広島大学学位請求論文

Existence and Construction of Balanced Incomplete Block Designs with Pairwise Additivity

(組加法性をもつ釣合い型不完備ブロック計画の存在性と構成法)

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主 論 文

Existence and Construction of Balanced Incomplete Block Designs with Pairwise Additivity

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This is devoted to review a collection of seven papers by Matsubara and Kageyama [26, 27, 28, 29], Matsubara et al. [30] and Sawa et al. [36, 37]. Among them five [26, 27, 30, 36, 37] have been already published, while two [28, 29] have been accepted for publication. This thesis is made by Kazuki Matsubara to contribute the advancement on combinatorics in pairwise additive BIB designs and to obtain a Ph.D. In fact, Chapter 2 introduces methods of construction of pairwise additive BIB designs produced in [26, 28, 30, 36], Chapter 3 focuses on results of the exact existence shown in [26, 28, 29], Chapter 4 presents results of the asymptotic existence provided in [29], Chapter 5 particularly gives the spectrum of additive BIB designs discussed in [36], Chapter 6 investigates cyclic designs as in [27], Chapter 7 deals with a decomposition problem of incidence matrices as in [37] and Chapter 8 considers applications of pairwise additivity discussed in [26, 36].

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Contents

Chapter 1

Introduction

Combinatorial design theory mainly deals with problems about existence, constructions and enumeration of the design with certain "balance" properties. Kirkman [23] proposed the problem called "Kirkman's schoolgirl problem" (i.e., resolvable balanced incomplete block (BIB) design) and discussed block designs mathematically. On the other hand, Fisher [16] and Yates [42, 43] proposed the usage of BIB designs in agricultural experiments. Many researches of design theory were originated in the work of statistical experiments developed by them. In fact, a BIB design is one of the main interest of combinatorial design theory. The Kirkman's problem is just a resolvable BIB design with $v = 15, k = 3, \lambda = 1$. Bose [9] investigated BIB designs comprehensively, using finite fields and finite geometries, and produced several methods to create many families of designs. After his works, many researchers have extensively investigated the constructions of BIB designs and an exact existence of many designs has been shown by applying the methods of construction in [9]. On the other hand, an asymptotic existence of pairwise balanced designs has been shown by Wilson [39, 40, 41] and an asymptotic existence of other combinatorial structures has also been shown by use of Wilson's theorem.

Furthermore, many types of combinatorial designs have been proposed. Some of them discussed here are resolvable BIB designs, cyclic BIB designs, nested BIB designs, pairwise balanced designs, group divisible designs and transversal designs. In many literatures, each of designs has been studied regarding their existence, methods of construction and relationships of each other. There have been also many results of their relationships to the other combinatorial structures, graph, Latin square, code and so on.

For a set of BIB designs, the concept of pairwise additivity discussed here is a generalization of that of compatibility of Steiner triple systems proposed by Colbourn and Rosa [13], who also have shown many applications of compatibility. However, there are not many progress on the existence of BIB designs with such a property until now. The reason seems to be from such situation that the compatibility has a strong property, and thus it is not easy to construct a design with compatibility. Thus it is worthwhile to study the pairwise additivity. In this thesis, existence, constructions and applications of a set of BIB designs with the pairwise additivity are dealt with from a combinatorial point of view.

In the present chapter, the concept of pairwise additivity is introduced and its elementary properties are discussed. We also introduce some other combinatorial structures like perpendicular arrays, ordered designs, self-orthogonal latin square, pairwise balanced designs, group divisible designs, transversal designs and difference matrices.

In Chapter 2, general methods of construction of pairwise additive BIB designs will be provided as direct and recursive manners. Constructions from other combinatorial structures will be also provided.

From a necessary condition for the existence of pairwise additive BIB designs with a block size k and a coincidence number λ, it follows that $\lambda = 1$ implies $k = 2, 3$. In this set-up, Chapter 3 will discuss the exact existence of pairwise additive BIB designs with $k = 2, 3$ and $\lambda = 1$. The complete existence of 2 pairwise additive BIB designs and 3 pairwise additive BIB designs with $k = 2$ and $\lambda = 1$ will be shown by different methods. On the other hand, for 2 pairwise additive BIB designs with $k = 3$ and $\lambda = 1$, the complete existence has not been yet shown, while their existence except possibly for 12 cases will be presented. Finally, general considerations of more pairwise additive BIB designs will also be discussed.

Chapter 4 will discuss an asymptotic existence of pairwise additive minimal BIB designs. An asymptotic existence of some classes will be shown by use of Wilson's theorem through elementary number theory.

Chapter 5 will be devoted to existence and constructions of additive BIB designs, i.e., s pairwise additive BIB designs with $v = sk$. Since additive BIB designs have a quite strong property on structure, special recursive constructions will be presented. Also, for small values of parameters, a table on existence of additive BIB designs will be provided.

Chapter 6 will discuss pairwise additive BIB designs having a cyclic property. Cyclic BIB designs are closely related to difference families and other combinatorial structure with a cyclic property. Thus some direct and recursive constructions of pairwise additive cyclic BIB designs will be discussed and some classes of such cyclic designs will be provided. On the other hand, non-existence of pairwise additive cyclic BIB designs will be shown.

Chapter 7 will discuss the existence of s incidence matrices of BIB designs decomposing the all-one matrix. If such s incidence matrices have pairwise additivity, additive BIB designs can be obtained. It will be shown that any resolvable BIB designs can provide a solution for such a decomposition ploblem. A class of solution with block size 3 each of which is not resolvable will be provided by Skolem's method. It will also be shown that some solutions of the decomposition problem can be obtained from pairwise additive BIB designs.

Chapter 8 will provide some applications of pairwise additive BIB designs. Some BIB designs with different block sizes will be obtained from pairwise additive BIB designs. Some classes of multiply nested BIB designs will also be obtained from pairwise additive BIB designs.

1.1 Preliminaries

The basic design in the thesis is a BIB design. At first, the definition, some properties and some examples of a BIB design are here provided. And then, some BIB designs with special properties which are here important are introduced.

Some notations are defined. Z^+ denotes a set of positive integers, Z_n denotes a set $\{0, 1, \ldots, n-1\}$ for a positive integer n, GF(n) denotes a finite field of order n, $\boldsymbol{I}_v$ denotes the identity matrix of order v, $\boldsymbol{J}_{e \times f}$ denotes the matrix of size $e \times f$ all of whose elements are 1, $\boldsymbol{J}_v = \boldsymbol{J}_{v \times v}$ and $\lfloor x \rfloor$ ($\lceil x \rceil$) means the largest (smallest) integer y such that $x \geq y$ ($x \leq y$).

Definition 1.1.1 A *balanced incomplete block* (BIB) *design* with parameters v, b, r, k, λ is a system $(V, \mathcal{B})$ with v points ($|V| = v$) and b blocks ($|\mathcal{B}| = b$) such that

(i) each point appears in r different blocks,

(ii) each block contains k different points,

(iii) each pair of different points appears in exactly λ blocks.

This is denoted by BIBD(v, b, r, k, λ) or B(v, k, λ).

It is known that the five parameters of the BIB design satisfy

$$vr = bk, \ \lambda(v-1) = r(k-1).$$

Let $\boldsymbol{N} = (n_{ij})$ be the $v \times b$ incidence matrix of a BIB design, where $n_{ij} = 1$ or 0 for all $i \ (= 1, ..., v)$ and $j \ (= 1, ..., b)$, according as the ith point occurs in the jth block or otherwise. Hence the incidence matrix $\boldsymbol{N}$ satisfies the conditions: (i) $\sum_{j=1}^{b} n_{ij} = r$ for all i, (ii) $\sum_{i=1}^{v} n_{ij} = k$ for all j, (iii) $\sum_{j=1}^{b} n_{ij} n_{i'j} = \lambda$ for all $i, i' \ (i \neq i') = 1, ..., v$. This implies that

$$\boldsymbol{N}\boldsymbol{N}^T = (r - \lambda)\boldsymbol{I}_v + \lambda \boldsymbol{J}_v. \tag{1.1}$$

Example 1.1.2 The following system $(Z_7, \mathcal{B})$ is a B$(7, 3, 1)$:

$$\begin{aligned} \mathcal{B} \ = \ & \{\{0,1,3\},\{1,2,4\},\{2,3,5\},\{3,4,6\}, \\ & \{4,5,0\},\{5,6,1\},\{6,0,2\}\}. \end{aligned}$$

Now some BIB designs with certain properties will be introduced.

Definition 1.1.3 When v is divisible by k, a BIBD(v, b, r, k, λ) is said to be *resolvable* if its b blocks can be grouped into r resolution sets of v/k blocks each such that every point appears in each resolution set precisely once.

Example 1.1.4 The following system $(Z_9, \mathcal{B})$ forms a resolvable B(9, 3, 1):

$$\begin{aligned} \mathcal{B} \ = \ & \{[\{0,1,2\},\{3,4,5\},\{6,7,8\}],[\{0,3,6\},\{1,4,7\},\{2,5,8\}], \\ & [\{0,4,8\},\{1,5,6\},\{2,3,7\}],[\{0,5,7\},\{1,3,8\},\{2,4,6\}]\}. \end{aligned}$$

where [ ] denotes a resolution set of blocks.

In a BIB design $(V, \mathcal{B})$, let σ be a permutation on V. For a block $B = \{v_1, \ldots, v_k\} \in \mathcal{B}$ and a permutation σ on V, let $B^\sigma = \{v_1^\sigma, \ldots, v_k^\sigma\}$. When $\mathcal{B} = \{B^\sigma | B \in \mathcal{B}\}$, σ is called an *automorphism* of $(V, \mathcal{B})$.

Definition 1.1.5 In a BIB design $(V, \mathcal{B})$, if there exists an automorphism of order $|V|$, then the BIB design is said to be *cyclic*.

In a cyclic $\mathrm{B}(v, k, \lambda)$, the set V of points can be identified with Z_v. The *block orbit* containing $B \in \mathcal{B}$ is a set of distinct blocks $B + i = \{v_1 + i, \ldots, v_k + i\} \pmod{v}$ for $i \in Z_v$. A block orbit is said to be *full* or *short* according as whether $|\{B+i|0 \le i \le v-1\}| = v$ or not. Choose an arbitrary block from each block orbit and call it an *initial block*. Example 1.1.2 is a cyclic $\mathrm{B}(7, 3, 1)$, that is, an initial block is $\{0, 1, 3\} \pmod 7$. Especially, $\mathrm{PC}(s)$ means a partial cycle (short block orbit) of order s, i.e., only $0, 1, \ldots, s-1$ are to be added to the initial block.

If a short block orbit contains a block $\{0, v/k, 2v/k, \ldots, (k-1)v/k\}$, then the short block orbit is said to be *regular*. Then any $\mathrm{B}(2t, 2, 1)$, $t \in Z^+$, must have a regular short initial block $\{a, a+t\}\mathrm{PC}(t)$ with an arbitrary $a \in Z_{2t}$. This fact will be used in Sections 6.6 and 6.7.

Preece [33] introduced the concept of a *nested BIB design* (NBIBD), denoted by $\mathrm{NB}(v; b_1, b_2; k_1, k_2)$, in the statistical experiment whose spectrum is referred to [31]. An $\mathrm{NB}(v; b_1, b_2; k_1, k_2)$ is a triple $(V, \mathcal{B}_1, \mathcal{B}_2)$ with v points $(|V| = v)$ and two systems of blocks $(|\mathcal{B}_i| = b_i)$, $i = 1, 2$, such that

(i) the first system is nested within the second, i.e., each block in $\mathcal{B}_2$ is partitioned into u subblocks of size k_1 and the resulting subblocks form $\mathcal{B}_1$, say, $b_1 = ub_2$ and $k_2 = uk_1$,

(ii) $(V, \mathcal{B}_i)$ is a BIB design with v points and b_i blocks of k_i points each, $i = 1, 2$.

Generally a *multiply nested BIB design* [31] is an $(m+1)$-tuple $(V, \mathcal{B}_1, \ldots, \mathcal{B}_m)$ with v points and m systems of blocks $(|\mathcal{B}_i| = b_i)$, $i = 1, \ldots, m$, such that

(i) the jth system is nested within the ith system, $i > j$,

(ii) for each i, $1 \le i \le m$, $(V, \mathcal{B}_i)$ is a BIB design with v points, b_i blocks of k_i points each.

Such a design is denoted by $\mathrm{MNB}(v; b_1, \ldots, b_m; k_1, \ldots, k_m)$. Especially when $m = 3$, it is called a *doubly nested BIB design*. This concept will be used in Section 8.2.

A nested BIB design $(V, \mathcal{B}_1, \mathcal{B}_2)$ is said to be *cyclic*, denoted by $\mathrm{CNB}(v; b_1, b_2; k_1, k_2)$, if both of $(V, \mathcal{B}_1)$ and $(V, \mathcal{B}_2)$ are cyclic with respect to the same automorphism $\sigma : i \longmapsto i + 1 \pmod{v}$ (cf. [21]).

Example 1.1.6 The following system $(Z_{12}, \mathcal{B}_1, \mathcal{B}_2)$ is a CNB$(12; 44, 22; 3, 6)$:

$$\mathcal{B} : \{0, 1, 3|2, 6, 8\}, \{4, 5, 9|7, 10, \infty\} \quad \text{mod } 11,$$

where each { } denotes initial blocks in $\mathcal{B}_2$ and | partitions into subblocks in $\mathcal{B}_1$.

For $v \equiv 1$ (or 0) (mod 4), a *whist tournament* Wh(v) is a schedule of $v(v-1)/4$ games (a, b, c, d), where the unordered pairs $\{a, c\}, \{b, d\}$ are called *partners*, the pairs $\{a, b\}$, $\{c, d\}$ *opponents of the first kind,* and the pairs $\{a, d\}, \{b, c\}$ *opponents of the second kind*, such that

(i) the games are arranged into v (or $v-1$) rounds of $(v-1)/4$ (or $v/4$) games each,

(ii) each player plays in exactly one game in each round,

(iii) each player has every other player as a partner exactly once,

(iv) each player has every other player as an opponent exactly twice.

A *triplewhist tournament* TWh(v) is a Wh(v) with, in the stead of the above condition (iv), another condition such that

(v) each player has every other player as an opponent of the first kind exactly once, and that of the second kind exactly once (cf. [7]).

Note [7] that a necessary and sufficient condition for the existence of Wh(v) is $v \equiv 0, 1$ (mod 4). When $v \equiv 1$ (mod 4), a Wh(v) is said to be *Z-cyclic* if the players are the elements of Z_v and the round $i+1$ is obtained from the round i by adding 1 (mod v) to each element. Since each of players and each of games can be regarded as a point and a block, respectively, it can be seen that any Z-cyclic Wh(v) with $v \equiv 1$ (mod 4) itself shows a CNB$(v; v(v-1)/2, v(v-1)/4; 2, 4)$ with initial blocks $\{a, c|b, d\}$. Some results of Z-cyclic Wh(v) will be used in Chapter 6. On the other hand, Z-cyclic Wh(v) with $v \equiv 0$ (mod 4) are also introduced in [7]. However, since its point set is $\{0, 1, \ldots, v-2\} \cup \{\infty\}$ and each of initial blocks is developed on Z_{v-1}, such designs cannot be cyclic. Hence this case is not discussed in Chapter 6.

1.2 Pairwise additive BIB designs

In a set of BIB designs, a property of pairwise additivity that is main in the thesis is introduced. Let $s = v/k$, where s need not be an integer unlike other parameters on block designs.

Definition 1.2.1 A set of ℓ BIBD(v, b, r, k, λ) is called ℓ *pairwise additive BIB designs*, denoted by ℓ PAB(v, k, λ), if ℓ corresponding incidence matrices $\boldsymbol{N}_1, \ldots, \boldsymbol{N}_\ell$ $(2 \leq \ell \leq s)$ of the BIB design satisfy that $\boldsymbol{N}_{i_1} + \boldsymbol{N}_{i_2}$ is the incidence matrix of a BIBD$(v^* = v = sk,\ b^* = b,\ r^* = 2r,\ k^* = 2k,\ \lambda^* = 2r(2k-1)/(sk-1))$ for any distinct $i_1, i_2 \in \{1, 2, ..., \ell\}$. When $\ell = s$, this is called *additive BIB designs*, denoted by AB(v, k, λ).

It is clear by Definition 1.2.1 that the existence of ℓ PAB(v, k, λ) implies the existence of ℓ' PAB(v, k, λ) for any $2 \leq \ell' < \ell$. Thus, for given parameters v, k, λ, the larger ℓ is, the more meaningful a construction problem of ℓ pairwise additive BIB designs is. In general a construction problem of additive BIB designs is the most difficult for given parameters v, k, λ.

Definition 1.2.2 In ℓ PAB(v, k, λ), if $\boldsymbol{N}_1, \ldots, \boldsymbol{N}_\ell$ are cyclic and the jth initial block of $\boldsymbol{N}_{i_1} + \boldsymbol{N}_{i_2}$ is a set-union of the jth initial blocks of $\boldsymbol{N}_{i_1}$ and $\boldsymbol{N}_{i_2}$ for any distinct $i_1, i_2 \in \{1, 2, ..., \ell\}$ and $1 \leq j \leq \lceil b/v \rceil$, then the ℓ PAB(v, k, λ) is said to be *cyclic*, denoted by ℓ PACB(v, k, λ). When $\ell = s$, this is called *additive cyclic BIB designs*, denoted by ACB(v, k, λ).

Note that, in pairwise additive BIB designs, all $\boldsymbol{N}_i$'s are incidence matrices of BIB designs with the same parameters. Further note that additive BIB designs satisfy the condition:

$$\boldsymbol{J}_{v \times b} = \sum_{i=1}^{s} \boldsymbol{N}_i. \tag{1.2}$$

Now, two examples are presented as follows.

Example 1.2.3 Developing the following blocks on Z_4 yields an ACB(4,2,1):

$$\begin{aligned} \boldsymbol{N}_1 &: \{0,1\}, \{0,2\}\text{PC}(2) \quad \bmod 4 \\ \boldsymbol{N}_2 &: \{2,3\}, \{1,3\}\text{PC}(2) \quad \bmod 4. \end{aligned}$$

Example 1.2.4 Developing the following blocks on Z_8 yields an ACB(8,2,1):

$$\begin{aligned}
\boldsymbol{N}_1 &: \{0,1\},\{0,2\},\{0,3\},\{0,4\}\mathrm{PC}(4) \pmod 8\\
\boldsymbol{N}_2 &: \{4,5\},\{1,3\},\{2,5\},\{2,6\}\mathrm{PC}(4) \pmod 8\\
\boldsymbol{N}_3 &: \{3,6\},\{6,7\},\{1,7\},\{1,5\}\mathrm{PC}(4) \pmod 8\\
\boldsymbol{N}_4 &: \{2,7\},\{4,5\},\{4,6\},\{3,7\}\mathrm{PC}(4) \pmod 8.
\end{aligned}$$

Furthermore, the following Lemma 1.2.5 shows a structural property of the pairwise additivity.

Lemma 1.2.5 [36] Let ℓ, s and t be positive integers with $1 \leq t \leq \ell \leq s$. Let $\boldsymbol{N}_1, \ldots, \boldsymbol{N}_\ell$ be incidence matrices of ℓ PAB(v,k,λ). Then, for any distinct $i_1, \ldots, i_t \in \{1, \ldots, \ell\}$, $\boldsymbol{N}_{i_1} + \ldots + \boldsymbol{N}_{i_t}$ is an incidence matrix of a B$(v, tk, \lambda t(tk-1)/(k-1))$.

Note that an application of this property will be discussed in Chapter 8.

1.3 Necessary conditions

It is known that necessary conditions for the existence of a B(v,k,λ) are

$$\lambda(v-1) \equiv 0 \pmod{k-1},\ \ \lambda v(v-1) \equiv 0 \pmod{k(k-1)}. \tag{1.3}$$

In pairwise additive B$(v,\ k,\ \lambda)$, a sum of any two incidence matrices yields a BIB design, the parameters of which are given by

$$v^* = v, b^* = b, r^* = 2r, k^* = 2k, \lambda^* = 2r(2k-1)/(v-1).$$

Observe that

$$\lambda^* = \frac{2r(2k-1)}{v-1} = \frac{2\lambda(2k-1)}{k-1}.$$

Since $k-1$ and $2k-1$ are relatively prime, as a necessary condition for the existence of pairwise additive BIB designs, the following can be obtained:

$$2\lambda \equiv 0 \pmod{k-1}. \tag{1.4}$$

In view of (1.4), it is seen that for pairwise additive $\mathrm{B}(v, k, \lambda)$,

$$\lambda \geq \begin{cases} (k-1)/2 & (k \text{ is odd}), \\ k-1 & (k \text{ is even}). \end{cases}$$

We now introduce the minimality of pairwise additive BIB designs.

