広島大学学位請求論文

Global existence and decay estimates for the nonlinear wave equations with space-time dependent dissipative term

（時空間に依存する消散項を持つ消散型波動方程式の大域的な解の存在と減衰評価）

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主論文

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Abstract

We study the global existence and decay estimates for nonlinear wave equations with a space-time dependent dissipative term in an exterior domain. The linear dissipative effect may vanish in a compact space region. Moreover the nonlinear terms need not to be divergence form. For getting the higher order energy estimates, we introduce an argument using the rescaling. The method is useful to control derivatives of the dissipation coefficient.

Key Words and Phrases. dissipative nonlinear wave equations, space-time variable coefficient, time decay estimates

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1 Introduction

Let $n \geq 1, d \geq 2$ and $\Omega = \mathbb{R}^d/\mathcal{O}$, where $\mathcal{O}$ is a star-shaped domain with a smooth and compact boundary $\partial\Omega$. Moreover we assume that $\mathcal{O}$ contains the origin. In this paper, we consider the initial-boundary value problem for nonlinear wave equations with the space-time dependent dissipative term:

$$\text{(DW)} \quad \begin{cases} (\partial_t^2 - \triangle + B(t,x)\partial_t)u(t,x) = F(\partial u, \partial^2 u) & (t,x) \in [0,\infty) \times \Omega, \\ u(0,x) = u_0(x), \quad \partial_t u(0,x) = u_1(x) & x \in \Omega, \\ u(t,x) = 0 & (t,x) \in [0,\infty) \times \partial\Omega, \end{cases}$$

where $u = (u^1, \cdots, u^n)$, $\nabla = (\partial_{x_1}, \cdots, \partial_{x_d})$ and $\partial = (\partial_t, \nabla)$. The initial data (u_0, u_1) belong to $H^L(\Omega) \times H^{L-1}(\Omega)$ and satisfy the compatibility condition of order $L-1$. $H^L(\Omega)$ is the Sobolev space in Ω. We make the following assumptions for the space-time dependent damping coefficient matrix $B(t,x) = (B_{pq}(t,x))_{p,q=1,\cdots,n}$:

(B0) B_{pq} belong to $\mathcal{B}^\infty([0,\infty) \times \Omega)$, where $\mathcal{B}^\infty$ is the function space of smooth functions with bounded derivatives.

(B1) $B(t,x)$ is nonnegative definite in $[0,\infty) \times \Omega$.

(B2) $\partial_t B(t,x)$ is nonpositive definite in $[0,\infty) \times \Omega$.

(B3) There exist $b_0 > 0$ and $R > 0$ such that

$$\sum_{p,q=1}^{n} B_{pq}(t,x)\eta_p\eta_q \geq b_0|\eta|^2 \quad (t \in [0,\infty), |x| \geq R, \eta \in \mathbb{R}^n).$$

By the assumption **(B3)**, dissipative term works on $|x| \geq R$. This means that the dissipative effect may vanish in a compact space region.

We treat quadratic nonlinear terms. In what follows, ∂_0 means ∂_t and $\partial_j(j = 1, 2, \cdots, d)$ means ∂_{x_j}. Assume that F is of the form

$$F(\partial u, \partial^2 u) = \left(\tilde{F}_i(\partial u) + \sum_{j=1}^{n} \sum_{0 \leq a,b \leq d} c_{ij}^{ab}(\partial u)\partial_a\partial_b u^j \right)_{i=1,\cdots,n},$$

which satisfy

$$c_{ij}^{ab} = c_{ji}^{ba}, \tag{1}$$

$$|D_\xi^\alpha \tilde{F}_i(\xi)| \leq C_{\alpha,p_1}|\xi|^{\max\{0,p_1-|\alpha|\}} \quad (\xi \in \mathbb{R}^n \times \mathbb{R}^{d+1}, \ \ |\alpha| \leq L-1) \tag{2}$$

and

$$|D_\xi^\alpha c_{ij}^{ab}(\xi)| \leq C_{\alpha,p_2}|\xi|^{\max\{0,p_2-1-|\alpha|\}} \quad (\xi \in \mathbb{R}^n \times \mathbb{R}^{d+1}, \ \ |\alpha| \leq L-1) \tag{3}$$

for some $p_l \geq 2$ $(l = 1, 2)$. The main objective of this paper is to prove the global existence and decay estimate to (DW).

In the case that the coefficient function B vanishes, (DW) become the nonlinear wave equations. Then it is well known that no matter how small the initial data, there do not exist globally defined smooth solutions in general (e.g.[3], [5], [7]). If F has the "Null condition" then (DW) has a global smooth solution for sufficiently smooth and small the initial data (e.g.[8], [15]).

In the case that the coefficient function $B \equiv Const > 0$, there are many results ([6], [10] etc.). For the case of linear or semilinear version, it is known that the asymptotic profile of the solution to (DW) is given by the corresponding solution of heat equation (e.g. [13], [14] etc.). Such a property is called the diffusion phenomenon. Recently, there are many research concerning the diffusion phenomenon for the nonuniform dissipation. When the dissipation depends on space variable $B = B(x)$, Todorova and Yordanov consider like $B = (1+|x|)^{-\gamma}$ to linear and semilinear version in [16]. They show that if $0 \leq \gamma < 1$ then the solution to (DW) have some decay estimates, indeed Wakasugi [17] confirms the diffusion phenomenon recently. When the dissipation depends on time variable $B = B(t)$, Wirth [19] proves that if $tB(t) \to +\infty$ and $B \in L^1$ then the solution to (DW) satisfies the decay estimate like corresponding solution to heat equation. On the other hand, Mochizuki [11] considers the scattering for the free wave equation when B depends on time-space variable. He prove that if there exist $\xi, \eta \in L^1$ and small ε such that

$$|B(t,x)| \leq \varepsilon\xi(|x|) + \eta(t), \quad \xi, \eta \geq 0, \ \xi' \leq 0, \ \ \xi'^2 \leq 2\xi\xi''$$

then the solutions to (DW) close to the free wave equation. We remark that $|B(t,x)| \leq (1+t)^{-\alpha}(1+|x|)^{-\beta},\ \alpha+\beta > 1$ is a sufficient condition of above the conditions.

Now we consider the nonuniform dissipative term which doesn't decay near infinity but vanishes in a compact region. Nakao [12] gets the energy decay estimates like $E(u(t)) = O((1+t)^{-1})$ when B depends on space variable only, where $E(u(t))$ is the standard energy of wave equations. Furthermore, Ikehata [2] get the decay estimates as $\|u(t)\|^2_{L^2(\Omega)} = O((1+t)^{-1})$ and $E(u(t)) = O((1+t)^{-2})$ with an additional condition for the initial data. Those results mostly deal with the linear and the semilinear problem. For quasilinear version with divergence form, Nakao [12] considers the equation

$$\partial_t^2 u - \mathrm{div}\{\sigma(|\nabla^2 u|)\nabla u\} + a(x)\partial_t u = 0, \tag{4}$$

where σ is a smooth function like $\sigma(x) = (1+|x|)^{-\frac{1}{2}}$. Then the nonlinearity order p satisfy $p \geq 3$. Besides, the author of the present paper deals with $p = 2$ and proves the decay estimates in [18]. Moreover there is no result when B depend space-time variables. In this paper, we assume the space-time dependent dissipation B effective near the space infinity, even if the nonlinear terms F have no "null condition" and "divergence form".

First, we get the global existence as follows:

Theorem 1.1. *Let $L \geq [\frac{d}{2}]+3$. Then there exists a small constant $\delta > 0$ such that if the initial data $(u_0, u_1) \in H^L(\Omega) \times H^{L-1}(\Omega)$ satisfy the compatibility condition of order $L-1$ and*

$$\|(u_0, u_1)\|_{H^L(\Omega)\times H^{L-1}(\Omega)} \leq \delta, \tag{5}$$

then there exists a unique global solution to (DW) *in $\bigcap_{j=0}^{L-1} C^j([0,\infty); H^{L-j}(\Omega)\cap H_0^1(\Omega)) \bigcap C^L([0,\infty); L^2(\Omega))$.*

In the proof of Theorem 1.1, we use higher order energies (see for instance [5],[15]) and the rescaling (see section 2). Note that if $B = Const > 0$, we can prove theorem 1.1 under the assumption $\|(\nabla u_0, u_1)\|_{H^{L-1}(\Omega)\times H^{L-1}(\Omega)} \leq \hat{\delta}$ instead of (5), i.e. the smallness of $\|u_0\|_{L^2(\Omega)}$ is also needed for the case of nonuniform dissipative term.

Next, we also prove the decay estimate as follows:

Theorem 1.2. *In addition to the assumptions in Theorem 1.1, we assume* (H1) *and* (H2).

(H1) $\|d_0(\cdot)\{B(0)u_0 + u_1\}\|_{L^2(\Omega)} < \infty$.

(H2) $\displaystyle\int_0^\infty \|d_0(\cdot)\partial_t B(s)\|_{L^\infty(\Omega)} ds < \infty,$

where $d_0 : \mathbb{R}^d \to \mathbb{R}$ is defined by

$$d_0(x) = \begin{cases} |x| & (d \geq 3), \\ |x| \log(A|x|) & (d = 2) \end{cases} \tag{6}$$

with a constant A satisfying $\inf_{x\in\Omega} A|x| \geq 2$. *Furthermore if $d = 2$, we also assume* (H3).

(H3) *There exists M such that* $\mathrm{supp} u_0 \cup \mathrm{supp} u_1 \subset \{x \in \Omega : |x| \leq M\}$.

Then the global solution u to (DW) *satisfy following estimates:*

$$\sum_{\mu=0}^{L-1} \|(\partial_t^\mu u(t), \partial_t^{\mu+1} u(t))\|^2_{H^{L-\mu}(\Omega)\times H^{L-\mu-1}(\Omega)} \leq E_0(1+t)^{-1} \tag{7}$$

and

$$\|(\nabla u(t), \partial_t u(t))\|^2_{L^2(\Omega)\times L^2(\Omega)} \leq E_0(1+t)^{-2}, \tag{8}$$

where E_0 is a constant depend on (u_0, u_1) and M.

Theorem 1.2 is the decay estimate which correspond to the result of Ikehata [2].

The paper is organized as follows. In section 2 we prepare some known lemmas and the rescaling function. In section 3 we prove the high energy estimates to (DW) and the theorem 1.1. In the proof, the rescaling argument plays an important role. In section 4 we prove the Theorem 1.2.

2 Preliminaries

We consider the rescaling to (DW). Let u be the solution to (DW). We define $v(t,x) = \lambda^{-1}u(\lambda t, \lambda x)\ (\lambda > 0)$, then v satisfies

$$\begin{aligned} & \partial_t^2 v(t,x) - \triangle v(t,x) = \lambda\left\{\partial_t^2 u(\lambda t, \lambda x) - \triangle u(\lambda t, \lambda x)\right\} \\ = \ & -\lambda B(\lambda t, \lambda x)\,\partial_t u(\lambda t, \lambda x) + \lambda F(\partial u(\lambda t, \lambda x), \partial^2 u(\lambda t, \lambda x)) \\ = \ & -\lambda B(\lambda t, \lambda x)\,\partial_t v(t,x) + F_\lambda(\partial v(t,x), \partial^2 v(t,x)), \end{aligned}$$

where $(F_\lambda)_i(\partial v, \partial^2 v) = (\tilde{F}_\lambda)_i(\partial v) + \sum_{j=1}^{n} \sum_{0\leq a,b\leq d} c_{ij}^{ab}(\partial v)\partial_a\partial_b v^j$ and $\tilde{F}_\lambda(\partial v) = \lambda\tilde{F}(\partial v)$.

So v is the solution to the following initial-boundary value problem $(\mathrm{DW})_\lambda$:

$$(\mathrm{DW})_\lambda \quad \begin{cases} (\partial_t^2 - \triangle + B_\lambda(t,x)\partial_t)v = F_\lambda(\partial v, \partial^2 v) & [0,\infty)\times\Omega_\lambda, \\ v(0,x) = v_0(x), \quad \partial_t v(0,x) = v_1(x) & x \in \Omega_\lambda, \\ v(t,x) = 0 & (t,x) \in [0,\infty)\times\partial\Omega_\lambda, \end{cases}$$

where $\Omega_\lambda = \{x : \lambda x \in \Omega\}$, $B_\lambda(t,x) = \lambda B(\lambda t, \lambda x)$, $v_0(x) = \lambda^{-1}u_0(\lambda x)$, $v_1(x) = u_1(\lambda x)$. Then B_λ satisfy following $(\mathbf{B1})_\lambda$-$(\mathbf{B3})_\lambda$ instead of **(B1)-(B3)**.

(B1)$_\lambda$ $B_\lambda(t,x)$ is nonnegative in $[0,\infty)\times\Omega_\lambda$,

(B2)$_\lambda$ $\partial_t B_\lambda(t,x)$ is nonpositive in $[0,\infty)\times\Omega_\lambda$,

(B3)$_\lambda$ There exist $b_0>0$ and $R>0$ such that

$$\sum_{p,q=1}^{n}(B_\lambda(t,x))_{pq}\eta_p\eta_q \geq \lambda b_0|\eta|^2 \qquad (t\in[0,\infty), |x|\geq \frac{R}{\lambda}, \eta\in\mathbb{R}^n).$$

Furthermore B_λ satisfies

(B4)$_\lambda$ $\|\partial^\alpha B_\lambda\|_{L^\infty([0,\infty)\times\Omega_\lambda)}\leq \lambda^{|\alpha|+1}\|\partial^\alpha B\|_{L^\infty([0,\infty)\times\Omega)}$.

We consider (DW)$_\lambda$ instead of (DW). Throughout this paper, $\|\cdot\|_p=\|\cdot\|_{L^p(\Omega_\lambda)}$, $\|\cdot\|_{H^l}=\|\cdot\|_{H^l(\Omega_\lambda)}$ and $\langle\cdot,\cdot\rangle$ stands for $L^2(\Omega_\lambda)$-inner product.

First, we introduce some known results. First we prepare following lemmas for estimating nonlinear terms.

Lemma 2.1 (Sobolev's lemma)**.** *There exists a constant $C_\lambda>0$ such that*

$$\|f\|_\infty\leq C_\lambda\|f\|_{H^{[\frac{d}{2}]+1}} \quad (f\in H^{[\frac{d}{2}]+1}(\Omega_\lambda)).$$

Lemma 2.2 (Elliptic estimate)**.** *There exists $C_\lambda>0$ such that for any $\varphi\in H^m(\Omega_\lambda)\cap H^1_0(\Omega_\lambda)$ with an integer $m\geq 2$, we have*

$$\sum_{|\alpha|=m}\|\nabla^\alpha\varphi\|_2\leq C_\lambda(\|\triangle\varphi\|_{H^{m-2}}+\|\nabla\varphi\|_2).$$

Next, we prepare the Poincare type inequality associated with B_λ.

Lemma 2.3 (Poincare type inequality)**.** *There exists a constant $C_1\geq 1/4$ such that*

$$\|f\|_2^2\leq\lambda^{-1}C_1\langle f,B_\lambda f\rangle+\lambda^{-2}C_1\|\nabla f\|_2^2 \quad (f\in H^1_0(\Omega_\lambda),\lambda>0). \tag{9}$$

Proof. We define $U_r=\{x\in\Omega_\lambda : |x|\leq r\}$. Using Poincare inequality, we obtain the following estimate:

$$\int_{U_r}|f(x)|^2dx\leq r^2\int_{U_r}|\nabla f(x)|^2dx \quad (f\in H^1_0(U_r)).$$

Let $f\in H^1_0(\Omega_\lambda)$ and $\rho\in C_0^\infty(\mathbb{R}^d)$ be a function satisfying $0\leq\rho\leq 1, \rho(x)=1(|x|\leq 1), \rho(x)=0(|x|\geq\frac{3}{2})$. We define $\rho_\lambda(x)=\rho(\frac{\lambda x}{R})$ for $\lambda>0$, then because of $\rho_\lambda f\in H^1_0(U_{\frac{2R}{\lambda}})$ we have

$$\begin{aligned}\|f\|_2^2 &= \int_{\Omega_\lambda}|\rho_\lambda(x)f(x)|^2dx+\int_{\Omega_\lambda}(1-|\rho_\lambda(x)|^2)|f(x)|^2dx\\ &= \int_{U_{\frac{2R}{\lambda}}}|\rho_\lambda(x)f(x)|^2dx+\int_{|x|\geq\frac{R}{\lambda}}(1-|\rho_\lambda(x)|^2)|f(x)|^2dx\end{aligned}$$

$$
\begin{aligned}
&\le \frac{4R^2}{\lambda^2}\int_{U_{\frac{2R}{\lambda}}} |\nabla(\rho_\lambda(x)f(x))|^2dx + \int_{|x|\ge\frac{R}{\lambda}} |f(x)|^2dx \\
&\le \frac{4R^2}{\lambda^2}\int_{\frac{R}{\lambda}\le|x|\le\frac{2R}{\lambda}} |\nabla\rho_\lambda(x)|^2|f(x)|^2dx + \frac{4R^2}{\lambda^2}\int_{U_{\frac{2R}{\lambda}}} |\rho_\lambda(x)|^2|\nabla f(x)|^2dx \\
&\quad + \int_{|x|\ge\frac{R}{\lambda}} |f(x)|^2dx \\
&\le \left(4R^2\|\nabla\rho\|_\infty^2+1\right)\int_{|x|\ge\frac{R}{\lambda}} |f(x)|^2dx + \frac{4R^2}{\lambda^2}\int_{\Omega_\lambda} |\nabla f(x)|^2dx \\
&\le \frac{4R^2\|\nabla\rho\|_\infty^2+1}{b_0\lambda}\langle f, B_\lambda f\rangle + \frac{4R^2}{\lambda^2}\|\nabla f\|_2^2.
\end{aligned}
$$

Hence we get (9). $\square$

Finally, we introduce Hardy inequality and Gagliardo-Nirenberg inequality. We need these in the proof of Theorem 1.2 in section 4.

Lemma 2.4 (Hardy inequality; see for instance [1])**.** *Let $d \ge 2$. There exists a constant $C_\lambda > 0$ such that any $f \in H_0^1(\Omega_\lambda)$ satisfies*

$$\left\| \frac{f}{d_0} \right\|_2 \le C_\lambda \|\nabla f\|_2,$$

where d_0 is defined by (6)*.*

Lemma 2.5 (Gagliardo-Nirenberg inequality)**.** *Assume $1 \le q < d$ and $\frac{1}{r} = \frac{1}{q} - \frac{1}{d}$. Then there exists a constant $C_\lambda > 0$ such that*

$$\|f\|_r \le C_\lambda \|\nabla f\|_q \quad (f \in C_0^\infty(\Omega_\lambda)).$$

3 Energy estimates

In this section, we give a proof of the high order energy estimates.

First we introduce some notations. The energy $E(v(t))$ and the higher order energies $Z_m(v(t)), Z(v(t))$ are defined by

$$E(v(t)) = \frac{1}{2}\{\|\partial_t v(t)\|_2^2 + \|\nabla v(t)\|_2^2\}, \tag{10}$$

$$Z_m(v(t)) = \sum_{\mu=0}^{L-1-m} \left\{ \|\nabla\partial_t^\mu v(t)\|_{H^m}^2 + \|\partial_t^{\mu+1} v(t)\|_{H^m}^2 \right\} \quad (0 \le m \le L-1) \tag{11}$$

and

$$Z(v(t)) = \sum_{m=0}^{L-1} Z_m(v(t)). \tag{12}$$

We sometimes omit t or $v(t)$. Note that $\|v\|_2^2 + Z_{L-1}(v) = \|(v, \partial_t v)\|_{H^L \times H^{L-1}}^2$ holds. The following result concerning local existence is standard:

Proposition 3.1 (Local existence; for instance [4] or [12]). *Let $L \geq \left[\frac{d}{2}\right]+3$ and assume that $\partial\Omega_\lambda$ is smooth. Furthermore we assume the initial date $(v_0, v_1) \in H^L(\Omega_\lambda)\times H^{L-1}(\Omega_\lambda)$ satisfies the compatibility condition of order $L-1$ associated with the* $(\mathrm{DW})_\lambda$. *Then there exists a unique local solution to* $(\mathrm{DW})_\lambda$ *in*

$$X^T := \bigcap_{j=0}^{L-1} C^j([0,T]; H^{L-j}(\Omega_\lambda) \cap H_0^1(\Omega_\lambda)) \bigcap C^L([0,T]; L^2(\Omega_\lambda)),$$

where T depend on $\|(v_0, v_1)\|_{H^L\times H^{L-1}}$.

We define the function spaces X_δ^T and X_δ as follows:

$$X_\delta^T = \left\{v \in X^T : \|v(t)\|_2^2 + Z(v(t)) \leq \delta^2 \quad (0 \leq t \leq T)\right\}$$

and

$$X_\delta = \left\{v \in X^\infty : \|v(t)\|_2^2 + Z(v(t)) \leq \delta^2 \quad (0 \leq t < \infty)\right\}.$$

This section, we prove the following proposition:

Proposition 3.2. *There exist $0 < \lambda < 1$ and $\delta = \delta(\lambda)$ such that the local solution $v \in X_\delta^T$ to* $(\mathrm{DW})_\lambda$ *satisfies*

$$\|v(t)\|_2^2 + Z(v(t)) + \int_0^t Z(v(s))ds \leq C_\lambda\|(v_0, v_1)\|_{H^L\times H^{L-1}}^2. \tag{13}$$

If Proposition 3.2 holds, then we can prove Theorem 1.1. Indeed, using Proposition 3.1 and Proposition 3.2, we can prove the unique global existence theorem to $(\mathrm{DW})_\lambda$ by standard continuation argument. Furthermore we put $u(t,x) := \lambda v(\lambda^{-1}t, \lambda^{-1}x)$, it is easy to see u satisfies the statement of Theorem 1.1.

In order to show Proposition 3.2, we only need the estimates of Z_0. Indeed we can prove the next lemma (see for instance [5].).

Lemma 3.3. *For any $\lambda > 0$ there exist $\delta = \delta(\lambda)$ and $C_\lambda > 0$ such that a local solution $v \in X_\delta^T$ to* $(\mathrm{DW})_\lambda$ *satisfy the following estimates:*

$$Z(v(t)) \leq C_\lambda Z_0(v(t)) \quad (t \in [0,T)) \tag{14}$$

and

$$Z(v(t)) \leq C_\lambda Z_{L-1}(v(t)) \quad (t \in [0,T)). \tag{15}$$

Proof. Let $v \in X_\delta^T$ is a solution to $(\mathrm{DW})_\lambda$. First, we prove (14). For $1 \leq m \leq L-1$, it hold that

$$\begin{aligned} Z_m &= \sum_{\mu=0}^{L-1-m} \left\{ \|\nabla\partial_t^\mu v\|_2^2 + \sum_{2\leq|a|\leq m+1} \|\partial_t^\mu \nabla^a v\|_2^2 + \|\partial_t^{\mu+1} v\|_{H^m}^2 \right\} \\ &\leq C\left(Z_0 + \sum_{\mu=0}^{L-1-m} \sum_{2\leq|a|\leq m+1} \|\partial_t^\mu\nabla^a v\|_2^2 + Z_{m-1} \right). \end{aligned}$$

Using Lemma 2.2 and $(\mathrm{DW})_\lambda$, we get

$$\sum_{\mu=0}^{L-1-m} \sum_{2\le|a|\le m+1} \|\partial_t^\mu \nabla^a v\|_2^2 \le C_\lambda \sum_{\mu=0}^{L-1-m} \left(\|\partial_t^\mu \triangle v\|_{H^{m-1}}^2 + \|\partial_t^\mu \nabla v\|_2^2\right)$$

$$\le C_\lambda \sum_{\mu=0}^{L-1-m} \left(\|\partial_t^{\mu+2} v\|_{H^{m-1}}^2 + \|\partial_t^\mu (B_\lambda \partial_t v)\|_{H^{m-1}}^2 + \|\partial_t^\mu F_\lambda\|_{H^{m-1}}^2 + \|\partial_t^\mu \nabla v\|_2^2\right).$$

It is easy to see that

$$\sum_{\mu=0}^{L-1-m} \|\partial_t^{\mu+2} v\|_{H^{m-1}}^2 \le Z_{m-1}, \quad \sum_{\mu=0}^{L-1-m} \|\partial_t^\mu \nabla v\|_2^2 \le Z_0$$

and

$$\sum_{\mu=0}^{L-1-m} \|\partial_t^\mu (B_\lambda \partial_t v)\|_{H^{m-1}}^2 \le C_\lambda Z_{m-1}.$$

Furthermore applying Lemma A.1 and Lemma A.2 to the nonlinear terms, we get

$$\begin{aligned} Z_m(v(t)) &\le C_\lambda Z_0(v(t)) + C_\lambda Z_{m-1}(v(t))) + C_\lambda Z(v(t))^2 \\ &\le C_\lambda Z_0(v(t)) + C_\lambda Z_{m-1}(v(t))) + C_\lambda \delta^2 Z(v(t)). \end{aligned}$$

Therefore we obtain inductively that

$$Z(v(t)) = \sum_{m=0}^{L-1} Z_m(v(t)) \le C_\lambda Z_0(v(t)) + C_\lambda \delta^2 Z(v(t)).$$

Choosing δ small enough such that $\delta^2 \le \frac{1}{2C_\lambda}$, we get (14).

Next, we prove (15). Using the same way as for the proof of (14), for $0 \le m \le L-3$, we have

$$\begin{aligned} Z_m &= \sum_{\mu=0}^{L-1-m} \left\{\|\nabla \partial_t^\mu v\|_{H^m}^2 + \|\partial_t^{\mu+1} v\|_{H^m}^2\right\} \\ &= \sum_{\mu=0}^{1} \|\nabla \partial_t^\mu v\|_{H^m}^2 + \sum_{\mu=2}^{L-1-m} \|\nabla \partial_t^\mu v\|_{H^m}^2 + \|\partial_t v\|_{H^m}^2 + \sum_{\mu=1}^{L-1-m} \|\partial_t^{\mu+1} v\|_{H^m}^2 \\ &\le C Z_{L-2}(v) \\ &+ C \sum_{\mu=2}^{L-1-m} \left(\|\nabla \partial_t^{\mu-2} \triangle v\|_{H^m}^2 + \|\nabla \partial_t^{\mu-2} (B_\lambda \partial_t v)\|_{H^m}^2 + \|\nabla \partial_t^{\mu-2} F_\lambda\|_{H^m}^2\right) \\ &+ C \sum_{\mu=1}^{L-1-m} \left(\|\partial_t^{\mu-1} \triangle v\|_{H^m}^2 + \|\partial_t^{\mu-1} (B_\lambda \partial_t v)\|_{H^m}^2 + \|\partial_t^{\mu-1} F_\lambda\|_{H^m}^2\right) \end{aligned}$$

$$\leq C_\lambda \left(Z_{L-2} + Z_{m+1} + Z_{m+2} + \delta^2 Z \right).$$

Similarly we can get $Z_{L-2} \leq C_\lambda Z_{L-1} + C_\lambda \delta^2 Z$ because $\sum_{\mu=0}^{1} \|\nabla \partial_t^\mu u\|_{H^{L-2}} + \|\partial_t u\|_{H^{L-2}} \leq \|\nabla u\|_{H^{L-1}} + 2\|\partial_t u\|_{H^{L-1}}$. Thus it follows that

$$Z(v(t)) = \sum_{m=0}^{L-1} Z_m(v(t)) \leq C_\lambda Z_{L-1}(v(t)) + C_\lambda \delta^2 Z(v(t)).$$

Choosing δ small enough depend on λ, we get (15). This completes the proof of Lemma 3.3. $\square$

We consider the estimates of $Z_0(v(t))$.

Lemma 3.4. *Let $\mu \leq L-1$ and $\lambda \leq 1$. Then there exists a constant $C > 0$ such that a local solution v to* $(\mathrm{DW})_\lambda$ *satisfies*

$$\frac{d}{dt} E(\partial_t^\mu v) + \langle \partial_t^{\mu+1} v, B_\lambda \partial_t^{\mu+1} v \rangle \leq C\lambda^2 Z_0(v) + \langle \partial_t^{\mu+1} v, \partial_t^\mu F_\lambda \rangle, \tag{16}$$

$$\begin{aligned} \frac{d}{dt}\{\langle \partial_t^\mu v, \partial_t^{\mu+1} v\rangle + \frac{1}{2}\langle \partial_t^\mu v, B_\lambda \partial_t^\mu v\rangle\} + \|\nabla \partial_t^\mu v\|_2^2 - \|\partial_t^{\mu+1} v\|_2^2 \\ \leq C\lambda^2 Z_0(v) + \langle \partial_t^\mu v, \partial_t^\mu F_\lambda \rangle \end{aligned} \tag{17}$$

and for any $K > 0$, it holds that

$$\begin{aligned} &\frac{d}{dt}\langle \partial_t^{\mu+1} v, [h; \nabla \partial_t^\mu v]\rangle \\ &+ \int_{\Omega_\lambda} \left(\frac{d\phi + |x|\phi'}{2} \right) |\partial_t^{\mu+1} v|^2 dx + \int_{\Omega_\lambda} \left(\frac{(2-d)\phi + |x|\phi'}{2} \right) |\nabla \partial_t^\mu v|^2 dx \\ &\leq \frac{1}{2} \int_{\partial\Omega_\lambda} h \cdot \sigma |\sigma \cdot \nabla \partial_t^\mu v|^2 dS + \frac{K}{4} \langle \partial_t^{\mu+1} v, B_\lambda \partial_t^{\mu+1} v\rangle \\ &\quad + \frac{\|B\|_{L^\infty([0,\infty)\times\Omega)} b_0^2 R^2}{\lambda K} \|\nabla \partial_t^\mu v\|_2^2 + C\lambda Z_0(v) + \langle \partial_t^\mu F_\lambda, [h; \nabla \partial_t^\mu v]\rangle, \end{aligned} \tag{18}$$

where σ is the outward pointing unit normal vector of $\partial\Omega_\lambda$,

$$\phi(r) = \begin{cases} b_0, & (r \leq \frac{R}{\lambda}) \\ \frac{b_0 R}{\lambda r}, & (r \geq \frac{R}{\lambda}) \end{cases}, \quad h(x) = x\phi(|x|) \tag{19}$$

and $[h; \nabla g] : \mathbb{R}^d \to \mathbb{R}^n$ is defined by

$$([h; \nabla g])^i(x) = h(x) \cdot \nabla g^i(x) \quad (i = 1, 2, \cdots, n)$$

for any $g : \mathbb{R}^d \to \mathbb{R}^n$.

