The Fundamentals of Economic Dynamics and Policy Analyses: Learning through Numerical Examples. Part Ⅲ. Stochastic Dynamic General Equilibrium

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The objective of this paper is to present a simple stochastic optimal growth model (Ramsey model), and calculate a stochastic dynamic general equilibrium (hereafter referred as a SDGE) of the model. (This part draws an example from Farmer (1999).) Then, we demonstrate how to simulate the movements of economic variables in the stochastic dynamic general equilibrium by using Matlab. The paper consists of 3 sections. A stochastic optimal growth model is presented in section 1. The stochastic dynamic general equilibrium of the model is calculated in section 2. The movements of economic variables in the stochastic dynamic general equilibrium are simulated by using Matlab in section 3.

1. Model.

The structure of the stochastic optimal growth model is almost same as that of the discrete time Ramsey model. (See Futamura (2013).) The only difference is that the production technology is affected by a random multiplicative shock. The model consists of the following three equations. For each $t = 0, 1, 2, \ldots$,

(1.1) $c_t + k_{t+1} = (1-\delta)k_t + s_t k_t^{\alpha}$, $0 < \alpha < 1$, $0 \leq \delta \leq 1$,

(1.2) $s_{t+1} = s_t^{\rho} v_t$, $0 \leq \rho < 1$,

(1.3) $E_t\left[\sum_{\tau=0}^{\infty} \beta^{\tau} u(c_{t+\tau})\right]$,

where

(1.4) $u(c_{t+\tau}) = \dfrac{(c_{t+\tau})^{1-\sigma}-1}{1-\sigma}$, $\sigma \neq 1$, $\sigma > 0$.

Equation (1.1) describes the resource constraint of period t. The right-hand side of (1.1) consists of output $s_t k_t^{\alpha}$ and capital $(1-\delta)k_t$ carried over to the next period t + 1. δ is the depreciation rate. The left-hand side of (1.1) describes the disposition of the resource in t, consisting of consumption c_t and the next period's capital k_{t+1}. The production technology s_t is affected by stochastic shock. Its evolution is described by equation (1.2). In (1.2), v_t is identically and independently distributed (i.i.d) random variable with mean $\bar{v}$ and variance σ_v^2. Equation (1.2) implies that the larger is ρ, the more persistent is the technological shock (although the effect diminishes as time goes by because $0 \leq \rho < 1$). At each time t, the history of technology $\{s_t, s_{t-1}, s_{t-2}, \ldots\}$ is known to public. Equation (1.3) is an expected value of lifetime utility where the expectation is conditional on the information available at t. The utility of consumption c_t in period t is given by equation (1.4). (1.4) implies that the utility $u(c_t)$ is an increasing and concave function of c_t with constant intertemporal elasticity of substitution between c_t and c_{t+1}. At each time t, a representative economic agent chooses $\{k_{t+\tau+1}, c_{t+\tau}; \tau = 0, 1, 2, \ldots\}$ to maximize the expected lifetime utility (1.3) subject to the resource constraint (1.1) and the evolution process (1.2) of the stochastic technology, given $\{k_t, s_t\}$ as predetermined variables. (The model is also described as a decentralized economy as we did in the analysis of the dynamic general equilibrium of (non-stochastic) Ramsey model.)

2. The Analysis of Stochastic Dynamic General Equilibrium.

The SDGE of the optimal growth model of section 1 is analyzed through the following steps.

Step 1. We derive a system of stochastic difference equations with respect to $\{c_t, k_{t+1}, s_{t+1}$; t = 0, 1, 2, ... $\}$ such that these variables satisfy the SDGE conditions. Equation (1.3) is rewritten as follows.

$$(2.1)\quad U \equiv E_t[\ u(c_t) + \beta u(c_{t+1}) + \beta^2 u(c_{t+2}) + \ldots]$$

In equation (2.1), by (1.1),

$$(2.2)\quad c_t = (1-\delta)k_t + s_t k_t^{\alpha} - k_{t+1},$$

$$(2.3)\quad c_{t+1} = (1-\delta)k_{t+1} + s_{t+1} k_{t+1}^{\alpha} - k_{t+2},$$

$$(2.4)\quad c_{t+2} = (1-\delta)k_{t+2} + s_{t+2} k_{t+2}^{\alpha} - k_{t+3}.$$

Therefore, the maximization of (1.3) subject to (1.1), given the exogenous forcing process (1.2), implies that at each t = 0, 1, 2, ... , $\{c_t, k_{t+1}\}$ must satisfy the following first-order condition.

$$(2.5)\quad \frac{\partial U}{\partial k_{t+1}} = E_t[\ u'(c_t)\frac{\partial c_t}{\partial k_{t+1}} + \beta u'(c_{t+1})\frac{\partial c_{t+1}}{\partial k_{t+1}}\ ]$$
$$= E_t[-c_t^{-\sigma} + \beta c_{t+1}^{-\sigma}(1-\delta + \alpha s_{t+1} k_{t+1}^{\alpha-1})] = 0$$

Equation (1.4) is used to derive the second line of (2.5). Equation (2.5) is a stochastic Euler equation. Define

$$(2.6)\quad \Psi^1(c_t,\ c_{t+1},\ k_{t+1},\ s_{t+1}) \equiv -c_t^{-\sigma} + \beta c_{t+1}^{-\sigma}(1-\delta + \alpha s_{t+1} k_{t+1}^{\alpha-1}),$$

$$(2.7)\quad \Psi^2(c_t,\ k_t,\ k_{t+1},\ s_t) \equiv (1-\delta)k_t + s_t k_t^{\alpha} - k_{t+1} - c_t,$$

$$(2.8)\quad \Psi^3(s_t,\ s_{t+1},\ v_t) \equiv -s_{t+1} + s_t^{\rho} v_t.$$

Then, at any t = 0, 1, 2, ... , given $\{k_t, s_t\}$, $\{c_t, k_{t+1}, s_{t+1}\}$ must satisfy the following system of first-order nonlinear stochastic difference equations.

$$(2.9)\quad E_t[\ \Psi^1(c_t,\ c_{t+1},\ k_{t+1},\ s_{t+1})] = 0$$

$$(2.10)\quad \Psi^2(c_t,\ k_t,\ k_{t+1},\ s_t) = 0$$

$$(2.11)\quad \Psi^3(s_t,\ s_{t+1},\ v_t) = 0$$

Step 2. Define the steady state of {(2.9), (2.10), (2.11)} as a triplet $\{\bar{c},\ \bar{k},\ \bar{s}\}$ that satisfies the following.

$$(2.12)\quad \Psi^1(\bar{c},\ \bar{c},\ \bar{k},\ \bar{s}) = -(\bar{c})^{-\sigma} + \beta(\bar{c})^{-\sigma}(1-\delta + \alpha\bar{s}(\bar{k})^{\alpha-1}) = 0$$

$$(2.13)\quad \Psi^2(\bar{c},\ \bar{k},\ \bar{k},\ \bar{s}) \equiv (1-\delta)\bar{k} + \bar{s}(\bar{k})^{\alpha} - \bar{k} - \bar{c} = 0$$

$$(2.14)\quad \Psi^3(\bar{s},\ \bar{s},\ \bar{v}) \equiv -\bar{s} + (\bar{s})^{\rho}\bar{v} = 0$$

