The Fundamentals of Economic Dynamics and Policy Analyses: Learning through Numerical Examples.
Part Ⅳ. Overlapping Generations Model

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An overlapping generations model is an applied dynamic general equilibrium model for which the lifecycle models are employed as main analytical tools. At any point in time, there are overlapping generations consisting of individuals born this year, individuals born last year, individuals born two years ago, and so on. As we saw in the analysis of lifecycle models, each individual makes an optimal consumption-saving plan to maximize lifetime utility over her/his lifecycle. For example, an individual with higher income in earlier stages and lower income in later stages of lifecycle will save in earlier stages to prepare for the consumption in later stages where the income is lower. In an economy consisting of overlapping generations, the aggregate variables are the sum of variables chosen by individuals at different stages of lifecycles. In this framework, demographic structure is an important element to determine macroeconomic performance. For this reason, overlapping generations models are used for analyzing economies in which there are intergenerational transactions such as social security systems, inheritances, bequests, and so on.

This paper consists of four sections. Section 1 presents dynamic general equilibrium analysis of an overlapping generations model in which each individual lives two periods lifecycle. The model is the simplest form of overlapping generations models. Section 2 presents an application of overlapping generations model to tax policy analysis. Section 3 presents an application of overlapping generations model to the analysis of the effects of changes in demographic structure on economic growth and welfare in an economy with pay-as-you-go public pension system. Section 4 briefly presents the Computable Dynamic General Equilibrium models which is often used for more realistic public policy analyses

1．Overlapping Generations Model with Two Periods Lifecycles.

Let us analyze a simple overlapping generations model consisting of individuals with two periods lifecycles.

1-1. Households.

Let t denotes discrete time periods, t = ... , $-2, -1, 0, 1, 2, ...$. An individual born at the beginning of period t lives two periods, t and t $+1$. Therefore, in each time period t, there are two overlapping generations of individuals, those who were born in the previous period t -1, and those who are born in this period t. (See figure 1.1.) Let N_t be the number of individuals born at the beginning of period t. Because the number of individuals born in t -1 is N_{t-1}, and the number of individuals in period t is N_t, the total number of individuals in period t is $N_{t-1} + N_t$. Let us assume the growth rate of "cohort" is n, i.e.,

$$(1.1) \quad N_t = (1 + n) N_{t-1} \ .$$

In period t, the individual is "young", and supplies one unit labor to earn labor income $w_t \times 1$, where w_t is

Figure 1.1   Overlapping Generations Model with Two Periods Lifecycles
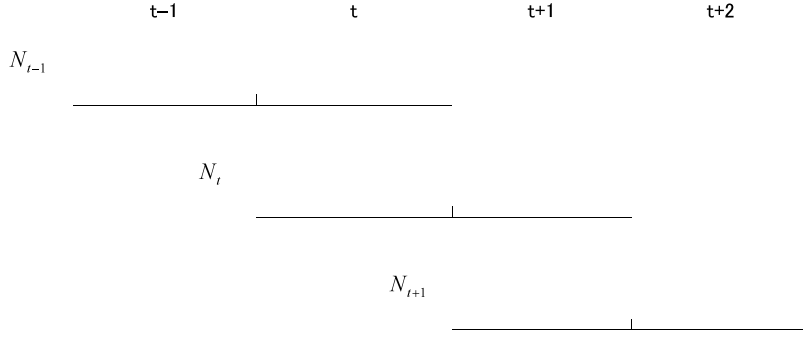
wage rate. The labor supply is assumed to be inelastic to the wage rate. (In more advanced and generalized models, however, labor supply can be modeled as an endogenous variable chosen by the individual.) The young individual consumes c_t^y and saves s_{t+1} subject to the following period t young budget constraint.

(1.2) $c_t^y + s_{t+1} = w_t$

We use the subscript $t+1$ for the saving s_{t+1} in period t because the saving is used to finance consumption c_{t+1}^o in the next period $t+1$ when the individual becomes "old". Furthermore, the saving in period t by young individual is employed as capital by firms in period $t+1$ who pays the interest rate r_{t+1} to each unit of capital. Therefore, the budget constraint of the old individual in period $t+1$ is expressed as follows.

(1.3) $c_{t+1}^o = (1 + r_{t+1}) s_{t+1}$

Let us assume that the individual's utility depends on young consumption c_t^y and old consumption c_{t+1}^o represented by the following logarithmic utility function.

(1.4) $u_t = \alpha \ln[c_t^y] + (1 - \alpha) \ln[c_{t+1}^o]$, $0 < \alpha < 1$.

In (1.4), $\alpha \in (0, 1)$ is a parameter measuring the relative importance of young consumption c_t^y to old consumption c_{t+1}^o. A representative individual born at the beginning of period t chooses young consumption c_t^y, old consumption c_{t+1}^o, and saving s_{t+1} to maximize utility $u_t(c_t^y, c_{t+1}^o)$ subject to the young budget constraint (1.2) and the old budget constraint (1.3). There are many ways to solve this constrained optimization problem. In the following, we employ the Lagrangean multiplier method. By eliminating the saving s_{t+1} from the young budget constraint (1.2) and the old budget constraint (1.3), we have the following "lifetime" budget constraint.

(1.5) $c_t^y + \dfrac{c_{t+1}^o}{1 + r_{t+1}} = w_t$

Define the Lagrangean function with respect to three unknowns $\{c_t^y, c_{t+1}^o, \lambda\}$ as follows.

(1.6) $L(c_t^y, c_{t+1}^o, \lambda) \equiv u_t(c_t^y, c_{t+1}^o) + \lambda [w_t - c_t^y - \dfrac{c_{t+1}^o}{1 + r_{t+1}}]$

In (1.6), λ is the Lagrangean multiplier which measures the marginal value of the lifetime budget constraint to the utility. The first-order conditions of the optimization problem are given as follows.

(1.7) $\dfrac{\partial L}{\partial c_t^y} = \dfrac{\partial u_t}{\partial c_t^y} - \lambda = 0$

(1.8) $\dfrac{\partial L}{\partial c_{t+1}^o} = \dfrac{\partial u_t}{\partial c_{t+1}^o} - \lambda \dfrac{1}{1+r_{t+1}} = 0$

(1.9) $\dfrac{\partial L}{\partial \lambda} = w_t - c_t^y - \dfrac{c_{t+1}^o}{1+r_{t+1}} = 0$

The optimal consumption plan $\{c_t^y, c_{t+1}^o\}$, together with the "shadow price" of lifetime budget constraint λ, is a solution to these first-order conditions that constitute a system of simultaneous equations with respect to $\{c_t^y, c_{t+1}^o, \lambda\}$. By (1.4), (1.7) and (1.8) are rewritten as follows.

(1.10) $\dfrac{\alpha}{c_t^y} = \lambda$

(1.11) $\dfrac{1-\alpha}{c_{t+1}^o} = \dfrac{\lambda}{1+r_{t+1}}$

By eliminating λ from (1.10) and (1.11), we have the following condition for the optimal consumption plan to satisfy.

(1.12) $\dfrac{\alpha}{c_t^y} \Big/ \dfrac{1-\alpha}{c_{t+1}^o} = 1 + r_{t+1}$

The left hand side of (1.12) is the marginal rate of substitution between young consumption c_t^y and old consumption c_{t+1}^o that is defined as follows.

(1.13) $-\dfrac{d c_{t+1}^o}{d c_t^y} = \dfrac{\partial u_t / \partial c_t^y}{\partial u_t / \partial c_{t+1}^o}$

Therefore, (1.12) implies that the slope of the indifference curve of the utility function must be equal to the slope $1 + r_{t+1}$ of the lifetime budget constraint in $\{c_t^y, c_{t+1}^o\}$-plane. In addition, (1.9) implies that the optimal consumption plan $\{c_t^y, c_{t+1}^o\}$ must satisfy the lifetime budget constraint. Therefore, the optimal consumption plan is depicted in figure 1.2 as a tangent point of an indifference curve and the graph of lifetime budget constraint.

Equations (1.9) and (1.12) are solved for the optimal consumption plan as functions of the wage rate w_t and the interest rate r_{t+1} as follows.

(1.14) $c_t^y = \alpha w_t$

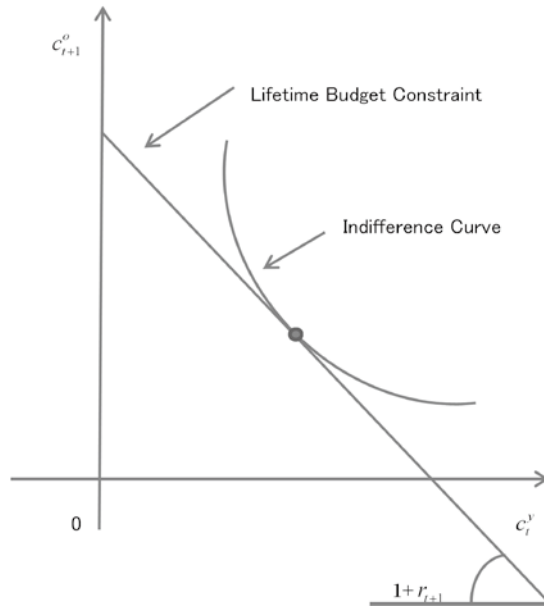(1.15) $c_{t+1}^o = (1-\alpha)(1+r_{t+1}) w_t$

By (1.14) and the young budget constraint (1.2), the optimal saving is solved as follows.

(1.16) $s_{t+1} = (1-\alpha) w_t$

(1.14) and (1.16) imply that in period t, the individual consumes $\alpha \times 100\%$ of labor income w_t and saves the rest. Therefore, α is the marginal propensity to consume and $1 - \alpha$ is the marginal propensity to save in this lifecycle model.

