広島大学学位請求論文

Fundamental biquandles of ribbon 2-knots and ribbon torus-knots with isomorphic fundamental quandles
(同型な基本カンドルをもつリボン 2 次元結び目とリボントーラス結び目の基本バイカンドル)

2014 年

広島大学大学院理学研究科
数学専攻

芦原 聡介

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主論文

Fundamental biquandles of ribbon 2-knots and ribbon torus-knots with isomorphic fundamental quandles

Sosuke Ashihara

Abstract

The fundamental quandles and biquandles are invariants of classical knots and surface knots. It is unknown whether there exist classical or surface knots which have isomorphic fundamental quandles and distinct fundamental biquandles. We show that ribbon 2-knots or ribbon torus-knots with isomorphic fundamental quandles have isomorphic fundamental biquandles. For this purpose, we give a method for obtaining a presentation of the fundamental biquandle of a ribbon 2-knot/torus-knot from its fundamental quandle.

1 Introduction

The fundamental quandles and the fundamental biquandles are invariants of classical knots, virtual knots and surface knots (cf. [2, 4, 5, 6, 10, 11]). For classical knots, the fundamental quandles can distinguish all knots up to orientations of knots and the ambient space. For surface knots, there exist distinct surface knots with isomorphic fundamental quandles (cf. [13]). By definition, the fundamental biquandles dominate the fundamental quandles. The fundamental biquandles are stronger than the fundamental quandles for some virtual knots (cf. [10]). It is unknown whether there exist classical knots or surface knots whose fundamental quandles are isomorphic but their fundamental biquandles are not. The following is our main theorem.

Theorem 1.1. *Two ribbon* 2*-knots or ribbon torus-knots with isomorphic fundamental quandles have isomorphic fundamental biquandles.*

Here a *ribbon* 2*-knot* (or a *ribbon torus-knot*, resp.) is a 2-sphere (or a torus, resp.) embedded in $\mathbb{R}^4$ that can be obtained from a trivial 2-link in $\mathbb{R}^4$ by adding 1-handles (cf. [12]).

To prove this theorem, we will associate with each quandle presentation $\langle S \mid R\rangle_q$ a biquandle presentation, $J_0(\langle S \mid R\rangle_q)$ (see Sec. 5).

Theorem 1.2. *If two quandle presentations $\langle S \mid R\rangle_q$ and $\langle S' \mid R'\rangle_q$ determine isomorphic quandles, then the two biquandle presentations $J_0(\langle S \mid R\rangle_q)$ and $J_0(\langle S' \mid R'\rangle_q)$ determine isomorphic biquandles.*

This theorem enables us to associate with each quandle Q a biquandle $J(Q)$. To be precise, if Q is a quandle presented by $\langle S \mid R\rangle_q$, then we define $J(Q)$ to be the biquandle presented by $J_0(\langle S \mid R\rangle_q)$.

Theorem 1.1 is a direct consequence of the following theorem.

Theorem 1.3. *For any ribbon 2-knot or a ribbon torus-knot F, the fundamental biquandle $BQ(F)$ is isomorphic to the biquandle $J(Q(F))$ obtained from the fundamental quandle $Q(F)$.*

To prove this theorem, we use Satoh's method for presenting a ribbon 2-knot/torus-knot by a virtual arc/knot diagram, [12].

This paper is organized as follows. In Sec. 2, we recall definitions of quandles, biquandles and the fundamental biquandle of a surface knot. In Sec. 3, we explain Satoh's method for presenting a ribbon 2-knot/torus-knot by a virtual arc/knot diagram. Sec. 4 deals with presentations of quandles and biquandles. In Sec. 5, we associate with each quandle presentation a biquandle presentation. In Sec. 6, we give a method for obtaining a presentation of the fundamental biquandle of a ribbon 2-knot/torus-knot directly from its virtual arc/knot presentation (Theorem 6.1) and prove Theorem 1.3.