Definition 1.3.1 Pairwise additive $\mathrm{B}(v, k, \lambda)$ are said to be *minimal* if $\lambda = (k-1)/2$ or $k-1$ according as k is odd or even.

The exact and the asymptotic existence of pairwise additive minimal BIB designs will be discussed in Chapters 3 and 4, respectively.

1.4 Other combinatorial structures

Some combinatorial structures which here play an important role are introduced.

A *perpendicular array* (PA), denoted by $\mathrm{PA}_d(g, s)$, is a matrix with g rows and $d\binom{s}{2}$ columns such that every unordered pair of an s-set appears in exactly d columns among every two rows, where $g \geq 1$ (cf. [8]). When $d = 1$, we simply write $\mathrm{PA}(g, s)$. If $(t_1 + 1, \ldots, t_g + 1)^T \pmod v$ is a column of the $\mathrm{PA}(g, v)$ for any column $(t_1, \ldots, t_g)^T$ of the $\mathrm{PA}(g, v)$, then the $\mathrm{PA}(g, v)$ is said to be *cyclic*, denoted by $\mathrm{CPA}(g, v)$. Choose an arbitrary column from each set of columns $(t_1 + i, \ldots, t_g + i)^T$, $1 \leq i \leq g$, and call it an *initial column*.

An *ordered design* (OD), denoted by $\mathrm{OD}_d(g, s)$, is a matrix with g rows and $2d\binom{s}{2}$ columns such that every ordered pair of an s-set appears in exactly d columns among every two rows, where $g \geq 1$ (cf. [8]). When $d = 1$, we simply write $\mathrm{OD}(g, s)$.

Let $A = (a_{ij})$ and $B = (b_{ij})$ are two latin squares of order v. The latin squares A and B are said to be *orthogonal* if all ordered pairs (a_{ij}, b_{ij}) are distinct. Latin squares $A_1, ..., A_\ell$ are called *mutually orthogonal latin squares of order* v, denoted by $\mathrm{MOLS}(v)$, if they are orthogonal in each pair. A *self-orthogonal latin square of order* v, denoted by $\mathrm{SOLS}(v)$, is a latin square that is orthogonal to its transpose. A set $\{L_1, ..., L_\ell\}$ of $\mathrm{SOLS}(v)$ is called ℓ $\mathrm{SOLS}(v)$, if $\{L_1, L_1^T, \ldots, L_\ell, L_\ell^T\}$ is a set of 2ℓ $\mathrm{MOLS}(v)$. A latin square $A = (a_{ij})$ is said to be *idempotent* if $a_{ii} = i$ where $1 \leq i \leq v$. It is known [17] that any latin squares in ℓ $\mathrm{SOLS}(v)$ can be made idempotent by renaming the symbol.

Some existence of 2 $\mathrm{SOLS}(v)$ is known as follows.

Lemma 1.4.1 [1] There are 2 SOLS(v) for any $v \geq 7$ except possibly for $v \in \{10, 12, 14, 18, 21, 22, 24, 30, 34\}$.

A *pairwise balanced design* (PBD) of order v with block sizes in a set K is a system $(V, \mathcal{B})$ with v points ($|V| = v$) and b blocks ($|\mathcal{B}| = b$) such that

(i) if $B \in \mathcal{B}$, then $|B| \in K$,

(ii) each pair of different points appears in exactly λ blocks of $\mathcal{B}$.

This is denoted by PBD(v, K, λ) (cf. [32]).

Some existence of PBD is known as follows.

Lemma 1.4.2 [12] There are PBD$(v, \{8, 9, 10\}, 1)$ for any positive integers $v \geq 583$.

Especially, for a set $P_{1,6}$ being a set of prime powers of form $6m + 1$, the existence of PBD$(v, P_{1,6}, 1)$ is shown as follows.

Lemma 1.4.3 ([13 ; Theorem 19.69]) Let $P_{1,6}$ be a set of prime powers of form $6m+1$ with a positive integer m. Then there are PBD$(v, P_{1,6}, 1)$ for any positive integers $v \equiv 1 \pmod 6$, except possibly for $6m + 1 \in \{55, 115, 145, 205, 235, 265, 319, 355, 391, 415, 445, 451, 493, 649, 667, 685, 697, 745, 781, 799, 805, 1315\}$.

A *group divisible design* of type $g_1^{a_1} g_2^{a_2} \ldots g_t^{a_t}$ and index $\{\lambda_1, \lambda_2\}$ with block sizes in a set K is a triple $(V, \mathcal{G}, \mathcal{B})$ such that

(i) V is a set of $a_1 g_1 + \ldots + a_t g_t$ points,

(ii) $\mathcal{G}$ is a partition of V into a_i groups of size g_i, $1 \leq i \leq t$,

(iii) $\mathcal{B}$ is a family of k-subsets (blocks) of V, $k \in K$,

(iv) every unordered pair of elements from the same group is contained in exactly λ_1 blocks,

(v) every unordered pair of elements from different groups is contained in exactly λ_2 blocks.

A group divisible design is used in the proof of Theorem 5.2.2.

A group divisible design of type n^k and index $\{0, \lambda\}$ with a block size k is said to be a *transversal design*, denoted by TD$_\lambda(k, n)$.

When $\lambda = 1$, we simply write TD(k, n), where $|\mathcal{B}| = n^2$

Since it is known [2] that the existence of a $\mathrm{TD}(k+2, n)$ is equivalent to the existence of k $\mathrm{MOLS}(n)$, the following result can be obtained for $k = 4$.

Lemma 1.4.4 [2] There exists a $\mathrm{TD}(6, n)$ for any $n \geq 5$ except for $n = 6$ and possibly for $n \in \{10, 14, 18, 22\}$.

A *difference matrix*, denoted by $\mathrm{DM}_d(g, s) = (a_{mn})$, based on a group $(G, *)$ of order s is a $g \times ds$ matrix such that the multiset $\{a_{mn} * a_{m'n}^{-1} | n = 1, \ldots, ds\}$ contains every element of G precisely d times for distinct $m, m' \in \{1, \ldots, g\}$. When $d = 1$, we simply write $\mathrm{DM}(g, s)$ (cf. [14]).

The difference matrices of symmetric type and of cyclic type will be defined in Sections 5.2 and 6.1 respectively, and they will be utilized there.

Chapter 2

Methods of construction

In this chapter, several methods of construction of pairwise additive BIB designs are provided.

2.1 Direct construction

The following fundamental and simple method is useful to construct pairwise additive BIB designs with specific parameters.

At first, perpendicular arrays are utilized.

Lemma 2.1.1 The existence of a PA(g, s) implies the existence of $\lfloor g/k \rfloor$ PAB$(v = s, k, k(k-1)/2)$ for any positive integer $k \leq g/2$.

Proof. Let PA$(g, s) = (p_{ij})$ for $1 \leq i \leq g$ and $1 \leq j \leq s(s-1)/2$. It can be shown that the following blocks yield incidence matrices of the required BIB design:

$$\boldsymbol{N}_t : \{p_{kt-k+1,j}, p_{kt-k+2,j}, \ldots, p_{kt}\}$$

where $1 \leq t \leq \lfloor g/k \rfloor$ and $1 \leq j \leq s(s-1)/2$. $\square$

On the other hand, some classes of PA(g, s) are obtained as Lemmas 2.1.2 and 2.1.3 show.

Lemma 2.1.2 There exists a PA(p^n, p^n) for any odd prime p and any positive integer n.

Proof. Let $\boldsymbol{x} = (x_1, \ldots, x_{p^n})^T$ be a $p^n \times 1$ column vector such that $x_i \neq x_j$ for distinct i and j. Put $y\boldsymbol{x} := (yx_1, \ldots, yx_{p^n})^T$ for y in GF(p^n) and

$$\boldsymbol{A} = \left(\alpha\boldsymbol{x}, \alpha^2\boldsymbol{x}, \ldots, \alpha^{\frac{p^n-1}{2}}\boldsymbol{x}\right),$$

where α is a primitive element of GF(p^n). Developing columns of $\boldsymbol{A}$ modulo GF(p^n) can yield a PA(p^n, p^n). $\square$

The structure of the above-mentioned array $\boldsymbol{A}$ is used in the proof of Theorem 2.4.1.

Lemma 2.1.3 [8] There exists a PA$(4, v)$ for any odd integer $v \geq 5$.

Hence Lemmas 2.1.1, 2.1.2 and 2.1.3 can provide some classes of pairwise additive BIB designs as Theorems 2.1.4 and 2.1.5 show.

Theorem 2.1.4 [30] There are AB$(p^n, p^{n-1}, p^{n-1}(p^{n-1}-1)/2)$ for any odd prime power p^n.

Theorem 2.1.5 There are 2 PAB$(v, 2, 1)$ and $\lfloor p^n/k \rfloor$ PAB$(p^n, k, k(k-1)/2)$ for any odd integer $v \geq 5$, any integer k and any odd prime power p^n.

Proof. Because of the existence of a PA$(4, v)$ and a PA(p^n, p^n) on account of Lemmas 2.1.2 and 2.1.3, the proof is completed by applying Lemma 2.1.1. $\square$

Next ordered designs are utilized to construct pairwise additive BIB designs for an even integer v.

Lemma 2.1.6 The existence of an OD(g, s) implies the existence of $\lfloor g/k \rfloor$ PAB$(v = s, k, k(k-1))$ for any positive integer $k \leq g/2$.

Proof. Let OD$(g, s) = (q_{ij})$ for $1 \leq i \leq g$ and $1 \leq j \leq s(s-1)$. It can be shown that the following blocks yield incidence matrices of the required BIB design:

$$\boldsymbol{N}_t : \{q_{kt-k+1,j}, q_{kt-k+2,j}, \ldots, q_{kt}\}$$

where $1 \leq t \leq \lfloor g/k \rfloor$ and $1 \leq j \leq s(s-1)$. $\square$

Similarly to Lemma 2.1.2, a class of OD(g, s) is obtained as Lemma 2.1.7 shows.

Lemma 2.1.7 [30] There exists an $\mathrm{OD}(2^n, 2^n)$ for any positive integer n.

Hence Lemmas 2.1.6 and 2.1.7 show the existence of additive BIB designs with $k = 2$ as Theorem 2.1.8 shows.

Theorem 2.1.8 [30] There are $\mathrm{AB}(2^n, 2, 2)$ for any positive integer n.

Note that additive BIB designs obtained in Theorems 2.1.4 and 2.1.8 are not minimal. However, Theorem 2.1.8 will be improved as in Theorem 5.1.2.

Finally, an individual new design on $Z_3 \times Z_3$ is given.

Lemma 2.1.9 There are AB(10,2,1).

Proof. A development of the following initial blocks on $Z_3 \times Z_3$ can yield incidence matrices $\boldsymbol{N}_1, \boldsymbol{N}_2, \ldots, \boldsymbol{N}_5$ of the required BIB design:

$$\begin{array}{rcl}
\boldsymbol{N}_1 & : & \{(1,1),(1,2)\},\{(1,1),(2,1)\},\{(1,1),(2,2)\},\{(0,0),\infty\}, \\
 & & \{(1,2),(2,1)\} \quad \mathrm{mod}\ (3,3) \\
\boldsymbol{N}_2 & : & \{(2,1),(2,2)\},\{(1,2),(2,2)\},\{(0,0),\infty\},\{(1,2),(2,1)\}, \\
 & & \{(1,1),(2,2)\} \quad \mathrm{mod}\ (3,3) \\
\boldsymbol{N}_3 & : & \{(0,1),(0,2)\},\{(0,0),\infty\},\{(0,1),(2,0)\},\{(0,1),(1,0)\}, \\
 & & \{(1,0),(2,0)\} \quad \mathrm{mod}\ (3,3) \\
\boldsymbol{N}_4 & : & \{(0,0),\infty\},\{(1,0),(2,0)\},\{(0,2),(1,0)\},\{(0,2),(2,0)\}, \\
 & & \{(0,1),(0,2)\} \quad \mathrm{mod}\ (3,3) \\
\boldsymbol{N}_5 & : & \{(1,0),(2,0)\},\{(0,1),(0,2)\},\{(1,2),(2,1)\},\{(1,1),(2,2)\}, \\
 & & \{(0,0),\infty\} \quad \mathrm{mod}\ (3,3).
\end{array}$$

□

Note that $\mathrm{AB}(10, 2, 1)$ mean 5 $\mathrm{PAB}(10, 2, 1)$. Further note that the existence of $\mathrm{AB}(10, 2, 1)$ will play an important role in Chapter 3.

2.2 Recursive construction

Some recursive constructions of pairwise additive BIB designs are here provided by use of $\mathrm{TD}(k\ell, n)$.

Theorem 2.2.1 The existence of ℓ PAB(v,k,λ), ℓ PAB(v',k,λ) and a TD$(k\ell,v)$ implies the existence of ℓ PAB(vv',k,λ).

Proof. Let $\mathcal{B}_h, \mathcal{B}'_h$, $1 \leq h \leq \ell$, be block sets of ℓ PAB(v,k,λ) and ℓ PAB(v',k,λ), respectively, as

$$\begin{aligned}\mathcal{B}_h &= \left\{\{x_{1,i}^{(h)}, x_{2,i}^{(h)}, \ldots x_{k,i}^{(h)}\} | 1 \leq i \leq \frac{\lambda v(v-1)}{k(k-1)}\right\}, \\ \mathcal{B}'_h &= \left\{\{y_{1,j}^{(h)}, y_{2,j}^{(h)}, \ldots, y_{k,j}^{(h)}\} | 1 \leq j \leq \frac{\lambda v'(v'-1)}{k(k-1)}\right\}\end{aligned}$$

and let $d(m,n)$, $1 \leq m \leq v^2$ and $1 \leq n \leq k\ell$, denote an element which occurs in both the mth block of a TD$(k\ell,v)$ and the nth group. Then it can be shown that the following ℓ incidence matrices yield the required ℓ pairwise additive BIB designs with vv' elements denoted by (s,t) for $1 \leq s \leq v$ and $1 \leq t \leq v'$:

$$\begin{aligned}\boldsymbol{N}_h \ : \ & \{(x_{1,i}^{(h)}, t), (x_{2,i}^{(h)}, t), \ldots, (x_{k,i}^{(h)}, t)\}, \\ & \{(d(m, hk-k+1), y_{1,j}^{(h)}), (d(m, hk-k+2), y_{2,j}^{(h)}), \\ & \ \ldots, (d(m, hk), y_{k,j}^{(h)})\},\end{aligned}$$

where $1 \leq h \leq \ell, 1 \leq i \leq \lambda v(v-1)/[k(k-1)], 1 \leq j \leq \lambda v'(v'-1)/[k(k-1)], 1 \leq t \leq v'$ and $1 \leq m \leq v^2$. $\square$

Corollary 2.2.2 The existence of ℓ PAB(v_i,k,λ) and TD$(k\ell, v_{i'})$ for $1 \leq i \leq t$ and $2 \leq i' \leq t$ implies the existence of ℓ PAB$(v_1 v_2 \cdots v_t, k, \lambda)$ for some t.

Proof. This result can be shown by applying Theorem 2.2.1 with ℓ PAB$(v_1 v_2 \cdots v_{i-1}, 2, 1)$, ℓ PAB(v_i, k, λ) and TD$(2\ell, v_i)$ for $2 \leq i \leq t$. $\square$

Another recursive method is presented.

Theorem 2.2.3 The existence of ℓ PAB$(v+1,k,\lambda)$, ℓ PAB(v',k,λ) and a TD$(k\ell,v)$ implies the existence of ℓ PAB$(vv'+1,k,\lambda)$.

Proof. Let $\mathcal{B}'_h$, $1 \leq h \leq \ell$, be a block set similarly to the proof of Theorem 2.2.1 and let $\mathcal{B}_h$, $1 \leq h \leq \ell$, be a block set of ℓ PAB$(v+1,k,\lambda)$, where

$$\mathcal{B}_h = \{\{x_{1,i}^{(h)}, x_{2,i}^{(h)}, \ldots, x_{k,i}^{(h)}\} | 1 \leq i \leq \frac{\lambda v(v+1)}{k(k-1)}\},$$

with $v+1$ elements $1, 2, \cdots, v$ and ∞. Also let $d(m,n)$, $1 \le m \le v^2$ and $1 \le n \le k\ell$, denote an element which occurs in both the mth block of a TD$(k\ell, v)$ and the nth group. Then the following ℓ incidence matrices can yield the required ℓ pairwise additive BIB designs with $vv'+1$ elements denoted by (s,t) for $1 \le s \le v$ and $1 \le t \le v'$, and ∞ $(= (\infty, t))$:

$$\begin{aligned}\boldsymbol{N}_h \quad : \quad & \{(x_{1,i}^{(h)}, t), (x_{2,i}^{(h)}, t), \ldots, (x_{k,i}^{(h)}, t)\}, \\ & \{(d(m, hk-k+1), y_{1,j}^{(h)}), (d(m, hk-k+2), y_{2,j}^{(h)}), \\ & \ldots, (d(m, hk), y_{k,j}^{(h)})\},\end{aligned}$$

where $1 \le h \le \ell, 1 \le i \le \lambda v(v+1)/[k(k-1)], 1 \le j \le \lambda v'(v'-1)/[k(k-1)], 1 \le t \le v'$ and $1 \le m \le v^2$. $\square$

Corollary 2.2.4 The existence of ℓ PAB(v_i, k, λ), ℓ PAB(v_t+1, k, λ) and TD$(k\ell, v_{i'})$ for $1 \le i \le t-1$ and $2 \le i' \le t$ implies the existence of ℓ PAB$(v_1 v_2 \cdots v_t + 1, k, \lambda)$ for some t.

Proof. This can be shown by applying Corollary 2.2.2 and Theorem 2.2.3 with ℓ PAB$(v_1 v_2 \cdots v_{i-1}, k, \lambda)$, ℓ PAB(v_i, k, λ), ℓ PAB(v_t+1, k, λ) and TD$(k\ell, v_{i'})$. Here $2 \le i \le t-1$ and $2 \le i' \le t$. $\square$

2.3 Construction from PBDs

A construction of ℓ PAB(v, k, λ) from PBD are here provided for $\ell \ge 2$.

The following fundamental and simple method is useful to construct pairwise additive BIB designs with specific parameters.

Lemma 2.3.1 The existence of a PBD(v, K, λ) and ℓ PAB$(v' = k, k', \lambda')$ for any $k \in K$ implies the existence of ℓ PAB$(v, k', \lambda\lambda')$.

Proof. Let B be a block of the PBD(v, K, λ). Then on the set B, all blocks with block size k' are formed by the ℓ PAB$(v' = k, k', \lambda')$. Hence it follows that these blocks obtained from each block of the PBD(v, K, λ) can yield the required BIB design. $\square$

Since a BIB design can be seen as a PBD with $K = \{k\}$, some classes of 2 PAB$(v, 2, 1)$ are constructed.

Lemma 2.3.2 [20] There exists a $\mathrm{B}(v,4,1)$ if and only if $v \equiv 1,4 \pmod{12}$.

Theorem 2.3.3 [26] Let $v \equiv 1,4 \pmod{12}$. Then there are 2 $\mathrm{PAB}(v,2,1)$.

Lemmas 1.4.2 and 2.3.1 also yield the following.

Theorem 2.3.4 [29] There are 4 $\mathrm{PAB}(v,2,1)$ for any integer $v \geq 583$.

As the next case of block sizes, $k=3$ is considered. A concept of $2m$ $\mathrm{PAB}(6m+1,3,1)$ has been discussed as a compatibly nested minimal partition in [13], which shows the existence of $\mathrm{PAB}(6m+1,3,1)$ as follows.

Lemma 2.3.5 ([13 ; Theorem 22.12]) Let $6m+1$ be an odd prime power for a positive integer m. Then there are $2m$ $\mathrm{PAB}(6m+1,3,1)$.