Equations (2.12), (2.13), and (2.14) are solved for the steady sate $\{\bar{c}, \bar{k}, \bar{s}\}$ as follows.

$$(2.15)\quad \bar{k} = \left[\frac{\alpha\beta\bar{s}}{1-\beta(1-\delta)}\right]^{\frac{1}{1-\alpha}}$$

$$(2.16)\quad \bar{c} = \bar{s}(\bar{k})^{\alpha} - \delta\bar{k}$$

$$(2.17)\quad \bar{s} = (\bar{v})^{\frac{1}{1-\rho}}$$

In addition, equation (2.12) is rewritten as follows.

$$(2.18)\quad \beta(1-\delta + \alpha\bar{s}(\bar{k})^{\alpha-1}) = 1$$

Step 3. We approximate equations {(2.9), (2.10), (2.11)} as a system of first-order linear stochastic difference equations. The first-order linear approximation of $\Psi^1(c_t, c_{t+1}, k_{t+1}, s_{t+1})$ at the steady state $(\bar{c}, \bar{c}, \bar{k}, \bar{s})$ is given by the following.

$$(2.19)\quad \Psi^1(c_t,\ c_{t+1},\ k_{t+1},\ s_{t+1}) \cong \Psi^1(\bar{c},\ \bar{c},\ \bar{k},\ \bar{s}) + (c_t - \bar{c})\frac{\partial\bar{\Psi}^1}{\partial c_t} + (c_{t+1} - \bar{c})\frac{\partial\bar{\Psi}^1}{\partial c_{t+1}} + (k_{t+1} - \bar{k})\frac{\partial\bar{\Psi}^1}{\partial k_{t+1}} + (s_{t+1} - \bar{s})\frac{\partial\bar{\Psi}^1}{\partial s_{t+1}}$$

Notice, by definition, $\Psi^1(\bar{c}, \bar{c}, \bar{k}, \bar{s}) = 0$ in equation (2.19). In (2.19), $\partial\bar{\Psi}^1/\partial c_t$ implies the partial differentiation of $\Psi^1(c_t, c_{t+1}, k_{t+1}, s_{t+1})$ with respect to c_t evaluated at the steady state $\{\bar{c}, \bar{c}, \bar{k}, \bar{s}\}$. Therefore, this term is calculated as follows.

$$(2.20)\quad \frac{\partial\bar{\Psi}^1}{\partial c_t} = \sigma(\bar{c})^{-\sigma-1}$$

The other derivatives in the right-hand side of (2.19) are calculated in the same manner as follows.

$$(2.21)\quad \frac{\partial\bar{\Psi}^1}{\partial c_{t+1}} = -\sigma(\bar{c})^{-\sigma-1}\beta(1-\delta + \alpha\bar{s}(\bar{k})^{\alpha-1}) = -\sigma(\bar{c})^{-\sigma-1}$$

Equation (2.18) is used for the calculation of (2.21).

$$(2.22)\quad \frac{\partial \overline{\Psi}^1}{\partial k_{t+1}} = (\bar{c})^{-\sigma}(\alpha-1)\,\alpha\beta\bar{s}(\bar{k})^{\alpha-2}$$

$$(2.23)\quad \frac{\partial \overline{\Psi}^1}{\partial s_{t+1}} = (\bar{c})^{-\sigma}\,\alpha\beta(\bar{k})^{\alpha-1}$$

Define the deviation of variable $x \in \{c_t, c_{t+1}, k_{t+1}, s_{t+1}\}$ from its steady state as follows.

$$(2.24)\quad \hat{x}_t \equiv (x_t - \bar{x})/\bar{x}$$

Then, equation (2.19) is rewritten as follows.

$$\begin{aligned}(2.25)\quad & \Psi^1(c_t,\ c_{t+1},\ k_{t+1},\ s_{t+1}) \\ & = \sigma(\bar{c})^{-\sigma}\hat{c}_t - \sigma(\bar{c})^{-\sigma}\hat{c}_{t+1} \\ & \quad - \sigma(\bar{c})^{-\sigma}[(1-\alpha)/\sigma]\alpha\beta\bar{s}(\bar{k})^{\alpha-1}\hat{k}_{t+1} \\ & \quad + \sigma(\bar{c})^{-\sigma}(1/\sigma)\alpha\beta\bar{s}(\bar{k})^{\alpha-1}\hat{s}_{t+1}\end{aligned}$$

By (2.25), equation (2.9) implies the following condition.

$$(2.26)\quad E_t[-\hat{c}_t + \hat{c}_{t+1} + a_1\hat{k}_{t+1} - a_2\hat{s}_{t+1}] = 0$$

In equation (2.26),

$$(2.27)\quad a_1 \equiv [(1-\alpha)/\sigma]\alpha\beta\bar{s}(\bar{k})^{\alpha-1},$$

$$(2.28)\quad a_2 \equiv (1/\sigma)\alpha\beta\bar{s}(\bar{k})^{\alpha-1}.$$

Equation (2.26) is rewritten further as follows.

$$\begin{aligned}(2.29)\quad & E_t[-\hat{c}_t + \hat{c}_{t+1} + a_1\hat{k}_{t+1} - a_2\hat{s}_{t+1}] \\ & = -\hat{c}_t + E_t[\hat{c}_{t+1}] + a_1 E_t[\hat{k}_{t+1}] - a_2 E_t[\hat{s}_{t+1}] \\ & = -\hat{c}_t + [\hat{c}_{t+1} + w^c_{t+1}] + a_1[\hat{k}_{t+1} + w^k_{t+1}] \\ & \quad - a_2[\hat{s}_{t+1} + w^s_{t+1}] \\ & = -\hat{c}_t + \hat{c}_{t+1} + a_1\hat{k}_{t+1} - a_2\hat{s}_{t+1} + w^c_{t+1} \\ & \quad + a_1 w^k_{t+1} - a_2 w^s_{t+1} \\ & = 0\end{aligned}$$

In equation (2.29),

$$(2.30)\quad w^c_{t+1} \equiv E_t[\hat{c}_{t+1}] - \hat{c}_{t+1},$$

$$(2.31)\quad w^k_{t+1} \equiv E_t[\hat{k}_{t+1}] - \hat{k}_{t+1},$$

$$(2.32)\quad w^s_{t+1} \equiv E_t[\hat{s}_{t+1}] - \hat{s}_{t+1}$$

are prediction errors. They are also said to be "innovations". When the expectations $\{E_t[\hat{c}_{t+1}], E_t[\hat{k}_{t+1}], E_t[\hat{s}_{t+1}];\ t = 0, 1, 2, \ldots\}$ are "rationally" formed, then $\{w^c_{t+1}, w^k_{t+1}, w^s_{t+1};\ t = 0, 1, 2, \ldots\}$ are i.i.d. random variables. In other words, they are "white noises". See Sargent (1987).