Proof. Let $0 \le \mu \le L-1$ and $\lambda \le 1$. First, we prove (16). Applying ∂_t^μ to (DW) and taking inner product it by $\partial_t^{\mu+1}u$, we have

$$\begin{aligned}
&\frac{d}{dt}E(\partial_t^\mu v) + \langle \partial_t^{\mu+1}v, B_\lambda \partial_t^{\mu+1}v\rangle \\
&= -\sum_{1\le\nu\le\mu}\binom{\mu}{\nu}\langle \partial_t^{\mu+1}v, \partial_t^\nu B_\lambda \partial_t^{\mu-\nu+1}v\rangle + \langle \partial_t^{\mu+1}v, \partial_t^\mu F_\lambda\rangle,
\end{aligned} \tag{20}$$

where we use

$$\partial_t^\mu v = 0 \quad on \quad \partial\Omega_\lambda. \tag{21}$$

Note that even if $\mu = 0$, (20) holds in the sense of the first term in the right-hand side to be zero. Since $\lambda \le 1$, the definition of Z_0 and condition $(\mathbf{B4})_\lambda$ imply that

$$-\sum_{1\le\nu\le\mu}\binom{\mu}{\nu}\langle \partial_t^{\mu+1}v, \partial_t^\nu B_\lambda \partial_t^{\mu-\nu+1}v\rangle \le CZ_0 \sum_{1\le\nu\le\mu}\|\partial_t^\nu B_\lambda\|_\infty \le C\lambda^2 Z_0,$$

thus we obtain (16).

Second, we prove (17). Applying ∂_t^μ to (DW) and taking inner product it by $\partial_t^\mu u$, we have

$$\begin{aligned}
&\frac{d}{dt}\{\langle \partial_t^\mu v, \partial_t^{\mu+1}v\rangle + \frac{1}{2}\langle \partial_t^\mu v, B_\lambda \partial_t^\mu v\rangle\} + \|\nabla\partial_t^\mu v\|_2^2 - \|\partial_t^{\mu+1}v\|_2^2 \\
&= \langle \partial_t^\mu v, \partial_t^{\mu+2}v\rangle + \langle \partial_t^\mu v, B_\lambda\partial_t^{\mu+1}v\rangle + \|\nabla\partial_t^\mu v\|_2^2 + \frac{1}{2}\langle \partial_t^\mu v, \partial_t B_\lambda \partial_t^\mu v\rangle \\
&= -\sum_{1\le\nu\le\mu}\binom{\mu}{\nu}\langle \partial_t^\mu v, \partial_t^\nu B_\lambda \partial_t^{\mu-\nu+1}v\rangle + \frac{1}{2}\langle \partial_t^\mu v, \partial_t B_\lambda \partial_t^\mu v\rangle + \langle \partial_t^\mu v, \partial_t^\mu F_\lambda\rangle \\
&\le C\lambda^2 Z_0 + \langle \partial_t^\mu v, \partial_t^\mu F_\lambda\rangle,
\end{aligned}$$

where we use assumption $(\mathbf{B2})_\lambda$. It means that (17) holds.

Finally, we prove (18). Applying ∂_t^μ to $(\mathrm{DW})_\lambda$ and taking inner product (DW) by $[h;\nabla\partial_t^\mu v]$ we obtain

$$\begin{aligned}
&\langle \partial_t^{\mu+2}v, [h;\nabla\partial_t^\mu v]\rangle - \langle \triangle\partial_t^\mu v, [h;\nabla\partial_t^\mu v]\rangle + \langle B_\lambda\partial_t^{\mu+1}v, [h;\nabla\partial_t^\mu v]\rangle \\
&= -\sum_{1\le\nu\le\mu}\binom{\mu}{\nu}\langle \partial_t^\nu B_\lambda\partial_t^{\mu-\nu+1}v, [h;\nabla\partial_t^\mu v]\rangle + \langle \partial_t^\mu F_\lambda, [h;\nabla\partial_t^\mu v]\rangle.
\end{aligned} \tag{22}$$

Noting

$$\nabla\partial_t^\mu v^k = \sigma\cdot\nabla\partial_t^\mu v^k\sigma \quad on\ \partial\Omega_\lambda, \tag{23}$$

we obtain

$$\begin{aligned}
&\langle \partial_t^{\mu+2}v, [h;\nabla\partial_t^\mu v]\rangle - \frac{d}{dt}\langle \partial_t^{\mu+1}v, [h;\nabla\partial_t^\mu v]\rangle = -\sum_{i=1}^d\sum_{k=1}^n\int_{\Omega_\lambda}\partial_t^{\mu+1}v^k\partial_i\partial_t^{\mu+1}v^k h^i dx \\
&= -\frac{1}{2}\int_{\Omega_\lambda} h\cdot\nabla|\partial_t^{\mu+1}v|^2 dx = \frac{1}{2}\int_{\Omega_\lambda}\mathrm{div}h|\partial_t^{\mu+1}v|^2 dx
\end{aligned}$$

and

$$
\begin{aligned}
& -\langle \triangle \partial_t^\mu v, [h; \nabla \partial_t^\mu v]\rangle \\
= \quad & \sum_{k=1}^{n} \int_{\Omega_\lambda} \nabla \partial_t^\mu v^k \cdot \nabla (h \cdot \nabla \partial_t^\mu v^k) dx - \sum_{k=1}^{n} \int_{\partial\Omega_\lambda} \sigma \cdot \nabla \partial_t^\mu v^k h \cdot \nabla \partial_t^\mu v^k dS \\
= \quad & \sum_{i,j=1}^{d} \sum_{k=1}^{n} \int_{\Omega_\lambda} \partial_j \partial_t^\mu v^k \partial_i \partial_t^\mu v^k \partial_i h^j dx + \frac{1}{2} \int_{\Omega_\lambda} h \cdot \nabla |\nabla \partial_t^\mu v|^2 dx \\
& - \int_{\partial\Omega_\lambda} h \cdot \sigma |\sigma \cdot \nabla \partial_t^\mu v|^2 dS \\
= \quad & \sum_{i,j=1}^{d} \sum_{k=1}^{n} \int_{\Omega_\lambda} \partial_j \partial_t^\mu v^k \partial_i \partial_t^\mu v^k \partial_i h^j dx - \frac{1}{2} \int_{\Omega_\lambda} \mathrm{div} h |\nabla \partial_t^\mu v|^2 dx \\
& - \frac{1}{2} \int_{\partial\Omega_\lambda} h \cdot \sigma |\sigma \cdot \nabla \partial_t^\mu v|^2 dS.
\end{aligned}
$$

Therefore we get from (22) that

$$
\begin{aligned}
& \frac{d}{dt} \langle \partial_t^{\mu+1} v, [h; \nabla \partial_t^\mu v]\rangle + \frac{1}{2} \int_{\Omega_\lambda} (|\partial_t^{\mu+1} v|^2 - |\nabla \partial_t^\mu v|^2) \mathrm{div} h dx \qquad (24) \\
& + \sum_{i,j=1}^{d} \sum_{k=1}^{n} \int_{\Omega_\lambda} \partial_j \partial_t^\mu v^k \partial_i \partial_t^\mu v^k \partial_i h^j dx \\
= \quad & \frac{1}{2} \int_{\partial\Omega_\lambda} h \cdot \sigma |\sigma \cdot \nabla \partial_t^\mu v|^2 dS - \langle B_\lambda \partial_t^{\mu+1} v, [h; \nabla \partial_t^\mu v]\rangle \\
& - \sum_{1 \le \nu \le \mu} \binom{\mu}{\nu} \langle \partial_t^\nu B_\lambda \partial_t^{\mu-\nu+1} v, [h; \nabla \partial_t^\mu v]\rangle + \langle \partial_t^\mu F_\lambda, [h; \nabla \partial_t^\mu v]\rangle.
\end{aligned}
$$

Now we remark that

$$
\begin{cases}
\dfrac{\partial h^i}{\partial x_j} = \delta_{ij} \phi(|x|) + \phi'(|x|) \dfrac{x_i x_j}{|x|}, \\
\mathrm{div} h(x) = d\phi(|x|) + \phi'(|x|)|x|, \\
\|h\|_\infty \le \dfrac{b_0 R}{\lambda} \quad \textit{and} \quad \|\nabla h\|_\infty \le 2b_0.
\end{cases} \qquad (25)
$$

Using (25) and $\phi'(r) \le 0$, we obtain

$$
\begin{aligned}
& \sum_{i,j=1}^{d} \sum_{k=1}^{n} \int_{\Omega_\lambda} \partial_j \partial_t^\mu v^k \partial_i \partial_t^\mu v^k \partial_i h^j dx \\
= \quad & \sum_{i=1}^{d} \sum_{k=1}^{n} \int_{\Omega_\lambda} |\partial_i \partial_t^\mu v^k|^2 \phi dx + \sum_{i,j=1}^{d} \sum_{k=1}^{n} \int_{\Omega_\lambda} \partial_j \partial_t^\mu v^k \partial_i \partial_t^\mu v^k \phi' \frac{x_i x_j}{|x|} dx
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{\Omega_\lambda} |\nabla \partial_t^\mu v|^2 \phi dx + \sum_{k=1}^n \int_{\Omega_\lambda} |x \cdot \nabla \partial_t^\mu v^k|^2 \phi' \frac{1}{|x|} dx \\
&\geq \int_{\Omega_\lambda} \{\phi + |x|\phi'\} |\nabla \partial_t^\mu v|^2 dx.
\end{aligned}
$$

These estimates and (24) imply that

$$
\begin{aligned}
&\frac{d}{dt} \langle \partial_t^{\mu+1} v, [h; \nabla \partial_t^\mu v] \rangle \\
&\quad + \int_{\Omega_\lambda} \left(\frac{(d\phi + |x|\phi')}{2} \right) |\partial_t^{\mu+1} v|^2 dx + \int_{\Omega_\lambda} \left(\frac{(2-d)\phi + |x|\phi'}{2} \right) |\nabla \partial_t^\mu v|^2 dx \\
&\leq \frac{1}{2} \int_{\partial\Omega_\lambda} h \cdot \sigma |\sigma \cdot \nabla \partial_t^\mu v|^2 dS - \langle B_\lambda \partial_t^{\mu+1} v, [h; \nabla \partial_t^\mu v] \rangle \\
&\quad - \sum_{1 \leq \nu \leq \mu} \binom{\mu}{\nu} \langle \partial_t^\nu B_\lambda \partial_t^{\mu-\nu+1} v, [h; \nabla \partial_t^\mu v] \rangle + \langle \partial_t^\mu F_\lambda, [h; \nabla \partial_t^\mu v] \rangle.
\end{aligned} \tag{26}
$$

Let we estimate for the right side of (26). We calculate

$$
\begin{aligned}
|\langle B_\lambda \partial_t^{\mu+1} v, [h; \nabla \partial_t^\mu v] \rangle| &= |\langle \sqrt{B_\lambda} \partial_t^{\mu+1} v, \sqrt{B_\lambda}[h; \nabla \partial_t^\mu v] \rangle| \\
&\leq \frac{K}{4} \|\sqrt{B_\lambda} \partial_t^{\mu+1} v\|_2^2 + \frac{1}{K} \|\sqrt{B_\lambda}[h; \nabla \partial_t^\mu v]\|_2^2 \\
&\leq \frac{K}{4} \langle \partial_t^{\mu+1} v, B_\lambda \partial_t^{\mu+1} v \rangle + \frac{\|B\|_{L^\infty([0,\infty)\times\Omega)} b_0^2 R^2}{\lambda K} \|\nabla \partial_t^\mu v\|_2^2,
\end{aligned} \tag{27}
$$

$$
\begin{aligned}
&\left| \sum_{1 \leq \nu \leq \mu} \binom{\mu}{\nu} \langle \partial_t^\nu B_\lambda \partial_t^{\mu-\nu+1} v, [h; \nabla \partial_t^\mu v] \rangle \right| \\
&\leq C \sum_{1 \leq \nu \leq \mu} \|\partial_t^\nu B_\lambda\|_\infty \|h\|_\infty \|\partial_t^{\mu-\nu+1} v\|_2 \|\nabla \partial_t^\mu v\|_2 \leq C\lambda Z_0.
\end{aligned} \tag{28}
$$

where we use $(\mathbf{B1})_\lambda, (\mathbf{B4})_\lambda$ and (25). Combining estimates (26) - (28), we get (18). This completes the proof of Lemma 3.4. □

We define $G(v(t))$ by

$$
\begin{aligned}
G(v(t)) &= \frac{C_0}{2\lambda} Z_0(v(t)) + \frac{b_0(2d-1)}{4} \sum_{\mu=0}^{L-1} \langle \partial_t^{\mu+1} v(t), \partial_t^\mu v(t) \rangle \\
&\quad + \frac{b_0(2d-1)}{8} \sum_{\mu=0}^{L-1} \langle \partial_t^\mu v(t), B_\lambda(t) \partial_t^\mu v(t) \rangle + \sum_{\mu=0}^{L-1} \langle \partial_t^{\mu+1} v(t), [h; \nabla \partial_t^\mu v(t)] \rangle,
\end{aligned} \tag{29}
$$

where

$$
C_0 = \max \left\{ 4b_0 R + \frac{C_1 b_0 (2d-1)}{2},\ d,\ \|B\|_{L^\infty([0,\infty)\times\Omega)} b_0^2 R^2 \times \frac{8}{b_0} \right\}. \tag{30}
$$

Lemma 3.5. *There exists a constant $0 < \lambda < 1$ such that the local solution v to* $(\mathrm{DW})_\lambda$ *satisfies*

$$\frac{d}{dt}G(v(t)) + \frac{b_0}{16}Z_0(v(t)) \tag{31}$$
$$\leq \frac{1}{2}\sum_{\mu=0}^{L-1}\int_{\partial\Omega_\lambda} h\cdot\sigma|\sigma\cdot\nabla\partial_t^\mu v|^2 dS + \bar{C}_\lambda\sum_{\mu=0}^{L-1}\langle\partial_t^\mu F_\lambda, \partial_t^{\mu+1}v + \partial_t^\mu v + [h;\nabla\partial_t^\mu v]\rangle,$$

where $\bar{C}_\lambda = \dfrac{C_0}{\lambda} + \dfrac{b_0(2d-1)}{4} + 1.$

Proof. Let $\mu \leq L-1$ and $K \geq \frac{d}{\lambda}$. Calculating $K \times (16) + \frac{b_0(2d-1)}{4} \times (17) + (18)$, we have

$$\frac{d}{dt}\left\{KE(\partial_t^\mu v) + \frac{b_0(2d-1)}{4}\langle\partial_t^{\mu+1}v, \partial_t^\mu v\rangle \right. \tag{32}$$
$$\left. + \frac{b_0(2d-1)}{8}\langle\partial_t^\mu v, B_\lambda\partial_t^\mu v\rangle + \langle\partial_t^{\mu+1}v, [h;\nabla\partial_t^\mu v]\rangle\right\}$$
$$+ K\langle\partial_t^{\mu+1}v, B_\lambda\partial_t^{\mu+1}v\rangle - \frac{b_0(2d-1)}{4}\|\partial_t^{\mu+1}v\|_2^2 + \int_{\Omega_\lambda}\left(\frac{d\phi + |x|\phi'}{2}\right)|\partial_t^{\mu+1}v|^2 dx$$
$$+ \frac{b_0(2d-1)}{4}\|\nabla\partial_t^\mu v\|_2^2 + \int_{\Omega_\lambda}\left(\frac{(2-d)\phi + |x|\phi'}{2}\right)|\nabla\partial_t^\mu v|^2 dx$$
$$\leq \frac{K}{4}\langle\partial_t^{\mu+1}v, B_\lambda\partial_t^{\mu+1}v\rangle + \frac{\|B\|_{L^\infty([0,\infty)\times\Omega)}b_0^2R^2}{\lambda K}\|\nabla\partial_t^\mu v\|_2^2$$
$$+ \frac{1}{2}\int_{\partial\Omega_\lambda} h\cdot\sigma|\sigma\cdot\nabla\partial_t^\mu v|^2 dS + C\lambda^2 Z_0\left(K + \frac{b_0(2d-1)}{4} + \frac{1}{\lambda}\right)$$
$$+ \left(K + \frac{b_0(2d-1)}{4} + 1\right)\langle\partial_t^\mu F_\lambda, \partial_t^{\mu+1}v + \partial_t^\mu v + [h;\nabla\partial_t^\mu v]\rangle.$$

Using $(\mathbf{B3})_\lambda$, $K \geq \frac{d}{\lambda}$, $\phi \geq 0$ and

$$r\phi'(r) = \begin{cases} 0, & (r < \frac{R}{\lambda}) \\ -\phi(r), & (r > \frac{R}{\lambda}), \end{cases}$$

we obtain

$$K\langle\partial_t^{\mu+1}v, B_\lambda\partial_t^{\mu+1}v\rangle - \frac{b_0(2d-1)}{4}\|\partial_t^{\mu+1}v\|_2^2 + \int_{\Omega_\lambda}\left\{\frac{d\phi + |x|\phi'}{2}\right\}|\partial_t^{\mu+1}v|^2 dx$$
$$\geq \frac{K}{2}\langle\partial_t^{\mu+1}v, B_\lambda\partial_t^{\mu+1}v\rangle + \int_{U_{\frac{R}{\lambda}}}\left(-\frac{b_0(2d-1)}{4} + \frac{db_0}{2}\right)|\partial_t^{\mu+1}v|^2 dx$$
$$+ \int_{|x|\geq\frac{R}{\lambda}}\left(\frac{\lambda b_0 K}{2} - \frac{b_0(2d-1)}{4} + \frac{d-1}{2}\phi\right)|\partial_t^{\mu+1}v|^2 dx$$
$$\geq \frac{K}{2}\langle\partial_t^{\mu+1}v, B_\lambda\partial_t^{\mu+1}v\rangle + \frac{b_0}{4}\|\partial_t^{\mu+1}v\|_2^2 \tag{33}$$

and

$$\frac{b_0(2d-1)}{4}\|\nabla\partial_t^\mu v\|_2^2+\int_{\Omega_\lambda}\frac{(2-d)\phi+|x|\phi'}{2}|\nabla\partial_t^\mu v|^2dx \tag{34}$$

$$\begin{aligned}
= &\int_{U_{\frac{R}{\lambda}}}\left(\frac{b_0(2d-1)}{4}+\frac{(2-d)b_0}{2}\right)|\nabla\partial_t^\mu v|^2dx\\
&+\int_{|x|\geq\frac{R}{\lambda}}\left(\frac{b_0(2d-1)}{4}+\frac{(1-d)}{2}\frac{b_0R}{\lambda|x|}\right)|\nabla\partial_t^\mu v|^2dx\geq\frac{b_0}{4}\|\nabla\partial_t^\mu v\|_2^2.
\end{aligned}$$

Combining (32), (33) and (34), we have

$$\begin{aligned}
&\frac{d}{dt}\left\{KE(\partial_t^\mu v)+\frac{b_0(2d-1)}{4}\langle\partial_t^{\mu+1}v,\partial_t^\mu v\rangle\right.\\
&\qquad\left.+\frac{b_0(2d-1)}{8}\langle\partial_t^\mu v,B_\lambda\partial_t^\mu v\rangle+\langle\partial_t^{\mu+1}v,[h;\nabla\partial_t^\mu v]\rangle\right\}\\
&+\frac{b_0}{4}\{\|\partial_t^{\mu+1}v\|_2^2+\|\nabla\partial_t^\mu v\|_2^2\}\\
&\leq\frac{\|B\|_{L^\infty([0,\infty)\times\Omega)}b_0^2R^2}{\lambda K}\|\nabla\partial_t^\mu v\|_2^2+\frac{1}{2}\int_{\partial\Omega_\lambda}h\cdot\sigma|\sigma\cdot\nabla\partial_t^\mu v|^2dS\\
&+C\lambda^2Z_0(v)\left(K+\frac{b_0(2d-1)}{4}+\frac{1}{\lambda}\right)\\
&+\left(K+\frac{b_0(2d-1)}{4}+1\right)\langle\partial_t^\mu F_\lambda,\partial_t^{\mu+1}v+\partial_t^\mu v+[h;\nabla\partial_t^\mu v]\rangle.
\end{aligned}$$

Let $K=\frac{C_0}{\lambda}$ and sum up μ from 0 to $L-1$. Then we get

$$\begin{aligned}
&\frac{d}{dt}G(v)+\frac{b_0}{8}Z_0(v)\leq C\lambda Z_0(v)\\
&+\frac{1}{2}\sum_{\mu=0}^{L-1}\int_{\partial\Omega_\lambda}h\cdot\sigma|\sigma\cdot\nabla\partial_t^\mu v|^2dS+\bar{C}_\lambda\sum_{\mu=0}^{L-1}\langle\partial_t^\mu F_\lambda,\partial_t^{\mu+1}v+\partial_t^\mu v+[h;\nabla\partial_t^\mu v]\rangle,
\end{aligned}$$

where we use (30). We choose small λ such that $C\lambda\leq\frac{b_0}{16}$. Then we obtain

$$\begin{aligned}
&\frac{d}{dt}G(v)+\frac{b_0}{16}Z_0(v)\\
&\leq\frac{1}{2}\sum_{\mu=0}^{L-1}\int_{\partial\Omega}h\cdot\sigma|\sigma\cdot\nabla\partial_t^\mu v|^2dS+\bar{C}_\lambda\sum_{\mu=0}^{L-1}\langle\partial_t^\mu F_\lambda,\partial_t^{\mu+1}v+\partial_t^\mu v+[h;\nabla\partial_t^\mu v]\rangle.
\end{aligned}$$

This completes the proof of Lemma 3.5. □

We choose λ so small that Lemma 3.5 may hold. Next Lemma shows the estimates of the nonlinear terms.

Lemma 3.6. *Let $\mu \leq L-1$. Then there exists a constant $C_\lambda > 0$ such that the local solution $v \in X_\delta^T$ to* $(\mathrm{DW})_\lambda$ *satisfies*

$$\langle \partial_t^{\mu+1} v, \partial_t^\mu F_\lambda \rangle \leq C_\lambda \delta Z_0 - \frac{1}{2} \sum_{i,j=1}^n \sum_{1\leq a,b\leq d} \frac{d}{dt} \int_{\Omega_\lambda} \partial_t^\mu \partial_a v^i \partial_t^\mu \partial_b v^j c_{ij}^{ab}(\partial v) dx \quad (35)$$
$$+ \sum_{i,j=1}^n \frac{1}{2} \frac{d}{dt} \int_{\Omega_\lambda} \partial_t^{\mu+1} v^i \partial_t^{\mu+1} v^j c_{ij}^{00}(\partial v) dx,$$

$$\langle \partial_t^\mu v, \partial_t^\mu F_\lambda \rangle \leq C_\lambda \delta Z_0 + \sum_{i,j=1}^n \sum_{0\leq b\leq d} \frac{d}{dt} \int_{\Omega_\lambda} \partial_t^\mu v^i \partial_t^\mu \partial_b v^j c_{ij}^{0b}(\partial v) dx \quad (36)$$

and

$$\langle [h; \nabla \partial_t^\mu v], \partial_t^\mu F_\lambda \rangle \leq C_\lambda \delta Z_0 + C_\lambda \delta \int_{\partial\Omega_\lambda} |h \cdot \sigma| |\sigma \cdot \nabla \partial_t^\mu v|^2 dS \quad (37)$$
$$+ \sum_{i,j=1}^n \sum_{0\leq b\leq d} \frac{d}{dt} \int_{\Omega_\lambda} h \cdot \nabla \partial_t^\mu v^i \partial_t^\mu \partial_b v^j c_{ij}^{0b}(\partial v) dx.$$

Proof. We may assume that $\delta < 1$. First, we prove (35). Using Lemma 3.3 and Lemma A.1, we have

$$\langle \partial_t^{\mu+1} v, \partial_t^\mu \tilde{F}_\lambda \rangle \leq \|\partial_t^{\mu+1} v\|_2 \|\partial_t^\mu \tilde{F}_\lambda\|_2 \leq C_\lambda \delta Z \leq C_\lambda \delta Z_0.$$

Furthermore we calculate that

$$\begin{aligned}
&\sum_{i,j=1}^n \sum_{0\leq a,b\leq d} \int_{\Omega_\lambda} \partial_t^{\mu+1} v^i \partial_t^\mu (c_{ij}^{ab}(\partial v) \partial_a \partial_b v^j) dx \\
&= \sum_{0\leq \nu\leq \mu-1} \binom{\mu}{\nu} \sum_{i,j=1}^n \sum_{0\leq a,b\leq d} \int_{\Omega_\lambda} \partial_t^{\mu+1} v^i \partial_t^\nu \partial_a \partial_b v^j \partial_t^{\mu-\nu} (c_{ij}^{ab}(\partial v)) dx \\
&- \sum_{i,j=1}^n \sum_{0\leq a,b\leq d} \int_{\Omega_\lambda} \partial_t^{\mu+1} v^i \partial_t^\mu \partial_b v^j \partial_a (c_{ij}^{ab}(\partial v)) dx \\
&- \sum_{i,j=1}^n \sum_{0\leq a,b\leq d} \int_{\Omega_\lambda} \partial_t^{\mu+1} \partial_a v^i \partial_t^\mu \partial_b v^j c_{ij}^{ab}(\partial v) dx \\
&+ \sum_{i,j=1}^n \sum_{0\leq a,b\leq d} \int_{\Omega_\lambda} \partial_a (\partial_t^{\mu+1} v^i \partial_t^\mu \partial_b v^j c_{ij}^{ab}(\partial v)) dx \\
&= J_1 + J_2 + \sum_{i,j=1}^n \sum_{0\leq a,b\leq d} \frac{1}{2} \int_{\Omega_\lambda} \partial_t^\mu \partial_a v^i \partial_t^\mu \partial_b v^j \partial_t (c_{ij}^{ab}(\partial v)) dx \\
&- \sum_{i,j=1}^n \sum_{0\leq a,b\leq d} \frac{1}{2} \frac{d}{dt} \int_{\Omega_\lambda} \partial_t^\mu \partial_a v^i \partial_t^\mu \partial_b v^j c_{ij}^{ab}(\partial v) dx
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i,j=1}^{n}\sum_{0\le b\le d}\frac{d}{dt}\int_{\Omega_\lambda}\partial_t^{\mu+1}v^i\partial_t^{\mu}\partial_b v^j c_{ij}^{0b}(\partial v)dx\\
&=J_1+J_2+J_3-\sum_{i,j=1}^{n}\sum_{1\le a,b\le d}\frac{1}{2}\frac{d}{dt}\int_{\Omega_\lambda}\partial_t^{\mu}\partial_a v^i\partial_t^{\mu}\partial_b v^j c_{ij}^{ab}(\partial v)dx\\
&\qquad+\sum_{i,j=1}^{n}\frac{1}{2}\frac{d}{dt}\int_{\Omega_\lambda}\partial_t^{\mu+1}v^i\partial_t^{\mu+1}v^j c_{ij}^{00}(\partial v)dx.
\end{aligned}
$$

We can estimate $|J_k|\le C_\lambda\delta Z_0$ $(k=1,2,3)$ from Lemma 3.3 and Lemma A.2. Therefore we get (35).

Second, we prove (36). Using Lemma 3.3 and Lemma A.1, we have

$$
\langle\partial_t^{\mu}v,\partial_t^{\mu}\tilde{F}_\lambda\rangle\le\|\partial_t^{\mu}v\|_2\|\partial_t^{\mu}\tilde{F}_\lambda\|_2\le C_\lambda\delta Z\le C_\lambda\delta Z_0.
$$

Furthermore we calculate that

$$
\begin{aligned}
&\sum_{i,j=1}^{n}\sum_{0\le a,b\le d}\int_{\Omega_\lambda}\partial_t^{\mu}v^i\partial_t^{\mu}(c_{ij}^{ab}(\partial v)\partial_a\partial_b v^j)dx\\
&=\sum_{0\le\nu\le\mu-1}\binom{\mu}{\nu}\sum_{i,j=1}^{n}\sum_{0\le a,b\le d}\int_{\Omega_\lambda}\partial_t^{\mu}v^i\partial_t^{\nu}\partial_a\partial_b v^j\partial_t^{\mu-\nu}(c_{ij}^{ab}(\partial v))dx\\
&-\sum_{i,j=1}^{n}\sum_{0\le a,b\le d}\int_{\Omega_\lambda}\partial_a\partial_t^{\mu}v^i\partial_t^{\mu}\partial_b v^j c_{ij}^{ab}(\partial v)dx\\
&-\sum_{i,j=1}^{n}\sum_{0\le a,b\le d}\int_{\Omega_\lambda}\partial_t^{\mu}v^i\partial_t^{\mu}\partial_b v^j\partial_a(c_{ij}^{ab}(\partial v))dx\\
&+\sum_{i,j=1}^{n}\sum_{0\le b\le d}\frac{d}{dt}\int_{\Omega_\lambda}\partial_t^{\mu}v^i\partial_t^{\mu}\partial_b v^j c_{ij}^{0b}(\partial v)dx\\
&=J_4+J_5+J_6+\sum_{i,j=1}^{n}\sum_{0\le b\le d}\frac{d}{dt}\int_{\Omega_\lambda}\partial_t^{\mu}v^i\partial_t^{\mu}\partial_b v^j c_{ij}^{0b}(\partial v)dx.
\end{aligned}
$$

We can estimate $|J_k|\le C_\lambda\delta Z_0$ $(k=4,5,6)$ from Lemma 3.3 and Lemma A.2. Therefore we get (36).