Lemmas 1.4.3, 2.3.1 and 2.3.5 can produce the following.

Theorem 2.3.6 ([13 ; Theorem 22.13]) There are 2 $\mathrm{PAB}(6m+1,3,1)$ for any positive integers m, except possibly for $6m+1 \in \{55, 115, 145, 205, 235, 265, 319, 355, 391, 415, 445, 451, 493, 649, 667, 685, 697, 745, 781, 799, 805, 1315\}$.

Theorem 2.3.6 will be improved as in Theorem 3.3.4.

2.4 Construction from nested BIB designs

Some constructions of pairwise additive BIB designs through nested BIB designs with a perpendicular array are here presented.

Theorem 2.4.1 Let s' $(\leq s)$ be an odd prime power and an $\mathrm{NB}(v = sk_1; b_1 = s'b_2, b_2; k_1, k_2 = s'k_1)$ exists. Then s' pairwise additive $\mathrm{BIBD}(v = sk_1, b = (s'-1)b_1/2, r = (s'-1)b_1/(2s), k = k_1, \lambda = (k_1-1)(s'-1)b_1/[2s(sk_1-1)])$ exist.

Proof. Since s' is an odd prime power, a $\mathrm{PA}(s',s')$ of size $s' \times [s'(s'-1)/2]$ exists on account of Lemma 2.1.2. Let the symbols in this $\mathrm{PA}(s',s')$ be from the set $\{1,\ldots,s'\}$. As shown in Lemma 2.1.2, we may assume that every symbol occurs in each row of the PA. For convenience of representation, let (m,n)-entry of the $\mathrm{PA}(s',s')$ be $p(m,n)$ $(1 \leq m \leq s', 1 \leq n \leq s'(s'-1)/2)$. For two systems of blocks, $|\mathcal{B}_i| = b_i$, $i = 1,2$, in the NBIBD, blocks are $B^{(h)} \in \mathcal{B}_2$ $(1 \leq h \leq b_2)$, and subblocks are $B_f^{(h)} \in \mathcal{B}_1$ $(1 \leq h \leq b_2, 1 \leq f \leq s')$. Here $\bigcup_{1\leq f\leq s'} B_f^{(h)} = B^{(h)}$.

Now let

$$\mathcal{B}_j^* = \left(\boldsymbol{B}_j^{(1)} : \cdots : \boldsymbol{B}_j^{(b_2)}\right)$$

where $\boldsymbol{B}_j^{(h)}$ are the juxtaposition of subblocks in $B^{(h)}$ with indices being entries of PA(s', s'), i.e.,

$$\boldsymbol{B}_j^{(h)} = \left(B_{p(j,1)}^{(h)} : \cdots : B_{p(j,s'(s'-1)/2)}^{(h)}\right), 1 \leq h \leq b_2, 1 \leq j \leq s'.$$

Then it follows that $(V, \mathcal{B}_j^*)$ is s' pairwise additive BIBD$(v = sk_1, b = (s'-1)b_1/2, r = (s'-1)b_1/(2s), k = k_1, \lambda = (k_1-1)(s'-1)b_1/[2s(sk_1-1)])$. $\square$

The following example illustrates Theorem 2.4.1 with an NB$(12; 66, 22; 2, 6)$.

Example 2.4.2 Developing the following blocks on Z_{11} gives an NB$(12; 66, 22; 2, 6)$ over $Z_{11} \cup \{\infty\}$:

$$\{\infty, 4|1, 3|5, 9\}, \{0, 6|7, 8|2, 10\} \quad \text{mod } 11$$

and take the following PA$(3, 3)$:

$$\begin{array}{ccc} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{array}$$

Now, by associating the symbols 1, 2 and 3 in this array with the first, second and third subblocks respectively in each of base blocks of the above NB$(12; 66, 22; 2, 6)$, 3 PAB$(12, 2, 1)$ can be obtained as taking the following incidence matrices:

$$\begin{array}{lll} \boldsymbol{N}_1 & : & \{\infty, 4\}, \{1, 3\}, \{5, 9\}, \{0, 6\}, \{7, 8\}, \{2, 10\} \quad \text{mod } 11 \\ \boldsymbol{N}_2 & : & \{5, 9\}, \{\infty, 4\}, \{1, 3\}, \{2, 10\}, \{0, 6\}, \{7, 8\} \quad \text{mod } 11 \\ \boldsymbol{N}_3 & : & \{1, 3\}, \{5, 9\}, \{\infty, 4\}, \{7, 8\}, \{2, 10\}, \{0, 6\} \quad \text{mod } 11. \end{array}$$

Since a resolvable B(sk, k, λ) can be regarded as a nested BIB design with $k_2 = sk_1 = v$ and the existence of a PA(s, s) with an odd prime power s is shown, a class of additive BIB designs can be obtained. Furthermore, by using an OD(s, s) obtained in Lemma 2.1.7 instead of a PA(s, s) in Theorem 2.4.1, another class of additive BIB designs can be obtained for a prime power s as follows.

Corollary 2.4.3 (i) For any odd prime power $s \geq 3$, if there exists a resolvable B(sk, k, λ), then there exist AB$(sk, k, (s-1)\lambda/2)$. (ii) For any prime power $s' \geq 3$, if there exists a resolvable B$(s'k, k, \lambda)$, then there exist AB$(s'k, k, (s'-1)\lambda)$.

On the other hand, it is known that resolvable B$(p^n, p^{n-1}, (p^{n-1} - 1)/(p-1))$ exist for any odd prime power p, by use of a finite affine geometry. Hence a class of additive minimal BIB designs can be obtained by Corollary 2.4.3.

Theorem 2.4.4 [36] There exist additive minimal B$(p^n, p^{n-1}, (p^{n-1} - 1)/2)$ for any odd prime power p.

2.5 Construction from self-orthogonal latin squares

A construction of pairwise additive BIB designs through self-orthogonal latin squares are here presented.

Here $\ell + 1$ PAB$(v, 2, 1)$ are obtained through ℓ SOLS(v) as follows.

Theorem 2.5.1 The existence of ℓ SOLS(v) implies the existence of $\ell+1$ PAB$(v, 2, 1)$ for a positive integer ℓ.

Proof. Let a set of 2ℓ idempotent MOLS(v) be $\{L_h, L_h^T | 1 \leq h \leq \ell\}$, where $L_h = (a_{ij}^{(2h-1)})$ and $L_h^T = (a_{ji}^{(2h-1)}) = (a_{ij}^{(2h)})$, derived from the ℓ SOLS(v). Let $\ell + 1$ incidence matrices of B$(v, 2, 1)$ be

$$\begin{array}{lcl} \boldsymbol{N}_0 & : & \{i, j\} \\ \boldsymbol{N}_h & : & \{a_{ij}^{(2h-1)}, a_{ij}^{(2h)}\} \end{array}$$

where $1 \leq i < j \leq v$. Then, for $1 \leq h < h' \leq \ell$, the $v(v-1)/2$ ordered pairs of $\{a_{ij}^{(2h-1)}, a_{ij}^{(2h'-1)}\}$ and $\{a_{ij}^{(2h)}, a_{ij}^{(2h')}\}$ are distinct from each other, because of the property of orthogonality of idempotent latin squares L_h and $L_{h'}$. Similarly, the $v(v-1)/2$ ordered pairs of $\{a_{ij}^{(2h-1)}, a_{ij}^{(2h')}\}$ and $\{a_{ij}^{(2h)}, a_{ij}^{(2h'-1)}\}$ are distinct from each other, because of the property of orthogonality of idempotent latin squares L_h and $L_{h'}^T$. Hence it follows that $\boldsymbol{N}_h + \boldsymbol{N}_{h'}$ forms a $\mathrm{B}(v, 4, 6)$. Finally, for $1 \le h \le \ell$, $\boldsymbol{N}_0 + \boldsymbol{N}_h$ can also form a $\mathrm{B}(v, 4, 6)$, because of the property of idempotent latin squares. Hence the $\ell + 1$ incidence matrices $\boldsymbol{N}_0, \boldsymbol{N}_h$, for $1 \le h \le \ell$, can yield the required design. $\square$

Note that 2ℓ idempotent $\mathrm{MOLS}(v)$ without self-orthogonality can also yield $\ell + 1$ $\mathrm{PAB}(v, 2, 2)$. However, since the $\mathrm{PAB}(v, 2, 2)$ is not minimal, this case is not discussed in the thesis.

Chapter 3

Pairwise additive BIB designs

When $\lambda = 1$ the condition (1.4) implies that $k = 2$ and 3. Thus the exact existence of pairwise additive minimal B$(v, k, 1)$ with $k = 2, 3$ is here discussed.

3.1 2 pairwise additive B$(v, 2, 1)$

Theorem 2.1.5 produces 2 PAB$(v, 2, 1)$ for an odd integer v. At first some constructions of 2 PAB$(v, 2, 1)$ are presented for even integer v.

Theorem 3.1.1 For any odd prime p, there are 2 PAB$(p + 1, 2, 1)$.

Proof. It follows that a development of the following initial blocks on Z_p can yield incidence matrices of the required BIB design:

$$\begin{array}{rcl} \boldsymbol{N}_1 & : & \{0, 1\}, \{0, \infty\}, \{0, \alpha^i\} \quad \bmod p \\ \boldsymbol{N}_2 & : & \{2, \infty\}, \{2, 3\}, \{2\alpha^i, 3\alpha^i\} \quad \bmod p \end{array}$$

with a primitive element α of Z_p and $1 \leq i \leq \frac{p-3}{2}$. $\square$

When $p = 11$ Theorem 3.1.1 illustrates Example 3.1.2.

Example 3.1.2 The following incidence matrices $\boldsymbol{N}_1, \boldsymbol{N}_2$ form 2 PAB(12, 2, 1).

$$\begin{array}{rcl} \boldsymbol{N}_1 & : & \{0, 1\}, \{0, \infty\}, \{0, 2\}, \{0, 4\}, \{0, 8\}, \{0, 5\} \quad \bmod 11 \\ \boldsymbol{N}_2 & : & \{2, \infty\}, \{2, 3\}, \{4, 6\}, \{8, 1\}, \{5, 2\}, \{10, 4\} \quad \bmod 11. \end{array}$$

Theorem 3.1.3 Let p (≥ 5) be an odd prime power. Then there are 2 PAB$(3p+1,2,1)$.

Proof. Let (s,t) and ∞ be $3p+1$ elements, where $0 \leq s \leq p-1$ and $0 \leq t \leq 2$. Then it follows that a development of the following initial blocks on GF$(p)\times Z_3$ can yield incidence matrices $\boldsymbol{N}_1, \boldsymbol{N}_2$ of the required 2 pairwise additive BIB designs:

$$\begin{array}{rcl} \boldsymbol{N}_1 & : & \{(\alpha^i,0),(\alpha^{i+1},1)\}, \{(\alpha^{i'},0),(-\alpha^{i'},0)\}, \\ & & \{(0,0),(0,1)\}, \{(0,2),\infty\} \quad \bmod (p,3) \\ \boldsymbol{N}_2 & : & \{(\alpha^{i+2},0),(\alpha^{i+3},1)\}, \{(\alpha^{i'+1},1),(-\alpha^{i'+1},1)\}, \\ & & \{(0,2),\infty\}, \{(0,0),(0,1)\} \quad \bmod (p,3) \end{array}$$

with a primitive element α of GF(p). Here $1 \leq i \leq p-1$ and $1 \leq i' \leq (p-1)/2$. $\square$

When $p = 7$, Theorem 3.1.3 illustrates Example 3.1.4.

Example 3.1.4 The following incidence matrices $\boldsymbol{N}_1, \boldsymbol{N}_2$ form 2 PAB$(22, 2,1)$.

$$\begin{array}{rcl} \boldsymbol{N}_1 & : & \{(3,0),(2,1)\}, \{(2,0),(6,1)\}, \{(6,0),(4,1)\}, \{(4,0),(5,1)\}, \\ & & \{(5,0),(1,1)\}, \{(1,0),(3,1)\}, \{(3,0),(4,0)\}, \{(2,0),(5,0)\}, \\ & & \{(6,0),(1,0)\}, \{(0,0),(0,1)\}, \{(0,2),\infty\} \quad \bmod (7,3) \\ \boldsymbol{N}_2 & : & \{(6,0),(4,1)\}, \{(4,0),(5,1)\}, \{(5,0),(1,1)\}, \{(1,0),(3,1)\}, \\ & & \{(3,0),(2,1)\}, \{(2,0),(6,1)\}, \{(2,1),(5,1)\}, \{(6,1),(1,1)\}, \\ & & \{(4,1),(3,1)\}, \{(0,2),\infty\}, \{(0,0),(0,1)\} \quad \bmod (7,3). \end{array}$$

Furthermore, for an even integer v, recursive constructions like Theorems 2.2.1 and 2.2.3 can yield 2 PAB$(v,2,1)$.

Now the main result of this section is shown.

Theorem 3.1.5 For any $v \geq 4$, there are 2 PAB$(v,2,1)$.

Proof. This is made by the mathematical induction on v. When $v = 4$, the required 2 PAB$(4,2,1)$ can be obtained by Example 1.2.3. Next, for any odd integer $v \geq 5$, Theorem 2.1.5 can produce 2 PAB$(v,2,1)$. Now assume that v is an even integer. (i) If $v-1$ is not divisible by 3 and not prime, 2 pairwise additive BIB designs can be obtained by applying

Lemma 1.4.4, Theorems 2.1.5 and 3.1.1 and Corollary 2.2.4. (ii) If $v-1$ is an odd prime, Theorem 3.1.1 can produce 2 pairwise additive BIB designs. (iii) If $v-1 = mp$ $(m \geq 5, p \geq 5)$ with an odd integer m and an odd prime power p, 2 pairwise additive BIB designs can be obtained by applying Theorem 2.2.3 with 2 PAB$(m, 2, 1)$, 2 PAB$(p+1, 2, 1)$ obtained by the mathematical induction and an TD$(4, p)$ obtained by applying Lemma 1.4.4. It is now seen that the remaining cases are (iv) $v-1 = 9, 27, 3q$, where q (≥ 5) is an odd prime.

(a) When $v-1 = 9$, Lemma 2.1.9 shows the result. (b) When $v-1 = 27$, i.e., $v = 28 = 4 \times 7$, Theorem 2.2.1 with Example 1.2.3, Lemma 1.4.4 and Theorem 2.1.5 shows the result. (c) When $v-1 = 3q$, Theorem 3.1.3 shows the result.

Therefore, 2 PAB$(v, 2, 1)$ can be constructed for any $v \geq 4$ $\square$

3.2 3 pairwise additive B$(v, 2, 1)$

The complete existence of 3 pairwise additive BIB designs is shown for all $v \geq 6$.

The existence of nested BIB designs with specific parameters is known below.

Theorem 3.2.1 [15] The necessary and sufficient condition for the existence of NB$(v; b_1 = v(v-1)/2, b_2 = v(v-1)/6; k_1 = 2, k_2 = 6)$ is that $v \equiv 0, 1 \pmod 3$ and $v \geq 6$.

Thus, Theorems 2.4.1 and 3.2.1 can show the following.

Theorem 3.2.2 There are 3 PAB$(v, 2, 1)$ for all $v \equiv 0, 1 \pmod 3$ and $v \geq 6$.

Furthermore, Lemma 1.4.1 and Theorem 2.5.1 with $\ell = 2$ can produce the following.

Theorem 3.2.3 There are 3 PAB$(v, 2, 1)$ for any $v \geq 7$ except possibly for $v \in \{10, 12, 14, 18, 21, 22, 24, 30, 34\}$.

The methods discussed here do not cover a case of $v = 14$. Hence it is individually constructed.

Lemma 3.2.4 There are 3 PAB(14,2,1).

Proof. A development of the following initial blocks on Z_{13} can yield three incidence matrices $\boldsymbol{N}_1, \boldsymbol{N}_2, \boldsymbol{N}_3$ of the required BIB design:

$$\begin{array}{lll} \boldsymbol{N}_1 & : & \{0,\infty\},\{0,1\},\{0,2\},\{0,3\},\{0,4\},\{0,5\},\{0,6\} \quad \bmod 13 \\ \boldsymbol{N}_2 & : & \{2,3\},\{2,\infty\},\{4,6\},\{6,9\},\{8,12\},\{10,2\},\{12,5\} \quad \bmod 13 \\ \boldsymbol{N}_3 & : & \{4,5\},\{4,6\},\{10,\infty\},\{12,2\},\{3,7\},\{7,12\},\{11,4\} \quad \bmod 13. \end{array}$$

□

Finally, the main result of this section is established.

Theorem 3.2.5 For any $v \geq 6$, there are 3 PAB$(v,2,1)$.

Proof. When $v \neq 6, 10, 12, 14, 18, 21, 22, 24, 30, 34$, Theorem 3.2.3 shows the existence of 3 PAB$(v,2,1)$. When $v = 6, 10, 12, 18, 21, 22, 24, 30, 34$, Theorem 3.2.2 shows the existence of 3 PAB$(v,2,1)$. When $v = 14$, the 3 PAB$(v,2,1)$ are given in Lemma 3.2.4. Hence the proof is completed. □

Note that the existence of 4 PAB$(v,2,1)$ was discussed in Theorem 2.3.4. However, it seems to be difficult to show the existence of such designs for any $v \geq 8$.

3.3 2 pairwise additive B$(v,3,1)$

By (1.3), the existence of PAB$(v,3,1)$ implies $v \equiv 1,3 \pmod 6$. The exact existence of 2 PAB$(v,3,1)$ with $v \equiv 1,3 \pmod 6$ is here discussed by providing direct and recursive methods of construction of PAB$(v,3,1)$. Finally, it is shown that there are 2 PAB$(v,3,1)$ for any $v \equiv 1 \pmod 6$ except possibly for 12 values.

Two classes of pairwise additive B$(v,3,1)$ are given as in Lemma 2.3.5 and Theorem 2.3.6. Also, for any integer $n \geq 2$, the existence of AB$(3^n,3,1)$ can be shown as follows.

Theorem 3.3.1 [35] There are AB$(3^n,3,1)$ for any positive integer $n \geq 2$.

For $v \equiv 1, 3 \pmod 6$, 15 is the smallest value of v for which the existence of 2 PAB$(v, 3, 1)$ is unknown in the above result. Hence at first this case is individually constructed here.

Lemma 3.3.2 There are 2 PAB(15,3,1).

Proof. It can be shown that a development of the following initial blocks on Z_7 with the index being fixed yields incidence matrices $\boldsymbol{N}_1, \boldsymbol{N}_2$ of the required BIB design:

$$\begin{aligned} \boldsymbol{N}_1 \;&:\; \{0_0, 1_1, 6_1\}, \{0_0, 2_1, 5_1\}, \{0_0, 3_1, 4_1\}, \\ &\quad\; \{1_0, 2_0, 4_0\}, \{0_0, 0_1, \infty\} \quad \text{mod } 7 \\ \boldsymbol{N}_2 \;&:\; \{0_1, 2_0, 5_0\}, \{0_1, 3_0, 4_0\}, \{0_1, 1_0, 6_0\}, \\ &\quad\; \{0_0, 0_1, \infty\}, \{1_1, 2_1, 4_1\} \quad \text{mod } 7 \end{aligned}$$

with 15 elements $\{i_j | i \in Z_7, j \in Z_2\} \cup \{\infty\}$. $\square$

Now more 2 PAB$(v, 3, 1)$ are obtained by recursive constructions.

Lemma 3.3.3 There are 2 PAB$(v, 3, 1)$ for $v \in \{235, 391, 445, 451, 649, 685, 745, 781, 799, 805\}$.

Proof. For $v = 235, 391, 445, 451, 649, 685, 745, 781, 799, 805,$ Theorem 2.2.3 with

$$\begin{aligned} (v, v') \;=\; & (26, 9), (30, 13), (12, 37), (30, 15), (8, 81), \\ & (36, 19), (24, 31), (60, 13), (42, 19), (12, 67) \end{aligned}$$

provides the required BIB designs, respectively, because 2 PAB$(v+1, 3, 1)$ and 2 PAB$(v', 3, 1)$ are obtained by use of Lemmas 2.3.5 and 3.3.2 and Theorem 3.3.1, and a TD$(6, v)$ is also obtained by Lemma 1.4.4. $\square$

Hence on account of Lemma 3.3.3, the following result can be obtained. This improves Theorem 2.3.6.