The first-order linear approximation of $\Psi^2(c_t, k_t, k_{t+1}, s_t)$ at the steady state $\{\bar{c}, \bar{k}, \bar{k}, \bar{s}\}$ is given by the following.

$$\begin{aligned}(2.33)\quad & \Psi^2(c_t,\ k_t,\ k_{t+1},\ s_t) \cong \Psi^2(\bar{c},\ \bar{k},\ \bar{k},\ \bar{s}) \\ & \quad + (c_t - \bar{c})\frac{\partial\overline{\Psi}^2}{\partial c_t} + (k_t - \bar{k})\frac{\partial\overline{\Psi}^2}{\partial k_t} \\ & \quad + (k_{t+1} - \bar{k})\frac{\partial\overline{\Psi}^2}{\partial k_{t+1}} + (s_t - \bar{s})\frac{\partial\overline{\Psi}^2}{\partial s_t}\end{aligned}$$

By following the same procedure we did above, equation (2.33) is rewritten as follows.

$$(2.34)\quad -\hat{k}_{t+1} + b_1\hat{k}_t + b_2\hat{s}_t - b_3\hat{c}_t = 0.$$

In equation (2.34),

$$(2.35)\quad b_1 \equiv 1/\beta,$$

$$(2.36)\quad b_2 = [(1/\beta) - (1-\delta)]/\alpha,$$

$$(2.37)\quad b_3 = \bar{c}/\bar{k}.$$

Equation (2.18) is used to obtain these expressions.

The first-order linear approximation of $\Psi^3(s_t, s_{t+1}, v_t)$ at the steady state $\{\bar{s}, \bar{s}, \bar{v}\}$ is expressed as follows.

$$\begin{aligned}(2.38)\quad & \Psi^3(s_t,\ s_{t+1},\ v_t) \cong \Psi^3(\bar{s},\ \bar{s},\ \bar{v}) \\ & \quad + (s_{t+1} - \bar{s})\frac{\partial\overline{\Psi}^3}{\partial s_{t+1}} + (s_t - \bar{s})\frac{\partial\overline{\Psi}^3}{\partial s_t} \\ & \quad + (v_t - \bar{v})\frac{\partial\overline{\Psi}^3}{\partial v_t}\end{aligned}$$

By following the same procedure we did above, (2.38) is rewritten as follows.

$$(2.39)\quad -\hat{s}_{t+1} + \rho\,\hat{s}_t + \hat{v}_t = 0$$

We summarize the above analysis as follows. At each t = 0, 1, 2, ... , given the initial (predetermined) variables $\{k_t, s_t\}$, the SDGE sequence of capital and consumption $\{k_{t+1}, c_t;\ t = 0, 1, 2, \cdots\}$ satisfy the following system of first-order linear difference equations.

$$(2.40)\quad -\hat{c}_t + \hat{c}_{t+1} + a_1\hat{k}_{t+1} - a_2\hat{s}_{t+1} + w^c_{t+1} + a_1 w^k_{t+1} - a_2 w^s_{t+1} = 0$$

$$(2.41)\quad -b_3\hat{c}_t + b_1\hat{k}_t + b_2\hat{s}_t - \hat{k}_{t+1} = 0$$

In equations (2.40) and (2.41), the sequence of stochastic production technology $\{\hat{s}_{t+1}\ ;\ t = 0, 1, 2, \ldots\}$ is generated by the following equation.

$$(2.42)\quad \rho\,\hat{s}_t - \hat{s}_{t+1} + \hat{v}_t = 0$$

Step 4. The above system of difference equations {(2.40), (2.41), (2.42)} is solved as follows. Equations (2.40), (2.41), and (2.42) are represented by the following matrix form.

$$(2.43)\quad \begin{bmatrix} -1 & 0 & 0 \\ -b_3 & b_1 & b_2 \\ 0 & 0 & \rho \end{bmatrix}\begin{bmatrix} \hat{c}_t \\ \hat{k}_t \\ \hat{s}_t \end{bmatrix} + \begin{bmatrix} 1 & a_1 & a_2 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} \hat{c}_{t+1} \\ \hat{k}_{t+1} \\ \hat{s}_{t+1} \end{bmatrix} + \begin{bmatrix} 0 & 1 & a_1 & a_2 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \hat{v}_t \\ w^c_{t+1} \\ w^k_{t+1} \\ w^s_{t+1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Equation (2.43) is rewritten as follows.

$$(2.44)\quad \begin{bmatrix} \hat{c}_t \\ \hat{k}_t \\ \hat{s}_t \end{bmatrix} = A\begin{bmatrix} \hat{c}_{t+1} \\ \hat{k}_{t+1} \\ \hat{s}_{t+1} \end{bmatrix} + B\begin{bmatrix} \hat{v}_t \\ w^c_{t+1} \\ w^k_{t+1} \\ w^s_{t+1} \end{bmatrix}$$

In equation (2.44),

$$(2.45)\quad A \equiv \begin{bmatrix} -1 & 0 & 0 \\ -b_3 & b_1 & b_2 \\ 0 & 0 & \rho \end{bmatrix}^{-1}\begin{bmatrix} 1 & a_1 & a_2 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & a_1 & -a_2 \\ b_3/b_1 & \dfrac{a_1 b_3 + 1}{b_1} & -\left[\dfrac{\rho a_2 b_3 + b_2}{\rho b_1}\right] \\ 0 & 0 & 1/\rho \end{bmatrix},$$

$$(2.46)\quad B \equiv \begin{bmatrix} -1 & 0 & 0 \\ -b_3 & b_1 & b_2 \\ 0 & 0 & \rho \end{bmatrix}^{-1}\begin{bmatrix} 0 & 1 & a_1 & a_2 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & a_1 & -a_2 \\ b_2/(\rho b_1) & b_3/b_1 & a_1 b_3/b_1 & -a_2 b_3/b_1 \\ -1/\rho & 0 & 0 & 0 \end{bmatrix}.$$

For the system of first-order linear difference equations to have a unique SDGE sequence converging (on average) to the steady state $\{\bar{c}, \bar{k}, \bar{s}\}$, the characteristic roots of coefficient matrix A must be such that two roots are larger than one in absolute value (corresponding to the state variables $\{k_{t+1}, s_{t+1}\}$) and the remaining root must be smaller than one in absolute value (corresponding to the control variable c_t). If the three characteristic roots are distinct, then the coefficient matrix A can be decomposed as follows.

$$(2.47)\quad A = Q\Lambda Q^{-1}$$

In equation (2.47),

$$(2.48)\quad \Lambda \equiv \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

is a diagonal matrix of characteristic roots $\{\lambda_1, \lambda_2, \lambda_3\}$, and

$$(2.49)\quad Q = [Q_1\ \ Q_2\ \ Q_3]$$

is a 3×3 matrix whose columns $\{Q_1, Q_2, Q_3\}$ are the characteristic vectors corresponding to each characteristic root. The characteristic roots are the solutions to the following equation.

$$(2.50)\quad \text{Det}\ |A - \Lambda| = 0$$

By substituting $b_1 = 1/\rho$ into A, (2.50) becomes as follows.

$$(2.51)\quad \text{Det}\begin{vmatrix} 1-\lambda & a_1 & -a_2 \\ \beta b_3 & \beta(a_1 b_3 + 1) - \lambda & -\beta(\rho a_2 b_3 + b_2)/\rho \\ 0 & 0 & (1/\rho) - \lambda \end{vmatrix} = \{(1/\rho) - 1\}\{(1-\lambda)[\beta(a_1 b_3 + 1) - \lambda] - \beta a_1 b_3\} = 0$$

Equation (2.51) implies one of the three characteristic roots is

$$(2.52)\quad \lambda = 1/\rho > 1.$$

Obviously, this implies the stability of the sequence of stochastic technology $\{s_t\ ;\ t = 0, 1, 2, \ldots\}$. By equation (2.51), the other two characteristic roots satisfy the following quadratic equation.