Finally, we prove (37), Using Lemma 3.3 and Lemma A.1, we have

$$
\langle[h;\nabla\partial_t^{\mu}v],\partial_t^{\mu}\tilde{F}_\lambda\rangle\le\|h\|_\infty\|\nabla\partial_t^{\mu}v\|_2\|\partial_t^{\mu}\tilde{F}_\lambda\|_2\le C_\lambda\delta Z\le C_\lambda\delta Z_0.
$$

We calculate that

$$
\begin{aligned}
&\sum_{i,j=1}^{n}\sum_{0\le a,b\le d}\int_{\Omega_\lambda}h\cdot\nabla\partial_t^{\mu}v^i\partial_t^{\mu}(c_{ij}^{ab}(\partial v)\partial_a\partial_b v^j)dx\\
&=\sum_{0\le\nu\le\mu-1}\sum_{i,j=1}^{n}\sum_{0\le a,b\le d}\int_{\Omega_\lambda}h\cdot\nabla\partial_t^{\mu}v^i\partial_t^{\nu}\partial_a\partial_b v^j\partial_t^{\mu-\nu}(c_{ij}^{ab}(\partial v))dx
\end{aligned}
$$

$$
\begin{aligned}
&-\sum_{i,j=1}^{n}\sum_{0\le a,b\le d}\int_{\Omega_\lambda} h\cdot\nabla\partial_t^\mu v^i\partial_t^\mu\partial_b v^j\partial_a(c_{ij}^{ab}(\partial v))dx\\
&-\sum_{i,j=1}^{n}\sum_{0\le a,b\le d}\int_{\Omega_\lambda} \partial_a h\cdot\nabla\partial_t^\mu v^i\partial_t^\mu\partial_b v^j c_{ij}^{ab}(\partial v)dx\\
&+\sum_{i,j=1}^{n}\sum_{0\le a,b\le d}\frac{1}{2}\int_{\Omega_\lambda} h\cdot\nabla(c_{ij}^{ab}(\partial v))\partial_t^\mu\partial_a v^i\partial_t^\mu\partial_b v^j dx\\
&+\sum_{i,j=1}^{n}\sum_{0\le a,b\le d}\frac{1}{2}\int_{\Omega_\lambda} (\mathrm{div}h)\partial_t^\mu\partial_a v^i\partial_t^\mu\partial_b v^j c_{ij}^{ab}(\partial v)dx\\
&+\sum_{i,j=1}^{n}\sum_{1\le a,b\le d}\int_{\partial\Omega_\lambda} \sigma_a h\cdot\nabla\partial_t^\mu v^i\partial_t^\mu\partial_b v^j c_{ij}^{ab}(\partial v)dS\\
&-\sum_{i,j=1}^{n}\sum_{1\le a,b\le d}\frac{1}{2}\int_{\partial\Omega_\lambda} h\cdot\sigma\partial_t^\mu\partial_a v^i\partial_t^\mu\partial_b v^j c_{ij}^{ab}(\partial v)dS\\
&+\sum_{i,j=1}^{n}\sum_{0\le b\le d}\frac{d}{dt}\int_{\Omega_\lambda} h\cdot\nabla\partial_t^\mu v^i\partial_t^\mu\partial_b v^j c_{ij}^{0b}(\partial v)dx\\
&= J_7+J_8+J_9+J_{10}+J_{11}+J_{12}+J_{13}\\
&\qquad+\sum_{i,j=1}^{n}\sum_{0\le b\le d}\frac{d}{dt}\int_{\Omega_\lambda} h\cdot\nabla\partial_t^\mu v^i\partial_t^\mu\partial_b v^j c_{ij}^{0b}(\partial v)dx.
\end{aligned}
$$

We can estimate $|J_k|\le C_\lambda\delta Z_0$ $(k=7,8,9,10,11)$ from Lemma 3.3 and Lemma A.2. Moreover using (23) and

$$
\|\partial v\|_{L^\infty(\partial\Omega_\lambda)}\le C_\lambda\|\partial v\|_{H^{\left[\frac{d-1}{2}\right]+1}(\partial\Omega_\lambda)}\le C_\lambda\|\partial v\|_{H^{\left[\frac{d-1}{2}\right]+2}}\le C_\lambda Z(v)
$$

(the second inequality follows from a trace theorem, see for instance:[9]), we get

$$
\begin{aligned}
J_k &\le C\sum_{i,j=1}^{n}\sum_{1\le a,b\le d}\|c_{ij}^{ab}(\partial v)\|_{L^\infty(\partial\Omega_\lambda)}\int_{\partial\Omega_\lambda}|h\cdot\sigma||\sigma\cdot\nabla\partial_t^\mu v|^2dS\\
&\le C_\lambda\delta\int_{\partial\Omega_\lambda}|h\cdot\sigma||\sigma\cdot\nabla\partial_t^\mu v|^2dS \qquad (k=12,13).
\end{aligned}
$$

Therefore we get (37). This completes the proof of Lemma 3.6. □

We define $\tilde{G}$ as follows:

$$
\begin{aligned}
\tilde{G}(v(t)) =& G(v(t))+\bar{C}_\lambda\sum_{\mu=0}^{L-1}\sum_{i,j=1}^{n}\Bigg\{\frac{1}{2}\sum_{1\le a,b\le d}\int_{\Omega_\lambda}\partial_t^\mu\partial_a v^i\partial_t^\mu\partial_b v^j c_{ij}^{ab}(\partial v)dx\\
&-\frac{1}{2}\int_{\Omega_\lambda}\partial_t^{\mu+1}v^i\partial_t^{\mu+1}v^j c_{ij}^{00}(\partial v)dx
\end{aligned}
$$

$$- \sum_{0\le b\le d} \left\{ \int_{\Omega_\lambda} \partial_t^\mu v^i \partial_t^\mu \partial_b v^j c_{ij}^{0b}(\partial v)dx + \int_{\Omega_\lambda} h\cdot \nabla \partial_t^\mu v^i \partial_t^\mu \partial_b v^j c_{ij}^{0b}(\partial v)dx \right\} \Bigg\}.$$

Then the following lemma holds.

Lemma 3.7. *Let $v \in X_\delta^T$ be the solution to* $(\mathrm{DW})_\lambda$ *and λ is sufficiently small to hold the Lemma 3.5. Then there exist a $\delta = \delta(\lambda)$ such that v satisfy*

$$\frac{d}{dt}\tilde{G}(v(t)) + \frac{b_0}{32} Z_0(v(t)) \le 0 \qquad (t \in [0,T]) \tag{38}$$

and

$$\tilde{G}(v(t)) \cong_\lambda \|v(t)\|_2^2 + Z_0(v(t)) \qquad (t\in[0,T]), \tag{39}$$

where comparability constant is independent of δ, t and T.

Proof. First, we prove (38). Using Lemma 3.5 and Lemma 3.6, we obtain

$$\begin{aligned}
&\frac{d}{dt}\tilde{G}(v(t)) + \frac{b_0}{16}Z_0 \\
&\le C_\lambda \delta Z_0 + \frac{1}{2}\sum_{\mu=0}^{L-1}\int_{\partial\Omega_\lambda} h\cdot\sigma|\sigma\cdot\nabla\partial_t^\mu v|^2 dS + C_\lambda\delta\sum_{\mu=0}^{L-1}\int_{\partial\Omega_\lambda}|h\cdot\sigma||\sigma\cdot\nabla\partial_t^\mu v|^2dS.
\end{aligned}$$

Because of $\mathbb{R}^d/\Omega_\lambda$ is star shaped, it holds that $h\cdot\sigma\le 0$ on $\partial\Omega_\lambda$. Then we can choose δ sufficiently small depend on λ such that (38) holds.

Next, we prove (39). It follows from (9) that

$$\begin{aligned}
&\left|\frac{b_0(2d-1)}{4}\sum_{\mu=0}^{L-1}\langle\partial_t^\mu v, \partial_t^{\mu+1}v\rangle\right| \qquad (40)\\
\le\ & \frac{b_0(2d-1)}{4}\sum_{\mu=0}^{L-1}\left\{\frac{\lambda}{4C_1}\|\partial_t^\mu v\|_2^2 + \frac{C_1}{\lambda}\|\partial_t^{\mu+1}v\|_2^2\right\}\\
\le\ & \frac{b_0(2d-1)}{4}\sum_{\mu=0}^{L-1}\left\{\frac{1}{4}\langle\partial_t^\mu v, B_\lambda\partial_t^\mu v\rangle + \frac{1}{4\lambda}\|\nabla\partial_t^\mu v\|_2^2 + \frac{C_1}{\lambda}\|\partial_t^{\mu+1}v\|_2^2\right\}\\
\le\ & \frac{b_0(2d-1)}{16}\sum_{\mu=0}^{L-1}\langle\partial_t^\mu v, B_\lambda\partial_t^\mu v\rangle + \frac{C_1 b_0(2d-1)}{4\lambda}Z_0
\end{aligned}$$

and

$$\left|\sum_{\mu=0}^{L-1}\langle\partial_t^{\mu+1}v, [h;\nabla\partial_t^\mu v]\rangle\right| \le \sum_{\mu=0}^{L-1}\|\partial_t^{\mu+1}v\|_2\|\nabla\partial_t^\mu v\|_2\|h\|_\infty \le \frac{b_0R}{\lambda}Z_0. \tag{41}$$

Using (40), (41), (30) and Lemma 2.3, we have

$$G(v) \tag{42}$$

$$\geq \frac{1}{\lambda}\left(\frac{C_0}{2} - b_0 R - \frac{C_1 b_0(2d-1)}{4}\right) Z_0 + \frac{b_0(2d-1)}{16}\sum_{\mu=0}^{L-1}\langle \partial_t^\mu v, B_\lambda \partial_t^\mu v\rangle$$

$$\geq \frac{b_0 R}{\lambda} Z_0 + \frac{b_0(2d-1)}{16}\sum_{\mu=0}^{L-1}\langle \partial_t^\mu v, B_\lambda \partial_t^\mu v\rangle \geq C_\lambda(\|v\|_2^2 + Z_0).$$

On the other hand, Lemma 2.1, Lemma 3.3 and Lemma A.2 imply that

$$\left|\int_{\Omega_\lambda} \partial_t^\mu \partial_a v^i \partial_t^\mu \partial_b v^j c_{ij}^{ab}(\partial v) dx\right| \leq \|\partial_a \partial_t^\mu v^i\|_2 \|\partial_t^\mu \partial_b v^j c_{ij}^{ab}(\partial v)\|_2 \leq C_\lambda \delta Z_0,$$

$$\left|\int_{\Omega_\lambda} \partial_t^{\mu+1} v^i \partial_t^{\mu+1} v^j c_{ij}^{00}(\partial v) dx\right| \leq \|\partial^{\mu+1} v^i\|_2 \|\partial_t^{\mu+1} v^j c_{ij}^{00}(\partial v)\|_2 \leq C_\lambda \delta Z_0,$$

$$\left|\int_{\Omega_\lambda} \partial_t^\mu v^i \partial_t^\mu \partial_b v^j c_{ij}^{0b}(\partial v) dx\right| \leq \|\partial_t^\mu v^i\|_2 \|\partial_t^\mu \partial_b v^j c_{ij}^{0b}(\partial v)\|_2 \leq C_\lambda \delta Z_0$$

and

$$\left|\int_{\Omega_\lambda} h \cdot \nabla \partial_t^\mu v^i \partial_t^\mu \partial_b v^j c_{ij}^{0b}(\partial v) dx\right| \leq \|h\|_\infty \|\nabla \partial_t^\mu v^i\|_2 \|\partial_t^\mu \partial_b v^j c_{ij}^{0b}(\partial v)\|_2 \leq C_\lambda \delta Z_0.$$

Since these estimates and (42) imply that we can choose $\delta = \delta(\lambda)$ to hold $\tilde{G}(v) \geq C_\lambda(\|v\|_2^2 + Z_0)$. It is clear that $\tilde{G}(v(t)) \leq C'_\lambda(\|v\|_2^2 + Z_0)$ is true. Thus it holds that (39). This completes the proof of Lemma 3.7. □

Proof of Proposition 3.2

Let λ and δ be sufficiently small to hold Lemma 3.5 and Lemma 3.7. Integrating (38) over $[0,t]$, we have

$$\tilde{G}(v(t)) + \frac{b_0}{32}\int_0^t Z_0(v(s)) ds \leq \tilde{G}(v(t))|_{t=0}.$$

Since (39) imply that

$$\|v(t)\|_2^2 + Z_0(v(t)) + \int_0^t Z_0(v(s)) ds \leq C_\lambda(\|v_0\|_2^2 + Z_0(v(t))|_{t=0}),$$

furthermore using Lemma 3.3, we get (13). This completes the proof of Proposition 3.2.

4 Decay Estimates

In this section, we prove Theorem 1.2. In what follows, λ and δ be sufficiently small to hold Theorem 1.1. Let $v \in X_\delta$ be the solution to $(\mathrm{DW})_\lambda$. Since (38) implies

$$\frac{d}{dt}\{(1+t)\tilde{G}(v(t))\} = \tilde{G}(v(t)) + (1+t)\frac{d}{dt}\tilde{G}(v(t)) \leq \tilde{G}(v(t)) - \frac{b_0}{32}(1+t)Z_0(v(t)).$$

Integrating the above estimate over $[0,t]$ and using (39), Lemma 2.3 and Proposition 3.2, we obtain

$$(1+t)\{\|v(t)\|_2^2 + Z(v(t))\} + \int_0^t (1+s)Z(v(s))ds \qquad (43)$$
$$\leq C_\lambda \|(v_0,v_1)\|_{H^L\times H^{L-1}}^2 + C_\lambda \int_0^t \langle v(s), B_\lambda(s)v(s)\rangle ds.$$

We want to the estimate for the second term in the right-hand side in (43). As is in Ikehata [2], we consider indefinite integral of v. We define

$$w(t,x) = \int_0^t v(s,x)ds. \qquad (44)$$

Then w satisfies

$$\begin{cases} (\partial_t^2 - \triangle + B_\lambda(t,x)\partial_t)w & \\ = \displaystyle\int_0^t (\partial_t B_\lambda v + F_\lambda)\, ds + B_\lambda(0)v_0 + v_1 & (t,x)\in[0,\infty)\times\Omega_\lambda, \\ w(0,x)=0, \partial_t w(0,x) = v_0(x) & x\in\Omega_\lambda, \\ w(t,x)=0 & (t,x)\in[0,\infty)\times\partial\Omega_\lambda. \end{cases} \qquad (45)$$

We remark that $\partial_t w = v$ and $E(w(t))$ is well-defined in $[0,\infty)$.

Lemma 4.1. *We assume that following* $(\mathbf{H1})_\lambda$ *and* $(\mathbf{H2})_\lambda$ *hold:*

$(\mathbf{H1})_\lambda$ $\|d_0(\cdot)\{B_\lambda(0)v_0+v_1\}\|_2 < \infty$,

$(\mathbf{H2})_\lambda$ $\displaystyle\int_0^\infty \|d_0(\cdot)\partial_t B_\lambda(s)\|_\infty ds < \infty$,

where d_0 *is defined in* (6)*. Then it holds following* (i) *and* (ii)*.*

(i) *When* $d\geq 3$*, there exists a constant* $E_0 = E_0(v_0,v_1)$ *such that*

$$\int_0^t \langle v, B_\lambda v\rangle ds \leq E_0 \qquad (46)$$

(ii) *When* $d=2$*, we assume also that* $(\mathbf{H3})_\lambda$ *holds.*

$(\mathbf{H3})_\lambda$ *There exists* $M>0$ *such that* $\mathrm{supp} v_0 \cup \mathrm{supp} v_1 \subset \{x\in\Omega_\lambda : |x| < \frac{M}{\lambda}\}$.

Then there exists $C_{\lambda,M}>0$ *such that*

$$\int_0^t \langle v, B_\lambda v\rangle ds \leq C_{\lambda,M}\|(v_0,v_1)\|_{H^L\times H^{L-1}}^2 + C_{\lambda,M}\left\{\int_0^t (1+s)Zds\right\}^2. \qquad (47)$$

Proof. Taking inner product (45) by $\partial_t w$, we have

$$\frac{d}{dt}E(w(t)) + \langle v(t), B_\lambda(t)v(t)\rangle$$
$$= \langle \partial_t w(t), B_\lambda(0)v_0 + v_1\rangle + \langle \partial_t w(t), \int_0^t \partial_t B_\lambda(s)v(s)ds\rangle + \langle \partial_t w(t), \int_0^t F_\lambda ds\rangle.$$

Integrating above equality over $[0,t]$, we obtain

$$E(w(t)) + \int_0^t \langle v(s), B_\lambda v(s)\rangle ds \le \frac{1}{2}\|v_0\|_2^2 + \langle w(t), B_\lambda(0)v_0 + v_1\rangle \tag{48}$$
$$+ \int_0^t \langle \partial_t w(s), \int_0^s \partial_t B_\lambda(r)v(r)dr\rangle ds + \int_0^t \langle \partial_t w(s), \int_0^s F_\lambda dr\rangle ds$$
$$= \frac{1}{2}\|v_0\|_2^2 + (A) + (B) + (C).$$

First, we estimate (A). Using Lemma 2.4, we get

$$(A) = \langle w, B_\lambda(0)v_0 + v_1\rangle \le \left\|\frac{w}{d_0(\cdot)}\right\|_2 \|d_0(\cdot)\{B_\lambda(0)v_0 + v_1\}\|_2$$
$$\le C_\lambda \|\nabla w\|_2 \|d_0(\cdot)\{B_\lambda(0)v_0 + v_1\}\|_2 \le \frac{1}{4}E(w(t)) + C_\lambda\|d_0(\cdot)\{B_\lambda(0)v_0 + v_1\}\|_2^2.$$

In particular, if $\mathrm{supp} v_0 \cup \mathrm{supp} v_1 \subset \{x \in \Omega_\lambda | |x| < M/\lambda\}$ then we have

$$\|d(\cdot)\{B_\lambda(0)v_0 + v_1\}\|_2^2 \le C_{\lambda,M}\|(v_0, v_1)\|_{H^L\times H^{L-1}}^2.$$

Second, we estimate (B). Using $(\mathrm{H2})_\lambda$ and (13), we calculate

$$(B) = \int_0^t \langle \partial_t w(s), \int_0^s \partial_t B_\lambda(r)v(r)dr\rangle ds$$
$$= \langle w(t), \int_0^t \partial_t B_\lambda v(s)ds\rangle - \int_0^t \langle w(s), \partial_t B_\lambda(s)v(s)\rangle ds$$
$$\le C \sup_{0\le s\le t}\left\|\frac{w(s)}{d_0(\cdot)}\right\|_2 \sup_{0\le s\le t}\|v(s)\|_2 \int_0^t \|d_0(\cdot)\partial_t B_\lambda(s)\|_\infty ds$$
$$\le C_\lambda \sup_{0\le s\le t}\|\nabla w(s)\|_2 \sup_{0\le s\le t}\|v(s)\|_2 \int_0^t \|d_0(\cdot)\partial_t B_\lambda(s)\|_\infty ds$$
$$\le \frac{1}{4}\sup_{0\le s\le t}E(w(s)) + C_\lambda \sup_{0\le s\le t}\|v(s)\|_2^2 \left\{\int_0^t \|d_0(\cdot)\partial_t B_\lambda(s)\|_\infty ds\right\}^2$$
$$\le \frac{1}{4}\sup_{0\le s\le t}E(w(s)) + C_\lambda\|(v_0, v_1)\|_{H^L\times H^{L-1}}^2.$$

Finally, we estimate (C). When $d \ge 3$, using Lemma 2.5, we have

$$\int_0^t \langle \partial_t w, \int_0^s F_\lambda dr\rangle ds = \langle w, \int_0^t F_\lambda ds\rangle - \int_0^t \langle w, F_\lambda\rangle ds$$

$$\leq 2 \sup_{0\leq s\leq t} \|w(s)\|_{\frac{2d}{d-2}} \int_0^t \|F_\lambda\|_{\frac{2d}{d+2}} ds \leq \frac{1}{4} \sup_{0\leq s\leq t} E(w(s)) + C_\lambda \left\{ \int_0^t \|F_\lambda\|_{\frac{2d}{d+2}} ds \right\}^2 .$$

Using Lemma 2.1, (13) and $p_l \geq 2$ $(l = 1, 2)$, we obtain

$$\int_0^t \|F_\lambda\|_{\frac{2d}{d+2}} ds \leq C_\lambda \int_0^t Z_0 ds \leq C_\lambda \|(v_0, v_1)\|^2_{H^L \times H^{L-1}} \tag{49}$$

(since $d > 2$ we have $\frac{2d}{d+2} > 1$.). Therefore we get

$$(C) \leq \frac{1}{4} \sup_{0\leq s\leq t} E(w(s)) + C\|(v_0, v_1)\|^2_{H^L \times H^{L-1}}.$$

On the other hand when $d = 2$, using Lemma 2.5 and Hölder inequality, we have

$$(C) = \int_0^t \langle \partial_t w, \int_0^s F_\lambda dr\rangle ds = \langle w, \int_0^t F_\lambda ds\rangle - \int_0^t \langle w, F_\lambda\rangle ds$$

$$\leq C_\lambda \sup_{0\leq s\leq t} \left\| \frac{w(s)}{|\cdot|} \right\|_r \left\| \int_0^t |\cdot| F_\lambda ds \right\|_{r'} \leq C_\lambda \sup_{0\leq s\leq t} \left\| \nabla \left(\frac{w(s)}{|\cdot|} \right) \right\|_q \int_0^t \||\cdot| F_\lambda\|_{r'} ds,$$

where $r \in (2.\infty)$, $\frac{1}{r} + \frac{1}{r'} = 1$ and $\frac{1}{r} = \frac{1}{q} - \frac{1}{2}$. From the assumption $\mathbf{(H3)}_\lambda$ and the finite speed of propagation, it holds that

$$\mathrm{supp} v(t) \cup \mathrm{supp} \partial_t v(t) \subset \{x \in \Omega_\lambda : |x| \leq M/\lambda + 2t\} \quad (t \in [0, \infty)). \tag{50}$$

Thus using (50) and considering the same way of (49), we get

$$\int_0^t \||\cdot| F_\lambda\|_{r'} ds \leq C_{\lambda,M} \int_0^t (1 + s) Z_0(v(s)) ds.$$

Furthermore using Lemma 2.4 and Hölder inequality, we obtain

$$\left\| \nabla \left(\frac{w}{|\cdot|} \right) \right\|_q \leq \left\| \frac{\nabla w}{|\cdot|} \right\|_q + \left\| \frac{w}{|\cdot|^2} \right\|_q \leq C_\lambda \|\nabla w\|_2 \left\| \frac{1}{|\cdot|} \right\|_r + C_\lambda \left\| \frac{w}{d_0} \right\|_2 \left\| \frac{d_0}{|\cdot|^2} \right\|_r$$

$$\leq C_\lambda \|\nabla w\|_2 \left(\left\| \frac{1}{|\cdot|} \right\|_r + \left\| \frac{d_0}{|\cdot|^{1+\varepsilon_0}} \right\|_\infty \left\| \frac{1}{|\cdot|^{1-\varepsilon_0}} \right\|_r \right),$$

where $\varepsilon_0 = \frac{r-2}{2r}$. Remember $0 \notin \Omega_\lambda$ and $r > r(1 - \varepsilon_0) > 2$, we get

$$\|\nabla w\|_2 \left(\left\| \frac{1}{|\cdot|} \right\|_r + \left\| \frac{d_0}{|\cdot|^{1+\varepsilon_0}} \right\|_\infty \left\| \frac{1}{|\cdot|^{1-\varepsilon_0}} \right\|_r \right) \leq C_\lambda \|\nabla w\|_2.$$

Above estimates and Lemma 3.3 imply that

$$(C) \leq C_{\lambda,M} \sup_{0\leq s\leq t} \|\nabla w(s)\|_2 \int_0^t (1 + s) Z(v(s)) ds$$

$$\leq \frac{1}{4} \sup_{0\leq s\leq t} E(w(s)) + C_{\lambda,M} \left\{ \int_0^t (1 + s) Z(v(s)) ds \right\}^2 .$$

Combining estimates for $(A), (B), (C)$ and (48), we get (46) and (47). This completes the proof of Lemma 4.1. □

Remark 4.2. *If F_λ has divergence form, we can prove* (i) *even if $d = 2$. Then we do not need to assume* $(\mathbf{H3})_\lambda$ *(See for instance: [18].).*

Proof of Theorem 1.2

It is easy to see that if $\{(u_0, u_1), B\}$ satisfy (**H1**), (**H2**) and (**H3**), then $\{(v_0, v_1), B_\lambda\}$ satisfy $(\mathbf{H1})_\lambda$, $(\mathbf{H2})_\lambda$ and $(\mathbf{H3})_\lambda$ respectively. Therefore when $d \geq 3$, combining (43) and (46), we get

$$(1+t)\{\|v(t)\|_2^2 + Z(v(t))\} + \int_0^t (1+s)Z(v(s))ds \leq E_0(v_0, v_1). \qquad (51)$$

The above estimate means (7).

When $d = 2$, combining (43) and (47), we get

$$\begin{aligned}&(1+t)\{\|v(t)\|_2^2 + Z(v(t))\} + \int_0^t (1+s)Z(v(s))ds \\ &\leq C_{\lambda,M}\|(v_0, v_1)\|_{H^L \times H^{L-1}}^2 + C_{\lambda,M}\left\{\int_0^t (1+s)Z(v(s))ds\right\}^2.\end{aligned}$$

The above estimate and $\|(v_0, v_1)\|_{H^L \times H^{L-1}}^2 \leq \delta^2$ imply

$$H(t) \leq C_{\lambda,M}\delta^2 + (H(t))^2,$$

where $H(t) = \int_0^t (1+s)Z(v(s))ds$. Because of $H(0) = 0$, we can choose a small δ depend on λ and M such that $H(t) \leq C_{\lambda,M}$ $(t \in [0, \infty))$. Therefore we obtain

$$(1+t)\{\|v(t)\|_2^2 + Z(v(t))\} + \int_0^t (1+s)Z(v(s))ds \leq C_{\lambda,M}\|(v_0, v_1)\|_{H^L \times H^{L-1}}^2 + C_{\lambda,M}^2.$$

This means (7). This completes the proof of (7).

Next, we prove (8). We calculate

$$\frac{d}{dt}\left\{(1+t)^2 E(v(t))\right\} = 2(1+t)E(v(t)) - (1+t)^2\langle \partial_t v, B_\lambda \partial_t v\rangle + (1+t)\langle \partial_t v, F_\lambda\rangle.$$

Integrating the above equality over $[0, t]$, we get

$$\begin{aligned}&(1+t)^2 E(v(t)) + \int_0^t (1+s)^2 \langle \partial_t v, B_\lambda \partial_t v\rangle ds \\ &\leq E(v(0)) + 2\int_0^t (1+s)E(v(s))ds + \int_0^t (1+s)^2\langle \partial_t v, F_\lambda\rangle ds \\ &\leq E(v(0)) + 2\int_0^t (1+s)Z_0 ds + C_\lambda \left\{\sup_{0\leq s\leq t}(1+s)^2 E(v(s))\right\}^{\frac{1}{2}} \int_0^t (1+s)Z_0 ds \\ &\leq E(v(0)) + 2\int_0^t (1+s)Z_0 ds + C_\lambda \left\{\int_0^t (1+s)Z_0 ds\right\}^2 + \frac{1}{2}\sup_{0\leq s\leq t}(1+s)^2 E(v(s)).\end{aligned}$$

Using the above estimate and (7), we obtain (8). This completes the proof of Theorem 1.2.

A Estimates of nonlinear terms

We show the estimates of the nonlinear terms F_λ.