Theorem 3.3.4 There are 2 PAB$(6m + 1, 3, 1)$ for any positive integer m, except possibly for $6m+1 \in \{55, 115, 145, 205, 265, 319, 355, 415, 493, 667, 697, 1315\}$.

Unfortunately, we cannot clear such 12 values displayed in Theorem 3.3.4. Furthermore, the existence of 2 PAB$(6m + 3, 3, 1)$ is not known except for $6m+3$ being 3^n and 15 as in Theorem 3.3.1 and Lemma 3.3.2.

3.4 General consideration

Note that most of methods obtained in previous sections can be used to construct ℓ PAB(v,k,λ), even if $\ell \geq 3$ and $\lambda \geq 2$. Thus many classes of ℓ PAB(v,k,λ) can be obtained. However, it is not easy to construct pairwise additive minimal BIB designs and to show the existence of such designs. On the other hand, the asymptotic existence of pairwise balanced designs has been shown by Wilson [41] for any given integers k and λ. Then in the same sense as [41] the asymptotic existence of pairwise additive minimal BIB designs will be discussed in the next chapter.

In PAB(v,k,λ), it is clear that the existence of ℓ PAB$(v,k,1)$ implies the existence of ℓ PAB(v,k,λ) for any $\lambda \geq 2$. For example, the complete existence of 3 PAB$(v,2,\lambda)$ for any $\lambda \geq 2$ is shown by the results given in Section 3.2. On the other hand, for $\ell \geq 4$, it is not easy to show the complete existence of ℓ PAB$(v,2,1)$. Especially, the existence of 3 PAB$(12,2,1)$ is known as in Example 2.4.2, while whether ℓ PAB$(12,2,1)$ with $\ell \in \{4,5,6\}$ exist or not is not known in literature.

It is also clear that the existence of ℓ PAB(v,k,λ) implies the existence of ℓ' $(\leq \ell)$ PAB(v,k,λ). Hence, for given integers v $(= sk)$, k and λ, it is the most difficult to construct AB(v,k,λ). This case will be discussed in Chapter 5.

Finally it is noted that we cannot find parameters v, k and λ in which ℓ (≥ 2) PAB(v,k,λ) with necessary conditions (1.3) and (1.4) do not exist.

Chapter 4

Asymptotic existence

In the previous chapter, the exact existence of pairwise additive BIB designs was provided. In this chapter, an asymptotic existence of ℓ pairwise additive minimal BIB designs with the block size k for any given positive integers ℓ and k is discussed. Especially, an asymptotic existence of PBD (Wison's theorem) and Lemma 2.3.1 are utilized.

4.1 Wilson's theorem

At first the existence of PBD(v, K, λ) is reviewed along with necessary and asymptotically sufficient conditions.

Let K be a set of positive integers and let

$$\alpha(K) = \gcd\{k - 1 | k \in K\},\ \beta(K) = \gcd\{k(k - 1) | k \in K\}.$$

Then necessary conditions for the existence of a PBD(v, K, λ) are known as follows.

Lemma 4.1.1 [40] Necessary conditions for the existence of a PBD(v, K, λ) are

$$\lambda(v - 1) \equiv 0 \pmod{\alpha(K)},\ \lambda v(v - 1) \equiv 0 \pmod{\beta(K)}. \tag{4.1}$$

Furthermore, the asymptotic existence is shown as follows.

Theorem 4.1.2 [41] The necessary conditions (4.1) for the existence of a PBD(v, K, λ) are asymptotically sufficient.

For any set K of positive integers and any positive integer λ, let $c(K, \lambda)$ denote the smallest integer such that there are PBD(v, K, λ) for every integer $v \geq c(K, \lambda)$ satisfying (4.1). Then note that Theorem 4.1.2 reveals the evidence of the existence of $c(K, \lambda)$. On the other hand, some explicit bounds for $c(K, \lambda)$ are provided as in Lemmas 1.4.2 and 1.4.3.

4.2 A class of pairwise additive B$(v, k, (k-1)/2)$

A necessary condition for the existence of pairwise additive B$(v, k, (k-1)/2)$ being minimal is here provided and then some classes of $(v-1)/k$ PAB$(v, k, (k-1)/2)$ and (k pairwise) AB$(v = k^2, k, (k-1)/2)$ are also provided.

Now the conditions (1.3) imply that necessary conditions for the existence of PAB$(v, k, (k-1)/2)$ are

$$v - 1 \equiv 0 \pmod 2, \;\; v(v-1) \equiv 0 \pmod{2k}.$$

Furthermore, the following is given.

Theorem 4.2.1 When k is an odd prime power, a necessary condition for the existence of PAB$(v, k, (k-1)/2)$ is

$$v \equiv 1, k \pmod{2k}. \tag{4.2}$$

Proof. Since $v(v-1) \equiv 0 \pmod{2k}$ and $\gcd(v, v-1) = 1$, when k is an odd prime power, it is shown that either $v \equiv 0$ or $v - 1 \equiv 0 \pmod k$. Hence $v - 1 \equiv 0 \pmod 2$ implies that $v \equiv 1, k \pmod{2k}$. $\square$

When k is an odd prime power, a class of PAB$(v, k, (k-1)/2)$ is obtained as follows. This observation shows a generalization of Lemma 2.3.5.

Theorem 4.2.2 Let both $2km + 1$ and k be odd prime powers for a positive integer m. Then there are $2m$ $(= (v-1)/k)$ PAB$(2km+1, k, (k-1)/2)$.

Proof. It can be shown that a development of the following initial blocks on GF$(2km + 1)$ yields incidence matrices $\boldsymbol{N}_1, \boldsymbol{N}_2, \ldots, \boldsymbol{N}_{2m}$ of the required BIB design:

$$\begin{array}{rcl} \boldsymbol{N}_i & : & \{\alpha^i, \alpha^{2m+i}, \alpha^{4m+i}, \ldots, \alpha^{2(k-1)m+i}\}, \\ & & \{\alpha^{i+1}, \alpha^{2m+i+1}, \alpha^{4m+i+1}, \ldots, \alpha^{2(k-1)m+i+1}\}, \\ & & \quad\vdots \\ & & \{\alpha^{i+m-1}, \alpha^{3m+i-1}, \alpha^{5m+i-1}, \ldots, \alpha^{(2k-1)m+i-1}\}, \end{array}$$

where α is a primitive element of $\mathrm{GF}(2km+1)$ and $1 \leq i \leq 2m$. $\square$

When $k = 3$, a class of $\mathrm{PAB}(v, 3, 1)$ is discussed in Theorem 3.3.1.

4.3 Asymptotic existence of minimal BIB designs

Let k be an odd prime power throughout this chapter. An asymptotic existence of $\mathrm{PAB}(v, k, (k-1)/2)$ is here discussed, and it is shown that the necessary conditions (4.2) for the existence of ℓ $\mathrm{PAB}(v, k, (k-1)/2)$ are asymptotically sufficient for a given positive integer ℓ ($\leq k$).

Dirichlet's theorem on primes is useful for the present discussion.

Theorem 4.3.1 (Dirichlet) If $\gcd(a, b) = 1$, then a set of integers of the following form

$$an + b, \ n = 1, 2, \ldots$$

contains infinitely many primes.

Now Theorem 4.3.1 yields the following.

Lemma 4.3.2 [19] For any positive even integer m, there are primes p and q for which $p \equiv q \equiv 1 \pmod{m}$ and $\gcd(p(p-1), q(q-1)) = m$.

In the proof of Lemma 4.3.2 (i.e., Lemma 3.4 in [19]), primes p and q are obtained by using Theorem 4.3.1. Thus Lemma 4.3.2 implies the existence of sufficiently large primes p and q as follows.

Lemma 4.3.3 For a given odd prime power k, there are primes p and q such that (a) $p > q > k^2$, (b) $p \equiv q \equiv 1 \pmod{2k}$, (c) $\alpha(K) = 2$ and (d) $\beta(K) = 2k$ for $K = \{p, q\} \cup \{k^2\}$.

Proof. Since k is an odd prime power, for an even integer $2k$, Lemma 4.3.2 provides primes p and q such that (a) $p > q > k^2$, (b) $p \equiv q \equiv 1 \pmod{2k}$ and $\gcd(p(p-1), q(q-1)) = 2k$. Hence it is seen that $\gcd(p-1, q-1) = 2k$, $\gcd(2k, k^2-1) = 2$ and $\gcd(2k, k^2(k^2-1)) = 2k$.

Now let $K = \{p, q\} \cup \{k^2\}$. Then

$$\begin{aligned}\alpha(K) &= \gcd\{p-1, q-1, k^2-1\} = 2,\\ \beta(K) &= \gcd\{p(p-1), q(q-1), k^2(k^2-1)\} = 2k,\end{aligned}$$

which imply (c) and (d). $\square$

Thus one of the main results of this chapter is now obtained through conditions (a), (b), (c) and (d) given in Lemma 4.3.3.

Theorem 4.3.4 For a given odd prime power k, (4.2) is a necessary and asymptotically sufficient condition for the existence of k PAB$(v, k, (k-1)/2)$.

Proof (sufficiency). Let p and q be primes as in Lemma 4.3.3 with $K = \{p, q\} \cup \{k^2\}$. Then conditions (c) and (d) show that there are PBD$(v, K, 1)$ for sufficiently large v satisfying (4.2), on account of Theorem 4.1.2. Conditions (a) and (b) show that there are $(p-1)/k$ $(\geq k)$ PAB$(p, k, (k-1)/2)$, $(q-1)/k$ $(\geq k)$ PAB$(q, k, (k-1)/2)$ and AB$(k^2, k, (k-1)/2)$, on account of Theorems 2.4.4 and 4.2.2. Hence the required design can be obtained on account of Lemma 2.3.1. $\square$

Unfortunately, by use of Theorem 4.3.4 we cannot show the existence of ℓ PAB$(v, k, (k-1)/2)$ for $\ell > k$, since an AB$(k^2, k, (k-1)/2)$ means k PAB$(k^2, k, (k-1)/2)$.

Next, for a given odd prime power k and a given positive integer ℓ, even if $\ell > k$, the existence of ℓ PAB$(v, k, (k-1)/2)$ is discussed for sufficiently large $v \equiv 1 \pmod{2k}$.

Lemma 4.3.5 For a given odd prime power k and a given positive integer ℓ, there are primes p and q such that (a) $p > q > k\ell$, (b) $p \equiv q \equiv 1 \pmod{2k}$ and (c) $\alpha(\{p, q\}) = \beta(\{p, q\}) = 2k$.

Proof. Since k is an odd prime power and ℓ is a positive integer, for a positive integer $2k$, Lemma 4.3.2 provides primes p and q such that (a) $p > q > k\ell$, (b) $p \equiv q \equiv 1 \pmod{2k}$ and $\gcd(p(p-1), q(q-1)) = 2k$.

Hence it is seen that $\gcd(p-1, q-1) = 2k$ and (c) holds. $\square$

Thus the following result is obtained through conditions (a), (b) and (c) as in Lemma 4.3.5.

Theorem 4.3.6 For a given odd prime power k and a given positive integer ℓ, there are ℓ PAB$(v, k, (k-1)/2)$ for sufficiently large $v \equiv 1 \pmod{2k}$.

Proof. Let p and q be primes as in Lemma 4.3.5. Then it follows from (c) that there are PBD$(v, \{p, q\}, 1)$ for sufficiently large $v \equiv 1 \pmod{2k}$, on account of Theorem 4.1.2. Also Theorem 4.2.2 along with conditions (a) and (b) shows that there are $(p-1)/k$ $(\geq \ell)$ PAB$(p, k, (k-1)/2)$ and $(q-1)/k$ $(\geq \ell)$ PAB$(q, k, (k-1)/2)$. Thus the required designs are obtained on account of Lemma 2.3.1. $\square$

Finally, the existence of ℓ PAB$(v, 3, 1)$ with sufficiently large $v \equiv 1, 3 \pmod 6$ is obtained, even if $\ell > k = 3$ as follows. This shows an extension of Theorem 4.3.4 with $k = 3$.

Theorem 4.3.7 For a given positive integer ℓ, even if $\ell > 3$, there are ℓ PAB$(v, 3, 1)$ with sufficiently large $v \equiv 1, 3 \pmod 6$.

Proof. Let n be a positive integer satisfying $3^n \geq 3\ell$. Then (3^{n-1} pairwise) AB$(3^n, 3, 1)$ are constructed by Theorem 3.3.1, and there are primes p and q such that $p > q > 3\ell$, $p \equiv q \equiv 1 \pmod 6$ and $\gcd(p(p-1), q(q-1)) = 6$, on account of Lemma 4.3.2. Furthermore, since $\alpha(K) = 2$ and $\beta(K) = 6$ with $K = \{p, q\} \cup \{3^n\}$, there are PBD$(v, \{p, q, 3^n\}, 1)$ for sufficiently large $v \equiv 1, 3 \pmod 6$, on account of Theorem 4.1.2. Hence ℓ PAB$(v, 3, 1)$ for sufficiently large $v \equiv 1, 3 \pmod 6$ can be constructed by use of Lemma 2.3.1 with $(p-1)/3$ $(\geq \ell)$ PAB$(p, 3, 1)$, $(q-1)/3$ $(\geq \ell)$ PAB$(q, 3, 1)$ and AB$(3^n, 3, 1)$. $\square$

Remark 4.3.8 Since Theorem 4.2.2 can be valid for a given odd integer k, Theorem 4.3.6 can be extended for a given odd integer k. On the other hand, when k is an even prime power, an asymptotic existence of pairwise additive minimal B$(v, 2^n, 2^n - 1)$ can be proved by utilizing several methods similar to Theorems 2.4.4, 4.2.2, 4.3.4 and 4.3.6.

Chapter 5

Additive BIB designs

Some existence and constructions of additive BIB designs are discussed. Especially, since additive BIB designs satisfy (1.2), the recursive constructions different from Chapter 2 are presented.

5.1 Additive B$(2^n, 2, 1)$

At first a construction of AB$(2^n, 2, 1)$ for any integer $n \geq 3$ is presented by use of Theorem 2.5.1.

Lemma 5.1.1 There are $2^{n-1} - 1$ SOLS(2^n) for any integer $n \geq 3$.

Proof. For each $\beta \in$GF(2^n), 2^n squares $L_\beta = (a_{xy}^{(\beta)})$ are defined by $a_{xy}^{(\beta)} = \beta x + (\beta + 1)y$ with a numbering of rows and columns as

$$0 := x_0, \alpha^1 := x_1, \ldots, \alpha^{2^n-1}(= 1) := x_{2^n-1}$$

in order, where α is a primitive element of GF(2^n). Then it is shown that $\{L_\beta\}_{\beta \in \mathrm{GF}(2^n)\setminus\{0,1\}}$ forms $2^{n-1} - 1$ SOLS(2^n). $\square$

Thus Theorems 2.5.1 and 5.1.1 provide a class of additive BIB designs.

Theorem 5.1.2 There exist AB$(2^n, 2, 1)$ for any positive integer $n \geq 2$.

5.2 Recursive construction of additive BIB designs

A recursive construction of additive minimal BIB designs is shown by use of difference matrices with symmetric property. A difference matrix

$\mathrm{DM}_d(s,g)$ on G is said to be *symmetric*, if given the mth and the m'th rows except a row of all-x for $x \in G$, and $\alpha, \beta \in G$,

$$|\{n|(a_{mn}, a_{m'n}) = (\alpha, \beta)\}| = |\{n|(a_{mn}, a_{m'n}) = (\beta, \alpha)\}|.$$

Such a difference matrix is denoted by $\mathrm{SDM}_d(s,g)$. When $d = 1$, this is simply denoted by $\mathrm{SDM}(s,g)$.

The following lemma shows a property of pairwise additive BIB designs.

Lemma 5.2.1 [36] Let $\boldsymbol{N}_1$ and $\boldsymbol{N}_2$ be incidence matrices of pairwise additive BIBD(sk, b, r, k, λ). Then it holds that

$$\boldsymbol{N}_1\boldsymbol{N}_2^T + \boldsymbol{N}_2\boldsymbol{N}_1^T = (\lambda^* - 2\lambda)(\boldsymbol{J}_v - \boldsymbol{I}_v) = \frac{2kr}{v-1}(\boldsymbol{J}_v - \boldsymbol{I}_v), \tag{5.1}$$

where λ^* is the coincidence number of a BIBD$(sk, b, 2r, 2k, \lambda^*)$.

We now present a recursive construction, which plays a key role on arguments in this Chapter.

Theorem 5.2.2 Let c and d be positive integers with $2\lambda c \equiv 0 \pmod{d(k-1)}$. Suppose that an $\mathrm{SDM}_c(s,s)$ based on a group of order s and a $\mathrm{PA}_d(s,s)$ exist. Then the existence of additive BIBD$(v = sk, b, r, k, \lambda)$ implies the existence of additive BIBD$(v^* = s^2k, b^* = cs[(s+1)r - s\lambda], r^* = c[(s+1)r - s\lambda], k^* = sk, \lambda^* = cr)$.

Proof. Let $G = \{g_1, \dots, g_s\}$ be a group of order s. Suppose that there exist additive BIBD$(v = sk, b, r, k, \lambda)$, say, $(V, \mathcal{B}_{g_i})$, on s incidence matrices $\boldsymbol{N}_{g_i}$, $i = 1, \dots, s$, an $\mathrm{SDM}_c(s,s) = (a_{mn})$ and $\mathrm{PA}_d(s,s)$ both are based on G. Also assume that there is no row with all entries x for some x on the $\mathrm{SDM}_c(s,s)$.

For given $g_i \in G$, take the $\mathrm{SDM}^{(i)} = (a_{mn}^{(i)})$ defined by $a_{mn}^{(i)} = g_i a_{mn}$ and then replace $a_{mn}^{(i)}$ by the submatrix $\boldsymbol{N}_{a_{mn}^{(i)}}$ to have an $sv \times scb$ incidence matrix. This SDM substitution step produces a group divisible design, say, D_{g_i}, of type v^s and index $\{cs\lambda, cr\}$.

Next, since $2\lambda c \equiv 0 \pmod{d(k-1)}$, it holds that $2c(r - s\lambda)/[d(s-1)]$ is a positive integer. Take a $\mathrm{PA}_d(s,s)$ based on G, and then replace every element g_i $(\in G)$ of the PA by the submatrix $\boldsymbol{J}_{v \times 2c(r-s\lambda)/[d(s-1)]}$ and all other elements by the zero-matrix $\boldsymbol{O}$ of size $v \times 2c(r - s\lambda)/[d(s-1)]$, to get an $sv \times d\binom{s}{2}2c(r - s\lambda)/[d(s-1)]$ incidence matrix. Hence, the PA substitution part gives a group divisible design, say, $\overline{D}_{g_i}$, of type v^s and index $\{cr - cs\lambda, 0\}$. Thus, each of s matrices $\boldsymbol{N}^*_{g_i}$ defined by

$$\boldsymbol{N}^*_{g_i} = \left(\boldsymbol{D}_{g_i} : \overline{\boldsymbol{D}}_{g_i}\right)$$

becomes an incidence matrix of a BIB design, where $\boldsymbol{D}_{g_i}$ and $\overline{\boldsymbol{D}}_{g_i}$ denote incidence matrices of designs D_{g_i} and $\overline{D}_{g_i}$, respectively.

It remains to show that these s BIB designs $D^*_{g_i} = (V^*, \mathcal{B}^*_{g_i})$ form additive BIB designs. Let $V^* = \{(\mu,\nu) | \mu = 1,\ldots,s, \nu = 1,\ldots,v\}$ be a point set of $D^*_{g_i}$, where the νth point in the μth group of D_{g_i} or $\overline{D}_{g_i}$ is represented by (μ,ν). For given t, t' with $t \neq t'$ and given $(\mu,\nu), (\mu',\nu')$, we shall count the number of blocks, containing (μ,ν) and (μ',ν'), of the BIB design corresponding to $\boldsymbol{N}^*_{g_t} + \boldsymbol{N}^*_{g_{t'}}$.