(2.53) $\lambda^2 - [\ 1+\beta(a_1 b_3+1)\ ]\ \lambda\ +\ \beta\ = 0$

Denote the solutions to (2.53) as $\{\lambda_1, \lambda_2\}$. Then, equation (2.53) implies the following.

(2.54) $\lambda_1\ +\ \lambda_2\ =\ 1+\beta(a_1 b_3+1)\ > 1$

(2.55) $\lambda_1 \lambda_2\ =\ \beta\ < 1$

Equations (2.54) and (2.55) together imply that both roots are positive, one root is larger than one while the other root is smaller than one. Denote the three characteristic roots of the coefficient matrix A as follows.

(2.56) $\lambda_1\ = (1/2)\{\ [\ 1+\beta(a_1 b_3+1)\ ]$
$\quad - ([1+\beta(a_1 b_3+1)]^2 - 4\beta)^{1/2}\ \} < 1$

(2.57) $\lambda_2\ = (1/2)\{\ [\ 1+\beta(a_1 b_3+1)\ ]$
$\quad + ([1+\beta(a_1 b_3+1)]^2 - 4\beta)^{1/2}\} > 1$

(2.58) $\lambda_3\ = 1/\rho\ > 1$

Then, the characteristic vectors $\{Q_1, Q_2, Q_3\}$ of each characteristic root are obtained as solutions to the following linear equations.

(2.59) $A Q_i = \lambda_i Q_i\ ,\quad Q_i = \begin{bmatrix} q_{1i} \\ q_{2i} \\ q_{3i} \end{bmatrix}\ ,\quad i\ = 1, 2, 3.$

(The three linear equations of (2.59) for three variables $\{q_{1i}, q_{2i}, q_{3i}\}$ are not linearly independent. When using Matlab to calculate a characteristic vector, the program normalizes the length of the vector to be one.) Transform the vector of variables $\{\hat{c}_t, \hat{k}_t, \hat{s}_t\}$ as follows to construct a vector of variables $\{z_{1t}, z_{2t}, z_{3t}\}$ that are linear combinations of $\{\hat{c}_t, \hat{k}_t, \hat{s}_t\}$.

(2.60) $Z_t\ \equiv \begin{bmatrix} z_{1t} \\ z_{2t} \\ z_{3t} \end{bmatrix} = Q^{-1} \begin{bmatrix} \hat{c}_t \\ \hat{k}_t \\ \hat{s}_t \end{bmatrix}$

Equation (2.60) implies

(2.61) $Q Z_t\ = \begin{bmatrix} \hat{c}_t \\ \hat{k}_t \\ \hat{s}_t \end{bmatrix}.$

Then, by multiplying Q^{-1} on both sides of equation (2.44), it is transformed as follows.

(2.62) $Q^{-1} \begin{bmatrix} \hat{c}_t \\ \hat{k}_t \\ \hat{s}_t \end{bmatrix} = Q^{-1} A \begin{bmatrix} \hat{c}_{t+1} \\ \hat{k}_{t+1} \\ \hat{s}_{t+1} \end{bmatrix} + Q^{-1} B \begin{bmatrix} \hat{v}_t \\ w^c_{t+1} \\ w^k_{t+1} \\ w^s_{t+1} \end{bmatrix}$

By (2.47) and (2.60), equation (2.62) is rewritten as follows.

(2.63) $Z_t\ =\ \Lambda Z_{t+1}\ +\ \Omega \begin{bmatrix} \hat{v}_t \\ w^c_{t+1} \\ w^k_{t+1} \\ w^s_{t+1} \end{bmatrix}$

In equation (2.63),

(2.64) $\Omega\ \equiv\ Q^{-1} B$.

Take the expectation on both sides of equation (2.63) conditional on the information available at t. Then, we have

(2.65) $Z_t\ =\ \Lambda E_t[Z_{t+1}]$

because

(2.66) $E_t \begin{bmatrix} \hat{v}_t \\ w^c_{t+1} \\ w^k_{t+1} \\ w^s_{t+1} \end{bmatrix} = 0.$

By (2.48) equation (2.65) implies the following.

(2.67a) $z_{1t}\ =\ \lambda_1 E_t[z_{1,t+1}]$

(2.67b) $z_{2t}\ =\ \lambda_2 E_t[z_{2,t+1}]$

(2.67c) $z_{3t}\ =\ \lambda_3 E_t[z_{3,t+1}]$

We can iterate equation (2.67a) into the future periods as follows.

(2.68) $z_{1,t+1}\ =\ \lambda_1 E_{t+1}[z_{1,t+2}]$

Take the expectation on both sides of (2.68) conditional on the information available in t. Then, we have the following.

(2.69) $E_t[z_{1,t+1}] =\ \lambda_1 E_t\{E_{t+1}[z_{1,t+2}]\}$
$\quad =\ \lambda_1 E_t[z_{1,t+2}]$

The second line is obtained by applying the law of iterated expectations. (See Sargent (1987).) By (2.67a) and (2.69), we have

$$(2.70)\quad z_{1t} = (\lambda_1)^2 E_t[z_{1,t+2}].$$

By repeating above process T times, we have

$$(2.71)\quad z_{1t} = (\lambda_1)^T E_t[z_{1,t+T}].$$

Because $0 < \lambda_1 < 1$, as $T \to \infty$, we have

$$(2.72)\quad z_{1t} = 0$$

as a general solution to the stochastic difference equation (2.67a). Denote

$$(2.73)\quad Q^{-1} \equiv \Gamma = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & \gamma_{33} \end{bmatrix}.$$

Then, by (2.72) and

$$(2.74)\quad Z_t = Q^{-1}\begin{bmatrix} \hat{c}_t \\ \hat{k}_t \\ \hat{s}_t \end{bmatrix} = \Gamma \begin{bmatrix} \hat{c}_t \\ \hat{k}_t \\ \hat{s}_t \end{bmatrix},$$

the first row of (2.74) is expressed as follows.

$$(2.75)\quad z_{1t} = \gamma_{11}\hat{c}_t + \gamma_{12}\hat{k}_t + \gamma_{13}\hat{s}_t = 0$$

Equation (2.75) is solved for $\hat{c}_t$ as follows.