Figure 1.2   The Optimal Consumption Plan

1-2. Firms.

Let us assume that in each period t, there are N_t firms. (The number of firms is irrelevant to the analysis because we will assume linear homogenous production function. The assumption that the number of firms is equal to the number of young workers, however, makes the analysis simpler because each firm employs one young worker in labor market equilibrium.) A representative firm in period t demands capital k_t^d and labor l_t^d to produce output y_t subject to the following Cobb-Douglas production function

$$(1.17) \quad y_t = A_t (k_t^d)^\beta (l_t^d)^{1-\beta} \ , \quad 0 < \beta < 1,$$

so as to maximize profit π_t which is defined as follows.

$$(1.18) \quad \pi_t = y_t - r_t k_t^d - w_t l_t^d$$

In (1.17), $\beta \in (0, 1)$ is a parameter measuring capital intensity relative to labor in production function. The technology level A_t is assumed to grow at rate g, i.e.,

$$(1.19) \quad A_{t+1} = (1 + g) A_t.$$

The first-order conditions for the profit maximization with respect to capital k_t^d and labor l_t^d imply the following expressions for the interest rate r_t and the wage rate w_t.

$$(1.20) \quad r_t = \beta A_t (k_t^d / l_t^d)^{\beta - 1}$$

$$(1.21) \quad w_t = (1 - \beta) A_t (k_t^d / l_t^d)^{\beta}$$

The Cobb-Douglas production function (1.17) implies the maximized profit π_t is zero because

$$(1.22) \quad y_t = \frac{\partial y_t}{\partial k_t^d} \times k_t^d + \frac{\partial y_t}{\partial l_t^d} \times l_t^d$$

$$= r_t k_t^d + w_t l_t^d$$

In addition, β is the share of income paid out to capital and $1 - \beta$ is the share of income paid out to labor because of the following relationship.

$$(1.23) \quad r_t k_t^d = \beta y_t$$

$$(1.24) \quad w_t l_t^d = (1 - \beta) y_t$$

1-3. Dynamic General Equilibrium Conditions.

There are three markets in this overlapping generations model; the market for output, the market for capital, and the market for labor. In a dynamic general equilibrium, the aggregate demands and the aggregate supplies are equated in all the three markets in every period. In the market for output in period t, the aggregate demand consists of the aggregate demand for consumption by young individuals $N_t c_t^y$, the aggregate demand for consumption by old individuals $N_{t-1} c_t^o$, and capital formation (investment) which is a difference between the aggregate saving by young individuals $N_t s_{t+1}$ and the aggregate dissaving by old individuals $N_{t-1} s_t$. On the other hand, the aggregate supply of output by firms is $N_t y_t$. Therefore, the equilibrium condition for output market in period t is expressed as follows.

$$(1.25) \quad Y_t = C_t + I_t \ ,$$

where $Y_t \equiv N_t y_t$, $C_t \equiv N_t c_t^y + N_{t-1} c_t^o$, and $I_t \equiv N_t s_{t+1} - N_{t-1} s_t$.

The capital market equilibrium condition in period t is expressed as follows.

$$(1.26) \quad N_t k_t^d = N_{t-1} s_t$$

In (1.26), the left hand side is the aggregate demand for capital by firms, and the right hand side is the aggregate supply of capital by old individuals born in the previous period $t - 1$.

Finally, the labor market equilibrium condition in period t is expressed as follows.

$$(1.27) \quad N_t l_t^d = N_t \times 1$$

In (1.27), the left hand side is the aggregate demand for labor by firms, and the right hand side is the aggregate supply of labor by young individuals each inelastically supplies one unit labor by assumption. (1.27) implies that each firm employs one labor in equilibrium, i.e.,

$$(1.28) \quad l_t^d = 1.$$

Definition: Dynamic General Equilibrium of the Overlapping Generations Model.

Let us assume the economy starts from period zero (t = 0). The saving s_0 by each individual born in the previous period t = -1 is taken as predetermined. Then, if the capital market equilibrium condition holds

in period t $= 0$, the capital employed by each firm k_0^d is also predetermined by

(1.29) $\ N_0 \, k_0^d \ = \ N_{-1} \, s_0$

which is rewritten as

(1.30) $\ k_0^d \ = \ s_0 \, / (1 + n)$

by using (1.1). Then, the consumption by each old individual in t $= 0$ is also predetermined because

(1.31) $\ c_0^o \ = \ (1 + r_0) \, s_0$

where the interest rate r_0 is

(1.32) $\ r_0 \ = \ \beta \, A_0 \, (k_0)^{\beta - 1}$.

Then for every time period t $= 0, 1, 2, \dots$, the dynamic general equilibrium is described by factor prices and resource allocations that satisfy the following three conditions;

(i) Given the sequence of factor prices $\{r_t, w_t \, ; \, t = 0, 1, 2, \dots \}$, the sequence of consumption-saving plan $\{c_t^y, c_{t+1}^o, s_{t+1} \, ; \, t = 0, 1, 2, \dots \}$ maximizes the utility (1.4) of each individual of generation t subject to the young budget constraint (1.2) and the old budget constraint (1.3).

(ii) Given the sequence of factor prices $\{r_t, w_t \, ; \, t = 0, 1, 2, \dots \}$, the sequence of capital and labor $\{k_t^d, l_t^d \, ; \, t = 0, 1, 2, \dots \}$ maximizes the profit (1.18) of each firm subject to the production function (1.17).

(iii) The sequence of factor prices $\{r_t, w_t \, ; \, t = 0, 1, 2, \dots \}$ clears all the markets {output market, capital market, labor market} in every period t.

Although there are three markets in each period, it can be shown when any two markets out of three are in equilibrium, so is the remaining one market. This is an example of Walras' law. The proof of the statement is given as follows.

[Proof : Walras' Law] In a dynamic general equilibrium, the aggregate supply of output in any period t satisfies the following relationship.

$$
\begin{aligned}
(1.33) \ \ Y_t \ &= \ N_t \, y_t \\
&= \ N_t \, (r_t \, k_t^d \ + \ w_t \, l_t^d) \\
&= \ r_t \, N_t \, k_t^d \ + \ w_t \, N_t \, l_t^d
\end{aligned}
$$

If the capital market equilibrium condition (1.26) and the labor market equilibrium condition (1.27) are satisfied, then (1.33) becomes

(1.34) $\ Y_t \ = \ r_t \, N_{t-1} \, s_t \ + \ w_t \, N_t \ $.

In period t, the aggregation of young budget constraint (1.2) over N_t young individuals gives the following.

(1.35) $\ N_t \, c_t^y \ + \ N_t \, s_{t+1} \ = \ N_t \, w_t$

Similarly, in period t, the aggregation of old budget constraint (1.3) over N_{t-1} old individuals gives the following.

(1.36) $N_{t-1} c_t^o = N_{t-1} (1 + r_t) s_t$

By adding (1.35) and (1.36), we have the following resource constraint for period t.

(1.37) $N_t c_t^y + N_{t-1} c_t^o + N_t s_{t+1} - N_{t-1} s_t = r_t N_{t-1} s_t + N_t w_t$

(1.34) and (1.37) imply that the output equilibrium condition (1.25) is also satisfied.

The dynamical system for the endogenous variables in the dynamic general equilibrium of this overlapping generations model is derived as follows. By (1.16) and (1.21), the capital market equilibrium condition in period t is rewritten as follows.

(1.38) $N_t k_t^d = N_{t-1} s_t$
$$= N_{t-1} (1-\alpha) w_{t-1}$$
$$= N_{t-1} (1-\alpha)(1-\beta) A_{t-1} (k_{t-1}^d / l_{t-1}^d)^\beta$$

Divide both sides of (1.38) by N_t and use (1.1) and (1.28) to derive the following.

(1.39) $k_t^d = \dfrac{1}{1+n} (1-\alpha)(1-\beta) A_{t-1} (k_{t-1}^d)^\beta$

By shifting the time index t one period forward, and dropping the superscript "d", we have the following first-order nonlinear difference equation with respect to k_t.

(1.40) $k_{t+1} = \dfrac{(1-\alpha)(1-\beta)}{1+n} A_t k_t^\beta$

Given s_0, which is predetermined by young individual in the previous period t $= -1$, the capital in period t $= 0$ is also predetermined by (1.30). Then, for t $= 0, 1, 2, ...$, given the initial capital k_0, (1.40) generates a dynamic general equilibrium sequence of capital $\{k_t ; t = 0, 1, 2, ... \}$.

Once the dynamic general equilibrium, sequence of capital is determined, the other endogenous variables are calculated as follows. For t $= 0, 1, 2, ...$,

(1.41) $r_t = \beta A_t k_t^{\beta-1}$: interest rate

(1.42) $w_t = (1-\beta) A_t k_t^\beta$: wage rate

(1.43) $s_{t+1} = (1-\alpha) w_t$: saving

(1.44) $c_t^y = \alpha w_t$: young consumption

(1.45) $c_{t+1}^o = (1+r_{t+1}) s_{t+1}$
$$= (1+r_{t+1})(1-\alpha) w_t : \text{old consumption}$$

(1.46) $u_t = \alpha \ln[c_t^y] + (1-\alpha) \ln[c_{t+1}^o]$: utility

(The utility of an individual born in period t $= -1$ is measured by her/his consumption in period t $= 0$

calculated by $c_0^o = (1 + r_0) s_0$ where s_0 is given as predetermined variable.)

(1.47) $y_t = A_t k_t^\beta$: output

(1.48) $Y_t = N_t y_t,$: aggregate output

The endogenous variables in the dynamic general equilibrium have the following properties that parallel those derived in Solow-Swan economic growth model with technological progress. (See Futamura (2013).) Define the aggregate capital K_t in period t as follows.

