundary blowup solution, which we have already proved in [28].

Theorem 4.1. *Let $k = n$. We assume that Ω is a bounded and strictly convex domain with boundary $\partial\Omega \in C^\infty$. Furthermore, we also assume that the conditions (A3) is satisfied and that $\limsup_{t\to\infty} g(t)t < \infty$. Then there exists a viscosity solution to (1.1)-(1.2).*

Theorem 4.2. *Let $k = n$. We assume that Ω is a bounded and strictly convex domain with boundary $\partial\Omega \in C^\infty$. Furthermore, we also assume that the conditions (A3), (B1), (B2) and*

(B5) *There exists a constant $\alpha > 0$ such that $H(t)/t^\alpha$ is non-decreasing for $t > 0$, where*

$$H(t) = \int_0^t \frac{s^n}{g(s)\,(1 + s^2)^{(n+2)/2}}\,ds, \tag{4.2}$$

are satisfied. Then there exist positive constants C_1, C_2 such that every solution u to (1.1)-(1.2) satisfies

$$C_1 \operatorname{dist}(x, \partial\Omega) \le \Psi(u(x)) \le C_2 \operatorname{dist}(x, \partial\Omega), \tag{4.3}$$

where Ψ is defined by

$$\Psi(t) = \int_t^\infty \frac{ds}{H^{-1}(F(s))}. \tag{4.4}$$

Next, we establish the uniqueness result for the case $k = n$.

Theorem 4.3. *Let $k = n$. We assume that the conditions in Theorem 4.2 are satisfied. Also, we assume that the conditions (C2) and*

(C3)' $\lim_{s\to+0} s\Psi^{-1}(s) = 0$, *where Ψ is defined in Theorem 4.2,*

are satisfied. Then the problem (1.1)-(1.2) has at most one viscosity solution.

The proof of this theorem is mostly the same as that of Theorem 3.1, so we omit it. Finally, we give some examples.

Example 4.1. Let $k = n$ and $p, q > 0$. Suppose Ω is a bounded and strictly convex domain with boundary $\partial\Omega \in C^\infty$. We consider again the same equations given before:

(i) $H_n[u] = u^p / \left(1 + |Du|^2\right)^{q/2}$ in Ω.

Theorem 2.1 implies that if $p > q \ge 1$, then there exists a boundary blowup solution, and it follows from Theorem 2.2 that any boundary blowup solution satisfies

$$C_1 \operatorname{dist}(x, \partial\Omega)^{-\frac{q-1}{p-q+2}} \le u(x) \le C_2 \operatorname{dist}(x, \partial\Omega)^{-\frac{q-1}{p-q+2}} \quad \text{near } \partial\Omega \tag{4.5}$$

for some constants $C_1, C_2 > 0$, provided $p \geq n$ and $p > q > 1$. Moreover, Theorem 2.2 implies that there exists at most one boundary blowup solution, provided $p \geq n$, $p > 2q - 3$ and $q > 1$.

(ii) $H_n[u] = e^{pu} / \left(1 + |Du|^2\right)^{q/2}$ in Ω.

One can see that there exists a unique boundary blowup solution which satisfies

$$u(x) = -\frac{q-1}{p} \log \operatorname{dist}(x, \partial\Omega) + O(1) \quad \text{near } \partial\Omega, \tag{4.6}$$

provided $q > 1$.

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Department of Mathematics, Graduate School of Science, Hiroshima University, 1-3-1 Kagamiyama, Higashi-Hiroshima city, Hiroshima 739-8526, Japan
E-mail address: `d113989@hiroshima-u.ac.jp`

Department of Mathematics, Graduate School of Science, Hiroshima University, 1-3-1 Kagamiyama, Higashi-Hiroshima city, Hiroshima 739-8526, Japan
E-mail address: `takimoto@math.sci.hiroshima-u.ac.jp`

A Bernstein type theorem for parabolic k-Hessian equations

Saori Nakamori[a,*], Kazuhiro Takimoto[a,1]

[a] *Department of Mathematics, Graduate School of Science, Hiroshima University, 1-3-1 Kagamiyama, Higashi-Hiroshima city, Hiroshima 739-8526, Japan*

Abstract

We are concerned with the characterization of entire solutions to the parabolic k-Hessian equation of the form $-u_t F_k(D^2u) = 1$ in $\mathbb{R}^n \times (-\infty, 0]$. We prove that for $1 \le k \le n$, any strictly convex-monotone solution $u = u(x,t) \in C^{4,2}(\mathbb{R}^n \times (-\infty, 0])$ to $-u_t F_k(D^2u) = 1$ in $\mathbb{R}^n \times (-\infty, 0]$ must be a linear function of t plus a quadratic polynomial of x, under some growth assumptions on u.

Keywords: Bernstein type theorem, Fully nonlinear equation, Parabolic Hessian equation, Pogorelov type lemma
2010 MSC: 35K55, 35B08, 35K96

1. Introduction

In the early 20th century, Bernstein [2] proved the following theorem; *If $f \in C^2(\mathbb{R}^2)$ and the graph of $z = f(x,y)$ is a minimal surface in $\mathbb{R}^3$, then f is necessarily a linear function of x and y.* This theorem gives the characterization of entire solutions to the minimal surface equation defined in the whole plane $\mathbb{R}^2$.

Many problems on the classification of entire solutions to PDEs have been extensively studied. We list some results concerning Bernstein type

* Corresponding author.
Email addresses: `d113989@hiroshima-u.ac.jp` (Saori Nakamori), `takimoto@math.sci.hiroshima-u.ac.jp` (Kazuhiro Takimoto)

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theorems for *fully nonlinear* equations. First, for Monge-Ampère equation, the following theorem is known.

Theorem 1.1. *Let $u \in C^4(\mathbb{R}^n)$ be a convex solution to*

$$\det D^2 u = 1 \quad \text{in } \mathbb{R}^n. \tag{1.1}$$

Then u is a quadratic polynomial.

This theorem was proved by Jörgens [15] for $n = 2$, by Calabi [6] for $n \leq 5$, and by Pogorelov [19] for arbitrary $n \geq 2$ (see also [7] for a simpler proof). Caffarelli [3] proved that the result holds for viscosity solutions (see also [4]). Moreover, Jian and Wang [14] obtained Bernstein type result for a certain Monge-Ampère equation in the half space $\mathbb{R}^n_+$.