By using the property of a $\mathrm{PA}_d(s,s)$, the contribution to the coincidence number in the PA substitution part is given as

$$\begin{cases} 2c(r - s\lambda) & \text{if } \mu = \mu', \\ 2c(r - s\lambda)/(s-1) & \text{otherwise.} \end{cases}$$

Hence, it remains to compute the contribution to the coincidence number in the SDM substitution part. Consider the following three cases that (i) $\mu = \mu', \nu \neq \nu'$, (ii) $\mu \neq \mu', \nu = \nu'$ and (iii) $\mu \neq \mu', \nu \neq \nu'$. Let $\boldsymbol{N}_{g_i}$'s be incidence matrices of $(V, \mathcal{B}_{g_i})$, $i = 1,\ldots,s$.
Case 1 : $\mu = \mu', \nu \neq \nu'$.

Observe that

$$\begin{aligned} & \sum_{\theta}^{sc} \left(\boldsymbol{N}_{a^{(t)}_{\mu\theta}} + \boldsymbol{N}_{a^{(t')}_{\mu\theta}}\right)\left(\boldsymbol{N}_{a^{(t)}_{\mu\theta}} + \boldsymbol{N}_{a^{(t')}_{\mu\theta}}\right)^T \\ = \;& sc\left\{\left(2r - \frac{2r(2k-1)}{v-1}\right)\boldsymbol{I}_v + \frac{2r(2k-1)}{v-1}\boldsymbol{J}_v\right\}. \end{aligned}$$

Thus, the contribution to the coincidence number in the SDM substitution part is $2rsc(2k-1)/(v-1)$, so that the total contribution to the coincidence number counts

$$\frac{2rsc(2k-1)}{v-1} + 2c(r - s\lambda) = \frac{2cr(2v-1)}{v-1}.$$

Case 2 : $\mu \neq \mu', \nu = \nu'$.

Let S and T be the multisets defined by

$$\begin{array}{rcl} S &=& \{(a_{\mu\theta}^{(t)}, a_{\mu'\theta}^{(t')})|t, t' \in \{i, i'\}, \theta = 1 \ldots, sc\}, \\ T &=& \{(a_{\mu\theta}^{(t)}, a_{\mu'\theta}^{(t')})|t, t' \in \{i, i'\}, \theta = 1 \ldots, sc, a_{\mu\theta}^{(t)} = a_{\mu'\theta}^{(t')}\}. \end{array}$$

Then it is obvious that $|S| = 4sc$ and $|T| = 4c$. By noting the condition (1.2), the contribution to the coincidence number in the SDM substitution part counts $4cr$, and thus total contribution to the coincidence number counts

$$4cr + \frac{2c(r - s\lambda)}{s-1} = \frac{2cr(2v-1)}{v-1}.$$

Case 3 : $\mu \neq \mu', \nu \neq \nu'$.

Since $|S| = 4sc$, $|T| = 4c$ and $S \supset T$, it follows that $|S \setminus T| = 4c(s-1)$. We claim that $S \setminus T$ can be partitioned into $2c(s-1)$ pairs, denoted by $\mathcal{P}$, as follows:

$$\begin{array}{rcl} S \setminus T &=& \bigcup \left\{ \left(a_{\mu\theta}^{(t)}, a_{\mu'\theta}^{(t')}\right), \left(a_{\mu'\theta'}^{(t)}, a_{\mu\theta'}^{(t')}\right) | a_{\mu\theta}^{(t)} = a_{\mu'\theta'}^{(t)}, a_{\mu'\theta}^{(t')} = a_{\mu\theta'}^{(t')} \right\}, \\ \mathcal{P} &=& \left\{ \left\{ \left(a_{\mu\theta}^{(t)}, a_{\mu'\theta}^{(t')}\right), \left(a_{\mu'\theta'}^{(t)}, a_{\mu\theta'}^{(t')}\right) \right\} | a_{\mu\theta}^{(t)} = a_{\mu'\theta'}^{(t)}, a_{\mu'\theta}^{(t')} = a_{\mu\theta'}^{(t')} \right\}. \end{array}$$

To derive this, it suffices to show that for any $x, y \in G$ with $x \neq y$, and for any $t, t' \in \{i, i'\}$,

$$|\{\theta|(a_{\mu\theta}^{(t)}, a_{\mu'\theta}^{(t')}) = (x, y)\}| = |\{\theta'|(a_{\mu'\theta'}^{(t)}, a_{\mu\theta'}^{(t')}) = (x, y)\}|$$

in the sense of multisets. In fact, this is valid since

$$\begin{array}{rcl} |\{\theta|(a_{\mu\theta}^{(t)}, a_{\mu'\theta}^{(t')}) = (x, y)\}| &=& |\{\theta|(a_{\mu'\theta}^{(t)}, g_{t'}(g_t)^{-1}a_{\mu\theta}^{(t)}) = (x, y)\}| \\ &=& |\{\theta'|(a_{\mu'\theta'}^{(t)}, a_{\mu\theta'}^{(t')}) = (x, y)\}|. \end{array}$$

By using such a partition $\mathcal{P}$,

$$\begin{aligned}
&\sum_{(a^{(t)}_{\mu\theta},a^{(t')}_{\mu'\theta})\in S} \boldsymbol{N}_{a^{(t)}_{\mu\theta}}\boldsymbol{N}^T_{a^{(t')}_{\mu'\theta}}\\
&= \sum_{(a^{(t)}_{\mu\theta},a^{(t')}_{\mu'\theta})\in T} \boldsymbol{N}_{a^{(t)}_{\mu\theta}}\boldsymbol{N}^T_{a^{(t')}_{\mu'\theta}} + \sum_{(a^{(t)}_{\mu\theta},a^{(t')}_{\mu'\theta})\in S\setminus T} \boldsymbol{N}_{a^{(t)}_{\mu\theta}}\boldsymbol{N}^T_{a^{(t')}_{\mu'\theta}}\\
&= \sum_{(a^{(t)}_{\mu\theta},a^{(t')}_{\mu'\theta})\in T} \boldsymbol{N}_{a^{(t)}_{\mu\theta}}\boldsymbol{N}^T_{a^{(t')}_{\mu'\theta}} + \sum_{\{(a^{(t)}_{\mu\theta},a^{(t')}_{\mu'\theta}),(a^{(t)}_{\mu'\theta'},a^{(t')}_{\mu\theta'})\}\in\mathcal{P}} (\boldsymbol{N}_{a^{(t)}_{\mu\theta}}\boldsymbol{N}^T_{a^{(t')}_{\mu'\theta}} + \boldsymbol{N}_{a^{(t)}_{\mu'\theta'}}\boldsymbol{N}^T_{a^{(t')}_{\mu\theta'}})\\
&= \sum_{(a^{(t)}_{\mu\theta},a^{(t')}_{\mu'\theta})\in T} \{(r-\lambda)\boldsymbol{I}_v + \lambda\boldsymbol{J}_v\} + \sum_{\{(a^{(t)}_{\mu\theta},a^{(t')}_{\mu'\theta}),(a^{(t)}_{\mu'\theta'},a^{(t')}_{\mu\theta'})\}\in\mathcal{P}} \left\{\frac{2kr}{v-1}(\boldsymbol{J}_v - \boldsymbol{I}_v)\right\},
\end{aligned}$$

where the second term after the last equality sign follows from (5.1). Hence, the total contribution counts

$$4c\lambda + \frac{4ckr(s-1)}{v-1} + \frac{2c(r-s\lambda)}{s-1} = \frac{2cr(2v-1)}{v-1}.$$

It is thus shown that for some $x \in G$, if there is no row with all entries x in the $\mathrm{SDM}_c(s,s)$, then $D^*_{g_i}$'s form additive BIB designs.

In the case that $\mathrm{SDM}_c(s,s)$ has an all-x row, since

$$\sum_{j\in Z_s} \boldsymbol{N}_i\boldsymbol{N}_j = \boldsymbol{N}_i\boldsymbol{J}_{b\times v} = r\boldsymbol{J}_v$$

for any $i \in Z_s$, the assertion can be shown similarly, and hence the proof of this case is omitted. □

In order to get additive BIB designs with small coincidence numbers, a symmetric difference matrix with small index is essentially required. The problem of finding a $\mathrm{DM}(s,s)$ for values of s in general appears to be difficult, since a $\mathrm{DM}(s,s)$ generates an $s \times s^2$ OA, the existence of which is known only for a prime power s. Some results on the existence of $\mathrm{SDM}_c(s,s)$ with $c \le 2$ are provided for a prime power s.

Lemma 5.2.3 Let $\boldsymbol{C}_d$ be a *circulant matrix* of order d defined by

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Then

(i) when s is a prime power, there exists an $\mathrm{SDM}_2(s, s)$,

(ii) when s is an odd prime, there exists an $\mathrm{SDM}(s, s)$.

Proof. Let us first note that the symbol x_i is similarly defined as in Theorem 5.1.1.

(i) Define an $s \times 2s$ matrix $\boldsymbol{P} = (p_{mn})$ based on $\mathrm{GF}(s)$ by

$$\boldsymbol{P} = \left(\begin{array}{cc|cc} \sum_{i=1}^{s-1} x_i \boldsymbol{C}_{s-1}^i & \boldsymbol{0}^T & \left(\sum_{i=1}^{s-1} x_i \boldsymbol{C}_{s-1}^i\right)^T & \boldsymbol{0}^T \\ \boldsymbol{0} & 0 & \boldsymbol{0} & 0 \end{array} \right).$$

Then $\boldsymbol{P}$ is an $\mathrm{SDM}_2(s, s)$.

(ii) For an odd prime s, define an $s \times s$ matrix $\boldsymbol{P} = (p_{mn})$ based on Z_s by

$$\boldsymbol{P} = \sum_{i=1}^{s} i^2 \boldsymbol{C}_s^i.$$

Then this matrix yields an $\mathrm{SDM}(s, s)$. In this case, there is no row all of whose elements are x for some x in Z_s. For given t with $1 \leq t \leq s-1$, since $(i+t)^2 - i^2 = t(2i+t)$, there is a unique solution i in Z_s for the equation $(i+t)^2 - i^2 \equiv 0 \pmod{s}$. Hence it holds that

$$\{(i+t)^2 - i^2 | i = 1, \ldots, s\} = Z_s$$

which completes the proof. $\square$

It is still unknown whether an $\mathrm{SDM}(s, s)$ exists or not for an odd prime power $s = p^m$ with $m \geq 2$.

Theorem 5.2.4 The existence of $\mathrm{AB}(v = sk, k, \lambda)$ implies the existence of additive BIB designs with parameters

(i) $v^* = s^2k, k^* = sk, \lambda^* = 2r$ for a prime power s,

(ii) $v^* = s^2k, k^* = sk, \lambda^* = r$ for an odd prime s.

Proof. Suppose that there exist AB$(v = sk, k, \lambda)$. Then this can be shown by the application of Theorem 5.2.2 to the PA constructed from Lemma 2.1.2 and SDM(s, s) constructed from Lemma 5.2.3. $\square$

The following example illustrates Theorem 5.2.4 (ii) with $s = 3$.

Example 5.2.5 Let $\boldsymbol{N}_1, \boldsymbol{N}_2, \boldsymbol{N}_3$ be incidence matrices of AB$(3k, k, \lambda)$ and take the following SDM$(3, 3)$:

$$\begin{array}{ccc} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{array}$$

Now, AB$(9k, k, \lambda)$ can be obtained by taking the following incidence matrices:

$$\begin{aligned}
\boldsymbol{N}_1^* &= \left(\begin{array}{ccc|ccc} \boldsymbol{N}_1 & \boldsymbol{N}_2 & \boldsymbol{N}_2 & \boldsymbol{J} & \boldsymbol{O} & \boldsymbol{O} \\ \boldsymbol{N}_2 & \boldsymbol{N}_1 & \boldsymbol{N}_2 & \boldsymbol{O} & \boldsymbol{J} & \boldsymbol{O} \\ \boldsymbol{N}_2 & \boldsymbol{N}_2 & \boldsymbol{N}_1 & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{J} \end{array}\right) \\
\boldsymbol{N}_2^* &= \left(\begin{array}{ccc|ccc} \boldsymbol{N}_2 & \boldsymbol{N}_3 & \boldsymbol{N}_3 & \boldsymbol{O} & \boldsymbol{J} & \boldsymbol{O} \\ \boldsymbol{N}_3 & \boldsymbol{N}_2 & \boldsymbol{N}_3 & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{J} \\ \boldsymbol{N}_3 & \boldsymbol{N}_3 & \boldsymbol{N}_2 & \boldsymbol{J} & \boldsymbol{O} & \boldsymbol{O} \end{array}\right) \\
\boldsymbol{N}_3^* &= \left(\begin{array}{ccc|ccc} \boldsymbol{N}_3 & \boldsymbol{N}_1 & \boldsymbol{N}_1 & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{J} \\ \boldsymbol{N}_1 & \boldsymbol{N}_3 & \boldsymbol{N}_1 & \boldsymbol{J} & \boldsymbol{O} & \boldsymbol{O} \\ \boldsymbol{N}_1 & \boldsymbol{N}_1 & \boldsymbol{N}_3 & \boldsymbol{O} & \boldsymbol{J} & \boldsymbol{O} \end{array}\right),
\end{aligned}$$

where $\boldsymbol{J}$ and $\boldsymbol{O}$ are all of size $v \times (r - 3\lambda)$.

Remark 5.2.6 In Theorem 5.2.4 (ii), $k = \lambda + 1$ if and only if $k^* = \lambda^* + 1$, and $k = 2\lambda + 1$ if and only if $k^* = 2\lambda^* + 1$. This fact means that additive minimal BIB designs can be also constructed from additive minimal BIB designs.

5.3 Existence status

The spectrum of additive BIB designs within the scope of parameters $s \geq 3$, $v \leq 100$, $r \leq 20$, $k \geq 2$ and $k > \lambda$ is given in the following two tables. Table 5.3.1 is a list of parameters s, v, b, r, k, λ for which additive BIB designs with $k > \lambda$ exist, along with source information on existence, where a resolvable $\mathrm{B}(v, k, \lambda)$ is denoted by $\mathrm{RB}(v, k, \lambda)$. Table 5.3.2 is a list of admissible parameters of BIB designs for which the existence of additive BIB designs is not known, along with other information on existence and resolvability of the BIB design itself.

Table 5.3.1

Additive BIB designs with $s \geq 3, v = sk \leq 100, r \leq 20, k \geq 2$ and $k > \lambda$

No.	s	v	b	r	k	λ	Source
1	3	6	15	5	2	1	$\{1,4\},\{2,3\},\{0,\infty\}$ mod 5
2	3	9	12	4	3	1	Theorem 2.4.4, $p = 3$ and $n = 2$
3	3	9	24	8	3	2	2-copies of No.2
4	3	12	33	11	4	3	Corollary 2.4.3, $\mathrm{RB}(12,4,3)$
5	3	18	51	17	6	5	Corollary 2.4.3, $\mathrm{RB}(18,6,5)$
6	3	21	60	20	7	6	Corollary 2.4.3, $\mathrm{RB}(21,7,6)$
7	3	27	39	13	9	4	Corollary 2.4.3, $\mathrm{RB}(27,9,4)$
8	4	8	28	7	2	1	Example 1.2.4
9	4	12	44	11	3	2	Theorem 2.4.1 with Example 1.1.6
10	4	16	60	15	4	3	Corollary 2.4.3, $\mathrm{RB}(16,4,1)$
11	5	10	45	9	2	1	Lemma 2.1.9
12	5	15	70	14	3	2	Corollary 2.4.3, $\mathrm{RB}(15,3,1)$
13	5	25	60	12	5	2	Theorem 2.4.4, $p = 5$ and $n = 2$
14	8	16	120	15	2	1	Theorem 5.1.2
15	9	27	117	13	3	1	Theorem 3.3.1

Note (see Lemma 2.1.9 and [35]) that two designs of Nos.11 and 15 given in Table 5.3.1 are shown to be additive after the publication of [36], in which the two designs were listed unknown.

Table 5.3.2

Unknown additive BIB designs with $s \geq 3, v = sk \leq 100, k \geq 2, r \leq 20$ and $k > \lambda$

No.	s	v	b	r	k	λ	Existence	Resolvability
1	3	15	42	14	5	4	Yes	No
2	3	21	30	10	7	3	Yes	No
3	3	33	48	16	11	5	Yes	No
4	3	39	57	19	13	6	?	No
5	4	20	76	19	5	4	Yes	Yes
6	5	15	35	7	3	1	Yes	Yes
7	5	20	95	19	4	3	Yes	Yes
8	5	35	85	17	7	3	Yes	??
9	6	12	66	11	2	1	Yes	Yes
10	6	18	102	17	3	2	Yes	Yes
11	7	14	91	13	2	1	Yes	Yes
12	7	21	70	10	3	1	Yes	Yes
13	7	21	140	20	3	2	Yes	Yes
14	7	35	119	17	5	2	Yes	Yes
15	9	18	153	17	2	1	Yes	Yes

Chapter 6

Pairwise additive cyclic BIB designs

The existence of pairwise additive BIB designs with cyclic property is discussed. Since two arrays are utilized for constructing cyclic BIB designs, some properties of the arrays are provided. Especially, the special array defined here are used in direct and recursive constructions. Constructions from cyclic nested BIB designs and difference families are also presented. Furthermore, non-existence of pairwise additive cyclic BIB designs is shown.

6.1 Properties of the arrays

Two arrays with four rows are introduced to derive direct and recursive constructions of 2 PACB$(v, 2, 1)$, and some properties of the arrays are discussed.

Denote $Z_v^* = \{1, 2, \ldots, v-1\}$. Let a multiset $Z_{(v,\lambda)}^* = \{1, \ldots, 1, 2, \ldots, 2, \ldots, v-1, \ldots, v-1\}$ that contains each element of Z_v^* λ times, Z_v' be a subset of Z_v^* such that $Z_v^* = \{a, -a | a \in Z_v'\} \pmod{v}$ (hence $|Z_v'| = (v-1)/2$), and a multiset $Z'_{(v,\lambda)}$ be a subset of $Z^*_{(v,\lambda)}$ such that $Z^*_{(v,\lambda)} = \{a, -a | a \in Z'_{(v,\lambda)}\} \pmod{v}$ (hence $|Z'_{(v,\lambda)}| = (v-1)\lambda/2$).

Now, let v be an odd integer and $(v-1)/2$ initial columns of a CPA$(4, v)$ be $(a(1, n), a(2, n), a(3, n), a(4, n))^T$, $1 \le n \le (v-1)/2$, with $a(m, n) \in Z_v$ for $1 \le m \le 4$. Then it is shown that

$$Z_v' = \{a(i, n) - a(j, n) \pmod{v} | 1 \le n \le \frac{v-1}{2}\} \tag{6.1}$$

for each of $1 \leq i < j \leq 4$.

A *cyclic difference matrix* on Z_v, denoted by CDM$(4, v)$, is defined as a $4 \times v$ array $(a(m, n))$, $a(m, n) \in Z_v$, $1 \leq m \leq 4$, that satisfies

$$Z_v = \{a(i, n) - a(j, n) \pmod{v} | 1 \leq n \leq v\} \tag{6.2}$$

for each of $1 \leq i < j \leq 4$ (cf. [18]).