(1.49) $K_t \equiv N_t k_t$

Then, $k_t = K_t/N_t$ is capital-labor ratio. Define the capital-labor ratio in "efficiency labor unit" as follows.

(1.50) $\tilde{k}_t \equiv K_t / \left[A_t^{1/(1-\beta)} N_t \right]$

Then, the capital-labor ratio k_t and the capital-labor ratio in efficiency labor unit $\tilde{k}_t$ satisfies the following relationship.

(1.51) $\tilde{k}_t = A_t^{-1/(1-\beta)} k_t$ or $k_t = A_t^{1/(1-\beta)} \tilde{k}_t$.

Multiply both sides of (1.40) with $A_{t+1}^{-1/(1-\beta)}$ and use (1.19) to transform the equation as follows.

(1.52) $A_{t+1}^{-1/(1-\beta)} k_{t+1} = A_{t+1}^{-1/(1-\beta)} \dfrac{(1-\alpha)(1-\beta)}{1+n} A_t k_t^\beta$

$$= \left[(1+g) A_t \right]^{-1/(1-\beta)} \frac{(1-\alpha)(1-\beta)}{1+n} A_t k_t^\beta$$

$$= \left[\frac{(1-\alpha)(1-\beta)}{(1+n)(1+g)^{1/(1-\beta)}} \right] (A_t^{-1/(1-\beta)} k_t)^\beta$$

By (1.51), equation (1.52) implies the following first-order nonlinear difference equation with respect to capital-labor ratio in efficiency labor unit $\tilde{k}_t$ given as follows.

(1.53) $\tilde{k}_{t+1} = \Gamma \tilde{k}_t^\beta$, $0 < \beta < 1$,

where

(1.54) $\Gamma \equiv \dfrac{(1-\alpha)(1-\beta)}{(1+n)(1+g)^{1/(1-\beta)}}$.

The dynamical system (1.53) has the following two properties;
(i) (1.53) has a unique steady state $\tilde{k}_s$ where

(1.55) $\tilde{k}_s = \Gamma^{1/(1-\beta)}$.

(ii) The steady state $\tilde{k}_s$ is globally stable. If the initial capital-labor ratio in efficiency labor unit $\tilde{k}_0 = A_0^{-1/(1-\beta)} k_0$ is smaller (larger) than the steady state $\tilde{k}_s$, then (1.53) generates a monotonically increasing (decreasing) sequence $\{\tilde{k}_t ; t = 0, 1, 2, ... \}$ converging to the steady state. By using (1.19), the capital-labor ratio k_t is rewritten as follows.

(1.56) $k_t = A_t^{1/(1-\beta)} \widetilde{k}_t$

$\qquad = \left[(1+g)^t A_0 \right]^{1/(1-\beta)} \widetilde{k}_t$

$\qquad = \gamma^t A_0^{1/(1-\beta)} \widetilde{k}_t \;,$

where

(1.57) $\gamma \equiv (1+g)^{1/(1-\beta)}.$

(1.56) implies that the growth rate of capital-labor ratio k_t in the steady state is $\gamma - 1$.

　　The output y_t in period t is rewritten as follows.

(1.58) $y_t = A_t k_t^{\beta}$

$\qquad = A_t [A_t^{1/(1-\beta)} \widetilde{k}_t]^{\beta}$

$\qquad = A_t^{1/(1-\beta)} \widetilde{k}_t^{\beta}$

$\qquad = \gamma^t A_0^{1/(1-\beta)} \widetilde{k}_t^{\beta}$

(1.58) implies that the growth rate of output in the steady state is also $\gamma - 1$.

　　By using (1.1) and (1.58), the aggregate output is rewritten as follows.

(1.59) $Y_t = N_t \, y_t$

$\qquad = [(1+n)^t N_0] [\gamma^t A_0^{1/(1-\beta)} \widetilde{k}_t^{\beta}]$

$\qquad = [(1+n)\gamma]^t N_0 A_0^{1/(1-\beta)} \widetilde{k}_t^{\beta}$

(1.59) implies that the growth rate of aggregate output Y_t in the steady state is $(1+n)\,\gamma - 1$.

　　The interest rate is rewritten as follows.

(1.60) $r_t = \beta A_t k_t^{\beta-1}$

$\qquad = \beta A_t [A_t^{1/(1-\beta)} \widetilde{k}_t]^{\beta-1}$

$\qquad = \beta \widetilde{k}_t^{\beta-1}$

(1.60) implies that the interest rate becomes constant in the steady state.

　　In the steady state, the growth rate of the wage rate w_t, the growth rate of young consumption c_t^y, and the growth rate of old consumption c_{t+1}^o are same and equal to $\gamma - 1$ because $w_t = (1-\beta) A_t k_t^{\beta} = (1-\beta) y_t$, $c_t^y = \alpha w_t$, and $c_{t+1}^o = (1+r_{t+1})(1-\alpha) w_t$.

1-4. Numerical Example.

The dynamic general equilibrium of above overlapping generations model can be numerically simulated through the following steps.

Step 1. Specify the values for parameters and initial conditions. For example, we specify the values as follows.

$\alpha = 0.5$; the weight of the utility of young consumption in the utility function (1.4). (The weight of the

utility of old consumption in the utility function is $1 - \alpha = 0.5$.)

$\beta = 0.3$; the weight of capital in the Cobb-Douglas production function (1.17).

$n = 0.1$; the rate of change in the cohort population.

$g = 0.1$; the growth rate of technological progress.

$s_0 = 1.1$; the saving in the initial period $t = 0$, predetermined by young individual in the previous period $t = -1$.

$k_0 = s_0/(1 + n) = 1.0$; initial capital.

$A_0 = 10$; initial level of technology.

$N_0 = 100$; initial cohort population.

Given these values, the initial capital-labor ratio in efficiency labor unit $\tilde{k}_0$ and the steady state capital-labor ratio in efficiency labor unit $\tilde{k}_s$ are

$$(1.61) \quad \tilde{k}_0 = 0.0373 \quad \text{and} \quad \tilde{k}_s = 0.1603.$$

(1.61) implies that the dynamic general equilibrium sequence of capital-labor ratio in efficiency labor unit $\{\tilde{k}_t ; t = 0, 1, 2, \dots \}$ is a monotonically increasing sequence converging to the steady state. In the steady state, capital-labor ratio k_t, output y_t, wage rate w_t, young consumption c_t^y, and old consumption c_{t+1}^o all grow at the same rate $\gamma - 1 = 0.1459$. The aggregate output Y_t grows at $(1 + n)\gamma - 1 = 0.2604$. The interest rate is constant at $r = 1.0804$.

In the overlapping generations model where each individual lives two periods, one period may corresponds to 30~40 years. Therefore, a growth rate x in the model implies $(1 + x)^{1/40} - 1 \sim (1 + x)^{1/30} - 1$ annual growth rate. By the same reason, the interest rate r in the model implies $(1 + r)^{1/40} - 1 \sim (1 + r)^{1/30} - 1$ annual interest rate.

Step 2. Calculate the endogenous variables for period $t = 0$ as follows.

$$(1.62) \quad r_0 = \beta A_0 k_0^{\beta-1}$$

$$(1.63) \quad w_0 = (1 - \beta) A_0 k_0^{\beta}$$

$$(1.64) \quad s_1 = (1 - \alpha) w_0$$

$$(1.65) \quad c_0^o = (1 + r_0) s_0$$

$$(1.66) \quad c_0^y = \alpha w_0$$

$$(1.67) \quad y_0 = A_0 k_0^{\beta}$$

$$(1.68) \quad Y_0 = N_0 y_0$$

Step 3. Update the capital-labor ratio and the capital-labor ratio in efficiency labor unit for period $t = 1$ as follows.

$$(1.69) \quad k_1 = \frac{(1-\alpha)(1-\beta)}{1+n} A_0 k_0^{\beta}$$

$$(1.70) \quad \tilde{k}_1 = \Gamma \tilde{k}_0^{\beta} \ , \quad \Gamma \equiv \frac{(1-\alpha)(1-\beta)}{(1+n)(1+g)^{1/(1-\beta)}}$$

Then, calculate the endogenous variables for period $t = 1$ as follows.

$$(1.71) \quad A_1 = (1+g) A_0$$

$$(1.72) \quad N_1 = (1+n) N_0 \ .$$

$$(1.73) \quad r_1 = \beta A_1 k_1^{\beta-1}$$

$$(1.74) \quad w_1 = (1-\beta) A_1 k_1^{\beta}$$

$$(1.75) \quad s_2 = (1-\alpha) w_1$$

$$(1.76) \quad c_1^o = (1+r_1) s_1$$

$$(1.77) \quad c_1^y = \alpha w_1$$

$$(1.78) \quad y_1 = A_1 k_1^{\beta}$$

$$(1.79) \quad Y_1 = N_1 y_1$$

The utility of an individual born in period $t = 0$ is calculated as follows.

$$(1.80) \quad u_0 = \alpha \ln [c_0^y] + (1-\alpha) \ln [c_1^o]$$

Step 4. Update the capital-labor ratio and the capital-labor ratio in efficiency labor unit for period $t = 2$ as follows.

$$(1.81) \quad k_2 = \frac{(1-\alpha)(1-\beta)}{1+n} A_1 k_1^{\beta}$$

$$(1.82) \quad \tilde{k}_2 = \Gamma \tilde{k}_1^{\beta}$$

Then, calculate the endogenous variables for period $t = 2$ by repeating step 3.

Step 5. Repeat the above steps to generate the sequence of endogenous variables in the dynamic general equilibrium $\{\tilde{k}_t, k_t, r_t, w_t, s_t, c_t^y, c_t^o, u_t, y_t, Y_t \ ; t = 0, 1, 2, ... \}$.