Here we note that the convexity assumption in Theorem 1.1 is quite natural, since Monge-Ampère operator $\det D^2u$ is degenerate elliptic for convex functions so that we usually seek solutions in the class of convex functions when we deal with Monge-Ampère equation.

Later, Bao, Chen, Guan and Ji [1] extended this result to the so-called *k-Hessian equation* of the form

$$F_k(D^2u) = 1 \quad \text{in } \mathbb{R}^n, \tag{1.2}$$

for $1 \leq k \leq n$. Here $F_k(D^2u)$ is defined by

$$F_k(D^2u) = S_k(\lambda_1, \ldots, \lambda_n), \tag{1.3}$$

where, for a C^2 function u, $\lambda_1, \ldots, \lambda_n$ denote the eigenvalues of the Hessian matrix D^2u, and S_k denotes the k-th elementary symmetric function, that is

$$S_k(\lambda_1, \ldots, \lambda_n) = \sum \lambda_{i_1} \cdots \lambda_{i_k}, \tag{1.4}$$

where the sum is taken over all increasing k-tuples, $1 \leq i_1 < \cdots < i_k \leq n$.

Laplace operator Δu and Monge-Ampère operator $\det D^2u$ correspond respectively to the special cases $k = 1$ and $k = n$ in (1.3). Hence, the class of k-Hessian equations includes important PDEs which arise in physics and geometry. Here we remark that (1.3) is a linear operator for $k = 1$ while it is a fully nonlinear operator for $k \geq 2$. It is much harder to study the intermediate case $2 \leq k \leq n-1$. Though, there are a number of papers concerning the analysis of k-Hessian equation, such as the solvability of the Dirichlet problem, see [5, 9, 12, 20, 21, 22, 23, 24, 25] for example.

Bao, Chen, Guan and Ji [1] proved the following Bernstein type theorem for k-Hessian equation (1.2).

Theorem 1.2. *Let $1 \le k \le n$ and $u \in C^4(\mathbb{R}^n)$ be a strictly convex solution to* (1.2). *Suppose that there exist constants $A, B > 0$ such that for all $x \in \mathbb{R}^n$,*

$$u(x) \ge A|x|^2 - B. \tag{1.5}$$

Then u is a quadratic polynomial.

In this theorem, for the case $k = n$ which corresponds to Monge-Ampère equation, the assumption (1.5) can be removed, due to Theorem 1.1. Furthermore, for the case $k = 1$ which corresponds to Poisson equation $\Delta u = 1$, the assumption (1.5) can also be removed. It is because the classical convex solution to $\Delta u = 1$ in $\mathbb{R}^n$ must be quadratic, as it follows almost straightforward from Liouville's theorem for harmonic functions.

Next, Gutiérrez and Huang [11] extended Theorem 1.1 to the parabolic analogue of Monge-Ampère equation

$$-u_t \det D^2 u = 1 \quad \text{in } \mathbb{R}^n \times (-\infty, 0]. \tag{1.6}$$

Here D^2u means the matrix of second partial derivatives with respect to x. This type of equation was firstly proposed by Krylov [16].

The function $u = u(x,t) : \mathbb{R}^n \times (-\infty, 0] \to \mathbb{R}$ is said to be *convex-monotone* if it is convex in x and non-increasing in t. We state Bernstein type theorem for (1.6) which Gutiérrez and Huang [11] proved.

Theorem 1.3. *Let $u \in C^{4,2}(\mathbb{R}^n \times (-\infty, 0])$ be a convex-monotone solution to* (1.6). *Suppose that there exist constants $m_1 \ge m_2 > 0$ such that for all $(x,t) \in \mathbb{R}^n \times (-\infty, 0]$,*

$$-m_1 \le u_t(x,t) \le -m_2. \tag{1.7}$$

Then u has the form $u(x,t) = -mt + p(x)$ where $m > 0$ is a constant and p is a quadratic polynomial.

We note that Xiong and Bao [28] have recently obtained Bernstein type theorems for more general parabolic Monge-Ampère equations, such as $u_t = (\det D^2u)^{1/n}$ and $u_t = \log\det D^2u$. However, as far as we know, Bernstein type theorems for parabolic fully nonlinear equations are known only for the parabolic Monge-Ampère equations.

In this paper, we are concerned with the parabolic analogue of k-Hessian equation of the following form

$$-u_t F_k(D^2u) = 1 \quad \text{in } \mathbb{R}^n \times (-\infty, 0], \tag{1.8}$$

for $1 \le k \le n$. Here $F_k(D^2u)$ is the k-Hessian operator defined in (1.3). We call (1.8) "parabolic k-Hessian equation" in this paper. For the special case $k = n$, (1.8) reduces to the parabolic Monge-Ampère equation (1.6). We shall obtain Bernstein type theorem for (1.8).

This paper is divided as follows. In Section 2, we state our main result and give the strategy for the proof. In Section 3, we prove Pogorelov type lemma, which is used later. Section 4 is devoted to the proof of the main result. Finally, in Section 5, we state some remarks and open problems.

2. Main result

The function $u = u(x,t) : \mathbb{R}^n \times (-\infty, 0] \to \mathbb{R}$ is said to be *strictly convex-monotone* if u is strictly convex in x and decreasing in t. Here is our main result of this paper.

Theorem 2.1. *Let $1 \le k \le n$ and $u \in C^{4,2}(\mathbb{R}^n \times (-\infty, 0])$ be a strictly convex-monotone solution to (1.8). Suppose that there exist constants $m_1 \ge m_2 > 0$ such that for all $(x,t) \in \mathbb{R}^n \times (-\infty, 0]$,*

$$-m_1 \le u_t(x,t) \le -m_2, \tag{2.1}$$

and that there exist constants $A, B > 0$ such that for all $x \in \mathbb{R}^n$,

$$u(x,0) \ge A|x|^2 - B. \tag{2.2}$$

Then u has the form $u(x,t) = -mt + p(x)$ where $m > 0$ is a constant and p is a quadratic polynomial.

Remark 2.1. For the case $k = n$ which corresponds to the parabolic Monge-Ampère equation (1.6), the assumption (2.2) can be removed, due to Theorem 1.3.

The proof of this theorem will be given in subsequent sections. Here we give the strategy for the proof:

Step 1. Derivation of a local gradient estimate of u.
Step 2. Pogorelov type lemma.
Step 3. Combining these results and Evans-Krylov type theorem, we obtain local α-Hölder estimates of D^2u and u_t.