$$(2.76)\quad \hat{c}_t = -(\gamma_{12}/\gamma_{11})\hat{k}_t - (\gamma_{13}/\gamma_{11})\hat{s}_t$$

By substituting (2.76) into (2.41), we have the following first-order stochastic difference equation with respect to $\hat{k}_t$.

$$\begin{aligned}(2.77)\quad \hat{k}_{t+1} &= b_1\hat{k}_t + b_2\hat{s}_t - b_3\hat{c}_t \\ &= b_1\hat{k}_t + b_2\hat{s}_t - b_3[-(\gamma_{12}/\gamma_{11})\hat{k}_t - (\gamma_{13}/\gamma_{11})\hat{s}_t] \\ &= [b_1 + b_3(\gamma_{12}/\gamma_{11})]\hat{k}_t + [b_2 + b_3(\gamma_{13}/\gamma_{11})]\hat{s}_t\end{aligned}$$

We can summarize the analysis of the SDGE as follows. For t = 0, 1, 2, ... , given $\{\hat{k}_0, \hat{s}_0\}$, the SDGE sequence of capital and stochastic technology $\{\hat{k}_{t+1}, \hat{s}_{t+1}\}$ is generated by the following system of first-order linear stochastic difference equations.

$$(2.78)\quad \hat{k}_{t+1} = [b_1 + b_3(\gamma_{12}/\gamma_{11})]\hat{k}_t + [b_2 + b_3(\gamma_{13}/\gamma_{11})]\hat{s}_t$$

$$(2.79)\quad \hat{s}_{t+1} = \rho\,\hat{s}_t + \hat{v}_t$$

The SDGE sequence of consumption is generated by the following.

$$(2.80)\quad \hat{c}_t = -(\gamma_{12}/\gamma_{11})\hat{k}_t - (\gamma_{13}/\gamma_{11})\hat{s}_t$$

In equations (2.78), (2.79), and (2.80), the variables are measured as deviations from their steady state values, i.e.,

$$(2.81)\quad \hat{k}_t = (k_t - \bar{k})/\bar{k},$$

$$(2.82)\quad \hat{s}_t = (s_t - \bar{s})/\bar{s},$$

$$(2.83)\quad \hat{c}_t = (c_t - \bar{c})/\bar{c},$$

$$(2.84)\quad \hat{v}_t = (v_t - \bar{v})/\bar{v}.$$

In equation (2.84), $\{v_t$; t = 0, 1, 2, ... } is a sequence of i.i.d. shocks.

3. Simulating for the SDGE.

In this section, we explain the simulation methods for the SDGE of the stochastic optimal growth model, and provide Matlab codes for implementing the simulation.

3.1. Simulation Methods.

The SDGE of the stochastic optimal growth model is simulated through the following steps.

Step 1. Specify the values of model parameters $\{\alpha, \beta, \delta, \sigma, \rho\}$ and the initial values of state variables $\{k_0, s_0\}$. As an example, we assign the following values.

$\alpha = 0.3$; the capital's share in the production function.

$\beta = 0.9$; the subjective discount factor of future utilities.

$\delta = 0.1$; the capital depreciation rate.

$\sigma = 2$; the elasticity of substitution between the consumption in period t and the consumption in period t + 1.

$\rho = 0.9$; AR(1) coefficient of the sequence of stochastic technology $\{s_{t+1}$; t = 0, 1, 2, ... }. See step 2 below for further explanations.

$k_0 = 20$; initial capital.

$s_0 = 10$; initial technology.

Step 2. Specify the stochastic distribution of the i.i.d. shocks $\{v_t$; t = 0, 1, 2, ... } to the production technology. The main objective of SDGE analyses is to build a model economy that can generate variables that fit well the actual economic data. (Then, we can simulate the model to generate SDGE variables for alternative parameter values, or to analyze the effects of economic policies.) The choice of stochastic distribution of the i.i.d. shocks, hence, is made to suit this objective. Equation (1.2) implies that v_t cannot be negative because output cannot be negative. This is one the restrictions imposed on v_t. Take the natural-log on both sides of (1.2) to obtain the following expression.

$$(3.1)\quad \ln s_{t+1} = \rho \ln s_t + \ln v_t$$

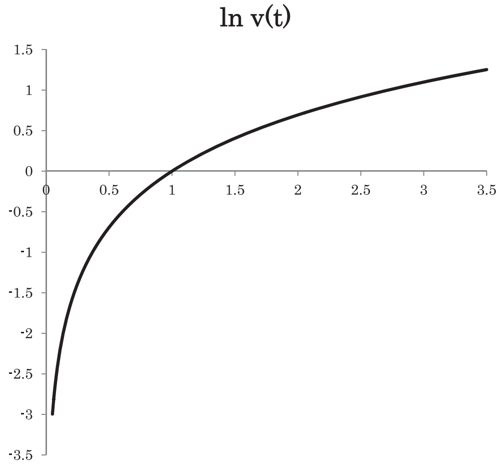
Equation (3.1) implies that $\ln s_t$ is subject to the first-order auto-regressive process (AR(1)) with i.i.d. shock $\ln v_t$. (See Enders (2009) for the introductory time series analysis. Although (1.2) implies both (2.42) and (3.1), (3.1) is an exact relationship, while (2.42) is a linear approximation. In fact, (2.9) and (2.10) can be log-linearized as well, consistent with (3.1), to yield a system of stochastic first-order linear difference equations with respect to $\{\ln c_t, \ln k_t, \ln s_t\}$. (See Campbell (1994). Taylor and Uhlig (1990) compares the fitness of different approximation methods to the SDGE models.) The AR(1) process is "stable" when $|\rho| < 1$. While v_t is restricted to be positive, $\ln v_t$ can take values in $(-\infty, +\infty)$. (See figure 1.)

Assume $\ln v_t$ is normally distributed with mean μ and variance σ^2, i.e.,

$$(3.2)\quad \ln v_t \sim N(\mu, \sigma^2).$$

Then, v_t is said to be a random variable generated by the log-normal distribution with mean $\bar{v}$ and variance

Figure 1. The Graph of $\ln v_t$.



σ_v^2 where

$$(3.3)\quad \bar{v} = \exp[\mu + \sigma^2/2],$$

$$(3.4)\quad \sigma_v^2 = \exp[2\mu + \sigma^2][\exp(\sigma^2) - 1].$$

(See Sydsaeter, Strom, and Berck (2011) for a reference.) As an example we specify the mean μ and the variance σ^2 of $\ln v_t$ as follows. First, assume that the steady state $\bar{s}$ of the stochastic technology s_t to be 10, i.e.,

$$(3.5)\quad \bar{s} = 10.$$

($\bar{s} = 10$ implies that the steady state value is equal to the initial value $s_0 = 10$.) Because $\rho = 0.9$, equation (2.14)implies that the mean of v_t is

$$(3.6)\quad \bar{v} = (\bar{s})^{1-\rho} = 1.259.$$

In addition, assume that the variance of v_t is 5% of the mean $\bar{v}$, i.e.,

$$(3.7)\quad \sigma_v^2 = 0.05 \times \bar{v} = 0.06295.$$

Then, by (3.3) and (3.4), we can solve the following simultaneous equations with respect to the mean μ and the variance σ^2 of $\ln v_t$.

$$(3.8)\quad 1.259 = \exp[\mu + \sigma^2/2],$$

$$(3.9)\quad 0.06295 = \exp[2\mu + \sigma^2][\exp(\sigma^2) - 1].$$

The solutions are

(3.10) μ = 0.2108,

(3.11) σ^2 = 0.0389.