The result of the numerical simulation is summarized in Table 1.1. As predicted, the capital-labor ratio in efficiency labor unit $\tilde{k}_t$ starts at $\tilde{k}_0 = 0.0373$ and converges monotonically and quickly toward the steady state $\tilde{k}_s = 0.1603$. The movement of $\tilde{k}_t$ is depicted in figure 1.3. Figure 1.4 depicts the movement of capital-

Table 1.1　Simulation of Overlapping Generations Model with Two Periods Lifecycles

t	$A(t)$	$N(t)$	$\tilde{k}(t)$	$k(t)$	$r(t)$	$w(t)$	$s(t)$
0	10.00	100.00	0.0373	1.00	3.00	7.00	1.10
1	11.00	110.00	0.1035	3.18	1.47	10.90	3.50
2	12.10	121.00	0.1406	4.95	1.18	13.69	5.45
3	13.31	133.10	0.1542	6.22	1.11	16.12	6.84
4	14.64	146.41	0.1585	7.33	1.09	18.63	8.06
5	16.11	161.05	0.1598	8.47	1.08	21.40	9.31
6	17.72	177.16	0.1602	9.73	1.08	24.54	10.70
7	19.49	194.87	0.1603	11.15	1.08	28.12	12.27
8	21.44	214.36	0.1603	12.78	1.08	32.23	14.06
9	23.58	235.79	0.1603	14.65	1.08	36.93	16.11
10	25.94	259.37	0.1603	16.79	1.08	42.32	18.46
11	28.53	285.31	0.1603	19.23	1.08	48.49	21.16
12	31.38	313.84	0.1603	22.04	1.08	55.56	24.24

t	$c^y(t)$	$c^o(t)$	$u(t)$	$y(t)$	$Y(t)$	$\Delta y(t)/y(t)$	$\Delta Y(t)/Y(t)$
0	3.50	4.40	1.70	10.00	1000.00		
1	5.45	8.64	2.09	15.57	1712.33	0.5567	0.7123
2	6.84	11.90	2.30	19.55	2366.09	0.2562	0.3818
3	8.06	14.45	2.46	23.03	3065.72	0.1779	0.2957
4	9.31	16.84	2.60	26.61	3896.28	0.1554	0.2709
5	10.70	19.40	2.74	30.57	4923.27	0.1487	0.2636
6	12.27	22.27	2.87	35.05	6210.15	0.1467	0.2614
7	14.06	25.53	3.01	40.18	7829.32	0.1461	0.2607
8	16.11	29.26	3.15	46.04	9869.11	0.1459	0.2605
9	18.46	33.52	3.28	52.76	12439.76	0.1459	0.2605
10	21.16	38.41	3.42	60.45	15679.77	0.1459	0.2605
11	24.24	44.02	3.55	69.27	19763.58	0.1459	0.2605
12	27.78	50.44	3.69	79.37	24910.98	0.1459	0.2604

alpha = 0.5, beta = 0.3, n = 0.1, g = 0.1, A(0) = 10, N(0) = 100, k(0) = 1.
s(0) = (1 + n)*k(0) = 1.1.
kxs = 0.160347, kx(0) = 0.0372759, Gamma = 0.277679.
gamma = 1.14586, (1 + n)*gamma = 1.26045.

labor ratio k_t, and figure 1.5 depicts the movement of utility u_t of each individual born in period t. Because young consumption c_t^y and old consumption c_{t+1}^o grow at the same rate $\gamma = (1 + g)^{1/(1-\beta)}$, and because the utility function is assumed to be of logarithmic form, the slope of the graph of u_t is approximately equal to $g/(1-\beta)$. ($\ln \gamma^t = t \times \ln \gamma = t \times \ln[(1+g)^{1/(1-\beta)}] = t \times (1/(1-\beta)) \times \ln(1+g) \cong t \times (1/(1-\beta)) \times g$.)

Figure 1.6 depicts the movement of output y_t, and figure 1.7 depicts the movement of the growth rate of output $(y_{t+1} - y_t)/y_t$. As predicted, the growth rate of y_t converges monotonically and quickly toward the

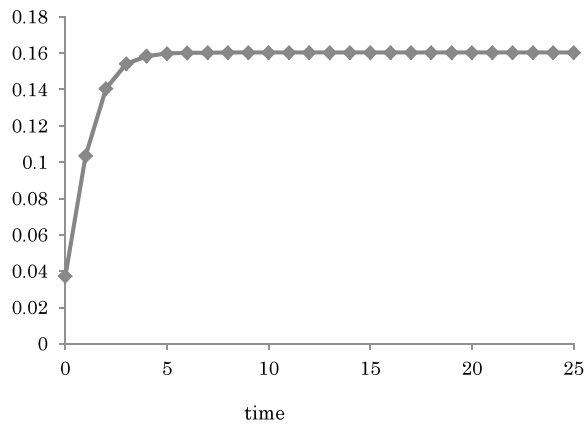Figure 1.3　The Graph of Capital-Labor Ratio in Efficiency Labor Unit $\tilde{k}_t$

time

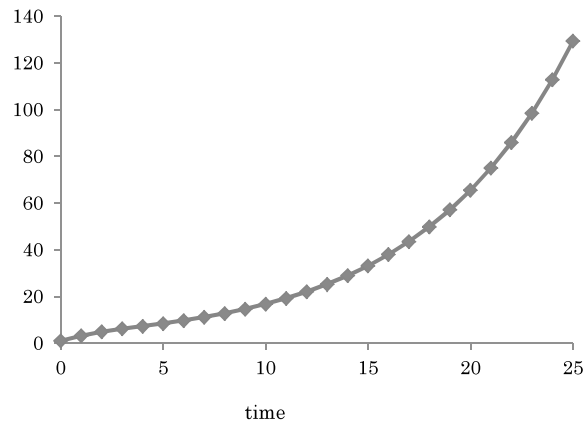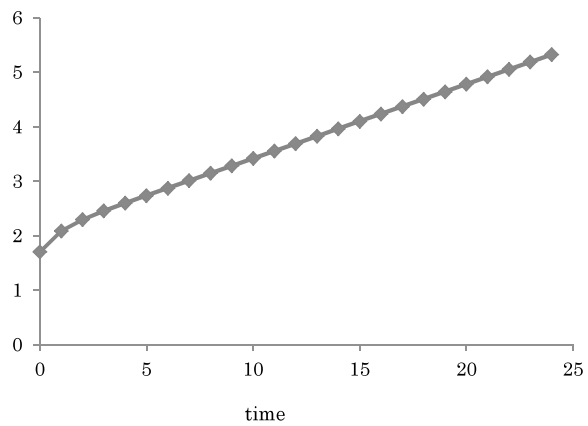Figure 1.4　The Graph of Capital-Labor Ratio k_t

time

Figure 1.5　The Graph of Utility u_t

time

steady state growth rate $\gamma - 1 = 0.1459$. Figure 1.8 depicts the movement of aggregate output Y_t, and figure 1.9 depicts the movement of the growth rate of output $(Y_{t+1} - Y_t)/Y_t$. As predicted, the growth rate of Y_t converges monotonically and quickly toward the steady state growth rate $(1 + n)\gamma - 1 = 0.2605$.
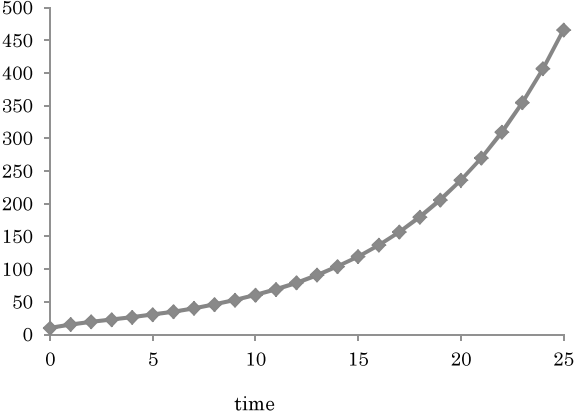
Figure 1.6   The Graph of Output y_t

time

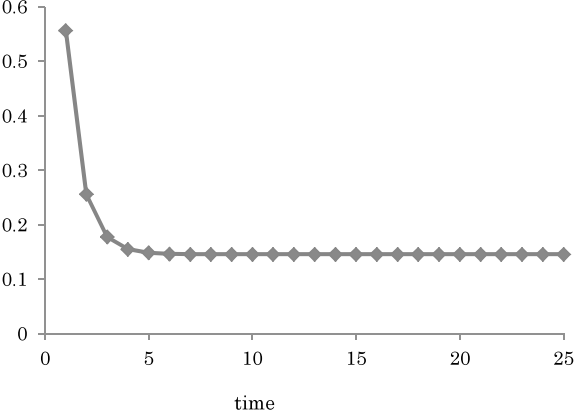Figure 1.7   The Graph of the Growth Rate of Output $(y_{t+1} - y_t)/y_t$

time

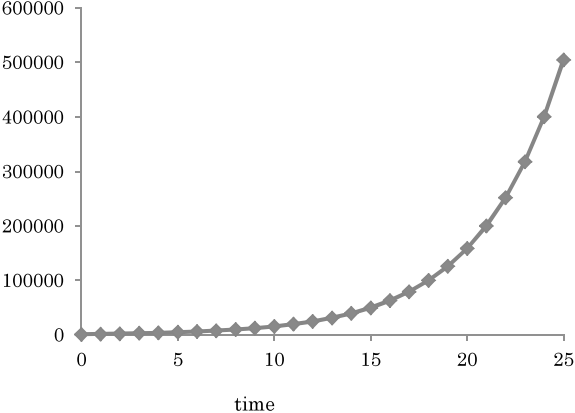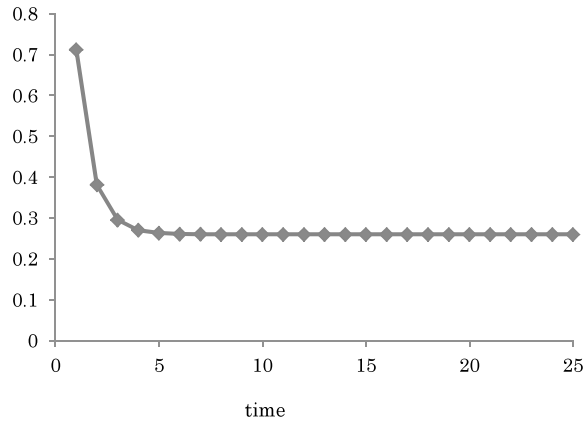Figure 1.8   The Graph of Aggregate Output Y_t

time

Figure 1.9　The Graph of the Growth Rate of Aggregate Output ($Y_{t+1} - Y_t$)/Y_t

2. Exercise: Is Sales Tax Regressive to Income?

In this exercise, we are going to investigate if the conventional wisdom that "sales tax is regressive to income" is true. For this purpose, we construct and analyze a two periods lifecycle model which is described as follows. There are two individuals, A and B. Each individual lives two periods. The optimal consumption-saving problem of A is described as follows.