Next, let $v \equiv 1 \pmod 4$ and $(v-1)/4$ initial blocks of a CNB$(v; (v-1)/2, (v-1)/4; 2, 4)$ which contains no short orbit be $\{a(1, n), a(2, n) | a(3, n), a(4, n)\}$, $1 \leq n \leq (v-1)/4$, with $a(m, n) \in Z_v$ for $1 \leq m \leq 4$. Then it is shown that

$$Z'_v = \{a(1, n) - a(2, n), a(3, n) - a(4, n) | 1 \leq n \leq \frac{v-1}{4}\}, \tag{6.3}$$

$$Z'_{(v,2)} = \bigcup_{i \in \{1,2\}, j \in \{3,4\}} \{a(i, n) - a(j, n) | 1 \leq n \leq \frac{v-1}{4}\}. \tag{6.4}$$

Furthermore, let initial blocks of a Z-cyclic TWh$(4t+1)$ be $\{a(1, n), a(2, n) | a(3, n), a(4, n)\}$, $1 \leq n \leq t$, with $a(m, n) \in Z_v$ for $1 \leq m \leq 4$. Then it is shown that

$$\begin{aligned} Z'_v &= \{a(1, n) - a(2, n), a(3, n) - a(4, n) | 1 \leq n \leq t\} && (6.5)\\ &= \{a(1, n) - a(3, n), a(2, n) - a(4, n) | 1 \leq n \leq t\} && (6.6)\\ &= \{a(1, n) - a(4, n), a(2, n) - a(3, n) | 1 \leq n \leq t\}. && (6.7) \end{aligned}$$

Also, letting v be an odd integer and $(v-1)/2$ initial blocks of $\boldsymbol{N}_1$ and $\boldsymbol{N}_2$ of 2 PACB$(v, 2, 1)$ be $\{a(1, n), a(2, n)\}$ and $\{a(3, n), a(4, n)\}$, $1 \leq n \leq (v-1)/2$, with $a(m, n) \in Z_v$ for $1 \leq m \leq 4$, respectively, it follows that

$$\begin{aligned} Z'_v &= \{a(1, n) - a(2, n) | 1 \leq n \leq \frac{v-1}{2}\} && (6.8)\\ &= \{a(3, n) - a(4, n) | 1 \leq n \leq \frac{v-1}{2}\}, && (6.9) \end{aligned}$$

$$Z'_{(v,4)} = \bigcup_{i \in \{1,2\}, j \in \{3,4\}} \{a(i, n) - a(j, n) | 1 \leq n \leq \frac{v-1}{2}\}. \tag{6.10}$$

Finally, a *special array* on Z_v for an odd integer v, denoted by SA$(4, v)$, is defined as a $4 \times (v-1)/2$ array $(a(m, n))$, $a(m, n) \in Z_v$, $1 \leq m \leq 4$, that satisfies

$$\begin{aligned} Z'_v &= \{a(1,n) - a(2,n) | 1 \le n \le \frac{v-1}{2}\} & (6.11)\\ &= \{a(3,n) - a(4,n) | 1 \le n \le \frac{v-1}{2}\}, & (6.12)\\ Z^*_v &= \{a(1,n) - a(3,n), a(2,n) - a(4,n) | 1 \le n \le \frac{v-1}{2}\} & (6.13)\\ &= \{a(1,n) - a(4,n), a(2,n) - a(3,n) | 1 \le n \le \frac{v-1}{2}\}. & (6.14)\end{aligned}$$

Then it follows from properties (6.8) to (6.14) that the initial blocks $\{a(1,n),\, a(2,n)\}$ and $\{a(3,n), a(4,n)\}$ yield 2 PACB$(v,2,1)$.

6.2 Direct construction with a cyclic property

It has been shown by Lemma 2.1.1 that the existence of a PA(g,v) implies the existence of $\lfloor g/2 \rfloor$ PAB$(v,2,1)$. Similarly, a class of 2 PACB$(v,2,1)$ can be constructed through CPA$(4,v)$ as the following shows.

Theorem 6.2.1 The existence of a CPA(g,v) implies the existence of $\lfloor g/2 \rfloor$ PACB$(v,2,1)$.

Proof. Let initial columns of the CPA(g,v) be $(a(1,n),\ldots,a(g,n))^T$ with $1 \le n \le (v-1)/2$. It can be shown by (6.1) that the following incidence matrices yield the required $\lfloor g/2 \rfloor$ PACB$(v,2,1)$:

$$\begin{array}{rcl} \boldsymbol{N}_1 & : & \{a(1,n), a(2,n)\} \quad \bmod v \\ \boldsymbol{N}_2 & : & \{a(3,n), a(4,n)\} \quad \bmod v \\ \vdots & & \vdots \\ \boldsymbol{N}_{\lfloor \frac{g}{2} \rfloor} & : & \{a(2\lfloor \frac{g}{2} \rfloor - 1, n), a(2\lfloor \frac{g}{2} \rfloor, n)\} \quad \bmod v \end{array}$$

for $1 \le n \le (v-1)/2$. $\square$

Lemma 6.2.2 Let $v \ge 5$ and $\gcd(v,6) = 1$. Then there exists a CPA$(4,v)$.

Proof. Since $v \ge 5$ and $\gcd(v,6) = 1$ imply that

$$\{2i | 1 \le i \le \frac{v-1}{2}\} = \{3i | 1 \le i \le \frac{v-1}{2}\} = Z'_v,$$

the following columns on Z_v can be seen to be initial columns of the required CPA$(4, v)$.

$$(0, i, 2i, 3i)^T,\ 1 \le i \le \frac{v-1}{2}.$$

□

Theorem 6.2.3 Let $v \ge 5$ and $\gcd(v, 6) = 1$. Then 2 PACB$(v, 2, 1)$ exist.

Proof. Because of the existence of a CPA$(4, v)$ shown by Lemma 6.2.2, the proof is complete by applying Theorem 6.2.1. □

Next, some individual examples (Examples 6.2.4, 6.2.5, 6.2.6, 6.2.7 and 6.2.8 next) which can be obtained by use of a computer are provided. Each of such examples cannot be obtained by the construction methods presented in the previous chapters.

In each example, the cyclic property is checked directly and the following procedure of checking PACB properties is taken. For two incidence matrices $\boldsymbol{N}_1$ with initial blocks $B_h^{(1)} = \{a_h, b_h\}$ and $\boldsymbol{N}_2$ with initial blocks $B_h^{(2)} = \{c_h, d_h\}$ for $1 \le h \le \lfloor v/2 \rfloor$, it can be checked that $\boldsymbol{N}_1 + \boldsymbol{N}_2$ with initial blocks $\{a_h, b_h, c_h, d_h\}$ is the incidence matrix of a B$(v, 4, 6)$. In fact, (i) for a full initial block $\{a, b\} \cup \{c, d\}$, let a multiset $\Delta_f(\{a, b\}, \{c, d\}) = \{e - f, f - e | e \in \{a, b\}, f \in \{c, d\}\}$ on Z_v, and (ii) for a short initial block $\{a, b\} \cup \{c, d\}$, let a multiset $\Delta_s(\{a, b\}, \{c, d\}) = \{a - f, f - a | f \in \{c, d\}\}$ on Z_v. If every non-zero element of Z_v occurs 4 times in $\bigcup_{h=1}^{(v-1)/2} \Delta_f(B_h^{(1)}, B_h^{(2)})$ for an odd integer v or every non-zero element of Z_v occurs 4 times in $\bigcup_{h=1}^{(v-2)/2} \Delta_f(B_h^{(1)}, B_h^{(2)}) \cup \Delta_s(B_{v/2}^{(1)}, B_{v/2}^{(2)})$ for an even integer v, then $\boldsymbol{N}_1 + \boldsymbol{N}_2$ forms the incidence matrix of a B$(v, 4, 6)$.

These symbols Δ_f for *full* initial blocks and Δ_s for *short* initial blocks play an important role in Section 6.7.

Example 6.2.4 2 PACB$(9, 2, 1)$ are given by developing the following blocks on Z_9:

$$\begin{aligned}
\boldsymbol{N}_1 &: \{0,1\}, \{0,2\}, \{0,3\}, \{0,4\} \quad \text{mod } 9 \\
\boldsymbol{N}_2 &: \{2,3\}, \{4,6\}, \{2,6\}, \{5,8\} \quad \text{mod } 9.
\end{aligned}$$

Example 6.2.5 2 PACB$(12,2,1)$ are given by developing the following blocks on Z_{12}:

$$\begin{aligned}
\boldsymbol{N}_1 &: \{0,1\},\{0,2\},\{0,3\},\{0,4\},\{0,5\},\{0,6\}\text{PC}(6) \mod 12\\
\boldsymbol{N}_2 &: \{3,6\},\{4,9\},\{2,4\},\{7,8\},\{2,6\},\{1,7\}\text{PC}(6) \mod 12.
\end{aligned}$$

Example 6.2.6 2 PACB$(15,2,1)$ are given by developing the following blocks on Z_{15}:

$$\begin{aligned}
\boldsymbol{N}_1 &: \{0,1\},\{0,2\},\{0,3\},\{0,4\},\{0,5\},\{0,6\},\{0,7\} \mod 15\\
\boldsymbol{N}_2 &: \{5,10\},\{4,13\},\{1,2\},\{8,12\},\{10,12\},\{9,12\},\{1,8\} \mod 15.
\end{aligned}$$

Example 6.2.7 2 PACB$(16,2,1)$ are given by developing the following blocks on Z_{16}:

$$\begin{aligned}
\boldsymbol{N}_1 &: \{0,1\},\{0,2\},\{0,3\},\{0,4\},\{0,5\},\{0,6\},\{0,7\},\\
&\quad \{0,8\}\text{PC}(8) \mod 16\\
\boldsymbol{N}_2 &: \{3,8\},\{3,9\},\{4,5\},\{7,10\},\{1,15\},\{8,12\},\{5,12\},\\
&\quad \{3,11\}\text{PC}(8) \mod 16.
\end{aligned}$$

Example 6.2.8 2 PACB$(24,2,1)$ are given by developing the following blocks on Z_{24}:

$$\begin{aligned}
\boldsymbol{N}_1 &: \{0,1\},\{0,2\},\{0,3\},\{0,4\},\{0,5\},\{0,6\},\{0,7\},\{0,8\},\\
&\quad \{0,9\},\{0,10\},\{0,11\},\{0,12\}\text{PC}(12) \mod 24\\
\boldsymbol{N}_2 &: \{15,18\},\{1,7\},\{5,6\},\{11,16\},\{21,8\},\{2,10\},\{11,15\},\{13,15\},\\
&\quad \{3,10\},\{6,15\},\{13,23\},\{2,14\}\text{PC}(12) \mod 24.
\end{aligned}$$

Next, take the following 4×13 array quoted from [4]:

$$\begin{pmatrix}
18 & 24 & 8 & 9 & 5 & 25 & 16 & 7 & 21 & 12 & 17 & 2 & 20\\
19 & 26 & 11 & 13 & 10 & 4 & 23 & 15 & 3 & 22 & 1 & 14 & 6\\
2 & 1 & 26 & 17 & 8 & 11 & 22 & 16 & 20 & 9 & 7 & 23 & 18\\
4 & 21 & 3 & 14 & 24 & 19 & 12 & 25 & 5 & 15 & 6 & 10 & 13
\end{pmatrix} = (a(m,n)),$$

say. This is an SA$(4,27)$ that satisfies properties (6.11) to (6.14). Hence the following example can be further presented.

Example 6.2.9 2 PACB$(27, 2, 1)$ are given by developing the following blocks on Z_{27}:

$$\begin{array}{lcll} \boldsymbol{N}_1 & : & \{a(1,n), a(2,n)\} & \text{mod } 27 \\ \boldsymbol{N}_2 & : & \{a(3,n), a(4,n)\} & \text{mod } 27 \end{array}$$

for $1 \leq n \leq 13$.

Remark 6.2.10 The existence of ACB$(v, 2, 1)$ is only known for $v = 4, 8$ as in Examples 1.2.3 and 1.2.4. In Section 6.7, the non-existence of ACB$(v, 2, 1)$ for $v = 12$ and $v \equiv 2 \pmod 4$ are also discussed.

6.3 Construction from cyclic nested BIB designs

At first it is pointed out that there are some classes of CNB$(4t+1; 2t(4t+1), t(4t+1); 2, 4)$ as follows.

Lemma 6.3.1 [7] There exists a CNB$(4t + 1; 2t(4t + 1), t(4t + 1); 2, 4)$ for $3 \leq t \leq 37$.

Lemma 6.3.2 [6] There exists a CNB$(3^{2m}; 3^{2m}(3^{2m} - 1)/2, 3^{2m}(3^{2m} - 1)/4; 2,\ 4)$ for any integer $m \geq 2$.

On the other hand, it is shown in Theorem 2.4.1 that 3 PAB$(v,\ 2, 1)$ can be constructed by an NB$(v; v(v-1)/2, v(v-1)/6; 2, 6)$. Similarly, it is shown that 2 PACB$(v, 2, 1)$ can be obtained from a cyclic nested BIB design.

Theorem 6.3.3 The existence of a CNB$(v; v(v - 1)/2, v(v - 1)/4; 2, 4)$ implies the existence of 2 PACB$(v, 2, 1)$.

Proof. Let the nth initial blocks of the CNB$(v; (v - 1)/4, (v - 1)/2; 2, 4)$ be

$$\{a(1, n), a(2, n)|a(3, n), a(4, n)\},\ 1 \leq n \leq \frac{v(v-1)}{4}.$$

Then it follows from properties (6.3) and (6.4) that the following incidence matrices yield the required 2 PACB$(v, 2, 1)$:

$$\begin{array}{lcll} \boldsymbol{N}_1 & : & \{a(1,n), a(2,n)\}, \{a(3,n), a(4,n)\} & \text{mod } v \\ \boldsymbol{N}_2 & : & \{a(3,n), a(4,n)\}, \{a(1,n), a(2,n)\} & \text{mod } v \end{array}$$

for $1 \leq n \leq v(v-1)/4$. $\square$

Hence Theorem 6.3.3 with Lemma 6.3.2 can produce the following.

Corollary 6.3.4 There are 2 PACB$(3^{2m}, 2, 1)$ for any integer $m \geq 2$.

6.4 Construction from cyclic difference families

A (vg, g, k, λ) *cyclic relative difference family*, denoted by (vg, g, k, λ)-CDF, is a family $\mathcal{F}$ of k-subsets of Z_{vg} with the property that a multiset of differences $\cup_{B\in\mathcal{F}}\Delta B$ is $Z^*_{(vg,\lambda)} \backslash vZ^*_{(vg,\lambda)}$, where $\Delta B = \{x_i - x_j, x_j - x_i | 0 \leq i < j \leq k-1\}$ for $B = \{x_0, \ldots, x_{k-1}\}$ and $vZ^*_{(vg,\lambda)}$ is a multiset which contains each element of $\{v, 2v, \ldots, vg\}$ λ times (cf. [10]).

Some classes of $(vg, g, 4, 1)$-CDF are known as follows.

Lemma 6.4.1 [10, 11] There exist a $(243, 27, 4, 1)$-CDF and a $(2^{s+4}, 2^s, 4, 1)$-CDF for any integer $s \geq 2$.

On the other hand, 2 PACB$(v, 2, 1)$ are obtained from a cyclic relative difference family.

Theorem 6.4.2 The existence of a $(vg, g, 4, 1)$-CDF and 2 PACB$(g, 2, 1)$ implies the existence of 2 PACB$(vg, 2, 1)$.

Proof. Let 4-subsets of the $(vg, g, 4, 1)$-CDF on Z_{vg} be

$$\{a_i, b_i, c_i, d_i\}, \ 1 \leq i \leq \frac{(v-1)g}{12}$$

and let initial blocks of 2 PACB$(g, 2, 1)$ on $vZ_{vg} = \{0, v, 2v, \ldots, (g-1)v\}$ be

$$\begin{aligned} \boldsymbol{N}_1 &: \{x_j, y_j\}, \ 1 \leq j \leq \lfloor \frac{g}{2} \rfloor \\ \boldsymbol{N}_2 &: \{z_j, w_j\}, \ 1 \leq j \leq \lfloor \frac{g}{2} \rfloor. \end{aligned}$$

Then it follows that developing the following initial blocks on Z_v yields the required 2 PACB$(v, 2, 1)$:

$$\begin{aligned} \boldsymbol{N}_1 \ &: \ \{a_i, b_i\}, \{a_i, c_i\}, \{a_i, d_i\}, \{c_i, b_i\}, \{b_i, d_i\}, \{d_i, c_i\}, \\ &\quad\ \{x_j, y_j\} \mod vg \\ \boldsymbol{N}_2 \ &: \ \{c_i, d_i\}, \{d_i, b_i\}, \{b_i, c_i\}, \{d_i, a_i\}, \{c_i, a_i\}, \{b_i, a_i\}, \\ &\quad\ \{z_j, w_j\} \mod vg \end{aligned}$$

for $1 \leq i \leq (v-1)g/12$ and $1 \leq j \leq \lfloor g/2 \rfloor$. $\square$

Finally, a class of 2 PACB$(v, 2, 1)$ can be given by use of Theorem 6.4.2.

Theorem 6.4.3 There are 2 PACB$(2^m, 2, 1)$ for any positive integer $m \not\equiv 1 \pmod 4$.

Proof. There exist $(2^{s+4}, 2^s, 4, 1)$-CDF for $s \geq 2$ on account of Lemma 6.4.1. Hence Theorem 6.4.2 shows that the existence of 2 PACB$(2^s, 2, 1)$ implies the existence of 2 PACB$(2^{s+4}, 2, 1)$ for $s \geq 2$. On the other hand, 2 PACB$(2^t, 2, 1)$ with $t = 2, 3$ and 4 are obtained as in Examples 1.2.3, 1.2.4 and 6.2.7. Thus, the proof is complete. $\square$

Remark 6.4.4 If 2 PACB$(2^5, 2, 1)$ could be constructed, then the condition on m in Theorem 6.4.3 would be removed.

6.5 Recursive construction with a cyclic property

At first an existence of cyclic difference matrices is reviewed.

Lemma 6.5.1 [18] There exists a CDM$(4, v)$ for any odd integer $v \geq 5$ and $\gcd(v,\ 27) \neq 9$.

Some recursive constructions of cyclic BIB designs with some regular short orbits and of cyclic nested BIB designs with no short orbit are provided by using a CDM(g, v) in [21]. Next, some similar methods are presented.

Theorem 6.5.2 Let $v \geq 5$ and $v' \geq 5$ be odd integers. Then the existence of 2 PACB$(v, 2, 1)$, 2 PACB$(v', 2, 1)$ and a CDM $(4, v')$ implies the existence of 2 PACB$(vv', 2, 1)$.

Proof. Let sets of initial blocks of 2 PACB$(v, 2, 1)$ and 2 PACB$(v', 2, 1)$ be

$$\{\{x_i^{(h)}, y_i^{(h)}\} | 1 \leq i \leq \frac{v-1}{2}\},\ \{\{z_j^{(h)}, w_j^{(h)}\} | 1 \leq j \leq \frac{v'-1}{2}\},\ h = 1, 2,$$

respectively. Let A be the CDM$(4, v')$ with $a(m, n)$ as the (m, n)-entry for $1 \leq m \leq 4$ and $1 \leq n \leq v'$. Then, since each block orbit of 2

PACB$(v,2,1)$ and 2 PACB$(v',2,1)$ is full for odd integers $v \geq 5$ and $v' \geq 5$, it can be shown by (6.2) that the following two incidence matrices yield the required 2 PACB$(v,2,1)$ on $Z_{vv'}$:

$$\begin{aligned} \boldsymbol{N}_1 &: \{x_i^{(1)} + a(1,n)v, y_i^{(1)} + a(2,n)v\}, \{z_j^{(1)}v, w_j^{(1)}v\} \quad \bmod vv' \\ \boldsymbol{N}_2 &: \{x_i^{(2)} + a(3,n)v, y_i^{(2)} + a(4,n)v\}, \{z_j^{(2)}v, w_j^{(2)}v\} \quad \bmod vv' \end{aligned}$$

for $1 \leq i \leq (v-1)/2, 1 \leq j \leq (v'-1)/2$ and $1 \leq n \leq v'$. □

Now, Theorem 6.5.2 produces a new class of 2 PACB$(v,2,1)$.

Corollary 6.5.3 There are 2 PACB$(3^m,2,1)$ for any integer $m \geq 2$.

Proof. Examples 6.2.4 and 6.2.9 show the existence of 2 PACB$(9,2,1)$ and 2 PACB$(27,2,1)$, respectively. Corollary 6.3.4 and Lemma 6.5.1 can produce 2 PACB$(3^{2s},2,1)$ with $s \geq 2$ and CDM$(4,27)$, respectively. Now, for $v = 9 \cdot 27, 3^{2s} \cdot 27$ $(s \geq 2)$, Theorem 6.5.2 yields 2 PACB$(v,2,1)$. Hence 2 PACB$(v,2,1)$ can be constructed for $v = 3^2, 3^3, 3^{2s}, 3^5, 3^{2s+3}$ with $s \geq 2$. □

Theorem 6.5.4 Let $v \geq 5$ and $\gcd(v,6) = 1$. Then there are 2 PACB $(3^m v,2,1)$ for any integer $m \geq 2$.