Lemma A.1. *Let $v \in X_\delta^T$, $\delta \le 1$ and $|\alpha| \le L-1$. Then there exists $C_\lambda > 0$ such that*

$$\|\partial^\alpha \tilde{F}_\lambda(\partial v(t))\|_2 \le C_\lambda Z(v(t)) \quad (t \in [0,T]). \tag{52}$$

Proof. If $|\alpha| = 0$, it is easy to see from Lemma 2.1 that we prove (52). Let $1 \le |\alpha| \le L-1$. From the chain rule, we have

$$\begin{aligned}
&\partial^\alpha \tilde{F}_\lambda(\partial v)\\
&= \lambda \sum_{\alpha_1+\cdots+\alpha_l=\alpha} C_{\alpha_1,\cdots,\alpha_l} \sum_{\substack{1\le i_1,\cdots,i_l\le n\\ 0\le j_1,\cdots,j_l\le d}} D_{i_1j_1}\cdots D_{i_lj_l}\tilde{F}(\partial v)\partial^{\alpha_1}\partial_{j_1}v^{i_1}\cdots\partial^{\alpha_l}\partial_{j_l}v^{i_l},
\end{aligned}$$

where $D_{ij} = D_{\xi_{ij}}$. When $l \ge 2$, we choose $2 < q_i \le \infty$ $(i = 1, 2, \cdots, l)$ satisfying

$$\sum_{k=1}^l \frac{1}{q_k} = \frac{1}{2} \quad \textit{and} \quad \frac{1}{2} \le \frac{1}{q_k} + \frac{L-1-|\alpha_k|}{d}. \tag{53}$$

Then it holds that

$$\begin{aligned}
&\|\partial^\alpha \tilde{F}_\lambda(\partial v)\|_2\\
&\le C_\lambda \sum_{\alpha_1+\cdots+\alpha_l=\alpha} \sum_{\substack{1\le i_1,\cdots,i_l\le n\\ 0\le j_1,\cdots,j_l\le d}} \|D_{i_1j_1}\cdots D_{i_lj_l}\tilde{F}(\partial v)\|_\infty \|\partial^{\alpha_1}\partial_{j_1}v^{i_1}\cdots\partial^{\alpha_l}\partial_{j_l}v^{i_l}\|_2\\
&\le C_\lambda \sum_{\alpha_1+\cdots+\alpha_l=\alpha} \sum_{\substack{1\le i_1,\cdots,i_l\le n\\ 0\le j_1,\cdots,j_l\le d}} \|\partial v\|_\infty^{\max\{0,p_1-l\}} \prod_{k=1}^l \|\partial^{\alpha_k}\partial_{j_k}v^{i_k}\|_{q_k}\\
&\le C_\lambda \sum_{\alpha_1+\cdots+\alpha_l=\alpha} \|\partial v\|_{H^{\left[\frac{d}{2}\right]+1}}^{\max\{0,p_1-l\}} \prod_{k=1}^l \|\partial\partial^{\alpha_k}v\|_{H^{L-1-|\alpha_k|}}\\
&\le C_\lambda (Z(v))^{\frac{\max\{l,p_1\}}{2}} \le C_\lambda Z(v),
\end{aligned}$$

where we use the generalized Hölder inequality and a well known embedding lemma like $L^q \subset H^s$ $(2 < q < \infty, \frac{1}{2} \le \frac{1}{q} + \frac{s}{d})$. Indeed it holds from $L \ge [d/2]+3$ that

$$\sum_{k=1}^l \left\{\frac{1}{2} - \frac{L-1-|\alpha_k|}{d}\right\} - \frac{1}{2} \le (l-1)\left\{\frac{1}{2} + \frac{1-L}{d}\right\} \le -\frac{l-1}{d} < 0,$$

thus we can choose q_k satisfying (53). When $l = 1$, we should choose $q_1 = 2$. This completes the proof of Lemma A.1. □

Lemma A.2. *Let $v \in X_\delta^T$, $\delta \le 1$, $1 \le |\alpha| \le L$, $|\beta| \le L-1$ and $|a+\beta| \le L+1$. For any i, j, a and b, it holds that*

$$\|\partial^\alpha v^i \partial^\beta (c_{ij}^{ab}(\partial v))\|_2 \le C_\lambda Z(v)$$

Proof. First, we assume $|\beta| = 1$. Then we have

$$\begin{aligned}\|\partial^\alpha v^i \partial^\beta (c_{ij}^{ab}(\partial v))\|_2 &\le \|\partial^\alpha v\|_2 \sum_{\substack{1\le k\le n\\ 0\le l\le d}} \|D_{lk} c_{ij}^{ab}(\partial v)\partial^\beta \partial_l v^k\|_\infty \\ &\le C\|\partial^\alpha v\|_2 \|\partial v\|_\infty^{p_2-2} \|\partial^\beta \partial v\|_\infty \le C_\lambda (Z(v))^{\frac{p_2}{2}} \le C_\lambda Z(v).\end{aligned}$$

In the same way, we can prove when $\beta = 0$.

Next, we assume $|\beta| \ge 2$. In the same way as the proof of Lemma (53), we obtain

$$\begin{aligned}&\|\partial^\alpha v^i \partial^\beta (c_{ij}^{ab}(\partial v))\|_2 \\ &\le C \sum_{\beta_1+\cdots+\beta_l=\beta} \sum_{\substack{1\le i_1,\cdots,i_l\le n\\ 0\le j_1,\cdots,j_l\le d}} \|\partial^\alpha v^i D_{i_1 j_1}\cdots D_{i_l j_l} c_{ij}^{ab}(\partial v)\partial^{\beta_1}\partial_{j_1} v^{i_1}\cdots \partial^{\beta_l}\partial_{j_l} v^{i_l}\|_2 \\ &\le C \sum_{\beta_1+\cdots+\beta_l=\beta} \sum_{\substack{1\le i_1,\cdots,i_l\le n\\ 0\le j_1,\cdots,j_l\le d}} \|D_{i_1 j_1}\cdots D_{i_l j_l} c_{ij}^{ab}(\partial v)\|_\infty \|\partial^\alpha v^i \partial^{\beta_1}\partial_{j_1} v^{i_1}\cdots \partial^{\beta_l}\partial_{j_l} v^{i_l}\|_2 \\ &\le C \sum_{\beta_1+\cdots+\beta_l=\beta} \sum_{\substack{1\le i_1,\cdots,i_l\le n\\ 0\le j_1,\cdots,j_l\le d}} \|\partial v\|_\infty^{\max\{0,p_2-1-l\}} \|\partial^\alpha v^i\|_{r_0} \prod_{k=1}^{l} \|\partial^{\beta_k}\partial_{j_k} v^{i_k}\|_{r_k} \\ &\le C_\lambda \sum_{\beta_1+\cdots+\beta_l=\beta} \|\partial v\|_{H^{[\frac{d}{2}]+1}}^{\max\{0,p_2-1-l\}} \|\partial^\alpha v\|_{H^{L-|\alpha|}} \prod_{k=1}^{l} \|\partial\partial^{\beta_k} v^{i_k}\|_{H^{L-1-|\beta_k|}} \\ &\le C_\lambda Z^{\frac{\max\{l+1,p_2+1\}}{2}} \le C_\lambda Z(v),\end{aligned}$$

where we choose $2 < r_k < \infty$ $(k = 0, 1, \cdots, l)$ satisfying

$$\sum_{k=0}^{l} \frac{1}{r_k} = \frac{1}{2}, \quad \frac{1}{2} \le \frac{1}{r_0} + \frac{L-|\alpha|}{d} \quad \text{and} \quad \frac{1}{2} \le \frac{1}{r_k} + \frac{L-|\beta_k|-1}{d} \ (1 \le k \le l). \tag{54}$$

Indeed it holds from $L \ge [d/2] + 3$ that

$$\frac{1}{2} - \frac{L-|\alpha|}{d} + \sum_{k=1}^{l} \left\{ \frac{1}{2} - \frac{L-1-|\beta_k|}{d} \right\} - \frac{1}{2} \le \frac{l}{2} + \frac{l(1-L)}{d} + \frac{1}{d} \le -\frac{-3l+2}{2d} < 0,$$

thus we can choose q_k satisfying (54). This completes the proof of Lemma A.2. $\square$

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参考論文

(1) Global existence and decay estimates for quasilinear wave equations with nonuniform dissipative term
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Global existence and decay estimates for quasilinear wave equations with nonuniform dissipative term

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Abstract

In this paper, we study a Cauchy problem for quasilinear wave equations with dissipative term in Sobolev space $H^L \times H^{L-1}$ ($L \geq [d/2]+3$). The coefficients of the dissipative term depends on space variables and may vanish in some compact region. In order to control the derivatives of the dissipative coefficients, we introduce a rescaling argument. Using the argument, we obtain a global existence theorem and decay estimates with additional assumptions for the initial data.

Key Words and Phrases. dissipative quasilinear wave equations, space variable coefficients, time decay estimates

2010 *Mathematics Subject Classification Numbers.* 35L72, 35L15.

1 Introduction

In this paper, we consider the following Cauchy problem for quasilinear wave equations with nonuniform dissipative term in $\mathbb{R}^d$ ($d \geq 1$) :

$$\text{(DW)} \quad \begin{cases} (\partial_t^2 - \triangle + B(x)\partial_t)u(t,x) = N[u,u](t,x), & (t,x) \in [0,\infty)\times\mathbb{R}^d, \\ u(0,x) = u_0(x), \quad \partial_t u(0,x) = u_1(x), & x \in \mathbb{R}^d, \end{cases}$$

where $u = (u^1, u^2, \cdots, u^d)$ is a vector valued function, $\triangle u(t,x) = (\triangle u^1, \triangle u^2, \cdots, \triangle u^d)$, $\partial_t u = (\partial_t u^1, \partial_t u^2, \cdots, \partial_t u^d)$ and $\partial_t^2 u = (\partial_t^2 u^1, \partial_t^2 u^2, \cdots, \partial_t^2 u^d)$. The initial data (u_0, u_1) in (DW) belong to $H^L \times H^{L-1}$, where H^L is the Sobolev space in $\mathbb{R}^d$. The coefficient function $B(x)$ is $d\times d$ symmetric matrix-valued function and quasilinear term $N[u,v](t,x)$ is defined by

$$N[u,v] = (N[u,v]^i)_{i=1,2,\cdots,d} = \left(\sum_{j,k,l,m,n=1}^{d} N_{lmn}^{ijk} \partial_l(\partial_m u^j \partial_n v^k) \right)_{i=1,2,\cdots,d}.$$

For B and N, we make the following assumptions:

(B0) $B(x) = (B_{pq}(x))_{p,q=1,2,\cdots,d}$ is $d \times d$ symmetric matrix-valued function whose components belong to $\mathcal{B}^\infty$, where $\mathcal{B}^\infty$ is a function space of smooth functions with bounded derivatives.

(B1) $B(x)$ is nonnegative, i.e.

$$\sum_{p,q=1}^{d} B_{pq}(x)\eta_p\eta_q \geq 0 \quad (\eta, x \in \mathbb{R}^d).$$

(B2) There exist $b_0 > 0$ and $R > 0$ such that

$$\sum_{p,q=1}^{d} B_{pq}(x)\eta_p\eta_q \geq b_0|\eta|^2 \quad (|x| \geq R, \eta \in \mathbb{R}^d).$$

(N0) $N^{ijk}_{lmn} \in \mathbb{R} \quad (i, j, k, l, m, n = 1, 2, \cdots, d)$.

(N1) $N^{ijk}_{lmn} = N^{jik}_{mln} = N^{ikj}_{lnm} \quad (i, j, k, l, m, n = 1, 2, \cdots, d)$.

If we assume $B = const > 0$, there are many results (see [3], [7], [9], [10] etc.). For general quasilinear version including N, Racke [13] shows that there exists a unique global solution and decay estimates when the initial data are sufficiently smooth and small. On the other hand, the global existence is not obvious when B is not positive constant. Indeed if B vanishes, (DW) becomes the quasilinear wave equations. Then it is well known that no matter how small the initial data, there does not exist globally defined smooth solution in general (e.g.[6]). However if the nonlinear term N has "Null Condition", then (DW) has a global smooth solution for sufficiently smooth and small initial data (e.g. [1], [8], [14]).

There are some results for nonuniform dissipative term which satisfies **(B1)-(B2)** ([2], [5], [11], [12] etc.). In particular, Nakao [12] obtains a global existence when the nonlinear term have second order derivatives. He assume that the order of the nonlinear term p satisfies $p \geq 3$. In this paper, we treat the case of $p = 2$, i.e. the quasilinear term $N[u, u]$ is a quadratic function. In order to treat the quadratic case, we introduce a rescaling argument and energy norms for controlling higher order derivatives. There norms are used for the nonlinear wave equations.

We prove the following global existence theorem:

Theorem 1.1. *Let $L \geq L_0 = [d/2] + 3$. Then there exists a small constant $\hat{\delta} > 0$ such that if the initial data $(u_0, u_1) \in H^L \times H^{L-1}$ satisfies*

$$\|u_0\|_{H^{L_0}}^2 + \|u_1\|_{H^{L_0-1}}^2 \leq \hat{\delta}, \tag{1}$$

then there exists a unique solution to (DW) in $\cap_{j=0}^{L} C^j([0,\infty); H^{L-j})$.

We note that $L \geq L_0 = [d/2] + 3$ is reasonable. Because we use the assumption to obtain a local solution in usual sense (see [10], [12], [16]). Furthermore the assumption is necessary to estimate the quasilinear term for the purpose of extending the solution. Indeed using product estimate (proposition 2.2) and embedding (proposition 2.1), we calculate as

$$\begin{aligned}
\|\partial_l(\partial_m u \partial_n u)\|_{H^{L-2}} &\leq C\left\{\|\nabla u\|_{H^{L-2}}\|\nabla^2 u\|_\infty + \|\nabla u\|_\infty \|\nabla^2 u\|_{H^{L-2}}\right\} \\
&\leq C\|\nabla u\|_{H^{L-1}}\|\nabla u\|_{H^{L_0-1}} \leq C\|\nabla u\|_{H^{L-1}}^2
\end{aligned}$$

(see lemma 5.1).

We also remark that the smallness of $\|u_0\|_{L^2}$ is needed to estimate the higher order derivatives of nonuniform dissipative term as $\nabla^\alpha(B(x)u)$ (see lemma 5.2). Indeed if $B = Const > 0$, we can prove theorem 1.1 under the assumption $\|\nabla u_0\|_{H^{L_0-1}}^2 + \|u_1\|_{H^{L_0-1}}^2 \leq \hat{\delta}$ instead of (1).

We show the decay estimates with the additional assumptions as follow:

Theorem 1.2. *In addition to the assumptions in theorem 1.1, we assume that one of the following* **(H1)-(H3)** *is held:*

(H1) *$d \geq 3$ and there exists $1 \leq p \leq \frac{2d}{d+2}$ such that $Bu_0 + u_1 \in L^p$,*

(H2) *$d \geq 3$ and $|\cdot|\{Bu_0 + u_1\} \in L^2$,*

(H3) *$d = 1$ or 2, $|\cdot|\{Bu_0 + u_1\} \in L^1$ and $\displaystyle\int_{\mathbb{R}^d} B(x)u_0(x) + u_1(x)dx = 0$.*

Then for any $\mu (0 \leq \mu \leq L - L_0)$, there exists a constant $E_0 > 0$ depending on (u_0, u_1) such that the global solution u to (DW) satisfies following estimates:

$$\|\partial_t^\mu u(t)\|_{H^{L-\mu}}^2 + \|\partial_t^{\mu+1} u(t)\|_{H^{L-\mu-1}}^2 \leq E_0(1+t)^{-2\mu-1}, \tag{2}$$

$$\|\nabla \partial_t^\mu u(t)\|_2^2 + \|\partial_t^{\mu+1} u(t)\|_2^2 \leq E_0(1+t)^{-2\mu-2}, \tag{3}$$

$$\|\partial_t^\mu u(t)\|_\infty^2 \leq E_0(1+t)^{-2\mu-1}. \tag{4}$$

Furthermore if $L > L_0$, it holds that

$$\|\triangle u(t)\|_2^2 \leq E_0(1+t)^{-3}. \tag{5}$$

Ikehata [5] treats the linear or semilinear cases and obtains the result corresponding to theorem 1.2 with the assumption **(H2)**. Thus theorem 1.2 means that even for the quasilinear case, the same type decay properties as in [5] are held. For the proof, we use the method which is introduced in [5]. However, to estimate the quasilinear term, we use the time weighted higher order estimates (see §4).

In this paper, the rescaling argument plays an important role (for the definition of the scaling, see §2). We have to control terms include $\nabla^\beta B(x)$ to calculate the higher order energy estimates. Using a usual way, we need the smallness of $\|\nabla^\beta B(x)\|_\infty$ or delicate calculation. However, using the rescaling, we can control the smallness of $\|\nabla^\beta B(x)\|_\infty$ by a scale parameter. Therefore we can expect to prove the higher order energy estimates with simple argument.

2 Preliminaries

Throughout this paper, $\|\cdot\|_p$ and $\|\cdot\|_{H^l}$ stand for the usual $L^p(\mathbb{R}^d)$-norm and $H^l(\mathbb{R}^d)$-norm. Furthermore, we adopt

$$\langle g, f\rangle = \sum_{i=1}^{d} \int_{\mathbb{R}^d} f^i(x) g^i(x) dx$$

as the usual $L^2(\mathbb{R}^d)$-inner product.

We consider the rescaling to (DW). Let u be the solution to (DW). We define $v(t,x) = \dfrac{1}{\lambda} u(\lambda t, \lambda x)\ (\lambda > 0)$, then v satisfies

$$\begin{aligned} \partial_t^2 v(t,x) - \triangle v(t,x) &= \lambda\left\{\partial_t^2 u(\lambda t, \lambda x) - \triangle u(\lambda t, \lambda x)\right\} \\ &= -\lambda B(\lambda x)\,\partial_t u(\lambda t, \lambda x) + \lambda N[u,u](\lambda t, \lambda x) \\ &= -\lambda B(\lambda x)\,\partial_t v(t,x) + N[v,v](t,x). \end{aligned}$$

Thus v is the solution to the following Cauchy problem $(\mathrm{DW})_\lambda$:

$$(\mathrm{DW})_\lambda \quad \begin{cases} (\partial_t^2 - \triangle + B_\lambda(x)\partial_t) v(t,x) = N[v,v](t,x), & (t,x) \in [0,\infty) \times \mathbb{R}^d, \\ v(0,x) = v_0(x), \quad \partial_t v(0,x) = v_1(x), & x \in \mathbb{R}^d, \end{cases}$$

where $B_\lambda(x) = \lambda B(\lambda x)$, $v_0(x) = u_0(\lambda x)/\lambda$, $v_1(x) = u_1(\lambda x)$. Now B_λ satisfies

$\mathbf{(B1)}_\lambda$ $B_\lambda(x)$ is nonnegative.

(B2)$_\lambda$ There exist $b_0 > 0$ and $R > 0$ such that

$$\sum_{p,q=1}^{d} (B_\lambda)_{pq}(x)\eta_p\eta_q = \sum_{p,q=1}^{d} \lambda B_{pq}(\lambda x)\,\eta_p\eta_q \geq \lambda b_0|\eta|^2 \ \left(|x| \geq \frac{R}{\lambda},\ \eta \in \mathbb{R}^d\right)$$

instead of **(B1)** and **(B2)**. Furthermore B_λ satisfies

(B3)$_\lambda$ $$\|\nabla^b B_\lambda\|_\infty \leq \lambda^{|b|+1}\|\nabla^b B\|_\infty.$$

We consider (DW)$_\lambda$ for $\lambda \leq 1$ instead of (DW).

We introduce the known results. First, we use the following two lemmas for estimating the nonlinearity terms. For the proof, see for example [16] section 13.

Lemma 2.1. *There exists a constant $C > 0$ such that*

$$\|f\|_\infty \leq C\|f\|_{H^{[\frac{d}{2}]+1}}.$$

Lemma 2.2. *Let $k \in \mathbb{Z}$ and $b, c \in \mathbb{Z}_+^d$ satisfy $|b| + |c| = k$. There exists $C > 0$ such that*

$$\begin{aligned}\|\nabla^b f \nabla^c g\|_2 &\leq C\|f\|_\infty\|\nabla^{b+c}g\|_2 + C\|\nabla^{b+c}f\|_2\|g\|_\infty \\ &\leq C\|f\|_\infty\|g\|_{H^k} + C\|f\|_{H^k}\|g\|_\infty.\end{aligned}$$

Next, we prepare the Poincare type inequality of B_λ for the proof of global existence.

Lemma 2.3. *(Poincare type inequality) There exists a constant $C_1 \geq 1/4$ such that*

$$\|f\|_2^2 \leq \frac{C_1}{\lambda}\langle f, B_\lambda f\rangle + \frac{C_1}{\lambda^2}\|\nabla f\|_2^2 \quad (\lambda > 0) \tag{6}$$

Proof. We define $U_r = \{x \in \mathbb{R}^d \,|\, |x| \leq r\}$. Using Poincare inequality (see [4]), we obtain the following estimate:

$$\int_{U_r} |f(x)|^2 dx \leq r^2 \int_{U_r} |\nabla f(x)|^2 dx.$$

Let $\rho \in C_0^\infty(\mathbb{R}^d)$ be a function satisfying $0 \leq \rho \leq 1, \rho(x) = 1(|x| \leq 1), \rho(x) = 0(|x| \geq \frac{3}{2})$. For any $f \in H^1(\mathbb{R}^d)$ and $\lambda > 0$ we define $\rho_\lambda(x) = \rho(\frac{\lambda x}{R})$, then because of $\rho_\lambda f \in H_0^1(U_{\frac{2R}{\lambda}})$ we have

$$\|f\|_2^2 = \int_{\mathbb{R}^d} |\rho_\lambda(x)f(x)|^2 dx + \int_{\mathbb{R}^d} (1 - |\rho_\lambda(x)|^2)f(x)|^2 dx$$

$$
\begin{aligned}
&= \int_{|x|\le \frac{2R}{\lambda}} |\rho_\lambda(x) f(x)|^2 dx + \int_{|x|\ge \frac{R}{\lambda}} (1-|\rho_\lambda(x)|^2)|f(x)|^2 dx \\
&\le \frac{4R^2}{\lambda^2}\int_{|x|\le \frac{2R}{\lambda}} |\nabla(\rho_\lambda(x) f(x))|^2 dx + \int_{|x|\ge \frac{R}{\lambda}} |f(x)|^2 dx \\
&\le \frac{4R^2}{\lambda^2}\int_{\frac{R}{\lambda}\le|x|\le \frac{2R}{\lambda}} |\nabla\rho_\lambda(x)|^2|f(x)|^2 dx + \frac{4R^2}{\lambda^2}\int_{|x|\le \frac{2R}{\lambda}} |\rho_\lambda(x)|^2|\nabla f(x)|^2 dx \\
&\quad + \int_{|x|\ge \frac{R}{\lambda}} |f(x)|^2 dx \\
&\le \left(4R^2\|\nabla\rho\|_\infty^2+1\right)\int_{|x|\ge \frac{R}{\lambda}} |f(x)|^2 dx + \frac{4R^2}{\lambda^2}\int_{\mathbb{R}^d} |\nabla f(x)|^2 dx \\
&\le \frac{4R^2\|\nabla\rho\|_\infty^2+1}{b_0\lambda}\langle f, B_\lambda f\rangle + \frac{4R^2}{\lambda^2}\|\nabla f\|_2^2.
\end{aligned}
$$

Hence we get (6). □

Finally, we introduce Hardy inequality and Gagliardo-Nirenberg inequality. For the proof, see for example [15] appendix A.

Lemma 2.4. *Let $d \ge 3$. There exists a constant $C > 0$ such that any $f \in H^1$ satisfies*

$$\left\| \frac{f}{|\cdot|} \right\|_2 \le C\|\nabla f\|_2. \tag{7}$$

Lemma 2.5. *Assume $1 \le q < d$ and $\frac{1}{p} = \frac{1}{q} - \frac{1}{d}$. Then there exists a constant $C > 0$ depend on p, q, d such that*

$$\|g\|_p \le C\|\nabla g\|_q. \tag{8}$$

3 Global existence

In this section we prove theorem 1.1. First we define some notations. For any $h, g : \mathbb{R}^d \to \mathbb{R}^d$, we define $[h; \nabla g] : \mathbb{R}^d \to \mathbb{R}^d$ as follows:

$$([h;\nabla g])^i(x) = h(x)\cdot\nabla g^i(x) \quad (i = 1, 2, \cdots, d).$$

The energy $E(u(t))$ and higher order energies $E_{\bar{L}}(u(t))$ of u are defined by

$$E(u(t)) = \frac{1}{2}\{\|\partial_t u(t)\|_2^2 + \|\nabla u(t)\|_2^2\} \tag{9}$$

and

$$E_{\bar{L}}(u(t)) = \sum_{|a|\le \bar{L}-1} E(\nabla^a u(t)). \tag{10}$$

Moreover we define

$$\tilde{N}[u,v,w](t)=\sum_{i,j,k,l,m,n=1}^{d} N_{lmn}^{ijk}\int_{\mathbb{R}^d}\partial_l u^i(t,x)\partial_m v^j(t,x)\partial_n w^k(t,x)dx \tag{11}$$

and

$$\tilde{E}_{\bar{L},\mu}(u(t))=E_{\bar{L}-\mu}(\partial_t^{\mu}u(t))+\sum_{|a|\le\bar{L}-\mu-1}\tilde{N}[\partial_t^{\mu}\nabla^a u,\partial_t^{\mu}\nabla^a u,u](t). \tag{12}$$

The function spaces $X_{\delta,T}$ and X_{δ} are defined by

$$X_{\delta,T}=\left\{u\in\cap_{j=0}^{L}C^j([0,T];H^{L-j})|\ E_{L_0}(u(t))\le\delta^2\quad(0\le t\le T)\right\} \tag{13}$$

and

$$X_{\delta}=\left\{u\in\cap_{j=0}^{L}C^j([0,\infty);H^{L-j})|\ E_{L_0}(u(t))\le\delta^2\quad(0\le t<\infty)\right\}. \tag{14}$$

Let $L_0\le\bar{L}\le L$, $\mu\le L-L_0$ and $\lambda>0$, we define $G_{\bar{L},\mu}(v(t))$ below.

$$\begin{aligned} & G_{\bar{L},\mu}(v(t)) \\ = \ & \frac{C_0}{\lambda}\tilde{E}_{\bar{L},\mu}(v(t))+\frac{b_0(2d-1)}{4}\sum_{|a|\le\bar{L}-\mu-1}\langle\partial_t^{\mu}\nabla^a v(t),\partial_t^{\mu+1}\nabla^a v(t)\rangle \\ & +\frac{b_0(2d-1)}{8}\sum_{|a|\le\bar{L}-\mu-1}\langle\partial_t^{\mu}\nabla^a v(t),B_{\lambda}\partial_t^{\mu}\nabla^a v(t)\rangle \\ & +\sum_{|a|\le\bar{L}-\mu-1}\langle\partial_t^{\mu+1}\nabla^a v(t),[h;\nabla\partial_t^{\mu}\nabla^a v](t)\rangle, \end{aligned} \tag{15}$$

where

$$C_0=\max\left\{\left(b_0Rd^2+\frac{C_1b_0(2d-1)}{2}\right)\times 4,\ d,\ 2\|B\|_{\infty}b_0^2R^2\times\frac{8}{b_0}\right\}, \tag{16}$$

and

$$\phi(r)=\begin{cases} b_0, & (r\le\frac{R}{\lambda}) \\ \frac{b_0R}{\lambda r}, & (r\ge\frac{R}{\lambda}) \end{cases}, \quad h(x)=x\phi(|x|). \tag{17}$$

We need the energy estimate to prove the global existence.

Lemma 3.1. *Let $L_0\le\bar{L}\le L$ and $\mu\le\bar{L}-L_0$. There exists a constant $C>0$ such that for any $\delta,\lambda,T>0$ and a local solution $v\in X_{\delta,T}$ to (DW)$_\lambda$ satisfy*

$$\frac{d}{dt}G_{\bar{L},\mu}(v(t))+\frac{b_0}{2}E_{\bar{L}-\mu}(\partial_t^{\mu}v(t)) \tag{18}$$

$$\leq \ \lambda C E_{\bar{L}-\mu}(\partial_t^\mu v(t)) + \frac{C}{\lambda} D_{\bar{L},\mu}(v(t)) + \frac{2\|B\|_\infty b_0^2 R^2}{C_0} E_{\bar{L}-\mu}(\partial_t^\mu v(t)),$$

where $C_0 > 0$ *is the constant give in* (16), *and*

$$D_{\bar{L},\mu}(v(t)) = E_{\bar{L}-\mu}^{\frac{1}{2}}(\partial_t^\mu v(t)) \cdot \left(\sum_{\nu=0}^{\mu} E_{L_0}^{\frac{1}{2}}(\partial_t^{\mu-\nu} v(t)) E_{\bar{L}-\nu}^{\frac{1}{2}}(\partial_t^\nu v(t)) \right). \qquad (19)$$

The proof of the lemma 3.1 is given in section 5. In what follows, assuming lemma 3.1, we derive the energy estimates for $(\mathrm{DW})_\lambda$.

3.1 Energy estimate

We prove the higher order energy estimates of $(\mathrm{DW})_\lambda$.