(2.1) $u^A = \alpha \ln[c_1^A - \bar{c}] + (1 - \alpha) \ln[c_2^A]$, $0 < \alpha < 1$; utility function.

(2.2) $(1+t)c_1^A + s^A = y^A$; first period budget constraint.

(2.3) $(1+t)c_2^A = (1+r)s^A$; second period budget constraint.

In (2.1), (2.2), and (2.3), u^A is utility, c_1^A is first period consumption, c_2^A is second period consumption, s^A is saving, y^A is first period income, t is sales tax rate, and r is interest rate. In (2.1), $\bar{c}$ is interpreted as the minimum consumption level that is necessary to sustain livelihood in the first period. (In fact, the first period consumption c_1^A must be above $\bar{c}$ because $\ln[c_1^A - \bar{c}] = -\infty$ at $c_1^A = \bar{c}$.) Given $\{y^A, r, t\}$, A chooses a consumption-saving plan $\{c_1^A, c_2^A, s^A\}$ that maximizes utility u^A subject to the first period budget constraint (2.2) and the second period budget constraint (2.3). The optimal consumption-saving problem of B, likewise, is described as follows.

(2.4) $u^B = \alpha \ln[c_1^B - \bar{c}] + (1 - \alpha) \ln[c_2^B]$, $0 < \alpha < 1$; utility function.

(2.5) $(1+t)c_1^B + s^B = y^B$; first period budget constraint.

(2.6) $(1+t)c_2^B = (1+r)s^B$; second period budget constraint.

In the following analysis, we assume $y^A = 200$ and $y^B = 400$ so that B is richer than A. In addition, we

assume $\bar{c} = 50$ so that the minimum consumption level in first period is same for A and B, and $\alpha = 0.5$ so that A and B have the same preference (utility function) with respect to first period consumption and second period consumption. We also assume $r = 0$ for analytical simplicity. (The analytical implications do not change even if $r \neq 0$.)

Question 1. Assume that the sales tax rate is zero ($t = 0$). Then, calculate the optimal consumption-saving plan of A $\{c_1^A, c_2^A, s^A\}$ that maximizes A's utility u^A subject to the first period budget constraint (2.2) and the second period budget constraint (2.3).

Answer: $c_1^A = ($ $), c_2^A = ($ $), s^A = ($ $)$.

Question 2. Assume $t = 0$. Then, calculate the optimal consumption-saving plan of B $\{c_1^B, c_B^A, s^B\}$ that maximizes B's utility u^B subject to the first period budget constraint (2.5) and the second period budget constraint (2.6).

Answer: $c_1^B = ($ $), c_2^B = ($ $), s^B = ($ $)$.

Question 3. Assume $t = 0$. Which one of the following options {(a), (b), (c), (d)} is true about the relationship between A's saving rate s^A/y^A and B's saving rate s^B/y^B?
(a) $s^A/y^A > s^B/y^B$
(b) $s^A/y^A = s^B/y^B$
(c) $s^A/y^A < s^B/y^B$
(d)   None of {(a), (b), (c)}.

Answer: (   )

Question 4. Assume that the sales tax rate is 100% ($t = 1$). Then, calculate the optimal consumption-saving plan of A $\{c_1^A, c_2^A, s^A\}$ that maximizes A's utility u^A subject to the first period budget constraint (2.2) and the second period budget constraint (2.3).

Answer: $c_1^A = ($ $), c_2^A = ($ $), s^A = ($ $)$.

Question 5. Assume $t = 1$. Then, calculate the optimal consumption-saving plan of B $\{c_1^B, c_B^A, s^B\}$ that maximizes B's utility u^B subject to the first period budget constraint (2.5) and the second period budget constraint (2.6).

Answer: $c_1^B = ($ $), c_2^B = ($ $), s^B = ($ $)$.

Question 6. Assume $t = 1$. Which one of the following options {(a), (b), (c), (d)} is true about the relationship between A's saving rate s^A/y^A and B's saving rate s^B/y^B?

(a) $s^A / y^A \; > \; s^B / y^B$

(b) $s^A / y^A \; = \; s^B / y^B$

(c) $s^A / y^A \; < \; s^B / y^B$

(d)   None of {(a), (b), (c)}.

Answer: (   )

Question 7. Assume $t = 1$. Which one of the following options {(a), (b), (c), (d)} is true about the relationship between A's first period sales tax burden ratio $t \times c_1^A / y^A$ and B's first period sales tax burden ratio $t \times c_1^B / y^B$?

(a) $t \times c_1^A / y^A \; > t \times c_1^B / y^B$

(b) $t \times c_1^A / y^A \; = t \times c_1^B / y^B$

(c) $t \times c_1^A / y^A \; < t \times c_1^B / y^B$

(d)   None of {(a), (b), (c)}.

Answer: (   )

Question 8. Assume $t = 1$. Which one of the following options {(a), (b), (c), (d)} is true about the relationship between A's lifetime sales tax burden ratio $(t \times c_1^A + t \times c_2^A)/y^A$ and B's lifetime sales tax burden ratio $(t \times c_1^B + t \times c_2^B)/y^B$?

(a) $(t \times c_1^A \; + t \times c_2^A)/ y^A \; > (t \times c_1^B \; + t \times c_2^B)/ y^B$

(b) $(t \times c_1^A \; + t \times c_2^A)/ y^A \; = (t \times c_1^B \; + t \times c_2^B)/ y^B$

(c) $(t \times c_1^A \; + t \times c_2^A)/ y^A \; < (t \times c_1^B \; + t \times c_2^B)/ y^B$

(d)   None of {(a), (b), (c)}.

Answer: (   )

Question 9. Apply above analyses to answer to the question that if sales tax is regressive to income. State explicitly the reason why sales tax has such properties in the answer.

Analyses: The above questions are analyzed as follows. The constrained utility optimization problem in general form is described as follows.

(2.7) $u \; = \; \alpha \ln [c_1 - \bar{c}] \; + (1 - \alpha) \ln [c_2]$; utility function.

(2.8) $(1 + t)c_1 \; + \; s \; = \; y$; first period budget constraint.

(2.9) $(1 + t)c_2 \; = \; (1 + r)s$; second period budget constraint.

In (2.7), (2.8), and (2.9), $\alpha = 0.5$ and $r = 0$ by assumption. By using (2.8) and (2.9), the utility is expressed as a function of saving s as a sole control variable as follows.

(2.10) $u = 0.5 \times \ln\left[\dfrac{y-s}{1+t} - \bar{c}\right] + 0.5 \times \ln\left(\dfrac{s}{1+t}\right)$

The first-order condition for the maximization of utility u with respect to saving s is $du/ds = 0$. The condition is solved for the optimal saving s^*. The solution is

(2.11) $s^* = \dfrac{1}{2}[y - (1+t)\bar{c}]$.

Then, by using (2.8) and (2.9), the optimal consumption plan $\{c_1^*, c_2^*\}$ is solved as follows.

(2.12) $c_1^* = (y - s^*)/(1+t)$

(2.13) $c_2^* = s^*/(1+t)$

By (2.11), the saving rate s^*/y is shown to be increasing in income y because

(2.14) $\dfrac{s^*}{y} = \dfrac{1}{2}\left[1 - \dfrac{(1+t)\bar{c}}{y}\right]$.

(2.14) implies

(2.15) $\dfrac{d}{dy}\left(\dfrac{s^*}{y}\right) > 0$.

(Notice if $\bar{c} = 0$, then the saving rate is $s^*/y = 1/2$ independent of income y.) Then, by using (2.12), the first period sales tax burden ratio $t \times c_1/y$ is shown to be decreasing in income y because

(2.16) $\dfrac{t c_1^*}{y} = \dfrac{t}{1+t}\left[1 - \dfrac{s^*}{y}\right]$.

By (2.15), (2.16) implies

(2.17) $\dfrac{d}{dy}\left(\dfrac{t c_1^*}{y}\right) < 0$.

(2.17) implies that "sales tax is regressive to income" if measured by the first period tax burden ratio.

The lifetime sales tax burden ratio, however, reveals the different aspect of sales tax. By eliminating saving from (2.8) and (2.9), we have the following lifetime budget constraint.

(2.18) $(1+t)c_1 + (1+t)c_2 = y$

The lifetime sales tax burden ratio, hence, is expressed as follows.