3. Pogorelov type lemma

We introduce some notation. First, if $D \subset \mathbb{R}^n \times (-\infty, 0]$ and $t \leq 0$, $D(t)$ is denoted by

$$D(t) = \{x \in \mathbb{R}^n \mid (x,t) \in D\}.$$

Let $D \subset \mathbb{R}^n \times (-\infty, 0]$ be a bounded set and $t_0 = \inf\{t \leq 0 \mid D(t) \neq \emptyset\}$. The parabolic boundary $\partial_p D$ of D is defined by

$$\partial_p D = \left(\overline{D(t_0)} \times \{t_0\}\right) \cup \bigcup_{t \leq 0} (\partial D(t) \times \{t\}),$$

where $\overline{D(t_0)}$ denotes the closure of $D(t_0)$ and $\partial D(t)$ denotes the boundary of $D(t)$. We say that the domain $D \subset \mathbb{R}^n \times (-\infty, 0]$ is a *bowl-shaped* domain if $D(t)$ is convex for each $t \in (-\infty, 0]$ and $D(t_1) \subset D(t_2)$ for $t_1 \leq t_2 \leq 0$.

Next, for $\lambda = (\lambda_1, \ldots, \lambda_n)$ and $1 \leq m \leq n$, we define

$$S_{m;i_1 i_2 \ldots i_j}(\lambda) = \begin{cases} S_m(\lambda)\,\big|_{\lambda_{i_1} = \lambda_{i_2} = \cdots = \lambda_{i_j} = 0} & \text{if } i_p \neq i_q \text{ for any } 1 \leq p < q \leq j, \\ 0 & \text{otherwise.} \end{cases}$$

In this section, we prove Pogorelov type lemma. This is an analogue of the result of Pogorelov [18], who derived interior C^2-estimates of a solution from C^1-estimates for Monge-Ampère equation. The idea of the proof of the following proposition is adapted from that of [8].

Proposition 3.1. *Let D be a bounded bowl-shaped domain in $\mathbb{R}^n \times (-\infty, 0]$ and $u \in C^{4,2}(\overline{D})$ a strictly convex-monotone solution to $-u_t F_k(D^2u) = 1$ in D with $u = 0$ on $\partial_p D$, which satisfies* (2.1) *in D. Then there exists a constant $C = C(n, k, m_2, \|u\|_{C^1(D)})$ such that*

$$\sup_{(x,t) \in D} |u(x,t)|^4 |D^2 u(x,t)| \leq C. \tag{3.1}$$

Proof. We consider the auxiliary function

$$\Psi(x,t;\xi) = (-u(x,t))^4 \varphi\left(\frac{|Du(x,t)|^2}{2}\right) D_{\xi\xi}u(x,t), \quad (x,t) \in \overline{D},\ |\xi| = 1,$$

where $\varphi(s) = (1 - s/M)^{-1/8}$ and $M = 2\sup_{(x,t)\in D} |Du(x,t)|^2$.

Then we can take a point $(x_0, t_0) \in \overline{D}$ and a unit vector $\xi_0 \in \mathbb{R}^n$ which satisfy

$$\Psi(x_0, t_0; \xi_0) = \max\{\Psi(x,t;\xi) \mid (x,t) \in \overline{D},\ |\xi| = 1\}.$$

The point (x_0, t_0) can be taken in $\overline{D}\backslash\partial_p D$ due to the boundary condition $u = 0$ on $\partial_p D$. Without loss of generality, we may assume $\xi_0 = e_1$ and $D^2u(x_0, t_0)$ is diagonal with $D_{11}u(x_0,t_0) \geq D_{22}u(x_0,t_0) \geq \cdots \geq D_{nn}u(x_0,t_0) > 0$. Then $\Psi = \Psi(x,t;e_1) = (-u(x,t))^4\varphi(|Du(x,t)|^2/2)D_{11}u(x,t)$ attains its maximum at (x_0,t_0) and the eigenvalues of $D^2u(x_0,t_0)$ are $\lambda = (\lambda_1,\ldots,\lambda_n) = (u_{11}(x_0,t_0),\ldots,u_{nn}(x_0,t_0))$. It is enough to consider the case $\lambda_1 = u_{11}(x_0,t_0) \geq 1$. Here and throughout the paper, we denoted D_iu by u_i, $D_{ij}u$ by u_{ij}, and so on.

Since Ψ attains its maximum at (x_0,t_0), direct calculation gives

$$(\log\Psi)_i = \frac{4u_i}{u} + \frac{\varphi_i}{\varphi} + \frac{u_{11i}}{u_{11}} = 0, \tag{3.2}$$

$$(\log\Psi)_{ii} = 4\left(\frac{u_{ii}}{u} - \frac{u_i^2}{u^2}\right) + \frac{\varphi_{ii}}{\varphi} - \frac{\varphi_i^2}{\varphi^2} + \frac{u_{11ii}}{u_{11}} - \frac{u_{11i}^2}{u_{11}^2} \leq 0, \tag{3.3}$$

$$(\log\Psi)_t = \frac{4u_t}{u} + \frac{\varphi_t}{\varphi} + \frac{u_{11t}}{u_{11}} \geq 0, \tag{3.4}$$

$$\varphi_i = \varphi'\left(\frac{|Du|^2}{2}\right)u_iu_{ii}, \tag{3.5}$$

$$\varphi_{ii} = \varphi''\left(\frac{|Du|^2}{2}\right)u_i^2u_{ii}^2 + \varphi'\left(\frac{|Du|^2}{2}\right)\left(u_{ii}^2 + \sum_{j=1}^n u_ju_{iij}\right), \tag{3.6}$$

$$\varphi_t = \varphi'\left(\frac{|Du|^2}{2}\right)\sum_{j=1}^n u_ju_{jt} \tag{3.7}$$

at (x_0,t_0), for $i = 1,\ldots,n$. We set $f(D^2u) = F_k(D^2u)^{1/k}$, then u satisfies

$$(-u_t)^{\frac{1}{k}} f(D^2u) = 1 \quad \text{in } \overline{D}. \tag{3.8}$$

Differentiating (3.8) with respect to x_γ (and using (3.8) itself) yields

$$-\frac{1}{k}(-u_t)^{-1}u_{\gamma t} + (-u_t)^{\frac{1}{k}} f_{ij}u_{ij\gamma} = 0. \tag{3.9}$$