Proof. Because of the existence of 2 PACB$(3^m,2,1)$ and 2 PACB$(v,2,1)$ shown by Corollary 6.5.3 and Theorem 6.2.3, the proof is complete by applying Theorem 6.5.2 with the CDM$(4,v)$ given by Lemma 6.5.1. □

Unfortunately, the above method (Theorem 6.5.2) of construction can be applied only for 2 PACB$(v,2,1)$ with no short orbit, that is, for v being an odd integer. Note that the recursive construction given in [21] of cyclic BIB designs with some regular short orbits cannot also be applied for the construction of 2 PACB$(v,2,1)$ with an even integer v. Next, another recursive construction of 2 PACB$(2t,2,1)$ with short initial blocks is considered.

Theorem 6.5.5 The existence of 2 PACB$(2t,2,1)$ and an SA$(4,v)$ implies the existence of 2 PACB$(2tv,2,1)$ for any integer $t \geq 2$ and any odd integer $v \geq 5$ with $\gcd(t,v) = 1$.

Proof. Let a set of initial blocks of 2 PACB$(2t, 2, 1)$ on Z_{2t} be

$$\{\{x_i^{(h)}, y_i^{(h)}\} | 1 \le i \le t-1\} \cup \{z^{(h)}, z^{(h)} + t\}\mathrm{PC}(t),\ h = 1, 2,$$

and let columns of the SA$(4, v)$ on Z_v be

$$(a(1,n), a(2,n), a(3,n), a(4,n))^T,\ 1 \le n \le \frac{v-1}{2}.$$

Then it can be shown by properties (6.11) to (6.14) that developing the following initial blocks on $Z_{2t} \times Z_v$ yields 2 PAB$(2tv, 2, 1)$ with $2tv$ elements denoted by (z, w) for $0 \le z \le 2t-1$ and $0 \le w \le v-1$:

$$\begin{array}{lll}
\boldsymbol{N}_1 & : & \{(x_i^{(1)}, 0), (y_i^{(1)}, 0)\}, \{(0, a(1,n)), (0, a(2,n))\}, \\
& & \{(x_i^{(1)}, a(1,n)), (y_i^{(1)}, a(2,n))\}, \{(x_i^{(1)}, a(2,n)), (y_i^{(1)}, a(1,n))\}, \\
& & \{(z^{(1)}, a(1,n)), (z^{(1)} + t, a(2,n))\}, \\
& & \{(z^{(1)}, 0), (z^{(1)} + t, 0)\}\mathrm{PC}\,(t, v) \quad \mathrm{mod}\ (2t, v) \\
\boldsymbol{N}_2 & : & \{(x_i^{(2)}, 0), (y_i^{(2)}, 0)\}, \{(0, a(3,n)), (0, a(4,n))\}, \\
& & \{(x_i^{(2)}, a(3,n)), (y_i^{(2)}, a(4,n))\}, \{(x_i^{(2)}, a(4,n)), (y_i^{(2)}, a(3,n))\}, \\
& & \{(z^{(2)}, a(3,n)), (z^{(2)} + t, a(4,n))\}, \\
& & \{(z^{(2)}, 0), (z^{(2)} + t, 0)\}\mathrm{PC}\,(t, v) \quad \mathrm{mod}\ (2t, v)
\end{array}$$

where $1 \le i \le t-1$, $1 \le n \le (v-1)/2$ and PC(t, v) means a partial cycle of order tv on $Z_{2t} \times Z_v$, i.e., only (s, u), $0 \le s \le t-1$ and $0 \le u \le v-1$ are to be added to the initial block.

Since $\gcd(2t, v) = 1$, the required 2 PACB$(2tv, 2, 1)$ on Z_{2tv} can be constructed by corresponding the element j to (z, w) for $0 \le j \le 2tv-1$, where $j \equiv z \pmod{2t}$ and $j \equiv w \pmod{v}$. $\square$

The following example illustrates Theorem 6.5.5 with $t = 2$ and $v = 5$.

Example 6.5.6 Consider an SA$(4, 5)$ on Z_5:

$$\begin{pmatrix} 0 & 1 & 4 & 2 \\ 0 & 2 & 3 & 4 \end{pmatrix}^T.$$

Since there are 2 PACB$(4, 2, 1)$ as in Example 1.2.3, 2 PACB$(20, 2, 1)$ are provided by developing the following initial blocks on $Z_4 \times Z_5$:

$$\begin{array}{rcl}
\boldsymbol{N}_1 & : & \{(0,0),(1,0)\},\{(0,0),(0,1)\},\{(0,0),(0,2)\},\{(0,0),(1,1)\}, \\
& & \{(0,0),(1,2)\},\{(0,1),(1,0)\},\{(0,2),(1,0)\},\{(0,0),(2,1)\}, \\
& & \{(0,0),(2,2)\},\{(0,0),(2,0)\}\mathrm{PC}(2,5) \quad \mathrm{mod}\ (4,5) \\
\boldsymbol{N}_2 & : & \{(2,0),(3,0)\},\{(0,4),(0,2)\},\{(0,3),(0,4)\},\{(2,4),(3,2)\}, \\
& & \{(2,3),(3,4)\},\{(2,2),(3,4)\},\{(2,4),(3,3)\},\{(1,4),(3,2)\}, \\
& & \{(1,3),(3,4)\},\{(1,0),(3,0)\}\mathrm{PC}(2,5) \quad \mathrm{mod}\ (4,5).
\end{array}$$

Hence 2 PACB$(20,2,1)$ on Z_{20} can be obtained by corresponding the element j to (z,w) for $0 \le j \le 19$, where $j \equiv z \pmod 4$ and $j \equiv w \pmod 5$. In fact, the following initial blocks on Z_{20} yields 2 PACB$(20,2,1)$:

$$\begin{array}{rcl}
\boldsymbol{N}_1 & : & \{0,5\},\{0,16\},\{0,12\},\{0,1\},\{0,17\},\{16,5\}, \\
& & \{12,5\},\{0,6\},\{0,2\},\{0,10\}\mathrm{PC}(10) \quad \mathrm{mod}\ (20) \\
\boldsymbol{N}_2 & : & \{10,15\},\{4,12\},\{8,4\},\{14,7\},\{18,19\},\{2,19\}, \\
& & \{14,3\},\{9,7\},\{13,19\},\{5,15\}\mathrm{PC}(10) \quad \mathrm{mod}\ (20).
\end{array}$$

6.6 2 pairwise additive cyclic B$(v,2,1)$

The existence of 2 PACB$(v,2,1)$ with $v \not\equiv 2 \pmod 4$ is discussed. At first, some classes of SA$(4,v)$ are provided to apply the recursive construction given in Theorem 6.5.2.

Lemma 6.6.1 Let $v \ge 5$ and $\gcd(v,6) = 1$. Then there exists an SA$(4,v)$.

Proof. Since $v \ge 5$ and $\gcd(v,6) = 1$ imply that

$$\{\pm i | 1 \le i \le \frac{v-1}{2}\} = \{\pm 2i | 1 \le i \le \frac{v-1}{2}\} = Z_v^*,$$

the following columns on Z_v can be seen to form the SA$(4,v)$:

$$(0,i,-i,2i)^T,\ 1 \le i \le \frac{v-1}{2}.$$

□

Next, an SA$(4, 4t+1)$ can be obtained from a Z-cyclic TWh(v) with $v = 4t+1$ as in Lemma 6.6.2. Furthermore, an SA$(4, v)$ can be obtained from a cyclic relative difference family as in Lemma 6.6.4. Especially, the SA$(4, 81)$ and the SA$(4, 243)$ constructed in Lemmas 6.6.3 and 6.6.5 below are utilized for the recursive construction of SA$(4, v)$.

Lemma 6.6.2 The existence of a Z-cyclic TWh$(4t+1)$ implies the existence of an SA$(4, 4t+1)$.

Proof. Let t games of the Z-cyclic TWh$(4t+1)$ be

$$(a(1,n), a(2,n), a(3,n), a(4,n)),\ 1 \le n \le t.$$

Then it is shown by properties (6.5) to (6.7) that the following columns yield an SA$(4, 4t+1)$:

$$\begin{pmatrix} a(1,n) & a(2,n) & a(3,n) & a(4,n) \\ a(4,n) & a(3,n) & a(2,n) & a(1,n) \end{pmatrix}^T$$

for $1 \le n \le t$. $\square$

Now, on account of Lemma 6.6.2 the Z-cyclic TWh(81) given in [4] can produce the following.

Lemma 6.6.3 There exists an SA$(4, 81)$.

Lemma 6.6.4 The existence of $(vg, g, 4, 1)$-CDF and an SA$(4, g)$ implies the existence of an SA$(4, vg)$.

Proof. Let initial blocks of the $(vg, g, 4, 1)$-CDF on Z_{vg} be

$$\{a_i, b_i, c_i, d_i\},\ 1 \le i \le \frac{(v-1)g}{12}$$

and let columns of the SA$(4, g)$ on $vZ_g = \{0, v, 2v, \ldots, (g-1)v\}$ be

$$(a(1,n), a(2,n), a(3,n), a(4,n))^T,\ 1 \le n \le \frac{g-1}{2}.$$

Then it follows from properties (6.11) to (6.14) that the following columns yield the SA$(4, vg)$:

$$\begin{pmatrix} a_i & a_i & a_i & c_i & b_i & d_i & a(1,n) \\ b_i & c_i & d_i & b_i & d_i & c_i & a(2,n) \\ c_i & d_i & b_i & d_i & c_i & b_i & a(3,n) \\ d_i & b_i & c_i & a_i & a_i & a_i & a(4,n) \end{pmatrix} \mod vg$$

for $1 \leq i \leq (v-1)g/12$ and $1 \leq n \leq \lfloor g/2 \rfloor$. $\square$

Thus, Lemmas 6.4.1 and 6.6.4 with the SA$(4, 27)$ displayed in Section 6.2 can produce the following.

Lemma 6.6.5 There exists an SA$(4, 243)$.

Now, a class of SA$(4, v)$ can be given by the recursive construction which is similar to Theorem 6.5.2.

Lemma 6.6.6 The existence of an SA$(4, v)$ and an SA$(4, v')$ implies the existence of an SA$(4, vv')$.

Proof. Let columns of the SA$(4, v)$ and the SA$(4, v')$ be

$$\begin{aligned} (a(1,n), a(2,n), a(3,n), a(4,n))^T, &\qquad 1 \leq n \leq \frac{v-1}{2}, \\ (a'(1,n'), a'(2,n'), a'(3,n'), a'(4,n'))^T, &\qquad 1 \leq n' \leq \frac{v'-1}{2}, \end{aligned}$$

respectively. Then the following columns yield the SA$(4, v)$:

$$\begin{pmatrix} a(1,n) & a'(1,n')v & a(1,n)+a'(1,n')v & a(1,n)+a'(2,n')v \\ a(2,n) & a'(2,n')v & a(2,n)+a'(2,n')v & a(2,n)+a'(1,n')v \\ a(3,n) & a'(3,n')v & a(3,n)+a'(3,n')v & a(3,n)+a'(4,n')v \\ a(4,n) & a'(4,n')v & a(4,n)+a'(4,n')v & a(4,n)+a'(3,n')v \end{pmatrix}$$

for $1 \leq n \leq (v-1)/2$ and $1 \leq n' \leq (v'-1)/2$. $\square$

Lemma 6.6.7 There are SA$(4, 3^m)$ for any integer $m \geq 3$.

Proof. For $m = 3, 4, 5$, SA$(4, 3^m)$ are given as in Section 6.2 and Lemmas 6.6.3 and 6.6.5. Hence, applying Lemma 6.6.6 repeatedly with $v = 3^3, 3^4, 3^5$ and $v' = 3^3$ shows the existence of SA$(4, 3^m)$ for any integer $m \geq 3$. $\square$

Finally, the main results of this section are established.

Theorem 6.6.8 There are 2 PACB$(v, 2, 1)$ for any odd integer $v \geq 5$ such that $\gcd(v, 9) \neq 3$.

Proof. Let $v(\geq 5)$ be an odd integer such that $\gcd(v, 9) \neq 3$. When $\gcd(v, 9) = 1$, since $\gcd(v, 6) = 1$, Theorem 6.2.3 shows the existence of 2 PACB$(v, 2, 1)$. When $\gcd(v, 9) = 9$, we can put $v = 3^n t$ with integers $n(\geq 2)$ and $t(\geq 1)$ such that $\gcd(t, 6) = 1$. Then Corollary 6.5.3 and Theorem 6.5.4 show the existence of 2 PACB$(v, 2, 1)$. Thus, the proof is complete. $\square$

Theorem 6.6.9 There are 2 PACB$(v, 2, 1)$ with $v = 2^m t$ for any integer $m \geq 2$ and any odd integer $t \geq 1$ such that $m \not\equiv 1 \pmod 4$ and $\gcd(t, 27) \neq 3, 9$.

Proof. Let $t(\geq 1)$ be an odd integer such that $\gcd(t, 27) \neq 3, 9$. Then we can put $t = 3^n t'$ with a non-negative integer $n \neq 1, 2$ and an odd integer $t'(\geq 1)$ such that $\gcd(t', 6) = 1$. Now there are 2 PACB$(2^m, 2, 1)$ for any positive integer $m \not\equiv 1 \pmod 4$ (see Theorem 6.4.3). Also there are SA$(4, 3^n)$ for $n \geq 3$ (see Lemma 6.6.7). Hence Theorem 6.5.5 shows the existence of 2 PACB$(2^m 3^n, 2, 1)$ for any positive integer $m \not\equiv 1$ (mod 4) and any non-negative integer $n \neq 1, 2$. By Lemma 6.6.1, when $t' \geq 5$, there are SA$(4, t')$, since $\gcd(t', 6) = 1$. Hence Theorem 6.5.5 shows the existence of 2 PACB$(2^m 3^n t', 2, 1)$ for any integer $m(\geq 2)$ and any odd integer $3^n t'(\geq 1)$ such that $m \not\equiv 1 \pmod 4$ and $\gcd(3^n t', 27) \neq 3, 9$. Thus, the proof is complete. $\square$

6.7 Non-existence of pairwise additive cyclic BIB designs

By considering all possible combinations of initial blocks, it is easily seen that there are no 2 PACB(6,2,1). However, for a given integer v, whether ℓ PACB$(v, 2, 1)$ exist or not is a difficult problem for general $\ell \leq v/2$. Here, it is shown that there are no 2 PACB$(v, 2, 1)$ for any $v \equiv 2$ (mod 4) and, incidentally, no ℓ PACB$(12, 2, 1)$ for $\ell \in \{5, 6\}$.

Theorem 6.7.1 There are no 2 PACB$(v, 2, 1)$ for any $v \equiv 2 \pmod 4$.

Proof. Assume that there exist 2 PACB$(v = 2t, 2, 1)$ $(V, \mathcal{B})$ with incidence matrices $\boldsymbol{N}_1$ and $\boldsymbol{N}_2$, where t is an odd integer. Since $\mathcal{B} = \{\{v_1, v_2\} | v_1, v_2 \in V, v_1 \neq v_2\}$ for any B$(v, 2, 1)$, without loss of generality, let initial blocks of $\boldsymbol{N}_1$ can be

$$B_i^{(1)} = \{0, i\}, 1 \le i \le t-1,\ B_t^{(1)} = \{0, t\}\mathrm{PC}(t) \bmod 2t,$$

initial blocks of $\boldsymbol{N}_2$ can be

$$B_i^{(2)} = \{a_i, b_i\},\ B_t^{(2)} = \{c, c+t\}\,\mathrm{PC}\,(t) \bmod 2t$$

and initial blocks of $\boldsymbol{N}_1 + \boldsymbol{N}_2$ can be

$$\{0, i, a_i, b_i\}, \{0, t, c, c+t\}\,\mathrm{PC}\,(t) \bmod 2t,$$

where $1 \le i \le t-1$. Further let $\Delta_f(B_i^{(1)}, B_i^{(2)})$ and $\Delta_s(B_t^{(1)}, B_t^{(2)})$ be multisets on Z_{2t} (see Section 6.2 for the meaning of notations). Then every non-zero element of Z_{2t} occurs 4 times in the multiset

$$\Delta = \Delta_f(B_1^{(1)}, B_1^{(2)}) \cup \ldots \cup \Delta_f(B_{t-1}^{(1)}, B_{t-1}^{(2)}) \cup \Delta_s(B_t^{(1)}, B_t^{(2)}).$$

In other words, the number of even elements of Z_v in Δ must be a multiple of 4. It is seen that exact 2 even elements occur in $\Delta_s(B_t^{(1)}, B_t^{(2)})$ and the number of even elements in each of $\Delta_f(B_i^{(1)}, B_i^{(2)})$, $1 \le i \le t-1$, is one of 0, 4 or 8. Hence the number of even elements in Δ is not a multiple of 4, which is a contradiction. $\Box$

Incidentally, another non-existence is shown.

Theorem 6.7.2 There are no 5 PACB$(12, 2, 1)$ and no ACB$(12, 2, 1)$.

Proof. Assume that there are 5 PACB$(12, 2, 1)$ with incidence matrices $\boldsymbol{N}_1, \ldots, \boldsymbol{N}_5$. Let $B_i^{(h)}$, $1 \le i \le 5$ and $1 \le h \le 5$, be the ith initial block of $\boldsymbol{N}_h$ and $B_6^{(h)} = \{c_h, c_h + 6\}$, $c_h \in Z_{12}$, $1 \le h \le 5$, be a short initial block of $\boldsymbol{N}_h$. Then, without loss of generality, we can let $c_1 = 0, c_2 = 2, c_3 = 4, c_4 = 1$ and $c_5 = 3$ by choosing an arbitrary block from the short block orbit with some replacement of subscripts.

Let $\Delta_f(B_i^{(h)}, B_i^{(h')})$ and $\Delta_s(B_6^{(h)}, B_6^{(h')})$ be multisets on Z_v similarly to Section 6.2. Then the number of even elements in $\Delta_s(B_6^{(h)}, B_6^{(h')})$ is (i) 4 if h, h' $(h \ne h') \in \{1, 2, 3\}$ and (ii) 0 if $h \in \{1, 2, 3\}$ and $h' \in \{4, 5\}$.

An initial block $B_i^{(h)}$ is said to be *even* or *odd*, according as the difference of the two elements in $B_i^{(h)}$ is even or odd. Then it is clear that each $\boldsymbol{N}_h$ includes exact 3 even initial blocks and the number of even elements in $\Delta_f(B_i^{(h)}, B_i^{(h')})$, $1 \leq h < h' \leq 5$, is (i) 0 or 8 if both two blocks are even and (ii) 4 if either of them is odd.

Now, every non-zero element of Z_v must occur 4 times, that is, there are 24 even elements and 20 odd elements in the multiset

$$\Delta_f(B_1^{(h)}, B_1^{(h')}) \cup \ldots \cup \Delta_f(B_5^{(h)}, B_5^{(h')}) \cup \Delta_s(B_6^{(h)}, B_6^{(h')})$$

for $1 \leq h < h' \leq 5$. Let $G = (g(m,n))$ be a matrix of order 5 and $\boldsymbol{x}_s$, $1 \leq s \leq 5$, be the sth row vector of G, where $g(m,n) = 1$ or 0, according as the nth initial block of $\boldsymbol{N}_m$ is even or odd. Then

$$\boldsymbol{x}_s \cdot \boldsymbol{x}_t = \begin{cases} 2 & \text{if } s = t, \\ 0, 2 & \text{if } s \in \{1,2,3\} \text{ and } t \in \{4,5\}, \\ 1 & \text{otherwise}, \end{cases} \tag{6.15}$$

where $\cdot$ is the usual inner product among row vectors.

The first three rows of G satisfying (6.15) must be one of the following under some permutation of columns:

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix},$$

and then the first four rows of G satisfying (6.15) can be further reduced only to the following:

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

It can be seen that there is no $\boldsymbol{x}_5$ such that $\boldsymbol{x}_5 \cdot \boldsymbol{x}_s = 0$ or 2 and $\boldsymbol{x}_5 \cdot \boldsymbol{x}_4 = 1$ for any $s \in \{1,2,3\}$. This implies that there does not exist the required matrix G satisfying (6.15). Hence there is no 5 PACB$(12,2,1)$ and then it is shown by the definition of additive cyclic BIB designs that there is no ACB$(12,2,1)$. $\square$

Note that the existence of 4 PACB$(12,2,1)$ and ℓ PAB$(12,2,1)$ with $\ell = 4,5,6$ are still unknown.