(2.19) $\dfrac{t c_1 + t c_2}{y} = \dfrac{t c_1 + t c_2}{(1+t)c_1 + (1+t)c_2}$

$= \dfrac{t}{1+t}$

(2.19) implies that sales tax is neither regressive nor progressive to income if measured by lifetime sales

tax burden ratio. The reason why sales tax has such a property is easy to understand. The sales tax is equivalent to a proportional income tax. The burden ratio of proportional income tax, by definition, is same for all individuals regardless of income level. In the lifecycle model, the lifetime budget constraint (2.18) is rewritten as follows.

(2.20) $c_1 + c_2 = \dfrac{y}{1+t}$

Therefore, an increase in sales tax ratio t causes real income $y/(1+t)$ to decrease. In fact, there is a proportional income tax rate θ that has the same effect as the sales tax has on income. Such a proportional income tax rate θ satisfies the following relationship.

(2.21) $(1 - \theta)y = \dfrac{y}{1+t}$

From (2.21), θ is calculated as

(2.22) $\theta = \dfrac{t}{1+t}$

which is nothing but lifetime sales tax burden ratio (2.19). (The proportional income tax burden ratio is $\theta y/y$ = θ. On the other hand, we can also express the lifetime sales tax burden ratio as $\{y - [y/(1+t)]\}/y$ because it causes the real income to decrease from y down to $y/(1+t)$. The lifetime sales tax burden ratio expressed in this way is equal to $t/(1+t)$.) By (2.22), if the sales tax rate is $t=1$ (100%), then the corresponding proportional income tax rate is $\theta = t/(1+t) = 0.5$. If $t = 0.1$ (10%), then $\theta = t/(1+t) = 0.1/1.1 \cong 0.091$.

Figure 2.1   Bequest between Generations
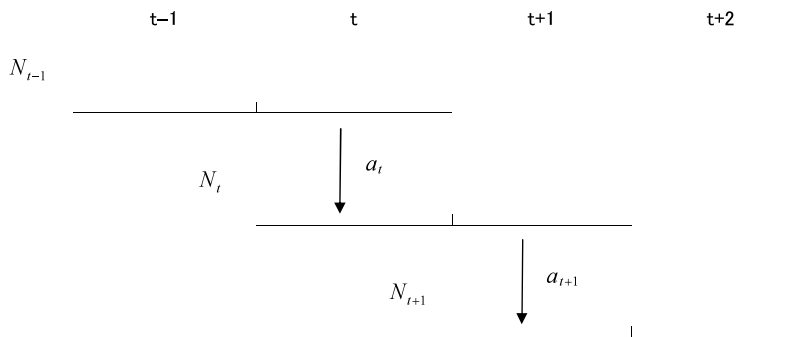
If each individual receives bequest from parents in the first period, and leaves bequest to children in the second period, we can still find a sales tax system which makes it equivalent to proportional income tax so that the sales tax is neither regressive nor progressive yet. If there are intergenerational transfers (bequests), (2.8) and (2.9) are rewritten as follows.

(2.23) $c_1 + s = y_1 + a_1$

(2.24) $c_2 + a_2 = y_2 + (1+r)s$

In (2.23), a_1 is a bequest received by the individual in the first period. In (2.24), a_2 is a bequest left by the

individual in the second period. (See figure 2.1.) By eliminating the saving s from (2.23) and (2.24), we have the following lifetime budget constraint.

(2.25) $\quad c_1 \;+\; \dfrac{c_2}{1+r} \;+\; \dfrac{a_2}{1+r} \;=\; y_1 \;+\; a_1 \;+\; \dfrac{y_2}{1+r}$

Therefore, for a "sales tax" imposed on $\{c_1, c_2, a_2\}$ at the same rate t, we can find a proportional "income tax" imposed on $\{y_1, a_1, y_2\}$ at the same rate θ which makes these two tax system equivalent.

People often criticize sales tax for its regressiveness to income level. This criticism is based on the following observation. "Statistics show that the saving rate of rich people is higher than poor people. Therefore, the ratio of sales tax burden to income is higher for poor people than for rich people because the former must consume out of income more than the latter". As we saw in the above analyses, however, this conventional wisdom is not necessarily true. The lifetime sales tax burden ratio may be neither regressive nor progressive because it may be equivalent to proportional income tax. To judge the legitimacy of the criticism against sales tax based on its regressiveness, above analysis based on the theoretical model must be rendered to empirical data analysis. In this field of research, Crawford, Keen, and Smith (2010) reported a result against the regressivity of sales tax.

3. Exercise: The Effects of Pay-as-you-go Public Pension System on Economic Growth and Welfare.

In this exercise, we are going to investigate the effects of pay-as-you-go public pension system on economic growth and welfare. For this purpose, we construct an overlapping generations model that incorporates public pension system. The structure of the model is described as follows. Let $t = 0, 1, 2, ...$ denote the index for discrete time periods. At the beginning of each period t, N_t individuals are born. The growth rate of the cohort population is n as follows.

(3.1) $\quad N_{t+1} = (1 + n) N_t$.

Each individual lives two periods. A representative individual born at the beginning of period t is "young" and is subject to the following young budget constraint.

(3.2) $\quad c_t^y \;+\; s_{t+1} \;=\; w_t \;-\; x_t$

The young individual inelastically supplies one unit labor and earn labor income $w_t \times 1$ where w_t is the wage rate. In addition, the young individual contributes premium x_t to public pension system. Therefore, the right hand side of (3.2) is labor income net of public pension premium. The left hand side of (3.2) implies that the individual consumes c_t^y and saves s_{t+1} to prepare for the consumption in the second period of her/his life. The young individual in period t becomes an old individual in the next period $t+1$, and is subject to the following old budget constraint.

(3.3) $\quad c_{t+1}^o = (1 + r_{t+1}) s_{t+1} \;+\; z_{t+1}$

In the right hand side of (3.3), r_{t+1} is the interest rate, and z_{t+1} is pension to the old individual. The saving

in period t by young individual is employed as capital by firms in period $t+1$ who pays the interest rate r_{t+1} to each unit of capital. Therefore, (3.3) implies that the old individual consumes c_{t+1}^o her/his old income which consists of the principal s_{t+1} and the interest income $r_{t+1}s_{t+1}$ that accrue to the saving, and pension z_{t+1}. The utility of the representative individual born at the beginning of period t is an increasing and concave function of young consumption c_t^y and old consumption c_{t+1}^o defined as follows.

(3.4) $u_t = \alpha \ln[c_t^y] + (1-\alpha) \ln[c_{t+1}^o], \quad 0 < \alpha < 1.$

Given $\{w_t, r_{t+1}, x_t, z_{t+1}\}$, the individual chooses an optimal consumption-saving plan $\{c_t^y, c_{t+1}^o, s_{t+1}\}$ that maximizes utility u_t subject to the young budget constraint (3.2) and the old budget constraint (3.3). (Notice if $\{x_t = 0, z_{t+1} = 0 ; t = 0, 1, 2, ... \}$, then this overlapping generations model is same as the one analyzed in section 1.)

Question 1. Express the optimal consumption-saving plan $\{c_t^y, c_{t+1}^o, s_{t+1}\}$ as functions of $\{w_t, r_{t+1}, x_t, z_{t+1}\}$.

In each period t, there are N_t firms. In period t, a representative firm, given the interest rate r_t and the wage rate w_t, demands capital k_t^d and labor l_t^d to maximize profit π_t which is defined as follows.

(3.5) $\pi_t = y_t - r_t k_t^d - w_t l_t^d$

In (3.5), y_t is output produced by the following production function.

(3.6) $y_t = A(k_t^d)^\beta (l_t^d)^{1-\beta}, \quad 0 < \beta < 1.$

For simplicity, we assume there is no technological progress so that the technology level A in (3.6) is constant.

Question 2. By solving the firm's profit maximization problem, express the interest rate r_t and the wage rate w_t as functions of capital k_t^d and labor l_t^d.

Although there are three markets {output market, capital market, labor market}, if two out of three markets are in equilibrium, the remaining one is in equilibrium as well by Walras' law. Therefore, we focus on the equilibrium conditions for capital market and labor market in the following analyses. The capital market equilibrium condition in period t is

(3.7) $N_t k_t^d = N_{t-1} s_t$.

The left hand side of (3.7) is the aggregate demand for capital by N_t firms, and the right hand side is the aggregate supply of capital by N_{t-1} old individuals born in the previous period $t-1$. The labor market equilibrium condition in period t is

(3.8) $N_t l_t^d = N_t \times 1$.

The left hand side of (3.8) is the aggregate demand for labor by N_t firms, and the right hand side is the aggregate supply of labor by N_t young individuals each inelastically supplies one unit labor.

The public pension is managed as a pay-as-you-go system. There is no fund. Therefore, the budget constraint of the pension system in each period t is

(3.9)　$N_{t-1} z_t = N_t x_t$.

The right hand side of (3.9) is the total contribution by N_t young individuals each pays pension premium x_t. The left hand side of (3.9) is the total pension payment to N_{t-1} old individuals each receives z_t. In the following analyses, assume that the pension is fixed at $z_t = z$, and the pension premium x_t is set to satisfy the budget constraint (3.9). By (3.1) and (3.9), this assumption implies that the pension premium is also constant at $x_t = x$ where

(3.10)　$x = z/(1 + n)$.

Because $1 + n = N_t/N_{t-1}$ is the ratio of young individuals to old individuals, an increase (a decrease) in the cohort growth rate n causes the pension premium to decrease (increase), for there are more (less) young individuals to support old individuals in the pay-as-you-go public pension system. (See figure 3.1.)