Next, generate an arbitrary number of random variables from a normal distribution with mean 0.2108 and variance 0.0389, i.e.,

(3.12) $\ln v_t \sim N(0.2108, 0.0389)$.

Finally, transform the random variables by

(3.13) $v_t = \exp[\ln v_t]$.

Step 3. Calculate the steady state $\{\bar{k}, \bar{c}, \bar{s}\}$ as follows.

(3.14) $\bar{k} = \left[\dfrac{\alpha\beta\bar{s}}{1-\beta(1-\delta)}\right]^{\frac{1}{1-\alpha}} = 44.3186$

(3.15) $\bar{c} = \bar{s}(\bar{k})^{\alpha} - \delta\bar{k} = 26.7553$

(3.16) $\bar{s} = (\bar{v})^{\frac{1}{1-\rho}} = 10$

Then, calculate the initial deviations $\{\hat{k}_0, \hat{s}_0\}$ from the steady state as follows.

(3.17) $\hat{k}_0 = (k_0 - \bar{k})/\bar{k} = -0.5487$

(3.18) $\hat{s}_0 = (s_0 - \bar{s})/\bar{s} = 0$

In addition calculate

(3.19) $\hat{v}_t = (v_t - \bar{v})/\bar{v}$.

Step 4. Calculate the coefficients of the SDGE system of difference equations {(2.40), (2.41)}.

(3.20) $a_1 = [(1-\alpha)/\sigma]\alpha\beta\bar{s}(\bar{k})^{\alpha-1}$

(3.21) $a_2 = (1/\sigma)\alpha\beta\bar{s}(\bar{k})^{\alpha-1}$

(3.22) $b_1 = 1/\beta$

(3.23) $b_2 = [(1/\beta)-(1-\delta)]/\alpha$

(3.24) $b_3 = \bar{c}/\bar{k}$

Then, calculate the characteristic roots $\{\lambda_1, \lambda_2, \lambda_3\}$ and the corresponding characteristic vectors $\{Q_1, Q_2, Q_3\}$ of the matrix A of coefficients where

(3.25) $A = \begin{bmatrix} 1 & a_1 & -a_2 \\ b_3/b_1 & \dfrac{a_1 b_3 + 1}{b_1} & -\left[\dfrac{\rho a_2 b_3 + b_2}{\rho b_1}\right] \\ 0 & 0 & 1/\rho \end{bmatrix}$.

They are calculated as follows.

(3.26) $\lambda_1 = 0.7753, \quad Q_1 = \begin{bmatrix} -0.2838 \\ 0.9589 \\ 0 \end{bmatrix}$

(3.27) $\lambda_2 = 1.1608, \quad Q_2 = \begin{bmatrix} 0.3821 \\ 0.9241 \\ 0 \end{bmatrix}$

(3.28) $\lambda_3 = 1/\rho = 1.11, \quad Q_3 = \begin{bmatrix} 0.4397 \\ 0.8914 \\ 0.1098 \end{bmatrix}$

Then, calculate the inverse matrix Γ of $Q = [Q_1\ Q_2\ Q_3]$. It is calculated as follows.

(3.29) $Q^{-1} \equiv \Gamma = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & \gamma_{33} \end{bmatrix}$

$$= \begin{bmatrix} -1.4704 & 0.608 & 0.9524 \\ 1.525 & 0.4516 & -9.773 \\ 0 & 0 & 9.1075 \end{bmatrix}$$

Step 5. Generate the SDGE sequence of capital and stochastic technology $\{\hat{k}_{t+1}, \hat{s}_{t+1}$; t = 0, 1, 2, ... } by recursively updating equations {(2.78), (2.79)} given the initial values $\{\hat{k}_0, \hat{s}_0\}$. At t = 0, $\{\hat{k}_1, \hat{s}_1\}$ is calculated as follows.

(3.30) $\hat{k}_1 = [b_1 + b_3(\gamma_{12}/\gamma_{11})]\hat{k}_0 + [b_2 + b_3(\gamma_{13}/\gamma_{11})]\hat{s}_0$

(3.31) $\hat{s}_1 = \rho\,\hat{s}_0 + \hat{v}_0$

Similarly, at t = 1, $\{\hat{k}_2, \hat{s}_2\}$ is calculated as follows.

(3.32) $\hat{k}_2 = [b_1 + b_3(\gamma_{12}/\gamma_{11})]\hat{k}_1 + [b_2 + b_3(\gamma_{13}/\gamma_{11})]\hat{s}_1$

(3.33) $\hat{s}_2 = \rho\,\hat{s}_1 + \hat{v}_1$

Repeat the process to generate $\{\hat{k}_{t+1}, \hat{s}_{t+1}$; t = 0, 1, 2, ... }. Then, the SDGE sequence of consumption $\{\hat{c}_t$; t = 0, 1, 2, ... } is calculated from

$$(3.34)\quad \hat{c}_t = -(\gamma_{12}/\gamma_{11})\hat{k}_t - (\gamma_{13}/\gamma_{11})\hat{s}_t .$$

Step 6. Calculate the SDGE sequence of capital and consumption $\{\hat{k}_{t+1}, \hat{c}_t$; t = 0, 1, 2, ... } by

$$(3.35)\quad k_{t+1} = (\hat{k}_{t+1} + 1)\bar{k} ,$$

$$(3.36)\quad c_t = (\hat{c}_t + 1)\bar{c} .$$

3.2 Matlab Codes for Simulating the SDGE.

The following table 1 states the relationship between the names of parameters and variables used in the Matlab codes and those in the stochastic optimal growth model of section 2.

Program 1.

```
(P1.1)  % Calculate eigen values and eigen vectors
        of Ax;
(P1.2)  % Parameters and initial values;
(P1.3)  alphax = 0.3;
(P1.4)  betax = 0.9;
(P1.5)  deltax = 0.1;
(P1.6)  sigmax = 2;
(P1.7)  sxs = 10;
(P1.8)  rhox = 0.9;
(P1.9)  kx0 = 20;
(P1.10) sx0 = 10;
(P1.11) % Calculating steady state {cxs, kxs}.
(P1.12) kxs =(alphax*betax*sxs/(1 - betax*(1 -
        deltax)))^(1/(1 - alphax));
(P1.13) cxs = sxs*(kxs^alphax) - deltax*kxs;
(P1.14) % Coefficients {a1, a2, b1, b2, b3}.
(P1.15) a1 =((1 - alphax)/sigmax)*alphax*betax*
        sxs*(kxs^(alphax - 1));
(P1.16) a2 =(1/sigmax)*alphax*betax*sxs*(kxs^
        (alphax - 1));
(P1.17) b1 = 1/betax;
(P1.18) b2 =(1/(alphax*betax))*(1 - betax*(1 -
        deltax));
(P1.19) b3 = sxs*(kxs^(alphax - 1)) - deltax;
(P1.20) % Eigen values and eigen vectors of Ax;
(P1.21) M1 = zeros(3, 3);
(P1.22) M1(1, 1)=-1;
(P1.23) M1(2, 1)=-b3;
(P1.24) M1(2, 2)= b1;
(P1.25) M1(2, 3)= b2;
(P1.26) M1(3, 3)= rhox;
(P1.27) M2 = zeros(3, 3);
(P1.28) M2(1, 1)=-1;
(P1.29) M2(1, 2)=-a1;
(P1.30) M2(1, 3)= a2;
(P1.31) M2(2, 2)= 1;
(P1.32) M2(3, 3)= 1;
```