Here, for $f = f(M)$ where $M = (m_{ij})_{1\leq i,j\leq n}$, we write $f_{ij} = \partial f/\partial m_{ij}$. Multiplying (3.9) by $(-u_t)^{-1/k}$, differentiating once more with respect to x_γ and multiplying $(-u_t)^{1/k}$, we obtain

$$-\left(\frac{1}{k}+1\right)\frac{u_{\gamma t}^2}{ku_t^2} + \frac{u_{\gamma\gamma t}}{ku_t} + (-u_t)^{\frac{1}{k}} f_{ii}u_{ii\gamma\gamma} + (-u_t)^{\frac{1}{k}} f_{ij,rs}u_{ij\gamma}u_{rs\gamma} = 0, \tag{3.10}$$

where $f_{ij,rs} = \partial^2 f/\partial m_{ij}\partial m_{rs}$. It follows from the calculation in [8, Section 4] that

$$f_{ij,rs}u_{ij\gamma}u_{rs\gamma} \leq -\frac{1}{k}\sum_{i,j=1}^{n} S_k(\lambda)^{\frac{1}{k}-1}S_{k-2;ij}(\lambda)u_{ij\gamma}^2 \tag{3.11}$$

at (x_0,t_0). By using (3.10) and (3.11), we get the inequality

$$\frac{u_{\gamma\gamma t}}{ku_t} + (-u_t)^{\frac{1}{k}} f_{ii}u_{ii\gamma\gamma} \geq (-u_t)^{\frac{1}{k}}\frac{1}{k}\sum_{i,j=1}^{n} S_k(\lambda)^{\frac{1}{k}-1}S_{k-2;ij}(\lambda)u_{ij\gamma}^2$$

at (x_0,t_0). Letting $\gamma = 1$ and multiplying $1/u_{11}$, we get at (x_0,t_0)

$$\frac{u_{11t}}{ku_tu_{11}} + (-u_t)^{\frac{1}{k}} f_{ii}\frac{u_{11ii}}{u_{11}} \geq (-u_t)^{\frac{1}{k}}\frac{1}{k}\sum_{i,j=1}^{n} S_k(\lambda)^{\frac{1}{k}-1}S_{k-2;ij}(\lambda)\frac{u_{1ij}^2}{u_{11}}. \tag{3.12}$$

Let L be the linearized operator of (3.8) at (x_0,t_0). Then one can write

$$L = \frac{1}{ku_t(x_0,t_0)}D_t + (-u_t(x_0,t_0))^{\frac{1}{k}} f_{ij}(D^2u(x_0,t_0))D_{ij}.$$

By (3.3) and (3.4), we obtain

$$\begin{aligned} L(\log\Psi) = (-u_t)^{\frac{1}{k}} f_{ii}&\left(4\left(\frac{u_{ii}}{u} - \frac{u_i^2}{u^2}\right) + \frac{\varphi_{ii}}{\varphi} - \frac{\varphi_i^2}{\varphi^2} + \frac{u_{11ii}}{u_{11}} - \frac{u_{11i}^2}{u_{11}^2}\right) \\ &+ \frac{1}{ku_t}\left(\frac{4u_t}{u} + \frac{\varphi_t}{\varphi} + \frac{u_{11t}}{u_{11}}\right) \leq 0. \end{aligned} \tag{3.13}$$

at (x_0, t_0). By substituting (3.6), (3.7), (3.9) and (3.12) into (3.13), we obtain

$$
\begin{aligned}
(-u_t)^{\frac{1}{k}} f_{ii} \left(4\left(\frac{u_{ii}}{u} - \frac{u_i^2}{u^2} \right) + \frac{\varphi''}{\varphi} u_i^2 u_{ii}^2 + \frac{\varphi'}{\varphi} u_{ii}^2 - \frac{\varphi'^2}{\varphi^2} u_i^2 u_{ii}^2 - \frac{u_{11i}^2}{u_{11}^2} \right) \\
+ (-u_t)^{\frac{1}{k}} \frac{1}{k} \sum_{i,j=1}^{n} S_k(\lambda)^{\frac{1}{k}-1} S_{k-2;ij}(\lambda) \frac{u_{1ij}^2}{u_{11}} + \frac{4}{ku} \le 0 \qquad (3.14)
\end{aligned}
$$

at (x_0, t_0).

Now we split into two cases.

(i) $u_{kk} \ge K u_{11}$, where $K > 0$ is a small constant to be determined later.

By (3.2) and (3.5), we have

$$
\frac{u_{11i}^2}{u_{11}^2} = \left(\frac{4u_i}{u} + \frac{\varphi_i}{\varphi} \right)^2 \le 2\left(\frac{16u_i^2}{u^2} + \frac{\varphi'^2 u_i^2 u_{ii}^2}{\varphi^2} \right) \qquad (3.15)
$$

at (x_0, t_0). Therefore (3.15) and the fact that the second term of the left hand side of (3.14) is non-negative yield

$$
(-u_t)^{\frac{1}{k}} f_{ii} \left(4\left(\frac{u_{ii}}{u} - \frac{9u_i^2}{u^2} \right) + \left(\frac{\varphi''}{\varphi} - \frac{3\varphi'^2}{\varphi^2} \right) u_i^2 u_{ii}^2 + \frac{\varphi'}{\varphi} u_{ii}^2 \right) + \frac{4}{ku} \le 0
$$

at (x_0, t_0). Since

$$
\sum_{i=1}^{n} f_{ii} u_{ii}^2 > f_{kk} u_{kk}^2 \ge \theta_1 \sum_{i=1}^{n} f_{ii} u_{11}^2 \qquad (3.16)
$$

holds for some constant $\theta_1 > 0$ (cf. [8]) and $\varphi''/\varphi - 3\varphi'^2/\varphi^2 \ge 0$, it can be derived by (3.15) that at (x_0, t_0)

$$
(-u_t)^{\frac{1}{k}} \theta_2 \sum_{i=1}^{n} f_{ii} u_{11}^2 - C(-u_t)^{\frac{1}{k}} \frac{1}{u^2} \sum_{i=1}^{n} f_{ii} + \frac{4}{u} \left(1 + \frac{1}{k} \right) \le 0,
$$

for some constant $\theta_2 > 0$. Here we used the fact that $\sum_{i=1}^{n} f_{ii}(D^2u) u_{ii} = f(D^2u) = (-u_t)^{-1/k}$ at (x_0, t_0), due to the homogeneity of f and (3.8). By multiplying $(-u)^8 \varphi^2$, we obtain

$$
(-u_t)^{\frac{1}{k}} \theta_2 \sum_{i=1}^{n} f_{ii} u_{11}^2 (-u)^8 \varphi^2 - C(-u_t)^{\frac{1}{k}} (-u)^6 \varphi^2 \sum_{i=1}^{n} f_{ii} - 4\left(1 + \frac{1}{k} \right) (-u)^7 \varphi^2 \le 0. \qquad (3.17)
$$