Figure 3.1　Pay-as-you-go Public Pension System
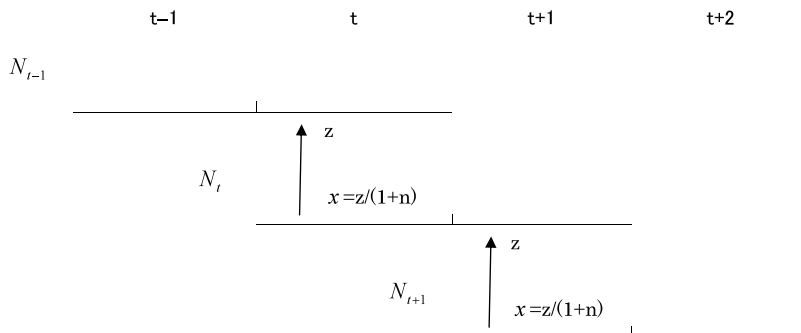
Question 3.　Derive the dynamical system with respect to capital-labor ratio k_t (the superscript "d" is omitted) in a dynamic general equilibrium of this overlapping generations model. (The dynamical system is same as the one derived in section 1 if there is no pension system so that $\{x_t = 0, z_{t+1} = 0 ; t = 0, 1, 2, ... \}$.)

By using the dynamical system with respect to k_t derived in question 3, readers are asked to numerically simulate the dynamic general equilibrium and analyze the effects of demographic changes on the dynamic general equilibrium variables of overlapping generations model with pay-as-you-go public pension system. For this purpose, we set the parameter values and the values of initial conditions as follows;
$\alpha = 0.5$; the weight of the utility of young consumption in the utility function (3.4). (The weight of the utility of old consumption in the utility function is $1 - \alpha = 0.5$.)
$\beta = 0.3$; the weight of capital in the Cobb-Douglas production function (3.6).
$k_0 = 5$; initial capital-labor ratio.
$A = 20$; the technology level in (3.6).
$N_0 = 100$; initial cohort population.

$z = 2$; pension per old individual.

In the following analyses, readers are asked to calculate endogenous variables in the dynamic general equilibrium for three different cases with respect to the cohort growth rate n; $\{n = 0.2, n = 0, n = -0.2\}$.

Question 4. Assume $n = 0.2$ so that the cohort size enlarges. Calculate the sequence of capital-labor ratio $\{k_t ; t = 0, 1, 2, \dots, T + 1\}$, where T is the last period of calculation, in the dynamic general equilibrium by following the steps presented below.

Step 1. Readers will notice that the dynamical system derived in question 3 is a first-order nonlinear difference equation with respect to capital-labor ratio k_t. Therefore, given the initial value $k_0 = 5$, readers may want to use programming software which is capable of solving the nonlinear equation with respect to k_1. (For example, "Solve" or "FindRoot" command is used to solve nonlinear equations by MATHEMATICA. Or, Newton's algorithm by FORTRUN fits to the purpose as well.)

Step 2. Given k_1 calculated in step 1, solve the same nonlinear equation for k_2.

Step 3. repeat above steps to calculate the dynamic general equilibrium sequence of capital-labor ratio $\{k_t ; t = 0, 1, 2, \dots, T + 1\}$

Question 5. Assume $n = 0.2$. Use the dynamic general equilibrium sequence of capital-labor ratio in question 4 to calculate the following dynamic general equilibrium variables; $\{r_t, w_t, c_t^y, c_{t+1}^o, u_t, y_t, Y_t ; t = 0, 1, 2, \dots, T\}$, where $Y_t = N_t y_t$ is aggregate output. In addition, calculate the old consumption in the initial period $t = 0$ by

(3.11) $c_0^o = (1 + r_0) s_0 + z$

where $s_0 = (1 + n) k_0$ by (3.7).

Question 6. Assume $n = 0$ so that the cohort size is constant at $\{N_t = 100 ; t = 0, 1, 2, \dots, T\}$. Apply the numerical simulation methods of question 4 and question 5 to calculate the dynamic general equilibrium sequence of capital-labor ratio $\{k_t ; t = 0, 1, 2, \dots, T + 1\}$, the sequence of variables in the dynamic general equilibrium $\{r_t, w_t, c_t^y, c_{t+1}^o, u_t, y_t, Y_t ; t = 0, 1, 2, \dots, T\}$, and c_0^o.

Question 7. Assume $n = -0.2$ so that the cohort size diminishes over time. Apply the numerical simulation methods of question 4 and question 5 to calculate the dynamic general equilibrium sequence of capital-labor ratio $\{k_t ; t = 0, 1, 2, \dots, T + 1\}$, the sequence of variables in the dynamic general equilibrium $\{r_t, w_t, c_t^y, c_{t+1}^o, u_t, y_t, Y_t ; t = 0, 1, 2, \dots, T\}$, and c_0^o.

Question 8. Calculate the steady state values of capital-labor ratio $k_t = k_{t+1} = k_s$ for the three cases with respect to cohort growth rate $\{n = 0.2, n = 0, n = -0.2\}$. (Readers may achieve this by solving the dynamical system with respect to capital-labor ratio of question 3 where k_t and k_{t+1} are replaced with a constant k_s.)

Question 9. Draw and overlap the graphs of dynamic general equilibrium sequences of capital-labor ratio

$\{k_t \; ; t = 0, 1, 2, \dots , T + 1\}$ for the three cases with respect to cohort growth rate $\{n = 0.2, n = 0, n = -0.2\}$ such that the horizontal axis measures time period $t = 0, 1, 2, \dots , T + 1$, and the vertical axis measures capital-labor ratio k_t.

Question 10. Draw and overlap the graphs of dynamic general equilibrium sequences of utility $\{u_t \; ; t = 0, 1, 2, \dots , T\}$ for the three cases with respect to cohort growth rate $\{n = 0.2, n = 0, n = -0.2\}$ such that the horizontal axis measures time period $t = 0, 1, 2, \dots , T + 1$, and the vertical axis measures utility u_t. Use the graphs to summarize the effects of demographic changes on the utility of each cohort in the overlapping generations model with pay-as-you-go public pension system.

Analyses: In this exercise, the optimal consumption-saving plan $\{c_t^y{}^*, c_{t+1}^o{}^*, s_{t+1}{}^*\}$ of a representative individual born in period t is solved as follows.

$$(3.12) \quad s_{t+1}{}^* = (1 - \alpha)(w_t - x) - \frac{\alpha z}{1 + r_{t+1}}$$

$$(3.13) \quad c_t^y{}^* = w_t - x - s_{t+1}{}^*$$

$$(3.14) \quad c_{t+1}^o = (1 + r_{t+1})s_{t+1}{}^* + z$$

where the interest rate r_t is equal to the marginal product of capital and the wage rate w_t is equal to the marginal product of labor.

$$(3.15) \quad r_t = \beta A k_t^{\beta - 1}$$

$$(3.16) \quad w_t = (1 - \beta) A k_t^{\beta}$$

In addition, the public pension premium x and the pension z satisfy

$$(3.17) \quad x = z / (1 + n).$$

The maximized utility is

$$(3.18) \quad u_t = \alpha \ln[c_t^y{}^*] + (1 - \alpha) \ln[c_{t+1}^o{}^*] \; .$$

The welfare level of an old individual born in period $t = -1$ is measured by her/his old consumption in period zero which is calculated by

$$(3.19) \quad c_0^o = (1 + r_0)s_0 + z$$

where s_0 satisfies

$$(3.20) \quad s_0 = (1 + n)k_0$$

by the capital market equilibrium condition (3.7). The sequence of capital-labor ratio $\{k_t \; ; t = 0, 1, 2, \dots , T + 1\}$ in dynamic general equilibrium satisfies the following nonlinear first-order difference equation.

(3.21) $0 = f(k_{t+1}, k_t)$

$$= k_{t+1} - \left\{ \frac{1}{1+n} \left[(1-\alpha)(1-\beta)Ak_t^{\beta} - \left(\frac{1-\alpha}{1+n} \right)z - \left(\frac{\alpha z}{1 + \beta Ak_{t+1}^{\beta-1}} \right) \right] \right\}$$

Given the initial capital-labor ratio k_0, (3.21) is solved for k_1. Then, given k_1, (3.21) is again used to solve for k_2. We repeat these steps to generate the sequence of capital-labor ratio in dynamic general equilibrium. Then, the result is used to calculate other endogenous variables in the dynamic general equilibrium. (Appendix shows the MATLAB codes for solving for the capital-labor ratio and other endogenous variables in the dynamic general equilibrium.)

Figure 3.2 depicts the graphs of dynamic general equilibrium capital-labor ratio $\{k_t$; t $= 0, 1, 2, \ldots ,$ T $+$ 1$\}$ for three cases with respect to cohort growth rate $\{n = 0.2, n = 0, n = -0.2\}$. (The last period is set at T $= 10$.) In this exercise, the speed of capital accumulation is faster the smaller is the cohort growth rate n. By setting $k_t = k_{t+1} = k_s$ in (3.21), we can also solve for the steady state capital-labor ratio k_s. The steady state capital-labor ratio k_s is also shown to be larger the smaller is the cohort growth rate n. (When $n =$ 0.2, $k_s = 10.8381$. When $n = 0$, $k_s = 13.9066$. When $n = -0.2$, $k_s = 18.8441$.) Figure 3.3 depicts the graphs of dynamic general equilibrium utility $\{u_t$; t $= 0, 1, 2, \ldots ,$ T$\}$ for three cases with respect to cohort growth rate $\{n = 0.2, n = 0, n = -0.2\}$. Because the cohort growth rate n is also the ratio of young individuals to old individuals $N_t/N_{t-1} = 1 + n$, when the pension per old individual z is fixed, the pension premium per young individual $x = z/(1+n)$ is larger the smaller is n. (There are less young individuals who support old individuals.) The smaller cohort growth rate, however, raises capita-labor ratio, and hence, raises output. Therefore, in this exercise, despite the higher pension premium imposition, the utility is higher the smaller is n as displayed by figure 3.3.