Table 1. The relationship between the names of parameters and variables used in the Matlab codes and those in the stochastic optimal growth model of section 2.

alphax	α	a1	a_1	vxs	$\bar{v}$
betax	β	a2	a_2	varvx	σ_v^2
deltax	δ	b1	b_1	mux	μ
sigmax	σ	b2	b_2	varx	σ^2
sxs	$\bar{s}$	b3	b_3	kxh	$\hat{k}$
rhox	ρ	Ax	A	sxh	$\hat{s}$
kx0	k_0	Q *	Q	cxh	$\hat{c}$
sx0	s_0	L	Λ	kx	k
kxs	$\bar{k}$	Qx **	Q	cx	c
cxs	$\bar{c}$	Gx	Γ		

*, ** ; The columns of matrix Q are rearranged in matrix Qx so that they correspond to the characteristic roots { $\lambda_1, \lambda_2, \lambda_3$}

```
(P1.33)   Ax = inv(M1)*M2;
(P1.34)   [Q, L]= eig(Ax);
```

In Program 1, (P1.1) ~ (P.34) are labels to identify each Matlab code. When you write Matlab programs, you do not have to put these labels.

Program 1 calculates the characteristic roots and the characteristic vectors of the coefficient matrix A. Lines (P1.1) ~ (P1.10) assign the values of parameters $\{\alpha = 0.3, \beta = 0.9, \delta = 0.1, \sigma = 2, \bar{s} = 10, \rho = 0.9\}$ and the initial conditions $\{k_0 = 20, s_0 = 10\}$. Lines (P.11) ~ (P1.13) calculate the steady state $\{\bar{k}, \bar{c}\}$. Lines (P1.14) ~ (P1.19) calculate the coefficients $\{a_1, a_2, b_1, b_2, b_3\}$. Lines (P1.20) ~ (P1.33) calculates the coefficient matrix A. In line (P1.33), inv(M1) calculates the inverse matrix of M1. Line (P1.34) calculates the characteristic roots and the characteristic vectors of the coefficient matrix A. In line (P1.34), eig(Ax) returns two outputs that are specified by the left-hand side [Q, L] where L is a matrix whose diagonal elements consist of characteristic roots of Ax, and Q is a matrix whose columns consist of characteristic vectors coressponding to each characteristic roots of Ax. Program 1 returns the following output.

```
Q =
      0.3821     -0.2838     0.4397
      0.9241      0.9589     0.8914
           0           0     0.1098
L =
      1.1608           0          0
           0      0.7753          0
           0           0     1.1111
```

We rearrange the columns of Q so that the first column corresponds to the characteristic vector $Q_1' =$ [-0.2838 0.9589 0] of the unstable characteristic root $\lambda_1 = 0.7753$, the second column corresponds to the characteristic vector $Q_2' =$[0.3821 0.9241 0] of the stable characteristic root $\lambda_2 = 1.1608$, and the third column corresponds to the characteristic vector $Q_3' =$ [0.4397 0.8914 0.1098] of the stable characteristic root $\lambda_3 = 1.11$.

Program 2.

```
(P2.1)    % The first-order dynamical system with respect to {kx(t), sx(t)}.
(P2.2)    global vxs varvx;
(P2.3)    % Parameters and initial values;
(P2.4)    alphax = 0.3;
(P2.5)    betax = 0.9;
(P2.6)    deltax = 0.1;
(P2.7)    sigmax = 2;
(P2.8)    sxs = 10;
(P2.9)    rhox = 0.9;
(P2.10)   kx0 = 20;
(P2.11)   sx0 = 10;
(P2.12)   Tx = 100;
(P2.13)   % Generating exogenous shocks to technology.
(P2.14)   vxs = sxs^(1 - rhox);
(P2.15)   varvx = 0.05*vxs;
(P2.16)   dist_ini = ones(2, 1);
(P2.17)   dist = fsolve(@vdist, dist_ini);
(P2.18)   mux = dist(1);
(P2.19)   varx = dist(2);
(P2.20)   Ex = zeros(Tx, 1);
(P2.21)   Ex = mux + (varx^0.5)*randn(Tx, 1);
(P2.22)   % Shocks to technology.
(P2.23)   vx = zeros(Tx, 1);
(P2.24)   for t = 1 : Tx ;
(P2.25)   vx(t)= exp(Ex(t));
(P2.26)   end;
(P2.27)   % Shocks to technology (Deviation from the mean).
(P2.28)   vxh = zeros(Tx, 1);
(P2.29)   for t = 1 : Tx ;
(P2.30)   vxh(t)=(vx(t) - vxs)/vxs;
(P2.31)   end;
(P2.32)   % Calculating steady state {cxs, kxs}.
(P2.33)   kxs =(alphax*betax*sxs/(1 - betax*(1 - deltax)))^(1/(1 - alphax));
(P2.34)   cxs = sxs*(kxs^alphax) - deltax*kxs;
(P2.35)   % Coefficients {a1, a2, b1, b2, b3}.
(P2.36)   a1 =((1 - alphax)/sigmax)*alphax*betax*sxs*(kxs^(alphax - 1));
(P2.37)   a2 =(1/sigmax)*alphax*betax*sxs*(kxs^(alphax - 1));
```

```
(P2.38)  b1 = 1/betax;
(P2.39)  b2 =(1/(alphax*betax))*(1 - betax*(1 - deltax));
(P2.40)  b3 = sxs*(kxs^(alphax - 1)) - deltax;
(P2.41)  % Eigen vector matrix.
(P2.42)  Qx = zeros(3, 3);
(P2.43)  Qx(1, 1)=-0.2838;
(P2.44)  Qx(1, 2)= 0.3821;
(P2.45)  Qx(1, 3)= 0.4397;
(P2.46)  Qx(2, 1)= 0.9584;
(P2.47)  Qx(2, 2)= 0.9241;
(P2.48)  Qx(2, 3)= 0.8914;
(P2.49)  Qx(3, 1)= 0;
(P2.50)  Qx(3, 2)= 0;
(P2.51)  Qx(3, 3)= 0.1098;
(P2.52)  Gx = inv(Qx);
(P2.53)  % Deviations from the steady state.
(P2.54)  kxh = zeros(Tx + 1, 1);
(P2.55)  sxh = zeros(Tx + 1, 1);
(P2.56)  kxh(1)=(kx0 - kxs)/kxs;
(P2.57)  sxh(1)=(sx0 - sxs)/sxs;
(P2.58)  for t = 1 : Tx ;
(P2.59)  kxh(t + 1)=(b1 + b3*(Gx(1, 2)/Gx(1, 1)))*kxh(t) + (b2 + b3*(Gx(1, 3)/Gx(1, 1)))*sxh(t);
(P2.60)  sxh(t + 1)= rhox*sxh(t) + vxh(t);
(P2.61)  end;
(P2.62)  cxh = zeros(Tx, 1);
(P2.63)  for t = 1 : Tx ;
(P2.64)  cxh(t)=-(Gx(1, 2)/Gx(1, 1))*kxh(t) - (Gx(1, 3)/Gx(1, 1))*sxh(t);
(P2.65)  end;
(P2.66)  % SDGE sequence of capital and consumption.
(P2.67)  kx = zeros(Tx + 1, 1);
(P2.68)  for t = 1 : Tx + 1;
(P2.69)  kx(t)=(kxh(t) + 1)*kxs;
(P2.70)  end;
(P2.71)  cx = zeros(Tx, 1);
(P2.72)  for t = 1 : Tx ;
(P2.73)  cx(t)=(cxh(t) + 1)*cxs;
(P2.74)  end;
```