Lemma 3.2. *Let* $L_0 \leq \bar{L} \leq L$ *and* $\mu \leq \bar{L} - L_0$. *There exists* $C > 0$ *such that if* $\delta, \lambda > 0$ *are sufficiently small, then the local solution* $v \in X_{\delta,T}$ *to* $(DW)_\lambda$ *satisfies*

$$\begin{aligned} \frac{1}{C}\{\lambda \|\partial_t^\mu v(t)\|_2^2 + \frac{1}{\lambda} E_{\bar{L}-\mu}(\partial_t^\mu v(t))\} &\leq G_{\bar{L},\mu}(\partial_t^\mu v(t)) \qquad (20) \\ &\leq C\{\lambda \|\partial_t^\mu v(t)\|_2^2 + \frac{1}{\lambda} E_{\bar{L}-\mu}(\partial_t^\mu v(t))\}. \end{aligned}$$

Proof. Let $v \in X_{\delta,T}$. First it holds that

$$\frac{1}{2} E_{\bar{L}-\mu}(\partial_t^\mu v(t)) \leq \tilde{E}_{\bar{L},\mu}(v(t)) \leq \frac{3}{2} E_{\bar{L}-\mu}(\partial_t^\mu v(t)). \qquad (21)$$

Because

$$\begin{aligned} & \left| \sum_{|a| \leq \bar{L}-\mu-1} \tilde{N}[\partial_t^\mu \nabla^a v, \partial_t^\mu \nabla^a v, v](t) \right| \\ \leq \ & C \sum_{|a| \leq \bar{L}-\mu-1} \|\nabla v(t)\|_\infty \|\nabla \partial_t^\mu \nabla^a v(t)\|_2^2 \\ \leq \ & C \sum_{|a| \leq \bar{L}-\mu-1} \|\nabla v(t)\|_{H^{[\frac{d}{2}]+1}} \|\nabla \partial_t^\mu \nabla^a v(t)\|_2^2 \\ \leq \ & \delta C E_{\bar{L}-\mu}(\partial_t^\mu v(t)), \end{aligned}$$

we choose δ sufficiently small depend on d and N, then we get (21). It follows from lemma 2.3 that

$$\left| \sum_{|a| \leq \bar{L}-\mu-1} \langle \partial_t^\mu \nabla^a v(t), \partial_t^{\mu+1} \nabla^a v(t) \rangle \right| \qquad (22)$$

$$
\begin{aligned}
&\leq \sum_{|a|\leq \bar{L}-\mu-1} \left\{ \frac{\lambda}{4C_1} \|\partial_t^{\mu} \nabla^a v(t)\|_2^2 + \frac{C_1}{\lambda} \|\partial_t^{\mu+1} \nabla^a v(t)\|_2^2 \right\} \\
&\leq \sum_{|a|\leq \bar{L}-\mu-1} \left\{ \frac{1}{4} \langle \partial_t^{\mu} \nabla^a v(t), B_\lambda \partial_t^{\mu} \nabla^a v(t) \rangle \right. \\
&\qquad \left. + \frac{1}{4\lambda} \|\nabla \partial_t^{\mu} \nabla^a v(t)\|_2^2 + \frac{C_1}{\lambda} \|\partial_t^{\mu+1} \nabla^a v(t)\|_2^2 \right\} \\
&\leq \frac{1}{4} \sum_{|a|\leq \bar{L}-\mu-1} \langle \partial_t^{\mu} \nabla^a v(t), B_\lambda \partial_t^{\mu} \nabla^a v(t) \rangle + \frac{2C_1}{\lambda} E_{\bar{L}-\mu}(\partial_t^{\mu} v(t)).
\end{aligned}
$$

Using

$$
\|h\|_\infty \leq \frac{b_0 R}{\lambda} \quad and \quad \|\nabla h\|_\infty \leq 2b_0, \tag{23}
$$

we get

$$
\begin{aligned}
&\left| \sum_{|a|\leq \bar{L}-\mu-1} \langle \partial_t^{\mu+1} \nabla^a v(t), [h; \nabla \partial_t^{\mu} \nabla^a v](t) \rangle \right| \qquad (24) \\
&\leq \sum_{|a|\leq \bar{L}-\mu-1} \sum_{k,j=1}^{d} \left| \int_{\mathbb{R}^d} \partial_t^{\mu+1} \nabla^a v^k(t) \partial_j \partial_t^{\mu} \nabla^a v^k(t) h^j dx \right| \\
&\leq \sum_{|a|\leq \bar{L}-\mu-1} \sum_{k,j=1}^{d} \|\partial_t^{\mu+1} \nabla^a v^k(t)\|_2 \|\partial_j \partial_t^{\mu} \nabla^a v^k(t)\|_2 \|h^j\|_\infty \\
&\leq d^2 \sum_{|a|\leq \bar{L}-\mu-1} \left\{ \frac{1}{2} \|\partial_t^{\mu+1} \nabla^a v(t)\|_2^2 + \frac{1}{2} \|\nabla \partial_t^{\mu} \nabla^a v(t)\|_2^2 \right\} \|h\|_\infty \\
&\leq \frac{b_0 R d^2}{\lambda} E_{\bar{L}-\mu}(\partial_t^{\mu} v(t)).
\end{aligned}
$$

Using (21), (22), (24) and (16) we have

$$
\begin{aligned}
G_{\bar{L},\mu}(v(t)) \;\geq\; & \frac{1}{\lambda} \left(\frac{C_0}{2} - b_0 R d^2 - \frac{C_1 b_0 (2d-1)}{2} \right) E_{\bar{L}-\mu}(\partial_t^{\mu} v(t)) \\
& + \frac{b_0(2d-1)}{16} \sum_{|a|\leq \bar{L}-\mu-1} \langle \partial_t^{\mu} \nabla^a v(t), B_\lambda \partial_t^{\mu} \nabla^a v(t) \rangle \\
\geq\; & \frac{1}{\lambda} \left(b_0 R d^2 + \frac{C_1 b_0 (2d-1)}{2} \right) E_{\bar{L}-\mu}(\partial_t^{\mu} v(t)) \\
& + \frac{b_0(2d-1)}{16} \sum_{|a|\leq \bar{L}-\mu-1} \langle \partial_t^{\mu} \nabla^a v(t), B_\lambda \partial_t^{\mu} \nabla^a v(t) \rangle.
\end{aligned}
$$

So there exists a constant C such that v satisfies

$$\frac{1}{\lambda}E_{\bar{L}-\mu}(\partial_t^\mu v(t)) + \sum_{|a|\le\bar{L}-\mu-1}\langle\partial_t^\mu\nabla^a v(t), B_\lambda\partial_t^\mu\nabla^a v(t)\rangle \le CG_{\bar{L},\mu}(v(t)). \tag{25}$$

Furthermore using (25) and lemma 2.3, we have

$$\begin{aligned}\|\partial_t^\mu v(t)\|_2^2 &\le \frac{C_1}{\lambda}\langle B_\lambda\partial_t^\mu v(t), \partial_t^\mu v(t)\rangle + \frac{C_1}{\lambda^2}\|\nabla\partial_t^\mu v\|_2^2 \qquad (26)\\ &\le \frac{C_1}{\lambda}\sum_{|a|\le\bar{L}-\mu-1}\langle B_\lambda\partial_t^\mu\nabla^a v(t), \partial_t^\mu\nabla^a v(t)\rangle + \frac{2C_1}{\lambda^2}E_{\bar{L}-\mu}(\partial_t^\mu v(t))\\ &\le \frac{2C_1}{\lambda}\left(\sum_{|a|\le\bar{L}-\mu-1}\langle B_\lambda\partial_t^\mu\nabla^a v(t), \partial_t^\mu\nabla^a v(t)\rangle + \frac{1}{\lambda}E_{\bar{L}-\mu}(\partial_t^\mu v(t))\right)\\ &\le \frac{2C_1C}{\lambda}G_{\bar{L},\mu}(v(t)).\end{aligned}$$

From (25) and (26), there exists a constant $C > 0$ such that v satisfies

$$\frac{1}{C}\left\{\lambda\|\partial_t^\mu v(t)\|_2^2 + \frac{1}{\lambda}E_{\bar{L}-\mu}(\partial_t^\mu v(t))\right\} \le G_{\bar{L},\mu}(v(t)). \tag{27}$$

On the other hand using (21), (22) and (24) , we get

$$\begin{aligned}&G_{\bar{L},\mu}(v(t)) \qquad (28)\\ \le\ & \frac{1}{\lambda}\left(\frac{3C_0}{2\lambda} + \frac{C_1b_0(2d-1)}{2} + b_0Rd^2\right)E_{\bar{L}-\mu}(\partial_t^\mu v(t))\\ &\quad + \frac{3b_0(2d-1)}{16}\sum_{|a|\le\bar{L}-\mu-1}\langle\partial_t^\mu\nabla^a v(t), B_\lambda\partial_t^\mu\nabla^a v(t)\rangle\\ \le\ & \frac{C}{\lambda}E_{\bar{L}-\mu}(\partial_t^\mu v(t)) + C\|B_\lambda\|_\infty\sum_{|a|\le\bar{L}-\mu-1}\|\partial_t^\mu\nabla^a v(t)\|_2^2\\ \le\ & \frac{C}{\lambda}E_{\bar{L}-\mu}(\partial_t^\mu v(t)) + \lambda C\|\partial_t^\mu v(t)\|_2^2.\end{aligned}$$

So combining (27) and (28), we get (20). This completes the proof of lemma 3.2. □

Lemma 3.3. *Let $L_0 \le \bar{L} \le L$. There exist sufficiently small constants λ and δ such that if $v \in X_{\delta,T}$ is a local solution to $(DW)_\lambda$ then v satisfies*

$$\frac{d}{dt}G_{\bar{L},0}(v(t)) + \frac{b_0}{8}E_{\bar{L}}(v(t)) \le 0. \tag{29}$$

Proof. Let $v \in X_{\delta,T}$ be a local solution to $(\mathrm{DW})_\lambda$. From (16), C_0 satisfies $C_0 \geq d$. Thus using lemma 3.1 for $\mu = 0$ and

$$D_{\bar{L},0}(v(t)) = E^{\frac{1}{2}}_{\bar{L}_0}(\partial_t^\mu v(t))E_{\bar{L}}(\partial_t^\mu v(t)) \leq \delta E_{\bar{L}}(\partial_t^\mu v(t)),$$

we have

$$\frac{d}{dt}G_{\bar{L},0}(v(t)) + \frac{b_0}{2}E_{\bar{L}}(v(t)) \leq C(\lambda + \frac{\delta}{\lambda})E_{\bar{L}}(v(t)) + \frac{2\|B\|_\infty b_0^2 R^2}{C_0}E_{\bar{L}}(v(t)).$$

From (16), we also obtain

$$\frac{2\|B\|_\infty b_0^2 R^2}{C_0} \leq \frac{b_0}{8}.$$

We choose sufficiently small constants λ and δ such that

$$C(\lambda + \frac{\delta}{\lambda}) \leq \frac{b_0}{4},$$

then we have

$$\frac{d}{dt}G_{\bar{L},0}(v(t)) + \frac{b_0}{8}E_{\bar{L}}(v(t)) \leq 0.$$

This completes the proof of lemma 3.3. □

Corollary 3.4. *Let $L_0 \leq \bar{L} \leq L$, and λ and δ be sufficiently small constants in lemma 3.3. Then there exists a constant C^* depending on λ such that for any $T > 0$ and a local solution $v \in X_{\delta,T}$ to $(DW)_\lambda$ satisfy*

$$\|v(t)\|_2^2 + E_{\bar{L}}(v(t)) + \int_0^t E_{\bar{L}}(v(s))ds \leq C^*\{\|v(0)\|_2^2 + E_{\bar{L}}(v(0))\} \tag{30}$$

and

$$\|v(t)\|^2_{H^{L_0}} + \|\partial_t v(t)\|^2_{H^{L_0-1}} \leq C^*\{\|v(0)\|^2_{H^{L_0}} + \|\partial_t v(0)\|^2_{H^{L_0-1}}\}. \tag{31}$$

Proof. The estimate (31) is verified by (30). Thus we should prove (30). Integrating (29) over $[0,t]$ we, get

$$G_{\bar{L},0}(v(t)) + \frac{b_0}{8}\int_0^t E_{\bar{L}}(v(s))ds \leq G_{\bar{L},\mu}(v(0)). \tag{32}$$

Then using lemma 3.2, there exists a constant $C > 0$ such that

$$\frac{1}{C}\{\lambda\|v(t)\|_2^2 + \frac{1}{\lambda}E_l(v(t))\} + \frac{b_0}{8}\int_0^t E_l(v(s))ds \leq C\{\lambda\|v(0)\|_2^2 + \frac{1}{\lambda}E_{\bar{L}}(v(0))\}.$$

We rearrange coefficient and define C^* depend on λ, it holds that (30).

□

3.2 Global existence

The following local existence theorem is known (see [10], [12], [16]).

Lemma 3.5. *Let* $L \geq L_0 = [d/2] + 3$, $\lambda > 0$ *and* $(v_0, v_1) \in H^L \times H^{L-1}$. *For any sufficiently small constant* $\varepsilon > 0$ *there exist constants* $0 < t_0$ *and* $0 < \eta \leq 1$ *such that if*

$$\|v_0\|_{H^{L_0}}^2 + \|v_1\|_{H^{L_0-1}}^2 \leq \eta\varepsilon^2 \tag{33}$$

then (DW)$_\lambda$ *has a unique local solution* $v \in \cap_{j=0}^{L} C^j([0, t_0]; H^{L-j})$ *and the* v *satisfies*

$$E_{L_0}(v(t)) \leq \varepsilon^2 \qquad (0 \leq t \leq t_0). \tag{34}$$

Using lemma 3.5 and corollary 3.4, we can prove a global existence theorem for (DW)$_\lambda$.

Theorem 3.6. *Let* $L \geq L_0 = [d/2] + 3$, λ *and* δ *are sufficiently small constants. There exists a small constant* $\delta^* > 0$ *such that if the initial data* $(v_0, v_1) \in H^L \times H^{L-1}$ *satisfy*

$$\|v_0\|_{H^{L_0}}^2 + \|v_1\|_{H^{L_0-1}}^2 \leq \delta^*, \tag{35}$$

then (DW)$_\lambda$ *has a unique global solution* $v \in X_\delta$.

Proof. Let λ and $\delta > 0$ are sufficiently small constants for which corollary 3.4 holds. Furthermore let $0 < \varepsilon \leq \delta$, $0 < t_0$ and $0 < \eta \leq 1$ be the constants given in lemma 3.5. Now we define

$$\delta^* = \min\left\{\eta\varepsilon^2, \frac{\eta\varepsilon^2}{C^*}\right\},$$

where the constant C^* is given in corollary 3.4. We assume that $(v_0, v_1) \in H^L \times H^{L-1}$ satisfy

$$\|v_0\|_{H^{L_0}}^2 + \|v_1\|_{H^{L_0-1}}^2 \leq \delta^*.$$

Lemma 3.5 yields that there exists $v \in \cap_{j=0}^{L} C^j([0, t_0]; H^{L-j})$ such that v is a unique local solution to (DW)$_\lambda$ and satisfies

$$E_{L_0}(v(t)) \leq \varepsilon^2 \leq \delta^2 \qquad (0 \leq t \leq t_0).$$

Because of $v \in X_{\delta,t_0}$ and corollary 3.4, it holds that

$$\|v(t)\|_{H^{L_0}}^2 + \|\partial_t v(t)\|_{H^{L_0-1}}^2 \leq C^*\{\|v_0\|_{H^{L_0}}^2 + \|v_1\|_{H^{L_0-1}}^2\} \leq \eta\varepsilon^2 \quad (t \in [0, t_0]).$$

Thus we can use lemma 3.5 in $t = t_0$. The solution v is uniquely extended to $\cap_{j=0}^{L} C^j([0, 2t_0]; H^{L-j})$ and satisfies

$$E_{L_0}(v(t)) \leq \varepsilon^2 \leq \delta^2 \qquad (0 \leq t \leq 2t_0).$$

Because of $v \in X_{\delta, 2t_0}$ we can use corollary 3.4 again. Then v satisfies

$$\|v(t)\|_{H^{L_0}}^2 + \|\partial_t v(t)\|_{H^{L_0-1}}^2 \leq C^* \{\|v_0\|_{H^{L_0}}^2 + \|v_1\|_{H^{L_0-1}}^2\} \leq \eta\varepsilon^2 \quad (t \in [0, 2t_0]).$$

Thus we can use lemma 3.5 in $t = 2t_0$.

Repeating this argument, we can uniquely extend v to a global solution to $(\mathrm{DW})_\lambda$, furthermore the v satisfies

$$E_{L_0}(v(t)) \leq \delta^2 \qquad (t \in [0, \infty)).$$

This completes the proof of theorem 3.6. □

Proof of theorem1.1

Let λ and δ^* are the small constants in theorem 3.6. We define $\hat{\delta} = \lambda^{d+1}\delta^*$ and assume the initial data $(u_0, u_1) \in H^L \times H^{L-1}$ satisfy

$$\|u_0\|_{H^{L_0}}^2 + \|u_1\|_{H^{L_0-1}}^2 \leq \hat{\delta}.$$

Now we define

$$v_0(x) = \frac{1}{\lambda} u_0(\lambda x), \quad v_1(x) = u_1(\lambda x),$$

then (v_0, v_1) satisfy

$$\begin{aligned}
& \|v_0\|_{H^{L_0}}^2 + \|v_1\|_{H^{L_0-1}}^2 \\
= & \sum_{|a| \leq L_0} \lambda^{2(|a|-1)} \int_{\mathbb{R}^d} |\nabla^a u_0(\lambda x)|^2 dx + \sum_{|a| \leq L_0 - 1} \lambda^{2|a|} \int_{\mathbb{R}^d} |\nabla^a u_1(\lambda x)|^2 dx \\
= & \sum_{|a| \leq L_0} \lambda^{2(|a|-1)-d} \int_{\mathbb{R}^d} |\nabla^a u_0(x)|^2 dx + \sum_{|a| \leq L_0 - 1} \lambda^{2|a|-d} \int_{\mathbb{R}^d} |\nabla^a u_1(x)|^2 dx \\
\leq & \lambda^{-d-1} \{ \sum_{|a| \leq L_0} \int_{\mathbb{R}^d} |\nabla^a u_0(x)|^2 dx + \sum_{|a| \leq L_0 - 1} \int_{\mathbb{R}^d} |\nabla^a u_1(x)|^2 dx \} \\
= & \lambda^{-d-1} \{ \|u_0\|_{H^{L_0}}^2 + \|u_1\|_{H^{L_0-1}}^2 \} \leq \delta^*.
\end{aligned}$$

From theorem 3.6, there exists a unique global solution v to $(\mathrm{DW})_\lambda$ in $\cap_{j=0}^{L} C^j([0, \infty); H^{L-j})$. We define

$$u(t, x) = \lambda v\left(\frac{t}{\lambda}, \frac{x}{\lambda}\right),$$

then $u \in \cap_{j=0}^{L} C^j([0,\infty); H^{L-j})$ and the u satisfies (DW).

As regard to uniqueness, if u and u' are solutions to (DW) then rescaling functions u_λ and u'_λ are solutions to (DW)$_\lambda$. From theorem 3.6 we got the uniqueness of (DW)$_\lambda$, so we obtain $u_\lambda = u'_\lambda$, thus $u = u'$.

□

4 Decay Estimates

The goal of this section is to show theorem 4.1. We say that f satisfies the property **(H1)**$'$, **(H2)**$'$ or **(H3)**$'$ if and only if

(H1)$'$ $d \geq 3$ and there exists $1 \leq p \leq \frac{2d}{d+2}$ such that $f \in L^p$,

(H2)$'$ $d \geq 3$ and $|\cdot|f \in L^2$,

(H3)$'$ $d = 1$ or 2 , $|\cdot|f \in L^1$ and $\displaystyle\int_{\mathbb{R}^d} f(x)dx = 0$.

We prove the decay estimates for (DW)$_\lambda$ as follow:

Theorem 4.1. *In addition to the assumptions in theorem 3.6, we assume that $B_\lambda v_0 + v_1$ satisfies one of the* ***(H1)'-(H3)'****. Then for any i $(0 \leq i \leq L - L_0)$, there exists a constant E_0 depending on λ, v_0 and v_1 such that the global solution $v \in X_\delta$ to (DW)$_\lambda$ satisfies*

$$(1+t)^{2i+1}\{\|\partial_t^i v(t)\|_2^2 + E_{L-i}(\partial_t^i v(t))\} + \int_0^t (1+s)^{2i+1} E_{L-i}(\partial_t^i v(s))ds \leq E_0 \quad (36)$$

and

$$(1+t)^{2i+2} E(\partial_t^i v(t)) + \int_0^t (1+s)^{2i+2} \langle \partial_t^{i+1} v(s), B_\lambda \partial_t^{i+1} v(s)\rangle \leq E_0. \quad (37)$$

Using theorem 4.1, we can prove theorem 1.2.

Proof of theorem 1.2

We assume that theorem 4.1 is true. From theorem 1.1 there exists a constant $\hat{\delta}$ such that if the initial data (u_0, u_1) satisfies

$$\|u_0\|_{H^{L_0}}^2 + \|u_1\|_{H^{L_0-1}}^2 \leq \hat{\delta},$$

then (DW) has a unique global solution $u \in \cap_{j=0}^{L} C^j([0,\infty); H^{L-j})$. Now we define $v = \dfrac{1}{\lambda} u(\lambda t, \lambda x)$ then v satisfies

$$(\mathrm{DW})_\lambda \quad \begin{cases} (\partial_t^2 - \triangle + B_\lambda(x)\partial_t)v(t,x) = N[v,v](t,x), & (t,x) \in [0,\infty) \times \mathbb{R}^d, \\ v_(0,x) = v_0(x), \quad \partial_t v_\lambda(0,x) = v_1, & x \in \mathbb{R}^d, \end{cases}$$

where $v_0(x) = \frac{1}{\lambda} u_0(\lambda x)$ and $v_1(x) = u_1(\lambda x)$. If $Bu_0 + u_1$ satisfies **(H1)**, **(H2)** or **(H3)**, then $B_\lambda v_0 + v_1$ satisfies **(H1)**$'$, **(H2)**$'$ or **(H3)**$'$. Thus we can use theorem 4.1. For any μ $(0 \le \mu \le L - L_0)$, there exists a constant E_0 depending on λ, v_0 and v_1 such that v satisfies

$$\|\partial_t^\mu v(\tau)\|_2^2 + E_{L-\mu}(\partial_t^\mu v(\tau)) \le E_0(1+\tau)^{-2\mu-1}$$

and

$$E(\partial_t^\mu v(\tau)) \le E_0(1+\tau)^{-2\mu-2}.$$

Replacing $v(t,x) = \dfrac{1}{\lambda} u(\lambda t, \lambda x)$ and $t = \lambda\tau$, we get (2) and (3).

Using (2) and lemma 2.1, the estimate (4) is clear. Thus we prove (5). Let $L_0 < L$. The global solution u to (DW) satisfies

$$\begin{aligned} \|\triangle u(t)\|_2 &= \|\partial_t^2 u(t) + \partial_t u(t) - N[u,u](t)\|_2 \\ &\le \|\partial_t^2 u(t)\|_2 + \|\partial_t u(t)\|_2 + \|N[u,u](t)\|_2. \end{aligned} \tag{38}$$

Because of $L_0 < L$, we can use (2) and (3) to $\mu = 1$. Then we obtain

$$\|\partial_t u\|_2^2 \le E_0(1+t)^{-3},$$

$$\|\partial_t^2 u\|_2^2 \le E_0(1+t)^{-4}.$$

Furthermore using (2) and (3), we obtain

$$\begin{aligned} \|N[u,u](t)\|_2^2 &\le \sum_{i,j,k,l,m,n=1}^{d} |N_{lmn}^{ijk}| \|\partial_l(\partial_m u^j \partial_n u^k)\|_2^2 \\ &\le C\|\nabla^2 u(t)\|_\infty^2 \|\nabla u(t)\|_2^2 \\ &\le CE_{L_0}(u(t))E(u(t)) \\ &\le CE_0(1+t)^{-3}. \end{aligned}$$

Thus the estimate (5) holds from (38). $\square$

4.1 Proof of theorem 4.1

Let v be the global solution to $(\mathrm{DW})_\lambda$ and define

$$w(t,x) = \int_0^t v(s,x)ds. \tag{39}$$

Then w satisfies

$$\begin{cases} \partial_t^2 w(t,x) - \triangle w(t,x) + B_\lambda(x)\partial_t w(t,x) & \\ \qquad = \displaystyle\int_0^t N[v,v](\tau,x)d\tau + B_\lambda(x)v_0(x) + v_1(x), & (t,x)\in[0,\infty)\times\mathbb{R}^d, \\ w(0,x) = 0, \partial_t w(0,x) = v_0(x), & x\in\mathbb{R}^d. \end{cases} \tag{40}$$

We remark $\partial_t w = v$ and $E_{L_0-1}(w(t))$ are well-defined in $[0,\infty)$ because of corollary 3.4.

We prove the energy estimate of w under the assumption which $B_\lambda v_0 + v_1$ satisfies one of the **(H1)**$'$-**(H3)**$'$. We prepare the next lemma.

Lemma 4.2. *Let $f \in H^{L_0-1}$ satisfies one of the **(H1)**$'$-**(H3)**$'$. Then for any $g \in H^{L_0-1}$, there exists a constant E_0 depending on f such that*

$$\langle g, f\rangle \le E_0\|\nabla g\|_{H^{L_0-2}}. \tag{41}$$

Proof. First we assume that f satisfies **(H1)**$'$. Then $f \in L^{\frac{2d}{d+2}}$ because $f \in H^{L_0-1} \subset L^\infty$ and $f \in L^p$ $(1 \le p \le \frac{2d}{d+2})$. On the other hand, it holds from lemma 2.5 that

$$\|g\|_{L^{\frac{2d}{d-2}}} \le C\|\nabla g\|_2.$$

Thus using Hölder inequality, we get

$$\langle g, f\rangle \le \|f\|_{L^{\frac{2d}{d+2}}}\|g\|_{L^{\frac{2d}{d-2}}} \le C\|f\|_{L^{\frac{2d}{d+2}}}\|\nabla g\|_2 \le C\|f\|_{L^{\frac{2d}{d+2}}}\|\nabla g\|_{H^{L_0-2}}.$$

This means (41).

Next we assume that f satisfies **(H2)**$'$. Then using lemma 2.4, we get

$$\langle g, f\rangle \le \||\cdot|f\|_2\left\|\frac{g}{|\cdot|}\right\|_2 \le C\||\cdot|f\|_2\|\nabla g\|_2 \le C\||\cdot|f\|_2\|\nabla g\|_{H^{L_0-2}}.$$

Thus we have (41).

Finally we assume that f satisfies **(H3)**$'$. Because of $g \in H^{L_0-1} \subset C^1(\mathbb{R}^d)$, we have

$$|g(x) - g(0)| \;=\; \left|\int_0^1 \frac{d}{d\theta}g(\theta x)d\theta\right| = \left|\int_0^1 x\cdot\nabla g(\theta x)d\theta\right| \le |x|\|\nabla g\|_\infty \quad \text{a.e.}x.$$

Then using $\int_{\mathbb{R}^d} f(x)dx = 0$, we obtain

$$\begin{aligned}|\langle g, f\rangle| &= \left|\int_{\mathbb{R}^d} (g(x) - g(0))f(x)dx\right| \leq \int_{\mathbb{R}^d} |(g(x) - g(0)||f(x)|dx \\ &\leq \|\nabla g\|_\infty \int_{\mathbb{R}^d} |x||f(x)|dx \leq C\|\nabla g\|_{H^{L_0-2}}\|| \cdot |f\|_1.\end{aligned}$$

Thus we get (41). □

We need the estimate $\int_0^\infty \|v(t)\|_2^2 dt < \infty$. In order to prove this property, we use the idea in Ikehata [5].