On the other hand, it holds that at (x_0, t_0)

$$
\begin{aligned}
\sum_{i=1}^{n} f_{ii}(D^2u) &= \sum_{i=1}^{n} \frac{1}{k} F_k(D^2u)^{\frac{1}{k}-1} \frac{\partial F_k}{\partial m_{ii}}(D^2u) \\
&= \frac{1}{k}(-u_t)^{1-\frac{1}{k}} \sum_{i=1}^{n} S_{k-1;i}(\lambda) \leq C(-u_t)^{1-\frac{1}{k}} u_{11}^{k-1}, \qquad (3.18)
\end{aligned}
$$

and that

$$
\begin{aligned}
\sum_{i=1}^{n} f_{ii}(D^2u) &\geq f_{nn}(D^2u) \\
&= \frac{1}{k} F_k(D^2u)^{\frac{1}{k}-1} \frac{\partial F_k}{\partial m_{nn}}(D^2u) \\
&\geq \frac{1}{k} F_k(D^2u)^{\frac{1}{k}-1} \theta_3 u_{11} \cdots u_{k-1,k-1} \geq C(-u_t)^{1-\frac{1}{k}} u_{11}^{k-1}, \qquad (3.19)
\end{aligned}
$$

for some constant $\theta_3 > 0$ (see [8, (3.2)]), by the hypothesis $u_{kk} \geq K u_{11}$. Substituting (3.18) and (3.19) into (3.17), we obtain

$$
\Psi^2 \leq C(-u)^6 \varphi^2 + \frac{(-u)^7 \varphi^2}{(-u_t) u_{11}^{k-1}} \leq C(n, k, m_2, \|u\|_{C^1(D)}),
$$

at (x_0, t_0). Therefore, for all $(x, t) \in D$ and $\xi \in \mathbb{R}^n$ with $|\xi| = 1$, $(-u)^4 u_{\xi\xi} \leq C$ holds, so that $(-u)^4 |D^2 u|$ can be estimated from above by some constant C.

(ii) $u_{kk} \leq K u_{11}$, that is, $u_{jj} \leq K u_{11}$ for $j = k, k+1, \ldots, n$.

By (3.2),

$$
\frac{u_{111}}{u_{11}} = -\left(\frac{\varphi_1}{\varphi} + \frac{4u_1}{u}\right), \quad \frac{u_i}{u} = -\frac{1}{4}\left(\frac{\varphi_i}{\varphi} + \frac{u_{11i}}{u_{11}}\right), \quad i = 2, \ldots, n \qquad (3.20)
$$

at (x_0,t_0). Substituting (3.20) into (3.14), we obtain

$$
\begin{aligned}
0 \geq\ & (-u_t)^{\frac{1}{k}} f_{11}\left(4\left(\frac{u_{11}}{u}-\frac{u_1^2}{u^2}\right)+\frac{\varphi''}{\varphi}u_1^2u_{11}^2+\frac{\varphi'}{\varphi}u_{11}^2-\frac{\varphi'^2}{\varphi^2}u_1^2u_{11}^2-\left(\frac{\varphi_1}{\varphi}+\frac{4u_1}{u}\right)^2\right)\\
&+(-u_t)^{\frac{1}{k}}\sum_{i=2}^n f_{ii}\left(\frac{4u_{ii}}{u}-\frac{1}{4}\left(\frac{\varphi_i}{\varphi}+\frac{u_{11i}}{u_{11}}\right)^2+\frac{\varphi''}{\varphi}u_i^2u_{ii}^2+\frac{\varphi'}{\varphi}u_{ii}^2-\frac{\varphi'^2}{\varphi^2}u_i^2u_{ii}^2-\frac{u_{11i}^2}{u_{11}^2}\right)\\
&+(-u_t)^{\frac{1}{k}}\frac{1}{k}\sum_{i,j=1}^n S_k(\lambda)^{\frac{1}{k}-1}S_{k-2;ij}(\lambda)\frac{u_{1ij}^2}{u_{11}}+\frac{4}{ku}\\
\geq\ & \left[(-u_t)^{\frac{1}{k}}\sum_{i=1}^n f_{ii}\left(\frac{4u_{ii}}{u}+\left(\frac{\varphi''}{\varphi}-\frac{3\varphi'^2}{\varphi^2}\right)u_i^2u_{ii}^2+\frac{\varphi'}{\varphi}u_{ii}^2\right)-36(-u_t)^{\frac{1}{k}}f_{11}\frac{u_1^2}{u^2}\right]\\
&+\left[-\frac{3}{2}(-u_t)^{\frac{1}{k}}\sum_{i=2}^n f_{ii}\frac{u_{11i}^2}{u_{11}^2}+(-u_t)^{\frac{1}{k}}\frac{1}{k}\sum_{i,j=1}^n S_k(\lambda)^{\frac{1}{k}-1}S_{k-2;ij}(\lambda)\frac{u_{1ij}^2}{u_{11}}\right]+\frac{4}{ku}\\
=:\ & I_1+I_2+\frac{4}{ku}, \qquad (3.21)
\end{aligned}
$$

at (x_0,t_0). First, I_1 can be estimated from below as

$$
\begin{aligned}
I_1 &\geq (-u_t)^{\frac{1}{k}}\theta_1 f_{11}u_{11}^2+\frac{4}{u}-C(-u_t)^{\frac{1}{k}}\frac{f_{11}}{u^2}\\
&\geq (-u_t)^{\frac{1}{k}}\frac{1}{2}\theta_1 f_{11}u_{11}^2+\frac{4}{u}, \qquad (3.22)
\end{aligned}
$$