Figure 3.2 The Graph of Capital-Labor Ratio k_t
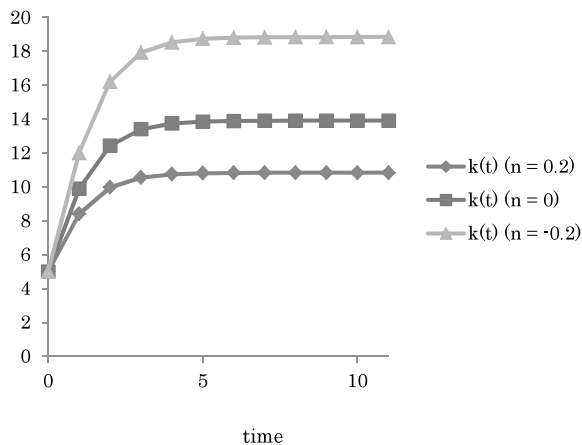
alpha $= 0.5$, beta $= 0.3$, A $= 20$, z $= 2$, k(0) $= 5$.

Figure 3.3   The Graph of Utility u_t

Figure 3.3   The Graph of Utility u_t
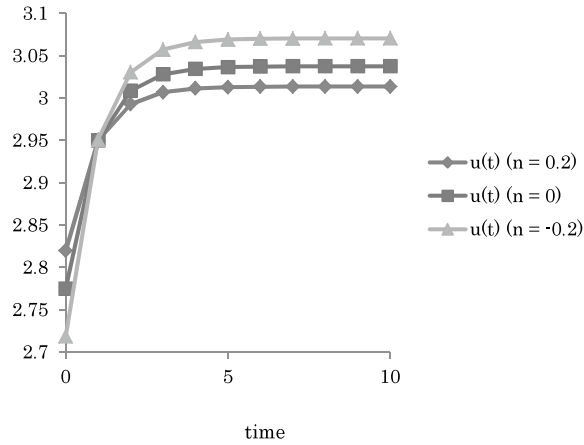
alpha $= 0.5$, beta $= 0.3$, A $= 20$, z $= 2$, k(0) $= 5$.

4．Computable Dynamic General Equilibrium Models.

The overlapping generations model in section 1, where each individual lives two periods, can be generalized as follows.

Suppose, each individual lives three periods. A representative individual born at the beginning of period t works, consumes, and saves in period t and period $t+1$, and retires, consumes, and dissaves in period $t+2$. For this individual, the budget constraints are expressed as follows.

$$(4.1)\quad c_t^1 + s_{t+1}^1 = w_t^1$$

$$(4.2)\quad c_{t+1}^2 + s_{t+2}^2 = w_{t+1}^2 + (1+r_{t+1})s_{t+1}^1$$

$$(4.3)\quad c_{t+2}^3 = (1+r_{t+2})s_{t+2}^2$$

(4.1) is "young" individual's budget constraint. c_t^1 is young consumption, s_{t+1}^1 is young saving, and w_t^1 is young labor income. (4.2) is "middle" individual's budget constraint. c_{t+1}^2 is middle consumption, s_{t+2}^2 is middle saving, and w_{t+1}^2 is middle labor income. (4.3) is "old" individual's budget constraint. c_{t+2}^3 is old consumption. (See figure 4.1.) In the overlapping generations model consisting of individuals each lives three periods, there are three overlapping generations in each period t. For example, in period t, there are N_{t-2} old individuals born two periods ago, N_{t-1} middle individuals born the previous period, and N_t young individuals born this period. The aggregate demand for consumption C_t in period t is

$$(4.4)\quad C_t = N_{t-2} \times c_t^3 + N_{t-1} \times c_t^2 + N_t \times c_t^1 \ .$$

The aggregate supply of labor L_t in period t is

(4.5) $\ L_t \ = \ N_{t-1} \times 1 \ + \ N_t \times 1 \ $.

The aggregate supply of capital S_t is

(4.6) $\ S_t \ = \ N_{t-2} \times s_t^2 \ + \ N_{t-1} \times s_t^1 \ $.

The definition of dynamic general equilibrium for this overlapping generations model with three periods lifecycles is same as that for overlapping generations model with two periods lifecycle. Each individual chooses optimal consumption-saving plan to maximize utility subject to budget constraints, each firm chooses factor demand to maximize profit, and every market clears.

A modeler may choose the length of lifecycle, depending on her/his analytical objective, and may construct overlapping generations models. For example, one may construct an overlapping generations model where each individual lives 80 periods (years) so that there are 80 overlapping generations in each period t.

Samuelson (1958) and Diamond (1965) contributed the progress in early stages of overlapping generations model analyses. Auerbach and Kotlikoff (1987) initiated this "computable dynamic general equilibrium model" to analyze effects of economic policies which involve intergenerational transactions such as social security system, deficit finance, bequest, etc. Although the computation is complex, Auerbach and Kotlikoff and their followers have reported many important results that have significant influence on policy makers and policy making processes.

Figure 4.1　Overlapping Generations Model with Three Periods Lifecycles
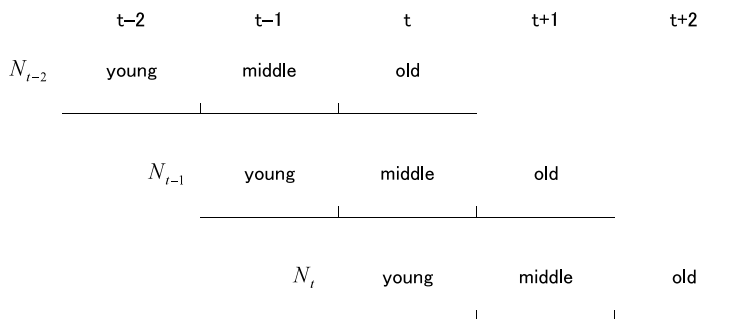
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Appendix. MATLAB Codes for Solving the Dynamic General Equilibrium Capital-Labor Ratio and Other Variables of Exercise in section 3..

```
(P1)     % DGE capital-labor ratio in a OLG model
(P2)     %  with Pay-as-you-go Public Pension System.
(P3)     global alpha beta A z n kx;
(P4)     % Parameter Values
(P5)     alpha = 0.5;
(P6)     beta = 0.3;
(P7)     A = 20;
(P8)     z = 2;
(P9)     n = 0.2;
(P10)    T = 11;
(P11)    % Solving nonlinear first-order difference equation
(P12)    % with respect to DGE capital-labor ratio k.
(P13)    k = zeros(T + 1, 1);
(P14)    k(1) = 5;
(P15)    for t = 1 : T;
(P16)    kx = k(t);
(P17)    options = optimset('Display', 'iter');
(P18)    k(t + 1) = fzero(@olg_k, [4 20], options);
(P19)    end;
(P20)    % DGE Variables.
(P21)    r = zeros(T + 1, 1);
(P22)    w = zeros(T + 1, 1);
(P23)    s = zeros(T + 1, 1);
(P24)    c_y = zeros(T, 1);
(P25)    c_o = zeros(T + 1, 1);
(P26)    u = zeros(T, 1);
(P27)    x = z/(1 + n);
(P28)    for t = 1 : T + 1 ;
(P29)    r(t) = beta*A*(k(t)^(beta-1));
(P30)    w(t) = (1-beta)*A*(k(t)^beta);
(P31)    end;
(P32)    s(1) = (1 + n)*k(1);
```

(P33)   c_o(1) = (1 + r(1))*s(1) + z;

(P34)   for t = 1 : T ;

(P35)   s(t + 1) = (1-alpha)*(w(t)-x) - (alpha*z/(1 + r(t+1)));

(P36)   c_y(t) = w(t) - x - s(t + 1);

(P37)   c_o(t + 1) = (1 + r(t+1))*s(t + 1) + z;

(P38)   u(t) = alpha*log(c_y(t)) + (1-alpha)*log(c_o(t + 1));

(P39)   end;

****

(P40)   function fk = olg_k(ky)

(P41)   global alpha beta A z n kx;

(P42)   fk1 = (1-alpha)*(1-beta)*A*(kx^beta);

(P43)   fk2 = (1-alpha)*z/(1 + n);

(P44)   fk3 = alpha*z/(1 + beta*A*(ky^(beta-1)));

(P45)   fk = ky - (1/(1 + n))*(fk1 - fk2 - fk3);

Lines (P1)~(P39) are the main program. Lines (P40)~(P45), named "olg_k.m", are sub-program which defines the function to be solved. (Omit the line names (P1) ~ (P45) when editing the codes in MATLAB editor.) The lines that begin with "%" are statements. "global" in lines (P3) and (P41) specify the parameters and variables used for the main program and the sub-program. Lines (P11)~(P19) solve recursively the nonlinear firs-order difference equation (3.21) with respect to the dynamic general equilibrium capital-labor ratio. For the options in "fzero" command, we specify the range of solution between 4 and 20. We know the initial value is $k_0 = 5$, and the steady state is $k_s = 10.838$ when the cohort growth rate is $n = 0.2$. Because $k_0 < k_s$, we also know that the sequence of dynamic general equilibrium capital-labor ratio is monotonically increasing and converging to the steady state. (Without this option, "fzero" command sometimes is unable to return the real-valued solution because the function to be solved might have complex-valued solution.) After solving for the dynamic general equilibrium capital-labor ratio, lines (P20)~(P39) calculate other dynamic general equilibrium variables $\{r_t, w_t, s_t, c_t^y, c_{t+1}^o, u_t, ; t = 0, 1, 2, ... , T\}$.