Program 2 specifies the stochastic distribution of i.i.d. shocks $\{v_t\ ;\ t = 0, 1, 2, \dots\}$ and calculate the SDGE sequence of capital and consumption $\{\hat{k}_{t+1}, \hat{c}_t\ ;\ t = 0, 1, 2, \dots\}$. In the program, time period t runs from 1 to Tx = 100. In program 2, line (P2.2) specifies the parameter names to be used in both main program and sub-program (program 3 which will be presented below). Lines (P2.3) ~ (P2.12), as in program 1, assign the values of parameters and the initial conditions. Lines (P2.13) ~ (P2.31) specify the stochastic distribution of the i.i.d. shocks $\{\ln v_t\ ;\ t = 0, 1, 2, \dots\}$. Lines (P2.14) and (P2.15) assign values of the mean $\bar{v}$ and the variance σ_v^2. (As mentioned before, the value of σ_v^2 is assumed to be 5% of $\bar{v}$.) Line (P2.17) calls sub-program vdist (program 3) which specifies the simultaneous equations to be solved with respect to $\{\mu, \sigma^2\}$ given the initial condition dist_ini. Line (P2.21) generates a vector whose elements are i.i.d random variables drawn from $N(\mu, \sigma^2)$. In line (P2.21), randn(Tx, 1) generates a Tx × 1 vector whose elements are drawn from N(0, 1). Lines (P2.22) ~ (P2.26) transform the vector of i.i.d. shocks $\{\ln v_t\ ;\ t = 0, 1, 2, \dots\}$ into $\{v_t\ ;\ t = 0, 1, 2, \dots\}$. Lines (P2.27) ~ (P2.31) generate a vector whose elements are i.i.d. shocks $\{\hat{v}_t = (v_t - \bar{v})/\bar{v}\ ;\ t = 0, 1, 2, \dots\}$ expressed as deviations from mean. Lines (P2.32) ~ (P2.34) calculate the steady state values $\{\bar{k}, \bar{c}\}$ of capital and consumption. Lines (P2.35) ~ (P2.40) calculate the coefficients $\{a_1, a_2, b_1, b_2, b_3\}$. Lines (P2.41) ~ (P2.51) specify the matrix $Q = [Q_1\ Q_2\ Q_3]$ consisting of the characteristic vectors of coefficient matrix A, and line (P2.52) calculate the inverse matrix $\Gamma \equiv Q^{-1}$. Lines (P2.53) ~ (P2.61) iterate the system of stochastic first-order difference equations to genrate the sequence of SDGE capital and stochastic technology $\{\hat{k}_{t+1}, \hat{s}_{t+1}\ ;\ t = 0, 1, 2, \dots\}$ expressed as deviations from the steady state $\{\bar{k}, \bar{s}\}$ given the initial condition $\{\hat{k}_0, \hat{s}_0\}$ specified in lines (P2.56) and (P2.57). Lines (P2.62) ~ (P2.65) calculate the sequence of SDGE consumption $\{\hat{c}_t\ ;\ t = 0, 1, 2, \dots\}$ expressed as deviations from the steady state $\bar{c}$. Lines (P2.66) ~ (P2.74) transform $\{\hat{k}_{t+1}, \hat{s}_{t+1}\ ;\ t = 0, 1, 2, \dots\}$ into the levels $\{k_{t+1}, s_{t+1}\ ;\ t = 0, 1, 2, \dots\}$.

Program 3.

(P3.1) % Subroutine to solve for the mean and variance of log-normal distribution.
(P3.2) function Fx = vdist(dist)
(P3.3) global vxs varvx;
(P3.4) Fx = ones(2, 1);
(P3.5) Fx(1)= exp(dist(1)+ dist(2)/2) - vxs;
(P3.6) Fx(2)= exp(2*dist(1)+ dist(2))*(exp (dist(2)) - 1) - varvx;

Program 3, as mentioned before, specifies simultaneous equations to be solved for the mean μ and the variance σ^2 of the i.i.d. shock $\ln v_t$.

The outputs of program 2 are summarized by figure 2 and figure 3. Figure 2 depicts the graph of capital $\{k_{t+1}$; $t = 0, 1, 2, ... \}$, and figure 3 depicts the graph of consumption $\{c_t$; $t = 0, 1, 2, ... \}$ in the SDGE.

Figure 2. The Graph of k_t.

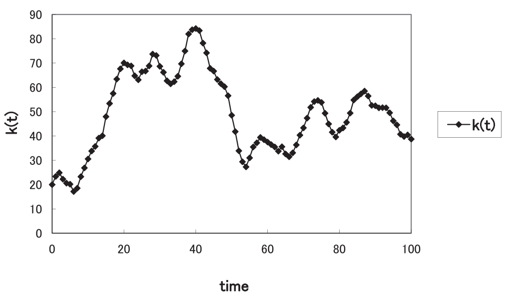
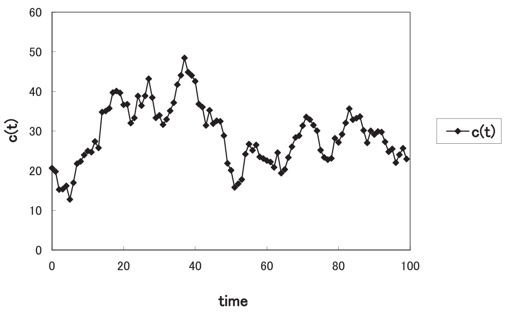


Figure 3. The Graph of c_t.



We may incorporate economic policies into the stochastic optimal growth model to analyze the effects of economic policies on the endogenous variables and social welfare (measured by the utility of a representative household) in SDGE. See King, Plosser, and Rebelo (1988) and King and Rebelo (1990) for further discussions.

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