provided $u(x_0,t_0)^2u_{11}(x_0,t_0)^2 \geq 2C/\theta_1$. If $u(x_0,t_0)^2u_{11}(x_0,t_0)^2 < 2C/\theta_1$, then (3.1) is obvious. Hence we may assume $u(x_0,t_0)^2u_{11}(x_0,t_0)^2 \geq 2C/\theta_1$ hereafter. Second, I_2 can be also estimated from below as

$$
\begin{aligned}
I_2 &\geq -\frac{3}{2}(-u_t)^{\frac{1}{k}}\frac{1}{k}S_k(\lambda)^{\frac{1}{k}-1}\sum_{i=2}^n S_{k-1;i}(\lambda)\frac{u_{11i}^2}{u_{11}^2}+2(-u_t)^{\frac{1}{k}}\frac{1}{k}S_k(\lambda)^{\frac{1}{k}-1}\sum_{i=2}^n S_{k-2;1i}(\lambda)\frac{u_{11i}^2}{u_{11}}\\
&= 2(-u_t)^{\frac{1}{k}}\frac{1}{k}S_k(\lambda)^{\frac{1}{k}-1}\left(\sum_{i=2}^n\left(S_{k-2;1i}(\lambda)-\frac{3}{4}\frac{S_{k-1;i}(\lambda)}{\lambda_1}\right)\frac{u_{11i}^2}{\lambda_1}\right)\geq 0, \quad (3.23)
\end{aligned}
$$

by using $\lambda_1 S_{k-2;1i}(\lambda) \geq 3S_{k-1;i}(\lambda)/4$ provided $K>0$ is sufficiently small (see [8, Lemma 3.1]).

Substituting (3.22) and (3.23) into (3.21), we obtain

$$
0 \geq (-u_t)^{\frac{1}{k}}\frac{1}{2}\theta_1 f_{11}u_{11}^2+\frac{4}{u}\left(1+\frac{1}{k}\right). \qquad (3.24)
$$

By multiplying $(-u)^4\varphi$, we get

$$0 \geq \frac{1}{2}\theta_1(-u_t)^{\frac{1}{k}} f_{11} u_{11}^2 (-u)^4 \varphi - C(-u)^3\varphi.$$

It follows from [8, Lemma 3.1] that $\lambda_1 S_{k-1;1}(\lambda) \geq \theta_4 S_k(\lambda)$ for some constant $\theta_4 > 0$, which implies that

$$f_{11}u_{11}^2 = \frac{1}{k} S_k(\lambda)^{\frac{1}{k}-1} S_{k-1;1}(\lambda)\lambda_1^2 \geq \frac{\theta_4}{k} S_k(\lambda)^{\frac{1}{k}}\lambda_1 = \frac{\theta_4}{k}(-u_t)^{-\frac{1}{k}} u_{11}. \tag{3.25}$$

Hence the inequality

$$0 \geq \frac{\theta_1\theta_4}{2k}(-u)^4\varphi u_{11} - C(-u)^3\varphi$$

holds at (x_0, t_0). Then we have

$$\Psi \leq C(-u)^3\varphi \leq C(n, k, m_2, \|u\|_{C^1(D)}),$$

at (x_0, t_0). Therefore, $(-u)^4|D^2u|$ can be estimated from above by some constant C. □

4. Proof of Theorem 2.1

Before giving a proof of Theorem 2.1, we introduce some notation. For a subset $D \subset \mathbb{R}^n \times (-\infty, 0]$, a function v defined on D and $\alpha \in (0,1)$, α-Hölder seminorm of v over D is denoted by

$$[v]_{\alpha,D} = \sup_{\substack{(x,t),(y,s)\in D,\\ (x,t)\neq(y,s)}} \frac{|v(x,t) - v(y,s)|}{(|x-y|^2 + |t-s|)^{\frac{\alpha}{2}}}. \tag{4.1}$$

Moreover, $\mathbb{S}^{n\times n}$ is defined to be the set of all symmetric $n \times n$ matrices, and $\mathbb{S}^{n\times n}_+$ is the set of all non-negative definite symmetric $n \times n$ matrices.

Let $u \in C^{4,2}(\mathbb{R}^n \times (-\infty, 0])$ be a strictly convex-monotone solution to (1.8), which satisfies the growth conditions (2.1) and (2.2). We may assume without loss of generality that $u(0,0) = 0$, $Du(0,0) = 0$, by considering $u(x,t) - u(0,0) - Du(0,0)\cdot x$ instead of $u(x,t)$. Then it can be seen by (2.2) that there exists a constant $\tilde{A} > 0$ such that $u(x,0) \geq \tilde{A}|x|^2$ for all $x \in \mathbb{R}^n$,

Let $R > 0$ be fixed. We define $v(x,t) = v_R(x,t) = u(Rx, R^2t)/R^2$. Then v is also a strictly convex-monotone classical solution to (1.8), and satisfies

$v_t(x,t) = u_t(Rx, R^2t)$ and $v_{ij}(x,t) = u_{ij}(Rx, R^2t)$. Moreover, it holds that for all $(x,t) \in \mathbb{R}^n \times (-\infty, 0]$,

$$-m_1 \le v_t(x,t) \le -m_2, \tag{4.2}$$

and that for all $x \in \mathbb{R}^n$,

$$v(x,0) \ge \tilde{A}|x|^2. \tag{4.3}$$

First, we shall obtain the local gradient estimate of the solution v. For $q > 0$, we set

$$\Omega_q = \{(x,t) \in \mathbb{R}^n \times (-\infty, 0] \mid v(x,t) < \tilde{A}q\}. \tag{4.4}$$

Then we can find that Ω_q is a bounded bowl-shaped domain and

$$\Omega_q(t) \subset \Omega_q(0) \subset B(0, \sqrt{q}), \tag{4.5}$$

due to (4.2), (4.3) and the strict parabolic-monotonicity of v. Now we establish the following estimate.

Lemma 4.1. *Let v and Ω_q be defined as above. Then there exists a constant $C > 0$, independent of q and R, such that for all $(x,t) \in \Omega_q$,*

$$|Dv(x,t)| \le C\sqrt{q}. \tag{4.6}$$

Proof. We note that $v(x,t)$ is strictly convex in x, and that $v(x,t) - \tilde{A}q = 0$ on $\partial_p \Omega_q$. From Newton-Maclaurin inequality it follows that $(F_k(M)/\binom{n}{k})^{1/k} \ge F_n(M)^{1/n}$ for all $M \in \mathbb{S}_+^{n\times n}$.