Proposition 4.3. *In addition to the assumptions theorem 3.6, we assume $B_\lambda v_0 + v_1$ satisfies one of the **(H1)**$'$-**(H3)**$'$. Then there exists a constant E_0 depending on λ, v_0 and v_1 such that the global solution $v \in X_\delta$ to $(DW)_\lambda$ satisfies*

$$\int_0^t \|v(s)\|_2^2 ds \leq E_0, \quad (t \in [0, \infty)). \tag{42}$$

Proof. Let v be the global solution to $(\mathrm{DW})_\lambda$ and w be defined by (39). Using (40), we have

$$\begin{aligned}&\frac{d}{dt}E_{L_0-1}(w(t)) \\ &= \sum_{|a|\leq L_0-2} \langle \partial_t \nabla^a w(t), \partial_t^2 \nabla^a w(t)\rangle - \sum_{|a|\leq L_0-2} \langle \partial_t \nabla^a w(t), \triangle \nabla^a w(t)\rangle \\ &= -\sum_{|a|\leq L_0-2} \langle \partial_t \nabla^a w(t), \nabla^a (B_\lambda \partial_t w(t))\rangle \\ &+ \sum_{|a|\leq L_0-2} \langle \partial_t \nabla^a w(t), \int_0^t \nabla^a N[v,v]d\tau\rangle + \sum_{|a|\leq L_0-2} \langle \partial_t \nabla^a w(t), \nabla^a (B_\lambda v_0 + v_1)\rangle \\ &= -\sum_{|a|\leq L_0-2} \langle \partial_t \nabla^a w(t), B_\lambda \partial_t \nabla^a w(t))\rangle \\ &- \sum_{1\leq|a|\leq L_0-2} \sum_{\substack{b\leq a \\ b\neq 0}} \binom{a}{b} \langle \partial_t \nabla^a w(t), \nabla^b B_\lambda \partial_t \nabla^{a-b} w(t))\rangle \\ &+ \sum_{|a|\leq L_0-2} \frac{d}{dt}\langle \nabla^a w(t), \int_0^t \nabla^a N[v,v]d\tau\rangle - \sum_{|a|\leq L_0-2} \langle \nabla^a w(t), \nabla^a N[v,v](t)\rangle \\ &+ \sum_{|a|\leq L_0-2} \frac{d}{dt}\langle \nabla^a w(t), \nabla^a (B_\lambda v_0 + v_1)\rangle.\end{aligned}$$

Then integrating it over $[0,t]$, we get

$$
\begin{aligned}
& E_{L_0-1}(w(t)) + \sum_{|a|\le L_0-2} \int_0^t \langle \nabla^a v(s), B_\lambda \nabla^a v(s)\rangle ds \qquad (43)\\
\le\ & \frac{1}{2}\|v_0\|^2_{H^{L_0-2}} - \sum_{1\le|a|\le L_0-2} \sum_{\substack{b\le a\\ b\ne 0}} \binom{a}{b} \int_0^t \langle \nabla^a v(s), \nabla^b B_\lambda \nabla^{a-b} v(s))\rangle ds \\
& + \sum_{|a|\le L_0-2} \langle \nabla^a w(t), \int_0^t \nabla^a N[v,v](s) ds\rangle \\
& - \sum_{|a|\le L_0-2} \int_0^t \langle \nabla^a w(s), \nabla^a N[v,v](s)\rangle ds + \sum_{|a|\le L_0-2} \langle \nabla^a w(t), \nabla^a (B_\lambda v_0 + v_1)\rangle \\
=\ & \frac{1}{2}\|v_0\|^2_{H^{L_0-2}} + A_1 + A_2 + A_3 + A_4.
\end{aligned}
$$

We estimate from A_1 to A_4. Using lemma 2.3 , $\mathbf{(B3)}_\lambda$ and smallness of λ, we get

$$
\begin{aligned}
|A_1| \ \le\ & \sum_{1\le|a|\le L_0-2} \sum_{\substack{b\le a\\ b\ne 0}} \binom{a}{b} \int_0^t \|\nabla^a v(s)\|_2 \|\nabla^b B_\lambda\|_\infty \|\nabla^{a-b} v(s)\|_2 ds \\
\le\ & \lambda^2 C \sum_{1\le|a|\le L_0-2} \sum_{\substack{b\le a\\ b\ne 0}} \int_0^t \|\nabla^a v(s)\|_2 \|\nabla^{a-b} v(s)\|_2 ds \\
\le\ & C \sum_{1\le|a|\le L_0-2} \left(\int_0^t \|\nabla^a v(s)\|_2^2 ds + \lambda^4 C \sum_{\substack{b\le a\\ b\ne 0}} \int_0^t \|\nabla^{a-b} v(s)\|_2^2 ds \right) \\
\le\ & C \int_0^t E_{L_0-2}(v(s)) ds \\
& + \lambda^2 C \sum_{1\le|a|\le L_0-2} \sum_{\substack{b\le a\\ b\ne 0}} \left\{ \int_0^t \langle \nabla^{a-b} v(s), B_\lambda \nabla^{a-b} v(s)\rangle ds \right. \\
& \qquad \left. + \int_0^t \|\nabla \nabla^{a-b} v(s)\|_2^2 ds \right\} \\
\le\ & C \int_0^t E_{L_0}(v(s)) ds + \frac{1}{4} \sum_{|a|\le L_0-2} \int_0^t \langle \nabla^a v(s), B_\lambda \nabla^a v(s)\rangle ds.
\end{aligned}
$$

Next we define

$$
M(t) = \sup_{0\le s\le t} E_{L_0-1}(w(s)). \qquad (44)
$$

Using $M(t)$ and lemma 2.2, we have

$$\begin{aligned}
&|A_2|\\
=&\sum_{|a|\le L_0-2}\left|\langle\nabla^a w(t),\int_0^t\nabla^a N[v,v](s)ds\rangle\right|\\
=&\sum_{|a|\le L_0-2}\left|\sum_{i,j,k,l,m,n=1}^d N^{ijk}_{lmn}\int_0^t\int_{\mathbb{R}^d}\nabla^a w^i(t)\partial_l\nabla^a(\partial_m v^j(s)\partial_n v^k(s))dxds\right|\\
=&\sum_{|a|\le L_0-2}\left|\sum_{i,j,k,l,m,n=1}^d N^{ijk}_{lmn}\int_0^t\int_{\mathbb{R}^d}\partial_l\nabla^a w^i(t)\nabla^a(\partial_m v^j(s)\partial_n v^k(s))dxds\right|\\
\le&\ C\sum_{|a|\le L_0-2}\int_0^t\|\nabla\nabla^a w(t)\|_2\sum_{j,k,m,n=1}^d\|\nabla^a(\partial_m v^j(s)\partial_n v^k(s))\|_2 ds\\
\le&\ C(M(t))^{\frac{1}{2}}\int_0^t E_{L_0}(v(s))ds
\end{aligned}$$

and

$$\begin{aligned}
&|A_3|\\
=&\sum_{|a|\le L_0-2}\left|\int_0^t\langle\nabla^a w(s),\nabla^a N[v,v](s)ds\rangle\right|\\
\le&\sum_{|a|\le L_0-2}\left|\sum_{i,j,k,l,m,n=1}^d N^{ijk}_{lmn}\int_0^t\int_{\mathbb{R}^d}\nabla^a w^i(s)\partial_l\nabla^a(\partial_m v^j(s)\partial_n v^k(s))dxds\right|\\
=&\sum_{|a|\le L_0-2}\left|\sum_{i,j,k,l,m,n=1}^d N^{ijk}_{lmn}\int_0^t\int_{\mathbb{R}^d}\partial_l\nabla^a w^i(s)\nabla^a(\partial_m v^j(s)\partial_n v^k(s))dxds\right|\\
\le&\ C\sum_{|a|\le L_0-2}\int_0^t\|\nabla\nabla^a w(s)\|_2\sum_{j,k,m,n=1}^d\|\nabla^a(\partial_m v^j(s)\partial_n v^k(s))\|_2 ds\\
\le&\ C(M(t))^{\frac{1}{2}}\int_0^t E_{L_0}(v(s))ds.
\end{aligned}$$

Using lemma 4.2, we get

$$\begin{aligned}
A_4 &= \langle w(t),B_\lambda v_0+v_1\rangle+\sum_{1\le|a|\le L_0-2}\langle\nabla^a w(t),\nabla^a(B_\lambda v_0+v_1)\rangle\\
&\le E_0\|\nabla w(t)\|_{H^{L_0-2}}+\sum_{1\le|a|\le L_0-2}\|\nabla^a w(t)\|_2\|\nabla^a(B_\lambda v_0+v_1)\|_2
\end{aligned}$$

$$\leq \quad E_0 E_{L_0-1}^{\frac{1}{2}}(w(t)) \leq E_0^2 + \frac{1}{4} E_{L_0-1}(w(t)),$$

where E_0 depend on λ, v_0 and v_1. Corollary 3.4 implies that there exists a constant $C^* > 0$ such that

$$\int_0^t E_{L_0}(v(s))ds \leq C^* \{\|v(0)\|_2^2 + E_{L_0}(v(0))\}.$$

From (43) and the estimates of A_1 - A_4, it holds that

$$E_{L_0-1}(w(t)) + \sum_{|a|\leq L_0-2} \int_0^t \langle \nabla^a v(s), B_\lambda \nabla^a v(s)\rangle ds \leq E_0 + E_0(M(t))^{\frac{1}{2}}. \quad (45)$$

From (45) we have

$$M(t) \leq E_0 + E_0(M(t))^{\frac{1}{2}} \quad (t \in [0, \infty)).$$

It means that $M(t)$ is bounded in $[0, \infty)$. Thus it holds from (45) that

$$\int_0^t \langle v(s), B_\lambda v(s)\rangle ds \leq E_0. \quad (46)$$

Finally using (46), lemma 2.3 and Corollary 3.4, we have

$$\int_0^t \|v(t)\|_2^2 ds \quad \leq \quad \frac{C}{\lambda} \int_0^t \langle v(s), B_\lambda v(s)\rangle ds + \frac{C}{\lambda^2} \int_0^t \|\nabla v(s)\|_2^2 ds \leq \frac{E_0}{\lambda} + \frac{E_0}{\lambda^2}.$$

Thus we get (42). □

Let prove theorem 4.1 by the induction. First we prove for $i = 0$.

Theorem 4.4. *In addition to the assumption theorem 3.6, we assume that $B_\lambda v_0 + v_1$ satisfies one of the* ***(H1)'-(H3)'****. Then there exists constant E_0 depending on λ, v_0 and v_1 such that the global solution $v \in X_\delta$ to $(DW)_\lambda$ satisfies*

$$(1+t)\left\{\|v(t)\|_2^2 + E_L(v(t))\right\} + \int_0^t (1+s)E_L(v(s))ds \leq E_0 \quad (47)$$

and

$$(1+t)^2 E(v(t)) + \int_0^t (1+s)^2 \langle \partial_t v(s), B_\lambda \partial_t v(s)\rangle \leq E_0. \quad (48)$$

Proof. First we prove (47). Using lemma 3.3 for $\bar{L} = L$, we obtain

$$\begin{aligned}\frac{d}{dt}\{(1+t)G_{L,0}(v(t))\} &= G_{L,0}(v(t)) + (1+t)\frac{d}{dt}G_{L,0}(v(t)) \\ &\leq G_{L,0}(v(t)) - \frac{b_0(1+t)}{8}E_L(v(t)).\end{aligned}$$

Integrating it over $[0,t]$ and using lemma 3.2, we get

$$\begin{aligned}&\frac{(1+t)}{C}\left\{\lambda\|v(t)\|_2^2 + \frac{1}{\lambda}E_L(v(t))\right\} + \frac{b_0}{8}\int_0^t (1+s)E_L(v(s))ds \\ \leq\ & C\left\{\lambda\|v(0)\|_2^2 + \frac{1}{\lambda}E_L(v(0)) + \lambda\int_0^t \|v(s)\|_2^2 ds + \frac{1}{\lambda}\int_0^t E_L(v(s))ds\right\}.\end{aligned}$$

Then there exists a constant E_0 which depends on λ, v_0 and v_1 such that

$$\begin{aligned}&(1+t)\left\{\|v(t)\|_2^2 + E_L(v(t))\right\} + \int_0^t (1+s)E_L(v(s))ds \\ \leq\ & E_0 + E_0\int_0^t \|v(s)\|_2^2 ds + E_0\int_0^t E_L(v(s))ds.\end{aligned} \tag{49}$$

Using proposition 4.3 and corollary 3.4, we get

$$\begin{aligned}&(1+t)\left\{\lambda\|v(t)\|_2^2 + E_L(v(t))\right\} + \int_0^t (1+s)E_L(v(s))ds \\ \leq\ & E_0 + E_0^2 + E_0C^*\{\|v(0)\|_2^2 + E_L(v(0))\}.\end{aligned}$$

Rearranging E_0 if we need, we get (47).

Next we prove (48). It holds that

$$\begin{aligned}&\frac{d}{dt}\left\{(1+t)^2E(v(t))\right\} \\ =\ & 2(1+t)E(v(t)) + (1+t)^2\frac{d}{dt}E(v(t)) \\ =\ & 2(1+t)E(v(t)) + (1+t)^2\{\langle\partial_t v(t), \partial_t^2 v(t)\rangle - \langle\partial_t v(t), \triangle v(t)\rangle\} \\ =\ & 2(1+t)E(v(t)) - (1+t)^2\langle\partial_t v(t), B_\lambda\partial_t v(t)\rangle + (1+t)^2\langle\partial_t v(t), N[v,v](t)\rangle.\end{aligned} \tag{50}$$

Using (50) and

$$\langle\partial_t v(t), N[v,v](t)\rangle \ \leq\ CE^{\frac{1}{2}}(v(t))E_L(v(t)),$$

we get

$$\frac{d}{dt}\left\{(1+t)^2E(v(t))\right\} + (1+t)^2\langle\partial_t v(t), B_\lambda\partial_t v(t)\rangle \tag{51}$$

$$\leq \ 2(1+t)E(v(t)) + C(1+t)^2 E^{\frac{1}{2}}(v(t))E_L(v(t)).$$

Now we define

$$M_0(t) = \sup_{0\leq s\leq t}(1+s)^2 E(v(s)). \tag{52}$$

Integrating (51) over $[0,\infty)$ and using (47), we obtain

$$\begin{aligned}
&(1+t)^2E(v(t)) + \int_0^t (1+s)^2\langle\partial_t v(s), B_\lambda \partial_t v(s)\rangle ds \qquad (53)\\
\leq\ & E(v(0)) + 2\int_0^t (1+s)E(v(s))ds + C\int_0^t (1+s)^2 E^{\frac{1}{2}}(v(s))E_L(v(s))ds\\
\leq\ & E(v(0)) + 2\int_0^t (1+s)E(v(s))ds + C(M_0(t))^{\frac{1}{2}}\int_0^t (1+s)E_L(v(s))ds\\
\leq\ & E_0 + E_0(M_0(t))^{\frac{1}{2}}.
\end{aligned}$$

From (53) we have

$$M_0(t) \leq E_0 + E_0(M_0(t))^{\frac{1}{2}},$$

which means that $M_0(t)$ is bounded in $[0,\infty)$. Therefore it holds from (53) that

$$(1+t)^2E(v(t)) + \int_0^t (1+s)^2\langle\partial_t v(s), B_\lambda\partial_t v(s)\rangle \leq E_0.$$

Thus we get (48). □

Next assuming the decay estimate of $\partial_t^i v$ for $0 \leq i \leq \mu-1$, we show the decay estimate of $\partial_t^\mu v$. For the purpose, we need the following lemma:

Lemma 4.5. *In addition to the assumption theorem 3.6, we assume that $B_\lambda v_0 + v_1$ satisfies one of the* ***(H1)$'$-(H3)$'$****. Let $1 \leq \mu \leq L - L_0$ and assume that for any $0 \leq i \leq \mu-1$ estimates* (36) *and* (37) *in theorem 4.1 hold. Then there exists a constant E_0 depending on λ, v_0 and v_1 such that the global solution $v \in X_\delta$ to $(DW)_\lambda$ satisfies*

$$\frac{d}{dt}G_{L,\mu}(v(t)) + \frac{b_0}{8}E_{L-\mu}(\partial_t^\mu v(t)) \leq E_0 \sum_{\nu=1}^{\mu} E_{L-\nu}(\partial_t^\nu v(t))(1+t)^{-2(\mu-\nu)-1}, \tag{54}$$

if λ and δ in theorem 3.6 are chosen small enough.

Proof. From (18) in lemma 3.1 and the assumption of induction, it follows that

$$D_{L,\mu}(v(t)) = E^{\frac{1}{2}}_{L-\mu}(\partial_t^\mu v(t)) \cdot \left(\sum_{\nu=0}^{\mu} E^{\frac{1}{2}}_{L_0}(\partial_t^{\mu-\nu} v(t))E^{\frac{1}{2}}_{L-\nu}(\partial_t^\nu v(t))\right)$$

$$
\begin{aligned}
&\leq \ \delta E_{L-\mu}(\partial_t^\mu v(t)) + E_{L-\mu}^{\frac{1}{2}}(\partial_t^\mu v(t)) \cdot \left(\sum_{\nu=0}^{\mu-1} E_{L_0}^{\frac{1}{2}}(\partial_t^{\mu-\nu} v(t)) E_{L-\nu}^{\frac{1}{2}}(\partial_t^\nu v(t)) \right) \\
&\leq \ 2\delta E_{L-\mu}(\partial_t^\mu v(t)) + \frac{1}{4\delta} \left(\sum_{\nu=0}^{\mu-1} E_{L_0}^{\frac{1}{2}}(\partial_t^{\mu-\nu} v(t)) E_{L-\nu}^{\frac{1}{2}}(\partial_t^\nu v(t)) \right)^2 \\
&\leq \ 2\delta E_{L-\mu}(\partial_t^\mu v(t)) + \frac{\mu}{4\delta} \sum_{\nu=0}^{\mu-1} E_{L_0}(\partial_t^{\mu-\nu} v(t)) E_{L-\nu}(\partial_t^\nu v(t)) \\
&\leq \ 2\delta E_{L-\mu}(\partial_t^\mu v(t)) + \frac{\mu E_0}{4\delta} \sum_{\nu=0}^{\mu-1} E_{L-(\mu-\nu)}(\partial_t^{\mu-\nu} v(t))(1+t)^{-2\nu-1} \\
&= \ 2\delta E_{L-\mu}(\partial_t^\mu v(t)) + \frac{\mu E_0}{4\delta} \sum_{\nu=1}^{\mu} E_{L-\nu}(\partial_t^\nu v(t))(1+t)^{-2(\mu-\nu)-1}.
\end{aligned}
$$

The above estimate and lemma 3.1 imply that

$$
\begin{aligned}
& \frac{d}{dt} G_{L,\mu}(v(t)) + \frac{b_0}{2} E_{L-\mu}(\partial_t^\mu v(t)) \qquad (55) \\
\leq \ & \lambda C E_{L-\mu}(\partial_t^\mu v(t)) + \frac{C}{\lambda} D_{L,\mu}(v(t)) + \frac{2\|B\|_\infty b_0^2 R^2}{C_0} E_{L-\mu}(\partial_t^\mu v(t)) \\
\leq \ & \left(\lambda C + \frac{2\delta C}{\lambda} + \frac{2\|B\|_\infty b_0^2 R^2}{C_0} \right) E_{L-\mu}(\partial_t^\mu v(t)) \\
& + \frac{C E_0}{\lambda\delta} \sum_{\nu=1}^{\mu} E_{L-\nu}(\partial_t^\nu v(t))(1+t)^{-2(\mu-\nu)-1}.
\end{aligned}
$$

From (16), it follows that

$$
\frac{2\|B\|_\infty b_0^2 R^2}{C_0} \leq \frac{b_0}{8}.
$$

Choosing λ and $\delta > 0$ in theorem 3.6 sufficiently small enough if necessary, we obtain

$$
\lambda C + \frac{2\delta C}{\lambda} \leq \frac{b_0}{4}.
$$

From these estimates and (55), it holds that there exists a constant E_0 such that

$$
\frac{d}{dt} G_{L,\mu}(v(t)) + \frac{b_0}{8} E_{L-\mu}(\partial_t^\mu v(t)) \leq E_0 \sum_{\nu=1}^{\mu} E_{L-\nu}(\partial_t^\nu v(t))(1+t)^{-2(\mu-\nu)-1}.
$$

This means (54). Hence we obtain lemma 4.5. □

We complete to prove theorem 4.1. When $i=0$ we already proved (theorem 4.4), thus we assume $1 \le \mu \le L-L_0$ and for any $0 \le i \le \mu-1$ satisfy theorem 4.1. The goal is to show (36) and (37) to $i=\mu$.

First we prove (36) for $i=\mu$. Lemma 4.5 yields that

$$
\begin{aligned}
& \frac{d}{dt}\{(1+t)^{2\mu+1}G_{L,\mu}(v(t))\} && (56)\\
= \;& (2\mu+1)(1+t)^{2\mu}G_{L,\mu}(v(t)) + (1+t)^{2\mu+1}\frac{d}{dt}G_{L,\mu}(v(t))\\
\le \;& (2\mu+1)(1+t)^{2\mu}G_{L,\mu}(v(t)) - (1+t)^{2\mu+1}\frac{b_0}{8}E_{L-\mu}(\partial_t^\mu v(t))\\
& +E_0\sum_{\nu=1}^{\mu}E_{L-\nu}(\partial_t^\nu v(t))(1+t)^{2\nu}.
\end{aligned}
$$

Integrating (56) over $[0,t]$, we get

$$
\begin{aligned}
& (1+t)^{2\mu+1}G_{L,\mu}(v(t)) + \frac{b_0}{8}\int_0^t (1+s)^{2\mu+1}E_{L-\mu}(\partial_t^\mu v(s))ds && (57)\\
\le \;& G_{L,\mu}(v(0)) + (2\mu+1)\int_0^t (1+s)^{2\mu}G_{L,\mu}(v(s))ds\\
& +E_0\sum_{\nu=1}^{\mu}\int_0^t (1+s)^{2\nu}E_{L-\nu}(\partial_t^\nu v(s))ds.
\end{aligned}
$$

From (20) and (56), there exists a constant E_0 depending on λ, v_0 and v_1 such that

$$
\begin{aligned}
& (1+t)^{2\mu+1}\{\|\partial_t^\mu v(t)\|_2^2 + E_{L-\mu}(\partial_t^\mu v(t))\} + \int_0^t (1+s)^{2\mu+1}E_{L-\mu}(\partial_t^\mu v(s))ds\\
\le \;& E_0 + E_0\int_0^t (1+s)^{2\mu}\|\partial_t^\mu v(s)\|_2^2 ds + E_0\int_0^t (1+t)^{2\mu}E_{L-\mu}(\partial_t^\mu v(s))ds\\
& +E_0\sum_{\nu=1}^{\mu-1}\int_0^t (1+s)^{2\nu}E_{L-\nu}(\partial_t^\nu v(s))ds. && (58)
\end{aligned}
$$

We estimate the right-side of (58). Using lemma 2.3 and (37) for $i=\mu-1$, we get

$$
\begin{aligned}
& \int_0^t (1+s)^{2\mu}\|\partial_t^\mu v(s)\|_2^2 ds && (59)\\
\le \;& \frac{C_1}{\lambda}\int_0^t (1+s)^{2\mu}\langle \partial_t^\mu v(s), B_\lambda \partial_t^\mu v(s)\rangle ds + \frac{C_1}{\lambda^2}\int_0^t (1+s)^{2\mu}\|\nabla \partial_t^\mu v(s)\|_2^2 ds
\end{aligned}
$$

$$\leq \frac{C_1E_0}{\lambda} + \frac{2C_1}{\lambda^2}\int_0^t (1+s)^{2\mu} E_{L-\mu}(\partial_t^\mu v(s))ds.$$

From the assumption of induction, it follows that (36) for $i = \nu$ with $\nu \leq \mu - 1$ hold, which yields that

$$\sum_{\nu=1}^{\mu-1} \int_0^t (1+s)^{2\nu} E_{L-\nu}(\partial_t^\nu v(s))ds \leq (\mu-1)E_0. \tag{60}$$

Using (58), (59) and (60), we obtain

$$\begin{aligned}
&(1+t)^{2\mu+1}\{\|\partial_t^\mu v(t)\|_2^2 + E_{L-\mu}(\partial_t^\mu v(t))\} + \int_0^t (1+s)^{2\mu+1} E_{L-\mu}(\partial_t^\mu v(s))ds \\
&\leq E_0 + E_0 \int_0^t (1+t)^{2\mu} E_{L-\mu}(\partial_t^\mu v(s))ds. \qquad (61)
\end{aligned}$$

We choose a constant t^* such that $2E_0(1+t^*)^{-1} \leq 1$ then we get

$$\begin{aligned}
& E_0 \int_0^t (1+s)^{2\mu} E_{L-\mu}(\partial_t^\mu v(s))ds \qquad (62)\\
= \ & E_0 \int_0^{t^*} (1+s)^{2\mu} E_{L-\mu}(\partial_t^\mu v(s))ds \\
& \qquad + E_0 \int_{t^*}^t (1+s)^{2\mu+1}(1+s)^{-1} E_{L-\mu}(\partial_t^\mu v(s))ds \\
\leq \ & \frac{E_0(1+t^*)^{2\mu+1}}{2\mu+1} \sup_{0\leq s\leq t^*} E_{L-\mu}(\partial_t^\mu v(s)) + \frac{1}{2}\int_0^t (1+t)^{2\mu+1} E_{L-\mu}(\partial_t^\mu v(s))ds.
\end{aligned}$$

Now $\dfrac{E_0(1+t^*)^{2\mu+1}}{2\mu+1} \sup_{0\leq s\leq t^*} E_{L-\mu}(\partial_t^\mu v(s))$ can include to E_0 because the constant is depended on λ, u_0 and u_1. Therefore using (61) and (62), we get (36) to $i = \mu$.

Next we prove (37) to $i = \mu$. For the solution v to $(\mathrm{DW})_\lambda$ holds that

$$\begin{aligned}
& \frac{d}{dt}\left\{(1+t)^{2\mu+2} E(\partial_t^\mu v(t))\right\} \\
= \ & (2\mu+2)(1+t)^{2\mu+1} E(\partial_t^\mu v(t)) + (1+t)^{2\mu+2}\frac{d}{dt}E(\partial_t^\mu v(t)) \\
\leq \ & (2\mu+2)(1+t)^{2\mu+1} E_{L-\mu}(\partial_t^\mu v(t)) - (1+t)^{2\mu+2}\langle \partial_t^{\mu+1} v(t), B_\lambda \partial_t^{\mu+1} v(t)\rangle \\
& +(1+t)^{2\mu+2}\langle \partial_t^{\mu+1} v(t), \partial_t^\mu N[v,v](t)\rangle.
\end{aligned}$$

Using the above estimate and

$$\langle \partial_t^{\mu+1} v(t), \partial_t^\mu N[v,v](t)\rangle$$

$$
\begin{aligned}
&\leq C\|\partial_t^{\mu+1}v(t)\|_2\sum_{\nu=0}^{\mu}\binom{\mu}{\nu}\|\partial_t^{\nu}\nabla^2 v(t)\|_\infty\|\partial_t^{\mu-\nu}\nabla v(t)\|_2\\
&\leq CE^{\frac{1}{2}}(\partial_t^{\mu}v(t))\sum_{\nu=0}^{\mu}E^{\frac{1}{2}}_{L-\nu}(\partial_t^{\nu}v(t))E^{\frac{1}{2}}_{L-(\mu-\nu)}(\partial_t^{\mu-\nu}v(t)),
\end{aligned}
$$

we obtain

$$
\begin{aligned}
&\frac{d}{dt}\left\{(1+t)^{2\mu+2}E(\partial_t^{\mu}v(t))\right\}+(1+t)^{2\mu+2}\langle\partial_t^{\mu+1}v(t),B_\lambda\partial_t^{\mu+1}v(t)\rangle\\
\leq\ & C(1+t)^{2\mu+1}E_{L-\mu}(\partial_t^{\mu}v(t)) \qquad (63)\\
&+C(1+t)^{2\mu+2}E^{\frac{1}{2}}(\partial_t^{\mu}v(t))\sum_{\nu=0}^{\mu}E^{\frac{1}{2}}_{L-\nu}(\partial_t^{\nu}v(t))E^{\frac{1}{2}}_{L-(\mu-\nu)}(\partial_t^{\mu-\nu}v(t)).
\end{aligned}
$$

Now we define

$$
M_\mu(t)=\sup_{0\leq s\leq t}(1+s)^{2\mu+2}E(\partial_t^{\mu}v(s)).
$$

Integrating (63) over $[0,t]$ and using (36) for $i=\nu$, we obtain

$$
\begin{aligned}
&(1+t)^{2\mu+2}E(\partial_t^{\mu}v(t))+\int_0^t(1+s)^{2\mu+2}\langle\partial_t^{\mu+1}v(s),B_\lambda\partial_t^{\mu+1}v(s)\rangle ds \qquad (64)\\
\leq\ & E(\partial_t^{\mu}v(0))+C\int_0^t(1+s)^{2\mu+1}E_{L-\mu}(\partial_t^{\mu}v(s))ds\\
&+C\int_0^t(1+s)^{2\mu+2}E^{\frac{1}{2}}(\partial_t^{\mu}v(s))\sum_{\nu=0}^{\mu}E^{\frac{1}{2}}_{L-\nu}(\partial_t^{\nu}v(s))E^{\frac{1}{2}}_{L-(\mu-\nu)}(\partial_t^{\mu-\nu}v(s))ds\\
\leq\ & E(\partial_t^{\mu}v(0))+C\int_0^t(1+s)^{2\mu+1}E_{L-\mu}(\partial_t^{\mu}v(s))ds+\frac{C}{2}(M_\mu(t))^{\frac{1}{2}}\\
&\times\sum_{\nu=0}^{\mu}\left\{\int_0^t(1+s)^{2\nu+1}E_{L-\nu}(\partial_t^{\nu}v(s))ds\right.\\
&\qquad\left.+\int_0^t(1+s)^{2(\mu-\nu)+1}E_{L-(\mu-\nu)}(\partial_t^{\mu-\nu}v(s))ds\right\}\\
\leq\ & E(\partial_t^{\mu}v(0))+CE_0+2(\mu+1)CE_0(M_\mu(t))^{\frac{1}{2}}.
\end{aligned}
$$

From (64), we obtain

$$
\begin{aligned}
&(1+t)^{2\mu+2}E(\partial_t^{\mu}v(t))+\int_0^t(1+s)^{2\mu+2}\langle\partial_t^{\mu+1}v(s),B_\lambda\partial_t^{\mu+1}v(s)\rangle ds \qquad (65)\\
&\leq E_0+E_0(M_\mu(t))^{\frac{1}{2}}
\end{aligned}
$$

Thus we have

$$M_\mu(t) \le E_0 + E_0(M_\mu(t))^{\frac{1}{2}}.$$

It means that $M_\mu(t)$ is bounded in $[0, \infty)$. Thus it holds from (65) that

$$(1+t)^{2\mu+2}E(\partial_t^\mu v(t)) + \int_0^t (1+s)^{2\mu+2}\langle \partial_t^{\mu+1}v(s), B_\lambda \partial_t^{\mu+1}v(s)\rangle ds \le E_0.$$

This means (37) for $i = \mu$. This completes the proof of theorem 4.1. $\square$

5 Proof of lemma 3.1

We prove lemma 3.1. First we prepare the estimates of the nonlinear term.