Chapter 7

Decomposition of all-one matrix

The existence (decomposition) of a set of BIB designs satisfying (1.2) given in Chapter 1 is discussed.

7.1 Method through resolvable BIB designs

For given parameters v, b, r, k and λ, does there exist a set of s matrices, say $\boldsymbol{N}_1, \ldots, \boldsymbol{N}_s$, each of which forms an incidence matrix of a BIBD(v, b, r, k, λ) satisfying (1.2)? Thus the present problem is to decompose $\boldsymbol{J}_{v\times b}$ into a sum of s incidence matrices of BIB designs with same given parameters. For any incidence matrix $\boldsymbol{N}_1$ of a B$(v = 2k, k, \lambda)$, i.e., $s = 2$, $\boldsymbol{N}_2 = \boldsymbol{J}_{v\times b} - \boldsymbol{N}_1$ is an incidence matrix of a B$(v = 2k, k, \lambda)$. Hence a complete answer to the decomposition problem for a B$(v = 2k, k, \lambda)$ is obtained. Thus the decomposition problem is essentially of interest when $s \geq 3$.

At first, a general result for a class of resolvable BIB designs is shown.

Theorem 7.1.1 Any resolvable BIBD$(v = sk, b, r, k, \lambda)$ has s incidence matrices satisfying the condition (1.2).

Proof. Let $B_i = \{B_{i1}, B_{i2}, \ldots, B_{is}\}$, $i = 1, \ldots, r$, be the ith resolution set of a resolvable B$(v = sk, k, \lambda)$. Identify a k-subset B_{ij} with zero-one column vector $\boldsymbol{B}_{ij}$ of length v such that $x \in B_{ij}$ if and only if the xth coordinate of $\boldsymbol{B}_{ij}$ equals one. For each $t = 1, \ldots, s$, the matrix $\boldsymbol{N}_t$ is defined by arranging $\boldsymbol{B}_{i,j+t-1}$ in the $((i-1)s + j)$th column, where the second subscript $j+t-1$ of $\boldsymbol{B}_{i,j+t-1}$ is reduced modulo s. Obviously, a set of $\boldsymbol{N}_t$ forms incidence matrices of a BIB design satisfying (1.2). $\square$

Thus a class of resolvable BIB designs solves the present problem. Once any concrete solution of resolvable BIB designs is found, the above-mentioned procedure produces different incidence matrices satisfying (1.2). Many resolvable BIB designs are available in literature (cf. [3, 22]).

7.2 Method through STS$(6m+3)$

The present problem is considered for a class of non-resolvable BIB designs.

Lemma 7.2.1 For a positive integer m, there exists a BIBD$(3(2m+1), (2m+1)(3m+1), 3m+1, 3, 1)$.

Proof. The method of construction of the BIB design is due to Skolem's method [38] (see also [5]). Let a set of points $V = \{x_i | x = 0, 1, ..., 2m;\ i = 0, 1, 2\}$ be arranged in a $3 \times (2m+1)$ array as

$$\begin{array}{lllll} 0_0 & 1_0 & 2_0 & \cdots & (2m)_0 \\ 0_1 & 1_1 & 2_1 & \cdots & (2m)_1 \\ 0_2 & 1_2 & 2_2 & \cdots & (2m)_2. \end{array} \tag{7.1}$$

For a pair $(x_i, y_i), x \neq y, i = 0, 1, 2$, form a block $\{x_i, y_i, z_{i+1}\}$ (called a type A) such that

$$x + y \equiv 2z \pmod{2m+1},$$

(subscripts reduced mod 3) and take more blocks $\{x_0, x_1, x_2\}$ (called a type B) for $0 \leq x \leq 2m$. Then there are $(2m+1)(3m+1)$ blocks in all. In fact, all blocks of type A are obtained by developing the following initial blocks on $Z_{2m+1} \times Z_3$:

$$\{0_0, 2_0, 1_1\}, \{0_0, 4_0, 2_1\}, \ldots, \{0_0, (2m)_0, m_1\}.$$

where a "development" of $\{x_{i_1}, y_{i_2}, z_{i_3}\}$ means that we get $3(2m+1)$ blocks as

$$\begin{array}{l} \{(x+\ell)_{i_1}, (y+\ell)_{i_2}, (z+\ell)_{i_3}\}, \\ \{(x+\ell)_{i_1+1}, (y+\ell)_{i_2+1}, (z+\ell)_{i_3+1}\}, \\ \{(x+\ell)_{i_1+2}, (y+\ell)_{i_2+2}, (z+\ell)_{i_3+2}\}, \end{array}$$

(mod $2m+1$) for $0 \le \ell \le 2m$ and subscripts are reduced mod 3. Note that a block $\{x_i, y_i, z_{i-1}\}$ (instead of $\{x_i, y_i, z_{i+1}\}$) is taken to produce the BIB design. $\square$

The BIB design constructed through the above procedure is called *Skolem type*.

Lemma 7.2.2 $\mathrm{B}(6m+3, 3, 1)$ of Skolem type is resolvable if and only if $m \equiv 1 \pmod 3$.

Proof. Assume that $\mathrm{B}(6m+3, 3, 1)$ of Skolem type is resolvable. Then it consists of $3m+1$ resolution sets. Since the number of blocks of type B equals $2m+1$, there exists at least one resolution set which consists only of blocks of type A. Pick up such a resolution set. A block of type A, say $\{x_i, y_i, ((x+y)/2)_{i+1}\}$, intersect the ith row of the array (7.1) at exactly two points. Denote by X_i, $i = 0, 1, 2$, the number of blocks of type A which intersect the ith row at exactly two points. Then it holds that

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} X_0 \\ X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 2m+1 \\ 2m+1 \\ 2m+1 \end{pmatrix}.$$

Solving these equations implies $X_i = (2m+1)/3$, $i = 0, 1, 2$. Since X_i is an integer, $2m+1 \equiv 0 \pmod 3$, that is, $m \equiv 1 \pmod 3$. The converse is obvious. $\square$

Theorem 7.2.3 For a positive integer $m \not\equiv 1 \pmod 3$, there exists a non-resolvable $\mathrm{B}(6m+3, 3, 1)$ has $2m+1$ incidence matrices satisfying the condition (1.2).

Proof. Consider $\mathcal{Z}/\sim$ with the equivalence relation

$$x \sim y \iff \begin{cases} x \equiv y & \text{if } x \equiv y \pmod{2m+1} \\ x + y \equiv 2m+1 & \text{if } x \not\equiv y \pmod{2m+1}. \end{cases}$$

Then it can be seen that elements in $\mathcal{Z}/\sim$ are even, i.e., 2, 4, ..., $2m$, and they are expressed by one of the follwing elements as

$$\begin{array}{llllll} x_1, & x_1 2, & x_1 2^2, & \cdots, & x_1 2^{p_1 - 1} & (x_1 2^{p_1} \equiv x_1 \pmod{2m+1}) \\ x_2, & x_2 2, & x_2 2^2, & \cdots, & x_2 2^{p_2 - 1} & (x_2 2^{p_2} \equiv x_2 \pmod{2m+1}) \\ \vdots & \vdots & \vdots & & \vdots & \\ x_f, & x_f 2, & x_f 2^2, & \cdots, & x_f 2^{p_f - 1} & (x_f 2^{p_f} \equiv x_f \pmod{2m+1}) \end{array} \tag{7.2}$$

where $x_2, x_3, \dots, x_f$ do not appear in the first row, the first two rows, …, the first $f-1$ rows, respectively, and then all elements are different each other, i.e., $x_i 2^j \not\equiv x_i$ for all j such that $0 \le j \le p_i$.

Now, for blocks, given by Skolem's method as in Lemma 7.2.1, $\{x_i,\ y_i,\ z_{i+1}\}$, $\{x_i,\ y_i,\ z_{i-1}\}$ where $x + y \equiv 2z \pmod{2m+1}$, when $x - y \sim d\ (\in \mathcal{Z}/\sim)$, these blocks are called d-blocks. Then the point set V in Lemma 7.2.1 can be expressed by the union of $2m+1-n$ d-blocks and n $2d$-blocks, where $n = (d, 2m+1)$. In fact, the $(2m+1-n)$ d-blocks are given by

$$\begin{array}{l}
\{0_0, d_0, (z_{i_0})_1\},\\
\{d_2, (2d)_2, (z_{i_1})_1\},\\
\{(2d)_0, (3d)_0, (z_{i_2})_1\},\\
\quad \cdots\\
\{((n'-3)d)_0, ((n'-2)d)_0, (z_{i_{n'-3}})_1\},\\
\{((n'-2)d + \frac{d}{2})_1, ((n'-1)d + \frac{d}{2})_1, (z_{i_{n'-2}})_2\}
\end{array}$$

and n $2d$-blocks are given by

$$\{((n'-2)d)_2, (n'd)_2, (z_{i_{n'-1}})_0\}$$

with the elements in a development of these blocks mod n, where $n' = (2m+1)/n$.

For further argument, put the arrangement (7.2) alternatively as

$$\begin{array}{llll}
a_{11}, & a_{12}, & \cdots, & a_{1p_1}\\
a_{21}, & a_{22}, & \cdots, & a_{2p_2}\\
\vdots & \vdots & & \vdots\\
a_{k1}, & a_{k2}, & \cdots, & a_{kp_k}.
\end{array}$$

Then it holds that $(a_{ij}, 2m+1) = n$ if and only if $(a_{ij'}, 2m+1) = n$. When $(a_{ij}, 2m+1) = n$, the union of $(2m+1-n)$ a_{ij}-blocks and n $a_{i,j+1}$-blocks presents a set X of symbols. These $(2m+1-n)$ a_{ij}-blocks play a role of initial blocks for incidence matrices $\boldsymbol{N}_1, \boldsymbol{N}_2, ..., \boldsymbol{N}_{2m+1-n}$, while n $a_{i,j+1}$-blocks play a role of initial blocks for incidence matrices $\boldsymbol{N}_{2m+2-n}, \boldsymbol{N}_{2m+3-n}, ..., \boldsymbol{N}_{2m+1}$. These should be developed under $1 \le i \le k$ and $1 \le j \le p_i$. Finally, to each of the above $\boldsymbol{N}_i$ add a block of type B, namely, $\{(i+j)_0,\ (i+j)_1,\ (i+j)_2\}$ for $0 \le j \le 2m$. Hence it can be shown that these incidence matrices satisfy (1.2). By Lemma 7.2.2, the BIB design is non-resolvable. □

The followings illustrates Theorem 7.2.3 with $m = 3$:

$\boldsymbol{N}_1$	$\boldsymbol{N}_2$	$\boldsymbol{N}_3$	$\boldsymbol{N}_4$	$\boldsymbol{N}_5$	$\boldsymbol{N}_6$	$\boldsymbol{N}_7$	
$0_0 6_0 3_1$	$6_2 5_2 2_1$	$5_0 4_0 1_1$	$4_2 3_2 0_1$	$3_0 2_0 6_1$	$5_1 4_1 1_2$	$2_2 0_2 1_0$	six 6-blocks, a 2-block
$0_0 2_0 1_1$	$2_2 4_2 3_1$	$4_0 6_0 5_1$	$6_2 1_2 0_1$	$1_0 3_0 2_1$	$4_1 6_1 5_2$	$3_2 0_2 5_0$	six 2-blocks, a 4-block
$0_0 4_0 2_1$	$4_2 1_2 6_1$	$1_0 5_0 3_1$	$5_2 2_2 0_1$	$2_0 6_0 4_1$	$1_1 5_1 3_2$	$6_2 0_2 3_0$	six 4-blocks, a 6-block
$0_0 0_1 0_2$	$1_0 1_1 1_2$	$2_0 2_1 2_2$	$3_0 3_1 3_2$	$4_0 4_1 4_2$	$5_0 5_1 5_2$	$6_0 6_1 6_2$	PC(7)

all of which are not resolvable.

Remark 7.2.4 Among BIB designs with $v = 3k$ (i.e., $s = 3$), a BIB design with parameters $v = 21, b = 30, r = 10, k = 7$ and $\lambda = 3$ cannot be resolvable [34]. It remains unknown whether or not the present decomposition problem can be solved for the BIB design.

7.3 Method through additive BIB designs

It is clear that the existence of AB(sk, k, λ) implies the existence of s incidence matrices of B(sk, k, λ) which gives a decomposition of all-one matrix by definition of additive BIB designs. Hence a construction of s incidence matrix with (1.2) through additive BIB designs, which is similar to Theorem 5.2.2, can be obtained without the existence of a symmetric difference matrix and a perpendicular array.

Theorem 7.3.1 Let $s = 2m + 1$ be an odd integer. If there exist additive BIBD$(sk, , b, r, k, \lambda)$, then there exist B(s^2k, sk, r) with the condition (1.2).

Proof. Let $\boldsymbol{C}_s$ be a circulant matrix defined in Lemma 5.2.3 and a matrix $\boldsymbol{A}_t = (a_{ij}^{(t)})$ based on Z_s, $1 \le t \le s$, be

$$\boldsymbol{A}_t = \sum_{h=1}^{m+1} h\boldsymbol{C}_s^h + \sum_{h=m+2}^{s} (2m + 2 - h)\boldsymbol{C}_s^h + t\boldsymbol{J}_s$$

and the incidence matrix $\boldsymbol{N}_t^*$ defined by

$$\boldsymbol{N}_t^* = [(\boldsymbol{N}_{a_{ij}^{(t)}}) : (\boldsymbol{J}_{ij})]$$

where $\boldsymbol{J}_{ij} = \delta_{i+t-1,j}\boldsymbol{J}_{sk\times(r-s\lambda)}$ and δ is the Kronecker delta.
Then $\boldsymbol{N}_t^*$ are s incidence matrices of the required design. $\square$

Note that, if starting BIB designs are not resolvable, it is not easy to show whether each BIB designs obtained by Theorem 7.3.1 is resolvable or not.

Chapter 8

Applications

Some applications of the property of pairwise additivity are discussed.

8.1 Construction of BIB designs

Some BIB designs with distinct block sizes can be obtained by ℓ pairwise additive BIB designs. For example, the fundamental method of construction of BIB designs can be obtained by the property in Lemma 1.2.5. Now, some series of BIB designs are provided.

Corollary 8.1.1 Let s be an odd integer $s \geq 3$. If there exist a resolvable $\mathrm{B}(sk, k, \lambda)$ and a $\mathrm{PA}(s, s)$, then there exists a $\mathrm{B}(sk, tk, \lambda t(s-1)(tk-1)/[2(k-1)])$ for any positive integer t with $2 \leq t \leq s-1$.

Proof. By applying Corollary 2.4.3 with a resolvable $\mathrm{B}(sk, k, \lambda)$ and a $\mathrm{PA}(s, s)$, $\mathrm{AB}(sk, k, (s-1)\lambda/2)$ can be obtained. Thus Lemma 1.2.5 yields the required design. $\square$

The following can be obtained by Theorem 3.3.1.

Corollary 8.1.2 There exist $\mathrm{B}(3^n, 3t, t(3t-1)/2)$ for any t with $2 \leq t \leq 3^{n-1}-1$.

Note that when $t = 2$, $\mathrm{B}(3^n, 6, 5)$ obtained by Corollary 8.1.2 satisfy the pairwise additivity and minimal property.

Next, a recursive construction of BIB designs, which is similar to Theorem 7.3.1, can be obtained from $\ell \geq (s+1)/2$ pairwise additive BIB designs instead of additive BIB designs. This fact was already mentioned in [35] as the following shows.

Theorem 8.1.3 [35] Let s be an odd integer. If there exist $\ell \geq (s+1)/2$ PAB(sk, k, λ), then there exists a B(s^2k, sk, r).

8.2 Construction of multiply nested BIB designs

The following lemma shows that multiply nested BIB designs can be obtained from pairwise additive BIB designs.

Lemma 8.2.1 Let ℓ be any positive integer with $2 \leq \ell \leq s$. If there are ℓ pairwise additive BIBD(sk, b, r, k, λ), then there exists an MNB$(sk; b_1 = 2^{m-1}b, b_2 = 2^{m-2}b, \ldots, b_m = b; k_1 = k, k_2 = 2k, \ldots, k_m = 2^{m-1}k)$ for any $2 \leq t \leq \ell$ and $m = \lfloor \log_2 t \rfloor + 1$.

By use of Lemma 8.2.1, Corollaries 2.2.2 and 2.2.4 and Theorem 2.3.4 can yield the following respective results.

Theorem 8.2.2 The existence of ℓ PAB$(v_i, 2, 1)$ and TD$(2\ell,\ v_{i'})$ for $1 \leq i \leq t$ and $2 \leq i' \leq t$ implies the existence of an MNB$(v_1v_2 \cdots\ v_t; 2^{m-1}b, 2^{m-2}b, \ldots, b;\ 2, 2^2, \ldots, 2^m)$, where $b = v_1v_2 \cdots v_t(v_1v_2 \cdots v_t - 1)/2$ and $m = \lfloor \log_2 \ell \rfloor + 1$.

Theorem 8.2.3 The existence of ℓ PAB$(v_i, 2, 1)$, ℓ PAB$(v_t + 1, 2, 1)$ and TD$(2\ell, v_{i'})$ for $1 \leq i \leq t-1$ and $2 \leq i' \leq t$ implies the existence of an MNB$(v_1v_2 \cdots v_t + 1;\ 2^{m-1}b, 2^{m-2}b, \ldots, b;\ 2, 2^2, \ldots, 2^m)$, where $b = v_1v_2 \cdots v_t(v_1v_2 \cdots v_t + 1)/2$ and $m = \lfloor \log_2 \ell \rfloor + 1$.

Theorem 8.2.4 There exists a doubly NB$(v; 2v(v-1), v(v-1), v(v-1)/2; 2, 4, 8)$ for $v \geq 583$.

Furthermore we have the following.

Theorem 8.2.5 Let $v = q, 8q+1, 9q+1, 10q$, where the prime factorization of q is $p_1^{d_1}p_2^{d_2}\ldots p_t^{d_t}$ with $p_i^{d_i} \geq 8$ for $1 \leq i \leq t$. Then there exists a doubly NB$(v; 2v(v-1), v(v-1), v(v-1)/2; 2, 4, 8)$.

Proof. Since $p_i^{d_i} \geq 8$ $(1 \leq i \leq t)$, there are 4 PAB$(p_i^{d_i}, 2, 1)$ (on account of Theorems 2.1.5 and 5.1.2) and 4 PAB$(10, 2, 1)$ (by Lemma 2.1.9) and

TD$(8, p_i^{d_i})$ (by Lemma 1.4.4). Hence the proof is complete by applying Corollaries 2.2.2 and 2.2.4. $\square$

Thus we can present a recursive construction of multiply nested BIB designs with $k_1 = 2$ by applying Theorems 8.2.2 and 8.2.3. Incidentally, this recursive construction can be applied to have pairwise additive BIB designs with block size $k \geq 3$ also (see Section 3.4).

As a generalization of pairwise additive BIB designs, t-designs with pairwise additivity may be considered. Some of methods here can be also applied to such t-designs. However, note that the property of pairwise additivity as shown in Lemma 1.2.5 does not always hold.

Concluding remark

We show some existence and constructions of pairwise additive BIB designs. However, there remain many open problems for the existence of pairwise additive BIB designs. For example, the existence of

(i) ℓ PAB$(12, 2, 1)$ for $\ell = 4, 5, 6$,

(ii) 2 PAB$(v, 3, 1)$ for $v = 55, 115, 145, 205, 265, 319, 355, 415, 493, 667, 697, 1315$,

(iii) 2 PAB$(v, 3, 1)$ for $v \equiv 3 \pmod 6$ and $v \neq 15, 3^n$,

(iv) AB$(v, 2, 1)$ in Table 5.3.2,

(v) 2 PACB$(32, 2, 1)$,

(vi) 2 PACB$(3p, 2, 1)$ for any odd prime p

may be unknown.

On the other hand, it also seems to be not easy to show non-existence of pairwise additive BIB designs by methods given in the present thesis. Nevertheless, the non-existence of PACB$(v, 2, 1)$ for $v \equiv 4 \pmod 8$ might be shown by methods similar to Section 6.7.

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