By Aleksandrov's maximum principle (cf. [10]), we obtain that at $(x_0, t) \in \Omega_q$,

$$\begin{aligned}
|v(x_0,t) - \tilde{A}q|^n &\le C\,(\operatorname{diam}\Omega_q(t))^{n-1}\operatorname{dist}(x_0, \partial\Omega_q(t))|\partial v(\Omega_q(t))| \\
&\le C(2\sqrt{q})^{n-1}\operatorname{dist}(x_0, \partial\Omega_q(t)) \int_{\Omega_q(t)} \det D^2 v(x,t)\,dx \\
&\le Cq^{\frac{n-1}{2}}\operatorname{dist}(x_0, \partial\Omega_q(t)) \int_{\Omega_q(t)} F_k(D^2 v(x,t))^{\frac{n}{k}}\,dx \\
&= Cq^{\frac{n-1}{2}}\operatorname{dist}(x_0, \partial\Omega_q(t)) \int_{\Omega_q(t)} (-v_t)^{-\frac{n}{k}}\,dx \\
&\le Cq^{\frac{n-1}{2}}\operatorname{dist}(x_0, \partial\Omega_q(t)) \cdot m_2^{-\frac{n}{k}} |B(0, \sqrt{q})| \\
&= Cq^{n-\frac{1}{2}}\operatorname{dist}(x_0, \partial\Omega_q(t)),
\end{aligned} \tag{4.7}$$

so that

$$|v(x_0,t) - \tilde{A}q| \leq Cq^{1-\frac{1}{2n}} \operatorname{dist}(x_0, \partial\Omega_q(t))^{\frac{1}{n}}. \tag{4.8}$$

Therefore for all $x_0 \in \Omega_{q/2}(t)$,

$$\tilde{A}q - \frac{1}{2}\tilde{A}q \leq \tilde{A}q - v(x_0,t) \leq Cq^{1-\frac{1}{2n}} \operatorname{dist}(x_0, \partial\Omega_q(t))^{\frac{1}{n}},$$

which implies the inequality

$$\operatorname{dist}(\Omega_{\frac{q}{2}}(t), \partial\Omega_q(t)) \geq Cq^{\frac{1}{2}}. \tag{4.9}$$

Therefore we can see that $|Dv(x,t)| \leq Cq^{1/2}$ for all $(x,t) \in \Omega_{q/2}$ by (4.9) and the convexity of v with respect to x. This ends the proof. $\square$

Especially, $|Dv(x,t)| \leq C$ for all $(x,t) \in \Omega_1$, in which C is independent of R. By applying (3.1) to the function $\tilde{A} - v(x,t)$, one obtains

$$\left(\tilde{A} - v(x,t)\right)^4 |D^2v(x,t)| \leq C$$

in Ω_1. This implies that

$$|D^2v(x,t)| \leq C \quad \text{in } \Omega_{1/2}. \tag{4.10}$$

The following Evans-Krylov type theorem is needed for the proof of Theorem 2.1. For the proof, see [11].

Theorem 4.2. *Let D and D' be bounded bowl-shaped domains which satisfy $D' \subset D$ and $\operatorname{dist}(D', \partial_p D) > 0$, and u be a $C^{4,2}(D)$ solution to the equation*

$$G(u_t, D^2u) = 0$$

in D, where $G = G(q,M)$ is defined for all $(q,M) \in \mathbb{R} \times \mathbb{S}^{n\times n}$ with $G(\cdot, M) \in C^1(\mathbb{R})$ for each $M \in \mathbb{S}^{n\times n}$, and $G \in C^2(\mathbb{R} \times X)$ for some $X \subset \mathbb{S}^{n\times n}$ which is a neighborhood of $D^2u(D)$. Suppose that:

(i) G is uniformly parabolic, i.e., there exist positive constants λ and Λ such that

$$-\Lambda \leq G_q(q,M) \leq -\lambda, \tag{4.11}$$
$$\lambda\|N\| \leq G(q, M+N) - G(q,M) \leq \Lambda\|N\|, \tag{4.12}$$

for all $q \in \mathbb{R}$ and M, $N \in \mathbb{S}^{n\times n}$ with $N \geq O$.

(ii) G is concave with respect to M.

If $\|u\|_{C^{2,1}(D)} \leq K$, then there exist positive constants C depending on λ, Λ, n, K, D, D' and $G(0,0)$, and $\alpha \in (0,1)$ depending on λ, Λ and n such that

$$\|u\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(D')} \leq C.$$

Then we prove the next lemma in order to use Theorem 4.2.

Lemma 4.3. *There exists a constant $C > 0$, independent of R, such that*

$$\mathrm{dist}(\Omega_{\frac{1}{8}}, \partial_p \Omega_{\frac{1}{2}}) \geq C. \tag{4.13}$$

Proof. Take $(x,t) \in \Omega_{1/8}$ arbitrarily. Then, putting $q = 1/4$ in (4.9), we obtain

$$\mathrm{dist}(\Omega_{\frac{1}{8}}(t), \partial\Omega_{\frac{1}{4}}(t)) \geq C', \tag{4.14}$$

where C' is a positive constant independent of R. We set $\delta = \min\{\tilde{A}/(4m_1), C'\}$. If $\mathrm{dist}((x,t),(x',t')) < \delta$, then $|x-x'| < C'$ and $|t-t'| < \tilde{A}/(4m_1)$, which imply that

$$\begin{aligned} v(x',t') &= v(x',t) + \int_t^{t'} v_t(x',s)ds \\ &\leq v(x',t) + m_1|t-t'| \\ &\leq \frac{1}{4}\tilde{A} + m_1 \cdot \frac{\tilde{A}}{4m_1} = \frac{1}{2}\tilde{A}, \end{aligned}$$

due to (4.14). Therefore $(x',t') \in \overline{\Omega_{1/2}}$ and this completes the proof. $\square$

We set $G(q,M) = (-q)^{1/k}F_k(M)^{1/k} - 1 = (-q)^{1/k}f(M) - 1$ for $(q,M) \in [-m_1, -m_2] \times X$, where

$$X = \left\{ M = (m_{ij}) \in \mathbb{S}_+^{n\times n} \;\middle|\; \frac{1}{m_1} \leq F_k(M) \leq \frac{1}{m_2},\ |m_{ij}| \leq C \text{ for } i,j = 1,\ldots,n \right\},$$

in which C is a constant appeared in (4.10).