Lemma 5.1. *(Estimates of the nonlinear term) Let $L_0 \le \bar{L} \le L$, $0 \le \mu \le \bar{L} - L_0$. Then there exists a constant $C > 0$ such that for any $T, \delta > 0, 0 < \lambda \le 1$ and the local solution $v \in X_{\delta,T}$ to $(DW)_\lambda$ satisfy*

$$\sum_{|a| \le \bar{L}-\mu-1} \tilde{N}[\partial_t^\mu \nabla^a v, \partial_t^\mu \nabla^a v, \partial_t v](t) \le CD_{\bar{L},\mu}(v(t)), \tag{66}$$

$$\sum_{|a|\le \bar{L}-\mu-1} \sum_{\nu=0}^{\mu-1} \binom{\mu}{\nu} \langle \partial_t^{\mu+1}\nabla^a v(t), N[\partial_t^\nu \nabla^a v, \partial_t^{\mu-\nu} v](t)\rangle \le CD_{\bar{L},\mu}(v(t)), \tag{67}$$

$$\sum_{|a|\le \bar{L}-\mu-1} \sum_{\substack{b+c=a \\ b,c\neq 0}} \langle \partial_t^{\mu+1}\nabla^a v(t), \partial_t^\mu N[\nabla^b v, \nabla^c v](t)\rangle \le CD_{\bar{L},\mu}(v(t)), \tag{68}$$

$$\sum_{|a|\le \bar{L}-\mu-1} \sum_{b+c=a} \binom{a}{b} \langle \partial_t^\mu \nabla^a v(t), \partial_t^\mu N[\nabla^b v, \nabla^c v](t)\rangle \le CD_{\bar{L},\mu}(v(t)) \tag{69}$$

and

$$\sum_{|a|\le \bar{L}-\mu-1} \langle \partial_t^\mu \nabla^a N[v,v](t), [h; \nabla \partial_t^\mu \nabla^a v(t)]\rangle \le \frac{C}{\lambda} D_{\bar{L},\mu}(v(t)), \tag{70}$$

where $\tilde{N}$ and $D_{\bar{L},\mu}$ are defined by (11) and (19)

Proof. First, we prove (66). Using lemma 2.1, we have

$$\begin{aligned} &\sum_{|a|\le \bar{L}-\mu-1} \tilde{N}[\partial_t^\mu \nabla^a v, \partial_t^\mu \nabla^a v, \partial_t v](t) \\ \le\ & C \sum_{|a|\le \bar{L}-\mu-1} \|\nabla \partial_t^\mu \nabla^a v(t)\|_2 \|\nabla \partial_t^\mu \nabla^a v(t)\|_2 \|\nabla \partial_t v(t)\|_\infty \end{aligned}$$

$$
\begin{aligned}
&\leq C \sum_{|a|\leq \bar{L}-\mu-1} \|\nabla\partial_t^\mu \nabla^a v(t)\|_2 \|\nabla\partial_t^\mu \nabla^a v(t)\|_2 \|\partial_t v(t)\|_{H^{[\frac{d}{2}]+2}} \\
&\leq C E^{\frac{1}{2}}_{\bar{L}-\mu}(\partial_t^\mu v(t)) E^{\frac{1}{2}}_{\bar{L}-\mu}(\partial_t^\mu v(t)) E^{\frac{1}{2}}_{L_0}(v(t)) \\
&\leq C E^{\frac{1}{2}}_{\bar{L}-\mu}(\partial_t^\mu v(t)) \sum_{\nu=0}^{\mu} E^{\frac{1}{2}}_{\bar{L}-\nu}(\partial_t^\nu v(t)) E^{\frac{1}{2}}_{L_0}(\partial_t^{\mu-\nu} v(t)) = C D_{\bar{L},\mu}(v(t)).
\end{aligned}
$$

Next, we prove (67). We remark that if $|a| \leq \bar{L} - \mu - 1$ and $0 \leq \nu \leq \mu - 1$ then $|a| + 1 \leq \bar{L} - \nu - 1$. Hence it follows that

$$
\begin{aligned}
&\sum_{|a|\leq \bar{L}-\mu-1} \sum_{\nu=0}^{\mu-1} \binom{\mu}{\nu} \langle \partial_t^{\mu+1}\nabla^a v(t), N[\partial_t^\nu \nabla^a v, \partial_t^{\mu-\nu} v](t)\rangle \\
\leq\ & C \sum_{|a|\leq \bar{L}-\mu-1} \|\partial_t^{\mu+1}\nabla^a v(t)\|_2 \sum_{\nu=0}^{\mu-1} \sum_{j,k,l,m,n=1}^{d} \|\partial_l(\partial_m \partial_t^\nu \nabla^a v^j(t) \partial_n \partial_t^{\mu-\nu} v^k(t))\|_2 \\
\leq\ & C \sum_{|a|\leq \bar{L}-\mu-1} \|\partial_t^{\mu+1}\nabla^a v(t)\|_2 \\
& \times \sum_{\nu=0}^{\mu-1} \sum_{j,k,l,m,n=1}^{d} \big\{ \|\partial_l \partial_m \partial_t^\nu \nabla^a v^j(t)\|_2 \|\partial_n \partial_t^{\mu-\nu} v^k(t)\|_\infty \\
& \qquad\qquad + \|\partial_m \partial_t^\nu \nabla^a v^j(t)\|_2 \|\partial_l \partial_n \partial_t^{\mu-\nu} v^k(t)\|_\infty \big\} \\
\leq\ & C E^{\frac{1}{2}}_{\bar{L}-\mu}(\partial_t^\mu v(t)) \sum_{\nu=0}^{\mu-1} \Big\{ E^{\frac{1}{2}}_{\bar{L}-\nu}(\partial_t^\nu v(t)) E^{\frac{1}{2}}_{L_0}(\partial_t^{\mu-\nu} v(t)) \\
& \qquad\qquad + E^{\frac{1}{2}}_{\bar{L}-\nu}(\partial_t^\nu v(t)) E^{\frac{1}{2}}_{L_0}(\partial_t^{\mu-\nu} v(t)) \Big\} \\
\leq\ & C E^{\frac{1}{2}}_{\bar{L}-\mu}(\partial_t^\mu v(t)) \sum_{\nu=0}^{\mu} E^{\frac{1}{2}}_{\bar{L}-\nu}(\partial_t^\nu v(t)) E^{\frac{1}{2}}_{L_0}(\partial_t^{\mu-\nu} v(t)) = C D_{\bar{L},\mu}(v(t)).
\end{aligned}
$$

Next, we prove (68). For any $|a| \leq \bar{L} - \mu - 1$, $b + c = a$ and $b, c \neq 0$ it hold that

$$
\|\partial_t^\mu N[\nabla^b v, \nabla^c v](t)\|_2 \leq C \sum_{\nu=0}^{\mu} E^{\frac{1}{2}}_{L_0}(\partial_t^{\mu-\nu} v(t)) E^{\frac{1}{2}}_{\bar{L}-\nu}(\partial_t^\nu v(t)). \tag{71}
$$

Because if $|a| \leq \bar{L} - \mu - 1$, $b + c = a$ and $b, c \neq 0$ then we can decompose c of the form: $c = c' + c''$, $|c'| = |c| - 1$, $|c''| = 1$. Then using lemma 2.1 and lemma 2.2 we obtain

$$
\|\partial_t^\mu(\partial_l \partial_m \nabla^b v^i(t) \partial_n \nabla^c v^k(t))\|_2
$$

$$
\begin{aligned}
&\le C\sum_{\nu=0}^{\mu}\|\partial_t^{\nu}\partial_l\partial_m\nabla^b v^j(t)\partial_t^{\mu-\nu}\partial_n\nabla^{c''}\nabla^{c'}v^k(t)\|_2\\
&\le C\sum_{\nu=0}^{\mu}\Big\{\|\nabla^2\partial_t^{\nu}v(t)\|_\infty\|\nabla^2\nabla^{b+c'}\partial_t^{\mu-\nu}v(t)\|_2\\
&\qquad\qquad +\|\nabla^2\nabla^{b+c'}\partial_t^{\nu}v(t)\|_2\|\nabla^2\partial_t^{\mu-\nu}v(t)\|_\infty\Big\}\\
&\le C\sum_{\nu=0}^{\mu}\Big\{\|\nabla^2\partial_t^{\nu}v(t)\|_{H^{[\frac{d}{2}]+1}}\|\nabla^2\nabla^{b+c'}\partial_t^{\mu-\nu}v(t)\|_2\\
&\qquad\qquad +\|\nabla^2\nabla^{b+c'}\partial_t^{\nu}v(t)\|_2\|\nabla^2\partial_t^{\mu-\nu}v(t)\|_{H^{[\frac{d}{2}]+1}}\Big\}\\
&\le C\sum_{\nu=0}^{\mu}\|\nabla^2\partial_t^{\mu-\nu}v(t)\|_{H^{[\frac{d}{2}]+1}}\|\nabla^2\nabla^{b+c'}\partial_t^{\nu}v(t)\|_2\\
&\le C\sum_{\nu=0}^{\mu}E_{L_0}^{\frac{1}{2}}(\partial_t^{\mu-\nu}v(t))E_{\bar{L}-\nu}^{\frac{1}{2}}(\partial_t^{\nu}v(t)).
\end{aligned}
$$

Similarly we obtain

$$
\|\partial_t^{\mu}(\partial_m\nabla^b v^i\partial_l\partial_n\nabla^c v^k)\|_2\le C\sum_{\nu=0}^{\mu}E_{L_0}^{\frac{1}{2}}(\partial_t^{\mu-\nu}v(t))E_{\bar{L}-\nu}^{\frac{1}{2}}(\partial_t^{\nu}v(t)).
$$

Then we get (71). It follows from (71) that

$$
\begin{aligned}
&\sum_{|a|\le\bar{L}-\mu-1}\sum_{\substack{b+c=a\\ b,c\ne0}}\langle\partial_t^{\mu+1}\nabla^a v(t),\partial_t^{\mu}N[\nabla^b v,\nabla^c v](t)\rangle\\
\le&\sum_{|a|\le\bar{L}-\mu-1}\sum_{\substack{b+c=a\\ b,c\ne0}}\|\partial_t\partial_t^{\mu}\nabla^a v(t)\|_2\|\partial_t^{\mu}N[\nabla^b v,\nabla^c v](t)\|_2\\
\le&\ CE_{\bar{L}-\mu}^{\frac{1}{2}}(\partial_t^{\mu}v(t))\sum_{\nu=0}^{\mu}E_{L_0}^{\frac{1}{2}}(\partial_t^{\mu-\nu}v(t))E_{\bar{L}-\nu}^{\frac{1}{2}}(\partial_t^{\nu}v(t))=CD_{\bar{L},\mu}(v(t)).
\end{aligned}
$$

Next, we prove (69). Using lemma 2.1 and lemma 2.2 we have

$$
\begin{aligned}
&\sum_{|a|\le\bar{L}-\mu-1}\sum_{b+c=a}\binom{a}{b}\langle\partial_t^{\mu}\nabla^a v(t),\partial_t^{\mu}N[\nabla^b v,\nabla^c v](t)\rangle\\
\le&\ C\sum_{|a|\le\bar{L}-\mu-1}\sum_{b+c=a}\left|\sum_{i,j,k,l,m,n=1}^{d}\int_{\mathbb{R}^d}\partial_t^{\mu}\nabla^a v^i\partial_t^{\mu}\partial_l(\partial_m\nabla^b v^j\partial_n\nabla^c v^k)dx\right|\\
=&\ C\sum_{|a|\le\bar{L}-\mu-1}\sum_{b+c=a}\left|\sum_{i,j,k,l,m,n=1}^{d}\int_{\mathbb{R}^d}\partial_l\partial_t^{\mu}\nabla^a v^i\partial_t^{\mu}(\partial_m\nabla^b v^j\partial_n\nabla^c v^k)dx\right|
\end{aligned}
$$

$$
\begin{aligned}
&\leq C \sum_{|a|\leq \bar{L}-\mu-1} \|\nabla\partial_t^\mu\nabla^a v(t)\|_2 \\
&\quad \times \sum_{\nu=0}^{\mu}\left\{\|\nabla\partial_t^{\mu-\nu}v(t)\|_\infty\|\nabla\nabla^a\partial_t^\nu v(t)\|_2+\|\nabla\nabla^a\partial_t^{\mu-\nu}v(t)\|_2\|\nabla\partial_t^\nu v(t)\|_\infty\right\} \\
&\leq C \sum_{|a|\leq \bar{L}-\mu-1} \|\nabla\partial_t^\mu\nabla^a v(t)\|_2 \sum_{\nu=0}^{\mu}\|\nabla\partial_t^{\mu-\nu}v(t)\|_{H^{[\frac{d}{2}]+1}}\|\nabla\nabla^a\partial_t^\nu v(t)\|_2 \\
&\leq CE^{\frac{1}{2}}_{\bar{L}-\mu}(\partial_t^\mu v(t))\sum_{\nu=0}^{\mu}E^{\frac{1}{2}}_{L_0}(\partial_t^{\mu-\nu}v(t))E^{\frac{1}{2}}_{\bar{L}-\mu}(\partial_t^\nu v)\leq CD_{\bar{L},\mu}(v(t)).
\end{aligned}
$$

Finally, we prove (70).

$$
\begin{aligned}
&\sum_{|a|\leq \bar{L}-\mu-1}\langle\partial_t^\mu\nabla^a N[v,v],[h;\nabla\partial_t^\mu\nabla^a v]\rangle \\
= &\sum_{|a|\leq \bar{L}-\mu-1}\sum_{b+c=a}\binom{a}{b}\langle\partial_t^\mu N[\nabla^b v,\nabla^c v],[h;\nabla\partial_t^\mu\nabla^a v]\rangle \\
= &\sum_{|a|\leq \bar{L}-\mu-1}\sum_{\substack{b+c=a\\ b,c\neq 0}}\binom{a}{b}\langle\partial_t^\mu N[\nabla^b v,\nabla^c v],[h;\nabla\partial_t^\mu\nabla^a v]\rangle \\
&+2\sum_{|a|\leq \bar{L}-\mu-1}\sum_{i,j,k,l,m,n=1}^{d}N^{ijk}_{lmn}\sum_{p=1}^{d}\int_{\mathbb{R}^d}\partial_t^\mu\partial_l(\partial_m\nabla^a v^j\partial_n v^k)h^p\partial_p\partial_t^\mu\nabla^a v^i dx \\
= &\sum_{|a|\leq \bar{L}-\mu-1}\sum_{\substack{b+c=a\\ b,c\neq 0}}\binom{a}{b}\langle\partial_t^\mu N[\nabla^b v,\nabla^c v],[h;\nabla\partial_t^\mu\nabla^a v]\rangle \\
&+2\sum_{|a|\leq \bar{L}-\mu-1}\sum_{i,j,k,l,m,n=1}^{d}N^{ijk}_{lmn} \\
&\qquad\times\sum_{p=1}^{d}\sum_{\nu=0}^{\mu-1}\binom{\mu}{\nu}\int_{\mathbb{R}^d}\partial_l(\partial_m\partial_t^\nu\nabla^a v^j\partial_n\partial_t^{\mu-\nu}v^k)h^p\partial_p\partial_t^\mu\nabla^a v^i dx \\
&+2\sum_{|a|\leq \bar{L}-\mu-1}\sum_{i,j,k,l,m,n=1}^{d}N^{ijk}_{lmn}\sum_{p=1}^{d}\int_{\mathbb{R}^d}\partial_l(\partial_m\partial_t^\mu\nabla^a v^j\partial_n v^k)h^p\partial_p\partial_t^\mu\nabla^a v^i dx \\
= &\sum_{|a|\leq \bar{L}-\mu-1}\sum_{\substack{b+c=a\\ b,c\neq 0}}\binom{a}{b}\langle\partial_t^\mu N[\nabla^b v,\nabla^c v],[h;\nabla\partial_t^\mu\nabla^a v]\rangle \\
&+2\sum_{|a|\leq \bar{L}-\mu-1}\sum_{i,j,k,l,m,n=1}^{d}N^{ijk}_{lmn}
\end{aligned}
$$

$$
\begin{aligned}
& \qquad \times \sum_{p=1}^{d} \sum_{\nu=0}^{\mu-1} \binom{\mu}{\nu} \int_{\mathbb{R}^d} \partial_l(\partial_m \partial_t^\nu \nabla^a v^j \partial_n \partial_t^{\mu-\nu} v^k) h^p \partial_p \partial_t^\mu \nabla^a v^i dx \\
& -2 \sum_{|a| \le \bar{L}-\mu-1} \sum_{i,j,k,l,m,n=1}^{d} N_{lmn}^{ijk} \sum_{p=1}^{d} \int_{\mathbb{R}^d} \partial_m \partial_t^\mu \nabla^a v^j \partial_n v^k \partial_l h^p \partial_p \partial_t^\mu \nabla^a v^i dx \\
& -2 \sum_{|a| \le \bar{L}-\mu-1} \sum_{i,j,k,l,m,n=1}^{d} N_{lmn}^{ijk} \sum_{p=1}^{d} \int_{\mathbb{R}^d} \partial_m \partial_t^\mu \nabla^a v^j \partial_n v^k h^p \partial_l \partial_p \partial_t^\mu \nabla^a v^i dx \\
= \; & \sum_{|a| \le \bar{L}-\mu-1} \sum_{\substack{b+c=a \\ b,c \neq 0}} \binom{a}{b} \langle \partial_t^\mu N[\nabla^b v, \nabla^c v], [h; \nabla \partial_t^\mu \nabla^a v] \rangle \\
& +2 \sum_{|a| \le \bar{L}-\mu-1} \sum_{i,j,k,l,m,n=1}^{d} N_{lmn}^{ijk} \\
& \qquad \times \sum_{p=1}^{d} \sum_{\nu=0}^{\mu-1} \binom{\mu}{\nu} \int_{\mathbb{R}^d} \partial_l(\partial_m \partial_t^\nu \nabla^a v^j \partial_n \partial_t^{\mu-\nu} v^k) h^p \partial_p \partial_t^\mu \nabla^a v^i dx \\
& -2 \sum_{|a| \le \bar{L}-\mu-1} \sum_{i,j,k,l,m,n=1}^{d} N_{lmn}^{ijk} \sum_{p=1}^{d} \int_{\mathbb{R}^d} \partial_m \partial_t^\mu \nabla^a v^j \partial_n v^k \partial_l h^p \partial_p \partial_t^\mu \nabla^a v^i dx \\
& - \sum_{|a| \le \bar{L}-\mu-1} \sum_{i,j,k,l,m,n=1}^{d} N_{lmn}^{ijk} \sum_{p=1}^{d} \int_{\mathbb{R}^d} \partial_p(\partial_m \partial_t^\mu \nabla^a v^j \partial_l \partial_t^\mu \nabla^a v^i) \partial_n v^k h^p dx \\
= \; & J_1 + J_2 + J_3 + J_4,
\end{aligned}
$$

where we use **(N1)**. For J_2 and J_3, it follows from (23) that

$$
\begin{aligned}
& |J_2| \\
\le \; & C \sum_{|a| \le \bar{L}-\mu-1} \sum_{i,j,k,l,m,n=1}^{d} \sum_{p=1}^{d} \sum_{\nu=0}^{\mu-1} \| \partial_l(\partial_m \partial_t^\nu \nabla^a v^j \partial_n \partial_t^{\mu-\nu} v^k) h^p \partial_p \partial_t^\mu \nabla^a v^i \|_1 \\
\le \; & C \sum_{|a| \le \bar{L}-\mu-1} \sum_{p=1}^{d} \sum_{\nu=0}^{\mu-1} \| \nabla^2 \partial_t^\nu \nabla^a v \|_2 \| \nabla \partial_t^{\mu-\nu} v \|_\infty \| h^p \|_\infty \| \partial_p \partial_t^\mu \nabla^a v \|_2 \\
& +C \sum_{|a| \le \bar{L}-\mu-1} \sum_{p=1}^{d} \sum_{\nu=0}^{\mu-1} \| \nabla \partial_t^\nu \nabla^a v \|_2 \| \nabla^2 \partial_t^{\mu-\nu} v \|_\infty \| h^p \|_\infty \| \partial_p \partial_t^\mu \nabla^a v \|_2 \\
\le \; & C \|h\|_\infty \sum_{\nu=0}^{\mu-1} E_{\bar{L}-\nu}^{\frac{1}{2}}(\partial_t^\nu v(t)) E_{\bar{L}_0}^{\frac{1}{2}}(\partial_t^{\mu-\nu} v(t)) E_{\bar{L}-\mu}^{\frac{1}{2}}(\partial_t^\mu v(t)) \le \frac{C}{\lambda} D_{\bar{L},\mu}(v(t))
\end{aligned}
$$

and

$$
\begin{aligned}
|J_3| &\le C \sum_{|a|\le \bar{L}-\mu-1} \sum_{i,j,k,l,m,n=1}^{d} \sum_{p=1}^{d} \|\partial_m \partial_t^\mu \nabla^a v^j \partial_n v^k \partial_l h^p \partial_p \partial_t^\mu \nabla^a v^i\|_1 \\
&\le C \sum_{|a|\le \bar{L}-\mu-1} \|\nabla \partial_t^\mu \nabla^a v\|_2 \|\nabla v\|_\infty \|\nabla h\|_\infty \|\nabla \partial_t^\mu \nabla^a v\|_2 \\
&\le C E^{\frac{1}{2}}_{\bar{L}-\mu}(\partial_t^\mu v(t)) E^{\frac{1}{2}}_{\bar{L}-\mu}(\partial_t^\mu v(t)) E^{\frac{1}{2}}_{L_0}(v(t)) \le C D_\mu(v(t)) \le \frac{C}{\lambda} D_{\bar{L},\mu}(v(t)).
\end{aligned}
$$

From (23) and (71), we get

$$
\begin{aligned}
|J_1| &\le C \sum_{|a|\le \bar{L}-\mu-1} \sum_{\substack{b+c=a \\ b,c\neq 0}} |\langle \partial_t^\mu N[\nabla^b v, \nabla^c v], [h; \nabla \partial_t^\mu \nabla^a v]\rangle| \\
&\le C \sum_{|a|\le \bar{L}-\mu-1} \sum_{\substack{b+c=a \\ b,c\neq 0}} \|\langle \partial_t^\mu N[\nabla^b v, \nabla^c v]\|_2 \|h\|_\infty \|\nabla \partial_t^\mu \nabla^a v\|_2 \\
&\le C\|h\|_\infty E^{\frac{1}{2}}_{\bar{L}-\mu}(\partial_t^\mu v(t)) \sum_{\nu=0}^{\mu} E^{\frac{1}{2}}_{L_0}(\partial_t^{\mu-\nu} v(t)) E^{\frac{1}{2}}_{\bar{L}-\nu}(\partial_t^\nu v(t)) \le \frac{C}{\lambda} D_{\bar{L},\mu}(v(t)).
\end{aligned}
$$

From (71), we also have

$$
\begin{aligned}
|J_4| &= \left| \sum_{|a|\le \bar{L}-\mu-1} \sum_{i,j,k,l,m,n=1}^{d} N^{ijk}_{lmn} \sum_{p=1}^{d} \int_{\mathbb{R}^d} \partial_p (\partial_m \partial_t^\mu \nabla^a v^j \partial_l \partial_t^\mu \nabla^a v^i) \partial_n v^k h^p dx \right| \\
&= \left| \sum_{|a|\le \bar{L}-\mu-1} \sum_{i,j,k,l,m,n=1}^{d} N^{ijk}_{lmn} \sum_{p=1}^{d} \int_{\mathbb{R}^d} \partial_m \partial_t^\mu \nabla^a v^j \partial_l \partial_t^\mu \nabla^a v^i \partial_p (\partial_n v^k h^p) dx \right| \\
&\le C \sum_{|a|\le \bar{L}-\mu-1} \sum_{i,j,k,l,m,n=1}^{d} \sum_{p=1}^{d} \int_{\mathbb{R}^d} |\partial_m \partial_t^\mu \nabla^a v^j \partial_l \partial_t^\mu \nabla^a v^i \partial_p \partial_n v^k h^p| dx \\
&+ C \sum_{|a|\le \bar{L}-\mu-1} \sum_{i,j,k,l,m,n=1}^{d} \sum_{p=1}^{d} \int_{\mathbb{R}^d} |\partial_m \partial_t^\mu \nabla^a v^j \partial_l \partial_t^\mu \nabla^a v^i \partial_n v^k \partial_p h^p| dx \\
&\le C \sum_{|a|\le \bar{L}-\mu-1} \|\nabla \partial_t^\mu \nabla^a v\|_2 \|\nabla \partial_t^\mu \nabla^a v\|_2 \left(\|\nabla^2 v\|_\infty \|h\|_\infty + \|\nabla v\|_\infty \|\nabla h\|_\infty \right) \\
&\le C \left(1 + \frac{1}{\lambda}\right) E^{\frac{1}{2}}_{\bar{L}-\mu}(\partial_t^\mu v(t)) E^{\frac{1}{2}}_{\bar{L}-\mu}(\partial_t^\mu v(t)) E^{\frac{1}{2}}_{L_0}(v(t)) \le \frac{C}{\lambda} D_{\bar{L},\mu}(v(t)).
\end{aligned}
$$

Hence we obtain (70), which completes the proof of lemma 5.1. □

Because of dissipation effect $(\mathbf{B2})_\lambda$, we are expected to decrease the energy in $|x| \gg 1$. Next lemma corresponds to this phenomenon.

Lemma 5.2. *Let $L_0 \le \bar{L} \le L$, $0 \le \mu \le \bar{L} - L_0$. Then there exists a constant $C > 0$ such that for any $T, \delta > 0$, $0 < \lambda \le 1$ and the local solution $v \in X_{\delta,T}$ to $(DW)_\lambda$ satisfy*

$$\begin{aligned} &\frac{d}{dt}\tilde{E}_{\bar{L},\mu}(v(t)) + \sum_{|a| \le \bar{L}-\mu-1} \langle \partial_t^{\mu+1}\nabla^a v(t), B_\lambda \partial_t^{\mu+1}\nabla^a v(t)\rangle \\ \le\ & C\lambda^2 E_{\bar{L}-\mu}(\partial_t^\mu v(t)) + CD_{\bar{L},\mu}(v(t)) \end{aligned} \tag{72}$$

and

$$\begin{aligned} &\frac{d}{dt}\sum_{|a| \le \bar{L}-\mu-1} \left\{ \langle \partial_t^\mu \nabla^a v(t), \partial_t^{\mu+1}\nabla^a v(t)\rangle + \frac{1}{2}\langle \partial_t^\mu \nabla^a v(t), B_\lambda \partial_t^\mu \nabla^a v(t)\rangle \right\} \\ &- \sum_{|a| \le \bar{L}-\mu-1} \|\partial_t^{\mu+1}\nabla^a v(t)\|_2^2 + \sum_{|a| \le \bar{L}-\mu-1} \|\nabla \partial_t^\mu \nabla^a v(t)\|_2^2 \\ \le\ & C\lambda^2 E_{\bar{L}-\mu}(\partial_t^\mu v(t)) + CD_{\bar{L},\mu}(v(t)). \end{aligned} \tag{73}$$

Proof. Let $L_0 \le \bar{L} \le L$ and $0 \le \mu \le \bar{L} - L_0$. First we show (72). We calculate

$$\frac{d}{dt}\tilde{E}_{\bar{L},\mu}(v(t)) = \sum_{|a| \le \bar{L}-\mu-1} \frac{d}{dt}E(\partial_t^\mu \nabla^a v(t)) + \sum_{|a| \le \bar{L}-\mu-1} \frac{d}{dt}\tilde{N}[\partial_t^\mu \nabla^a v, \partial_t^\mu \nabla^a v, v](t). \tag{74}$$

For the second terms of (74), for any $|a| \le \bar{L} - \mu - 1$, we need the following equality:

$$\begin{aligned} &\langle \partial_t^{\mu+1}\nabla^a v, \partial_t^\mu N[\nabla^a v, v]\rangle \\ =\ & \langle \partial_t^{\mu+1}\nabla^a v, N[\partial_t^\mu \nabla^a v, v]\rangle + \sum_{\nu=0}^{\mu-1}\binom{\mu}{\nu}\langle \partial_t^{\mu+1}\nabla^a v, N[\partial_t^\nu \nabla^a v, \partial_t^{\mu-\nu} v]\rangle \\ =\ & \sum_{i,j,k,l,m,n=1}^d N_{lmn}^{ijk}\int_{\mathbb{R}^d} \partial_t^{\mu+1}\nabla^a v^i \partial_l(\partial_m \partial_t^\mu \nabla^a v^j \partial_n v^k)dx \\ &+ \sum_{\nu=0}^{\mu-1}\binom{\mu}{\nu}\langle \partial_t^{\mu+1}\nabla^a v, N[\partial_t^\nu \nabla^a v, \partial_t^{\mu-\nu} v]\rangle \\ =\ & -\sum_{i,j,k,l,m,n=1}^d N_{lmn}^{ijk}\int_{\mathbb{R}^d} \partial_l \partial_t^{\mu+1}\nabla^a v^i \partial_m \partial_t^\mu \nabla^a v^j \partial_n v^k dx \end{aligned} \tag{75}$$

$$
\begin{aligned}
&+\sum_{\nu=0}^{\mu-1}\binom{\mu}{\nu}\langle\partial_t^{\mu+1}\nabla^a v, N[\partial_t^\nu\nabla^a v, \partial_t^{\mu-\nu}v]\rangle \\
= &-\frac{1}{2}\frac{d}{dt}\sum_{i,j,k,l,m,n=1}^{d} N_{lmn}^{ijk}\int_{\mathbb{R}^d}\partial_l\partial_t^\mu\nabla^a v^i\partial_m\partial_t^\mu\nabla^a v^j\partial_n v^k dx \\
&+\frac{1}{2}\sum_{i,j,k,l,m,n=1}^{d} N_{lmn}^{ijk}\int_{\mathbb{R}^d}\partial_l\partial_t^\mu\nabla^a v^i\partial_m\partial_t^\mu\nabla^a v^j\partial_n\partial_t v^k dx \\
&+\sum_{\nu=0}^{\mu-1}\binom{\mu}{\nu}\langle\partial_t^{\mu+1}\nabla^a v, N[\partial_t^\nu\nabla^a v, \partial_t^{\mu-\nu}v]\rangle \\
= &-\frac{1}{2}\frac{d}{dt}\tilde{N}[\partial_t^\mu\nabla^a v, \partial_t^\mu\nabla^a v, v] + \frac{1}{2}\tilde{N}[\partial_t^\mu\nabla^a v, \partial_t^\mu\nabla^a v, \partial_t v] \\
&+\sum_{\nu=0}^{\mu-1}\binom{\mu}{\nu}\langle\partial_t^{\mu+1}\nabla^a v, N[\partial_t^\nu\nabla^a v, \partial_t^{\mu-\nu}v]\rangle,
\end{aligned}
$$

where we use **(N1)**. To handle the first terms of (74), we apply $\partial_t^\mu\nabla^a$ to $(\mathrm{DW})_\lambda$. We get

$$
\partial_t^{\mu+2}\nabla^a v - \triangle\partial_t^\mu\nabla^a v + \nabla^a(B_\lambda\partial_t^{\mu+1}v) = \sum_{b+c=a}\binom{a}{b}\partial_t^\mu N[\nabla^b v, \nabla^c v], \tag{76}
$$

which yields that

$$
\begin{aligned}
&\sum_{|a|\le\bar{L}-\mu-1}\frac{d}{dt}E(\partial_t^\mu\nabla^a v) \qquad (77)\\
= &-\sum_{|a|\le\bar{L}-\mu-1}\langle\partial_t^{\mu+1}\nabla^a v, \nabla^a(B_\lambda\partial_t^{\mu+1}v)\rangle \\
&+\sum_{|a|\le\bar{L}-\mu-1}\sum_{b+c=a}\binom{a}{b}\langle\partial_t^{\mu+1}\nabla^a v, \partial_t^\mu N[\nabla^b v, \nabla^c v]\rangle.
\end{aligned}
$$