Since $G_q(q, M) = (-q)^{1/k-1}F_k(M)^{1/k}/k$, we see that there exist constants $\lambda, \Lambda > 0$ such that (4.11) holds in $[-m_1, -m_2] \times X$. Moreover, we can also see that (4.12) and (ii) in Theorem 4.2 holds in $[-m_1, -m_2] \times X$, due to [5].

Next we can extend G in $\mathbb{R} \times \mathbb{S}^{n\times n}$ so that G satisfies (i) and (ii) in Theorem 4.2 for different constants $\lambda, \Lambda > 0$ if necessary. Then we apply Theorem 4.2 to $G(v_t, D^2v) = 0$ in $\Omega_{1/2}$ and obtain that

$$\|v\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\Omega_{\frac{1}{8}})} \leq C.$$

Therefore it follows that $[D_{ij}v]_{\alpha,\Omega_{1/8}} \leq C$ for $i, j = 1, \ldots, n$ and $[v_t]_{\alpha,\Omega_{1/8}} \leq C$.

By substituting $v(x, t) = u(Rx, R^2t)/R^2$, we have

$$[D_{ij}u]_{\alpha,\{u(x,t)<\frac{\tilde{A}}{8}R^2\}} \leq CR^{-\alpha}, \tag{4.15}$$

$$[u_t]_{\alpha,\{u(x,t)<\frac{\tilde{A}}{8}R^2\}} \leq CR^{-\alpha}, \tag{4.16}$$

for any $R > 0$. This implies that for any bounded subset Ω of $\mathbb{R}^n \times (-\infty, 0]$, $[D_{ij}u]_{\alpha,\Omega} = 0$, and $[u_t]_{\alpha,\Omega} = 0$. Hence $D_{ij}u$ and u_t are constants in $\mathbb{R}^n \times (-\infty, 0]$ and this completes the proof of Theorem 2.1.

5. Final remarks

(i) Viscosity solutions

Here we consider whether Theorem 2.1 also holds for viscosity solutions to the parabolic k-Hessian equation (1.8). We can show the following proposition.

Proposition 5.1. *Let $1 \leq k \leq n$. Then there exists a convex-monotone viscosity solution $u \in C(\mathbb{R}^n \times (-\infty, 0])$ to (1.8), which does not have the form $u(x, t) = -mt + p(x)$ where $m \geq 0$ and p is a quadratic polynomial.*

Proof. Let $t_0 \geq 0$ be an arbitrary number. We set u by

$$u(x, t) = C(-t + t_0)^{\alpha}|x|^{\beta} \quad \text{in } \mathbb{R}^n \times (-\infty, 0], \tag{5.1}$$

where $\alpha = 1/(k + 1)$, $\beta = 2k/(k + 1)$ and

$$C = \left\{\alpha\beta^k \left[(\beta - 1)\binom{n-1}{k-1} + \binom{n-1}{k}\right]\right\}^{-\frac{1}{k+1}}. \tag{5.2}$$

Now we define $\binom{n-1}{n} = 0$. Then it can be easily seen that u is convex-monotone in $\mathbb{R}^n \times (-\infty, 0]$ and that u is a classical solution to (1.8) in $(\mathbb{R}^n \setminus \{0\}) \times (-\infty, 0]$.

For $t \leq 0$, there exists no $C^{2,1}$ function φ which touches u at $(0,t)$ from above, because $\beta < 2$. While, for any admissible $C^{2,1}$ function φ which touches u from below at $(0,t)$, $\varphi_t(0,t)$ must be 0, because $u(0,\cdot) \equiv 0$. This implies that $-\varphi_t(0,t)F_k(D^2\varphi)(0,t) = 0 \leq 1$. Therefore u is a viscosity solution to (1.8) in $\mathbb{R}^n \times (-\infty, 0]$. □

For $k = n$ which corresponds to the parabolic Monge-Ampère equation's case, the function u constructed above is almost the same as the one in [11]. We remark that this function u satisfies neither (2.1) nor (2.2), for arbitrary $t_0 \geq 0$. Also, it is not *strictly* convex-monotone. We would like to know whether Theorem 2.1 holds for viscosity solutions under the assumptions (2.1) and (2.2).

(ii) Other parabolic analogues of k-Hessian equation

In this paper we consider the parabolic k-Hessian equation of the form $-u_tF_k(D^2u) = 1$, and obtain Bernstein type theorem for this equation. But there are different parabolic analogues of k-Hessian equation which have been studied in the literature.

Ivochkina and Ladyzhenskaya [13] have studied the solvability of the first initial boundary value problem for

$$-u_t + F_k(D^2u)^{\frac{1}{k}} = \psi. \tag{5.3}$$

X.J. Wang [27] considered a following version of parabolic equation,

$$-u_t + \log F_k(D^2u) = \psi. \tag{5.4}$$

For the case $k = n$, (5.4) reduces to

$$-u_t + \log\det D^2u = \psi, \tag{5.5}$$

which was studied by G. Wang and W. Wang [26]. Moreover,

$$S_k(-u_t, \lambda_1, \ldots, \lambda_n) = \psi, \tag{5.6}$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of D^2u, i.e., $-u_tF_{k-1}(D^2u) + F_k(D^2u) = \psi$, was considered in [17].

Our next task is to obtain Bernstein type theorems for other parabolic analogues of k-Hessian equation.

(iii) Relaxing the assumptions : Growth conditions and convexity

We would like to remove growth conditions (1.5), (1.7), (2.1) and (2.2) in Theorems 1.2, 1.3 and 2.1 (or, to prove growth conditions are necessary). As we have stated in Section 1 and Remark 2.1 before, Theorem 1.2 remains valid without the growth condition (1.5) when $k = 1$ (the case of Poisson equation) and $k = n$ (the case of Monge-Ampère equation), and Theorem 2.1 is true without (2.2) when $k = n$. However, we do not know any more for other cases.

It is known that k-Hessian operator $F_k(D^2u)$ is degenerate elliptic for *k-convex functions*, the space of which is strictly wider than that of convex functions for $1 \leq k \leq n - 1$ (see [5] for the proof). Therefore, when we study k-Hessian equation, it is natural to seek solutions in the class of k-convex functions, rather than in the class of convex functions. It seems an interesting open problem whether Theorems 1.2 and 2.1 remain true if one replaces "strictly convex" by "strictly k-convex."

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