Combining (74), (75), (77) and lemma 5.1, we get

$$
\begin{aligned}
&\frac{d}{dt}\tilde{E}_{\bar{L},\mu}(\partial_t^\mu v) \qquad (78)\\
= &\sum_{|a|\le\bar{L}-\mu-1}\frac{d}{dt}E(\partial_t^\mu\nabla^a v) + \sum_{|a|\le\bar{L}-\mu-1}\frac{d}{dt}\tilde{N}[\partial_t^\mu\nabla^a v, \partial_t^\mu\nabla^a v, v] \\
= &-\sum_{|a|\le\bar{L}-\mu-1}\langle\partial_t^{\mu+1}\nabla^a v, \nabla^a(B_\lambda\partial_t^{\mu+1}v)\rangle
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{|a|\le\bar{L}-\mu-1}\sum_{b+c=a}\binom{a}{b}\langle\partial_t^{\mu+1}\nabla^a v,\partial_t^{\mu}N[\nabla^b v,\nabla^c v]\rangle\\
&-2\sum_{|a|\le\bar{L}-\mu-1}\langle\partial_t^{\mu+1}\nabla^a v,\partial_t^{\mu}N[\nabla^a v,v]\rangle+\sum_{|a|\le\bar{L}-\mu-1}\tilde{N}[\partial_t^{\mu}\nabla^a v,\partial_t^{\mu}\nabla^a v,\partial_t v]\\
&+2\sum_{|a|\le\bar{L}-\mu-1}\sum_{\nu=0}^{\mu-1}\binom{\mu}{\nu}\langle\partial_t^{\mu+1}\nabla^a v,N[\partial_t^{\nu}\nabla^a v,\partial_t^{\mu-\nu}v]\rangle\\
=\;&-\sum_{|a|\le\bar{L}-\mu-1}\langle\partial_t^{\mu+1}\nabla^a v,\nabla^a(B_\lambda\partial_t^{\mu+1}v)\rangle\\
&+\sum_{|a|\le\bar{L}-\mu-1}\sum_{\substack{b+c=a\\ b,c\ne0}}\binom{a}{b}\langle\partial_t^{\mu+1}\nabla^a v,\partial_t^{\mu}N[\nabla^b v,\nabla^c v]\rangle\\
&+\sum_{|a|\le\bar{L}-\mu-1}\tilde{N}[\partial_t^{\mu}\nabla^a v,\partial_t^{\mu}\nabla^a v,\partial_t v]\\
&+2\sum_{|a|\le\bar{L}-\mu-1}\sum_{\nu=0}^{\mu-1}\binom{\mu}{\nu}\langle\partial_t^{\mu+1}\nabla^a v,N[\partial_t^{\nu}\nabla^a v,\partial_t^{\mu-\nu}v]\rangle\\
\le\;&-\sum_{|a|\le\bar{L}-\mu-1}\langle\partial_t^{\mu+1}\nabla^a v,\nabla^a(B_\lambda\partial_t^{\mu+1}v)\rangle+CD_{\bar{L},\mu}(v).
\end{aligned}
$$

Since for $\lambda\le1$, we have

$$
\begin{aligned}
&-\sum_{|a|\le\bar{L}-\mu-1}\langle\partial_t^{\mu+1}\nabla^a v,\nabla^a(B_\lambda\partial_t^{\mu+1}v)\rangle\\
=\;&-\sum_{|a|\le\bar{L}-\mu-1}\langle\partial_t^{\mu+1}\nabla^a v,B_\lambda\partial_t^{\mu+1}\nabla^a v\rangle\\
&-\sum_{|a|\le\bar{L}-\mu-1}\sum_{\substack{b\le a\\ b\ne0}}\binom{a}{b}\langle\partial_t^{\mu+1}\nabla^a v,\nabla^b B_\lambda(x)\partial_t^{\mu+1}\nabla^{a-b}v\rangle\\
\le\;&-\sum_{|a|\le\bar{L}-\mu-1}\langle\partial_t^{\mu+1}\nabla^a v,B_\lambda\partial_t^{\mu+1}\nabla^a v\rangle\\
&+C\sum_{|a|\le\bar{L}-\mu-1}\sum_{\substack{b\le a\\ b\ne0}}\|\nabla^b B_\lambda\|_\infty\|\partial_t^{\mu+1}\nabla^a v\|_2\|\partial_t^{\mu+1}\nabla^{a-b}v\|_2\\
\le\;&-\sum_{|a|\le\bar{L}-\mu-1}\langle\partial_t^{\mu+1}\nabla^a v,B_\lambda\partial_t^{\mu+1}\nabla^a v\rangle+C\lambda^2E_{\bar{L}-\mu}(\partial_t^{\mu}v).
\end{aligned}
$$

Thus we obtain (72).

Next we prove (73). Using (76) and lemma 5.1 (69), we have

$$\frac{d}{dt}\left\{\sum_{|a|\le \bar{L}-\mu-1}\langle \partial_t^\mu \nabla^a v, \partial_t^{\mu+1}\nabla^a v\rangle + \frac{1}{2}\sum_{|a|\le \bar{L}-\mu-1}\langle \partial_t^\mu \nabla^a v, B_\lambda \partial_t^\mu \nabla^a v\rangle\right\} \tag{79}$$

$$\begin{aligned}
&= \sum_{|a|\le \bar{L}-\mu-1}\left\{\|\partial_t^{\mu+1}\nabla^a v\|_2^2 + \langle \partial_t^\mu \nabla^a v, \partial_t^{\mu+2}\nabla^a v\rangle + \langle \partial_t^\mu \nabla^a v, B_\lambda \partial_t^{\mu+1}\nabla^a v\rangle\right\}\\
&\le \sum_{|a|\le \bar{L}-\mu-1}\|\partial_t^{\mu+1}\nabla^a v\|_2^2 - \sum_{|a|\le \bar{L}-\mu-1}\|\nabla \partial_t^\mu \nabla^a v\|_2^2\\
&\quad -\sum_{\substack{b\le a\\ b\neq 0}}\binom{a}{b}\langle \partial_t^\mu \nabla^a v, \nabla^b B_\lambda \partial_t^{\mu+1}\nabla^{a-b}v\rangle + CD_{\bar{L},\mu}(v).
\end{aligned}$$

For $\lambda \le 1$, we have

$$\begin{aligned}
&\sum_{|a|\le \bar{L}-\mu-1}\sum_{\substack{b\le a\\ b\neq 0}}\binom{a}{b}|\langle \partial_t^\mu \nabla^a v, \nabla^b B_\lambda \partial_t^{\mu+1}\nabla^{a-b}v\rangle|\\
&= \sum_{1\le|a|\le \bar{L}-\mu-1}\sum_{\substack{b\le a\\ b\neq 0}}\binom{a}{b}|\langle \partial_t^\mu \nabla^a v, \nabla^b B_\lambda \partial_t^{\mu+1}\nabla^{a-b}v\rangle|\\
&\le \sum_{1\le|a|\le \bar{L}-\mu-1}\sum_{\substack{b\le a\\ b\neq 0}}\binom{a}{b}\|\nabla^b B_\lambda\|_\infty \|\partial_t^\mu \nabla^a v\|_2 \|\partial_t^{\mu+1}\nabla^{a-b}v\|_2 \le \lambda^2 C E_{\bar{L}-\mu}(\partial_t^\mu v),
\end{aligned}$$

which yields (73) from (79). This completes the proof of lemma 5.2. □

We can't expect the effect of dissipation in $|x| \le \frac{R}{\lambda}$ since $B_\lambda(x)$ may not strictly positive. So we use the local energy decay property in $(\mathrm{DW})_\lambda$. We lead the estimate which corresponding this property by using the argument of Nakao[11] and Ikehata[5].

Lemma 5.3. *Let $L_0 \le \bar{L} \le L$, $0 \le \mu \le \bar{L} - L_0$. Then there exists a constant $C > 0$ such that for any $K, T, \delta > 0$, $0 < \lambda \le 1$ and the local solution $v \in X_{\delta,T}$ to $(DW)_\lambda$ satisfy*

$$\frac{d}{dt}\left\{\sum_{|a|\le \bar{L}-\mu-1}\langle \partial_t^{\mu+1}\nabla^a v(t), [h; \nabla \partial_t^\mu \nabla^a v(t)]\rangle\right\} \tag{80}$$
$$+\sum_{|a|\le \bar{L}-\mu-1}\int_{\mathbb{R}^d}\left\{\frac{d\phi + |x|\phi'}{2}\right\}|\partial_t^{\mu+1}\nabla^a v(t,x)|^2 dx$$

$$
\begin{aligned}
&+\sum_{|a|\le \bar{L}-\mu-1}\int_{\mathbb{R}^d}\left\{\phi+|x|\phi'-\frac{d\phi+|x|\phi'}{2}\right\}|\nabla\partial_t^\mu\nabla^a v(t,x)|^2dx\\
\le\ & \lambda C E_{\bar{L}-\mu}(\partial_t^\mu v(t))+\frac{C}{\lambda}D_{\bar{L},\mu}(v(t))\\
&+\frac{K}{4}\sum_{|a|\le \bar{L}-\mu-1}\langle\partial_t^{\mu+1}\nabla^a v(t),B_\lambda\partial_t^{\mu+1}\nabla^a v(t)\rangle+\frac{2\|B\|_\infty b_0^2R^2}{\lambda K}E_{\bar{L}-\mu}(\partial_t^\mu v(t)),
\end{aligned}
$$

where $\phi=\phi(|x|), h$ *are defined by* (17).

Proof. Let $|a|\le\bar{L}-\mu-1$. We apply $\partial_t^\mu\nabla^a$ to $(\mathrm{DW})_\lambda$ and take inner product the equation by $[h;\nabla\partial_t^\mu\nabla^a v]$ we obtain

$$
\begin{aligned}
&\langle\partial_t^{\mu+2}\nabla^a v,[h;\nabla\partial_t^\mu\nabla^a v]\rangle-\langle\triangle\partial_t^\mu\nabla^a v,[h;\nabla\partial_t^\mu\nabla^a v]\rangle\\
&\quad+\langle\nabla^a(B_\lambda\partial_t^{\mu+1}v),[h;\nabla\partial_t^\mu\nabla^a v]\rangle=\langle\partial_t^\mu\nabla^a N[v,v],[h;\nabla\partial_t^\mu\nabla^a v]\rangle.
\end{aligned}\tag{81}
$$

Noting (81),

$$
\begin{aligned}
&\langle\partial_t^{\mu+2}\nabla^a v,[h;\nabla\partial_t^\mu\nabla^a v]\rangle\\
=\ &\frac{d}{dt}\langle\partial_t^{\mu+1}\nabla^a v,[h;\nabla\partial_t^\mu\nabla^a v]\rangle-\sum_{k=1}^d\sum_{i=1}^d\int_{\mathbb{R}^d}\partial_t^{\mu+1}\nabla^a v^k h^i\partial_i\partial_t^{\mu+1}\nabla^a v^k dx\\
=\ &\frac{d}{dt}\langle\partial_t^{\mu+1}\nabla^a v,[h;\nabla\partial_t^\mu\nabla^a v]\rangle-\frac{1}{2}\sum_{k=1}^d\sum_{i=1}^d\int_{\mathbb{R}^d}\partial_i|\partial_t^{\mu+1}\nabla^a v^k|^2h^i dx\\
=\ &\frac{d}{dt}\langle\partial_t^{\mu+1}\nabla^a v,[h;\nabla\partial_t^\mu\nabla^a v]\rangle+\frac{1}{2}\int_{\mathbb{R}^d}|\partial_t^{\mu+1}\nabla^a v|^2\mathrm{div}h dx
\end{aligned}
$$

and

$$
\begin{aligned}
&-\langle\triangle\partial_t^\mu\nabla^a v(t),[h;\nabla\partial_t^\mu\nabla^a v(t)]\rangle\\
&=\sum_{i,k=1}^d\int_{\mathbb{R}^d}\nabla\partial_t^\mu\nabla^a v^k\cdot\nabla(h^i\partial_i\partial_t^\mu\nabla^a v^k)dx\\
&=\sum_{i,k=1}^d\int_{\mathbb{R}^d}\nabla\partial_t^\mu\nabla^a v^k\cdot\nabla h^i\partial_i\partial_t^\mu\nabla^a v^k dx+\sum_{i,k=1}^d\int_{\mathbb{R}^d}\nabla\partial_t^\mu\nabla^a v^k\cdot\nabla\partial_i\partial_t^\mu\nabla^a v^k h^i dx\\
&=\sum_{i,j,k=1}^d\int_{\mathbb{R}^d}\partial_j\partial_t^\mu\nabla^a v^k\partial_j h^i\partial_i\partial_t^\mu\nabla^a v^k dx+\frac{1}{2}\sum_{i,k=1}^d\int_{\mathbb{R}^d}\partial_i|\nabla\partial_t^\mu\nabla^a v^k|^2h^i dx\\
&=\sum_{i,j,k=1}^d\int_{\mathbb{R}^d}\partial_j\partial_t^\mu\nabla^a v^k\partial_j h^i\partial_i\partial_t^\mu\nabla^a v^k dx-\frac{1}{2}\int_{\mathbb{R}^d}|\nabla\partial_t^\mu\nabla^a v|^2\mathrm{div}h dx,
\end{aligned}
$$

we obtain

$$\begin{aligned}
&\frac{d}{dt}\langle \partial_t^{\mu+1}\nabla^a v, [h;\nabla\partial_t^\mu\nabla^a v]\rangle + \frac{1}{2}\int_{\mathbb{R}^d}|\partial_t^{\mu+1}\nabla^a v|^2 \mathrm{div}h dx \\
&-\frac{1}{2}\int_{\mathbb{R}^d}|\nabla\partial_t^\mu\nabla^a v|^2\mathrm{div}h dx + \sum_{i,j,k=1}^d\int_{\mathbb{R}^d}\partial_j\partial_t^\mu\nabla^a v^k\partial_j h^i\partial_i\partial_t^\mu\nabla^a v^k dx \\
= \ & -\langle\nabla^a(B_\lambda\partial_t^{\mu+1}v), [h;\nabla\partial_t^\mu\nabla^a v]\rangle + \langle\partial_t^\mu\nabla^a N[v,v], [h;\nabla\partial_t^\mu\nabla^a v]\rangle.
\end{aligned} \tag{82}$$

Now we remark that it holds that

$$\frac{\partial h^i}{\partial x_j} = \delta_{ij}\phi(|x|) + \phi'(|x|)\frac{x_i x_j}{|x|} \tag{83}$$

and

$$\mathrm{div}h(x) = d\phi(|x|) + |x|\phi'(|x|) \quad (x\in\mathbb{R}^d). \tag{84}$$

Furthermore using $\phi'(r)\le 0$ we get

$$\begin{aligned}
&\sum_{i,j,k=1}^d\int_{\mathbb{R}^d}\partial_j\partial_t^\mu\nabla^a v^k\partial_j h^i\partial_i\partial_t^\mu\nabla^a v^k dx \\
&= \sum_{i,k=1}^d\int_{\mathbb{R}^d}\partial_i\partial_t^\mu\nabla^a v^k\phi\partial_t^\mu\partial_i\nabla^a v^k dx + \sum_{i,j,k=1}^d\int_{\mathbb{R}^d}\partial_j\partial_t^\mu\nabla^a v^k\phi'\frac{x_i x_j}{|x|}\partial_i\partial_t^\mu\nabla^a v^k dx \\
&= \int_{\mathbb{R}^d}|\nabla\partial_t^\mu\nabla^a v|^2\phi dx + \sum_{k=1}^d\int_{\mathbb{R}^d}|x\cdot\nabla\partial_t^\mu\nabla^a v^k|^2\phi'\frac{1}{|x|}dx \\
&\ge \int_{\mathbb{R}^d}\{\phi+|x|\phi'\}|\nabla\partial_t^\mu\nabla^a v|^2 dx.
\end{aligned}$$

This estimate and (82) imply that

$$\begin{aligned}
&\frac{d}{dt}\langle\partial_t^{\mu+1}\nabla^a v, [h;\nabla\partial_t^\mu\nabla^a v]\rangle + \int_{\mathbb{R}^d}\left\{\frac{(d\phi+|x|\phi')}{2}\right\}|\partial_t^{\mu+1}\nabla^a v|^2 dx \\
&+\int_{\mathbb{R}^d}\left\{\phi+|x|\phi'-\frac{d\phi+|x|\phi'}{2}\right\}|\nabla\partial_t^\mu\nabla^a v|^2 dx \\
&\le -\langle B_\lambda\partial_t^{\mu+1}\nabla^a v, [h;\nabla\partial_t^\mu\nabla^a v(t)]\rangle \\
&-\sum_{\substack{b\le a\\ b\ne 0}}\binom{a}{b}\langle\nabla^b B_\lambda\partial_t^{\mu+1}\nabla^{a-b}v, [h;\nabla\partial_t^\mu\nabla^a v]\rangle + \langle\partial_t^\mu\nabla^a N[v,v], [h;\nabla\partial_t^\mu\nabla^a v]\rangle.
\end{aligned} \tag{85}$$

Let estimate for the right side of (85). First, since B_λ is a nonnegative symmetric matrix, there exists a nonnegative symmetric matrix S_λ such that

$S_\lambda^2 = B_\lambda$. Using $\mathbf{(B3)}_\lambda$ and (23), for any $|a| \leq \bar{L} - \mu - 1$ and $K > 0$ we obtain

$$\begin{aligned}
& |\langle B_\lambda \partial_t^{\mu+1} \nabla^a v, [h; \nabla \partial_t^\mu \nabla^a v] \rangle| \\
= \ & |\langle S_\lambda \partial_t^{\mu+1} \nabla^a v, S_\lambda [h; \nabla \partial_t^\mu \nabla^a v] \rangle| \\
\leq \ & \frac{K}{4} \| S_\lambda \partial_t^{\mu+1} \nabla^a v \|_2^2 + \frac{1}{K} \| S_\lambda [h; \nabla \partial_t^\mu \nabla^a v] \|_2^2 \\
= \ & \frac{K}{4} \langle \partial_t^{\mu+1} \nabla^a v, B_\lambda \partial_t^{\mu+1} \nabla^a v \rangle + \frac{1}{K} \langle [h; \nabla \partial_t^\mu \nabla^a v], B_\lambda [h; \nabla \partial_t^\mu \nabla^a v] \rangle \\
\leq \ & \frac{K}{4} \langle \partial_t^{\mu+1} \nabla^a v, B_\lambda \partial_t^{\mu+1} \nabla^a v \rangle + \frac{1}{K} \| B_\lambda \|_\infty \| h \|_\infty^2 \| \nabla \partial_t^\mu \nabla^a v \|_2^2 \\
\leq \ & \frac{K}{4} \langle \partial_t^{\mu+1} \nabla^a v, B_\lambda \partial_t^{\mu+1} \nabla^a v \rangle + \frac{\| B \|_\infty b_0^2 R^2}{\lambda K} \| \nabla \partial_t^\mu \nabla^a v \|_2^2.
\end{aligned}$$

So it holds that

$$\begin{aligned}
& - \sum_{|a| \leq \bar{L} - \mu - 1} \langle B_\lambda \partial_t^{\mu+1} \nabla^a v, [h; \nabla \nabla^a \partial_t^\mu v] \rangle \qquad (86) \\
\leq \ & \sum_{|a| \leq \bar{L} - \mu - 1} \frac{K}{4} \langle \partial_t^{\mu+1} \nabla^a v, B_\lambda \partial_t^{\mu+1} \nabla^a v \rangle + \frac{2 \| B \|_\infty b_0^2 R^2}{\lambda K} E_{\bar{L} - \mu}(\partial_t^\mu v).
\end{aligned}$$

Second, using $\mathbf{(B3)}_\lambda$ and (23), for $\lambda \leq 1$ we have

$$\begin{aligned}
& \sum_{|a| \leq \bar{L} - \mu - 1} \sum_{\substack{b \leq a \\ b \neq 0}} \binom{a}{b} \langle \nabla^b B_\lambda \partial_t^{\mu+1} \nabla^{a-b} v, [h; \nabla \partial_t^\mu \nabla^a v] \rangle \qquad (87) \\
\leq \ & \sum_{|a| \leq \bar{L} - \mu - 1} \sum_{\substack{b \leq a \\ b \neq 0}} \binom{a}{b} \sum_{i,j,k=1}^{d} \left| \int_{\mathbb{R}^d} \nabla^b (B_\lambda)_{ij} \partial_t^{\mu+1} \nabla^{a-b} v^j h^k \partial_k \partial_t^\mu \nabla^a v^i dx \right| \\
\leq \ & \sum_{|a| \leq \bar{L} - \mu - 1} \sum_{\substack{b \leq a \\ b \neq 0}} \binom{a}{b} \sum_{i,j,k=1}^{d} \| \nabla^b B_\lambda \|_\infty \| \partial_t^{\mu+1} \nabla^{a-b} v^j \|_2 \| h^k \|_\infty \| \partial_k \nabla^a \partial_t^\mu v^i \|_2 \\
\leq \ & \lambda C E_{\bar{L} - \mu}(\partial_t^\mu v).
\end{aligned}$$

We already got the estimate of $\langle \partial_t^\mu \nabla^a N[v, v], [h; \nabla \partial_t^\mu \nabla^a v] \rangle$ in lemma 5.1. Combining estimates (85), (86), (87) and (70), we get (80). This completes the proof of lemma 5.3. $\square$

Proof of lemma 3.1

Let $K = \frac{C_0}{\lambda}$. Calculating $K \times (72) + \frac{b_0(2d-1)}{4} \times (73) + (80)$ we get

$$\frac{d}{dt} G_{\bar{L}, \mu}(v) + \sum_{|a| \leq \bar{L} - \mu - 1} \left\{ K \langle \partial_t^{\mu+1} \nabla^a v, B_\lambda \partial_t^{\mu+1} \nabla^a v \rangle \right. \qquad (88)$$

$$
-\frac{b_0(2d-1)}{4}\|\partial_t^{\mu+1}\nabla^a v\|_2^2+\int_{\mathbb{R}^d}\frac{d\phi+|x|\phi'}{2}|\partial_t^{\mu+1}\nabla^a v|^2dx\Big\}
$$
$$
+\sum_{|a|\le\bar{L}-\mu-1}\int_{\mathbb{R}^d}\left(\frac{b_0(2d-1)}{4}+\phi+|x|\phi'-\frac{d\phi+|x|\phi'}{2}\right)|\nabla\partial_t^\mu\nabla^a v|^2dx
$$
$$
\le C(\lambda^2K+\frac{\lambda^2 b_0(2d-1)}{4}+\lambda)E_{\bar{L}-\mu}(\partial_t^\mu v)+C(K+\frac{b_0(2d-1)}{4}+\frac{1}{\lambda})D_{\bar{L},\mu}(v)
$$
$$
+\frac{K}{4}\sum_{|a|\le\bar{L}-\mu-1}\langle\partial_t^{\mu+1}\nabla^a v,B_\lambda\partial_t^{\mu+1}\nabla^a v(t)\rangle+\frac{2\|B\|_\infty b_0^2R^2}{\lambda K}E_{\bar{L}-\mu}(\partial_t^\mu v).
$$

From (17), it holds that

$$
r\phi'(r)=\begin{cases}0, & (r\le\frac{R}{\lambda})\\ -\phi(r), & (r\ge\frac{R}{\lambda}).\end{cases}\tag{89}
$$

Using $(\mathbf{B2})_\lambda$, (89), $K\ge\frac{d}{\lambda}$ and $\phi\ge0$, we obtain

$$
\begin{aligned}
&K\langle\partial_t^{\mu+1}\nabla^a v,B_\lambda\partial_t^{\mu+1}\nabla^a v\rangle-\frac{b_0(2d-1)}{4}\|\partial_t^{\mu+1}\nabla^a v\|_2^2 \\
&+\int_{\mathbb{R}^d}\left\{\frac{d\phi+|x|\phi'}{2}\right\}|\partial_t^{\mu+1}\nabla^a v|^2dx\\
\ge\;&\frac{K}{2}\langle\partial_t^{\mu+1}\nabla^a v,B_\lambda\partial_t^{\mu+1}\nabla^a v\rangle+\int_{|x|\le\frac{R}{\lambda}}\left(-\frac{b_0(2d-1)}{4}+\frac{db_0}{2}\right)|\partial_t^{\mu+1}\nabla^a v|^2dx\\
&+\int_{|x|\ge\frac{R}{\lambda}}\left(\frac{\lambda b_0K}{2}-\frac{b_0(2d-1)}{4}+\frac{d-1}{2}\phi\right)|\partial_t^{\mu+1}\nabla^a v|^2dx\\
\ge\;&\frac{K}{2}\langle\partial_t^{\mu+1}\nabla^a v,B_\lambda\partial_t^{\mu+1}\nabla^a v\rangle+\frac{b_0}{4}\|\partial_t^{\mu+1}\nabla^a v(t)\|_2^2
\end{aligned}\tag{90}
$$

for any $|a|\le\bar{L}-\mu-1$. Since we have

$$
\begin{aligned}
&\int_{\mathbb{R}^d}\left(\frac{b_0(2d-1)}{4}+\phi+|x|\phi'-\frac{d\phi+|x|\phi'}{2}\right)|\nabla\partial_t^\mu\nabla^a v|^2dx\\
&=\int_{|x|\le\frac{R}{\lambda}}\left(\frac{b_0(2d-1)}{4}+b_0-\frac{db_0}{2}\right)|\nabla\partial_t^\mu\nabla^a v|^2dx\\
&+\int_{|x|\ge\frac{R}{\lambda}}\left(\frac{b_0(2d-1)}{4}-\frac{(d-1)}{2}\frac{b_0R}{\lambda|x|}\right)|\nabla\partial_t^\mu\nabla^a v|^2dx\\
&\ge\frac{3b_0}{4}\int_{|x|\le\frac{R}{\lambda}}|\nabla\partial_t^\mu\nabla^a v|^2dx+\int_{|x|\ge\frac{R}{\lambda}}\left(\frac{b_0(2d-1)}{4}-\frac{b_0(d-1)}{2}\right)|\nabla\partial_t^\mu\nabla^a v|^2dx\\
&\ge\frac{b_0}{4}\|\nabla\partial_t^\mu\nabla^a v\|_2^2.
\end{aligned}\tag{91}
$$

The estimates (88), (90) and (91) imply

$$\frac{d}{dt}G_{\bar{L},\mu}(v) + \sum_{|a|\le \bar{L}-\mu-1} \frac{K}{2}\langle \partial_t^{\mu+1}\nabla^a v, B_\lambda \partial_t^{\mu+1}\nabla^a v\rangle + \frac{b_0}{2}E_{\bar{L}-\mu}(\partial_t^\mu v)$$

$$\le C(\lambda^2 K + \frac{\lambda^2 b_0(2d-1)}{4} + \lambda)E_{L-\mu}(\partial_t^\mu v) + C(K + \frac{b_0(2d-1)}{4} + \frac{1}{\lambda})D_{\bar{L},\mu}(v)$$

$$+ \frac{K}{4}\sum_{|a|\le \bar{L}-\mu-1} \langle \partial_t^{\mu+1}\nabla^a v, B_\lambda \partial_t^{\mu+1}\nabla^a v\rangle + \frac{2\|B\|_\infty b_0^2 R^2}{\lambda K}E_{\bar{L}-\mu}(\partial_t^\mu v). \tag{92}$$

Finally using $\lambda \le 1$, $K = \frac{C_0}{\lambda}$ and $\langle \partial_t^{\mu+1}\nabla^a v(t), B_\lambda \partial_t^{\mu+1}\nabla^a v(t)\rangle \ge 0$, we get (18), which completes the proof of lemma 3.1. □

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