2 Quandles and biquandles

First we recall the definitions of a quandle and a biquandle.

Definition 2.1. A *quandle* is a set Q with two binary operations $(a,b)\mapsto a^b$ and $a^{\bar b}$ satisfying the following axioms (cf. [5, 6, 11]):

(Q1) For any $a\in Q$, $a^a=a$.

(Q2) For any $a,b\in Q$, $a^{b\bar b}=a^{\bar b b}=a$.

(Q3) For any $a,b,c\in Q$, $a^{bc}=a^{cb^c}$.

Here we use Fenn and Rourke's notation, i.e., a^{bc} means $(a^b)^c$ and a^{b^c} means $a^{(b^c)}$ (see. [5]). A *rack* is a set Q with two binary operations $(a,b)\mapsto a^b$ and $a^{\bar b}$ satisfying (Q2) and (Q3).

Consider a set B with four binary operations $(a,b)\mapsto a^b, a_b, a^{\bar b}$ and $a_{\bar b}$ satisfying the following:

(B1) For any fixed $b\in B$, the maps $a\mapsto a^b$, a_b, $a^{\bar b}$ and $a_{\bar b}$ are bijections.

Then four binary operations $(a,b)\mapsto a^{b^{-1}}$, $a_{b^{-1}}$, $a^{\bar b^{-1}}$ and $a_{\bar b^{-1}}$ on B are defined by the following rules. For every $a,b\in X$, $a^{bb^{-1}}=a^{b^{-1}b}=a$, $a_{bb^{-1}}=a_{b^{-1}b}=a$, $a^{\bar b\bar b^{-1}}=a^{\bar b^{-1}\bar b}=a$ and $a_{\bar b\bar b^{-1}}=a_{\bar b^{-1}\bar b}=a$.

Definition 2.2. A *biquandle* is a set B with four binary operations $(a,b)\mapsto a^b, a_b, a^{\bar b}$ and $a_{\bar b}$ satisfying (B1) and the following axioms (see [4, 7, 10]):

(B2) For every $a\in B$, we have $a_{a^{-1}}=a^{a_{a^{-1}}}$ and $a^{a^{-1}}=a_{a^{a^{-1}}}$.

(B3) For every $a, b \in B$, we have $a = a^{b\overline{b_a}}$, $b = b_{a\overline{a^b}}$, $a = a^{\bar{b}b_{\bar{a}}}$ and $b = b_{\bar{a}a^{\bar{b}}}$.

(B4) For every $a, b, c \in B$, we have $a^{bc} = a^{c_b b^c}$, $c_{ba} = c_{a^b b_a}$ and $(b_a)^{c_{a^b}} = (b^c)_{a^{c_b}}$.

Using eight binary operations on a set, we may restate the definition of a biquandle as follows.

Definition 2.3. A *biquandle* is a set B with eight binary operations $(a, b) \mapsto a^b, a_b, a^{\bar{b}}, a_{\bar{b}}, a^{b^{-1}}, a_{b^{-1}}, a^{\bar{b}^{-1}}$ and $a_{\bar{b}^{-1}}$ satisfying (B2), (B3), (B4) and

(B1′) For every $a, b \in B$, $a^{bb^{-1}} = a^{b^{-1}b} = a$, $a_{bb^{-1}} = a_{b^{-1}b} = a$, $a^{\bar{b}\bar{b}^{-1}} = a^{\bar{b}^{-1}\bar{b}} = a$ and $a_{\bar{b}\bar{b}^{-1}} = a_{\bar{b}^{-1}\bar{b}} = a$.

The fundamental quandles of classical knots, virtual knots and surface knots were defined in [5, 6, 11]. The fundamental biquandles were defined in [4, 10] for classical knots and virtual knots, and in [2] for surface knots.

For the convenience of the reader, we recall the definition of the fundamental biquandle of a surface knot following [2].

Let F be a surface knot in $\mathbb{R}^4$, i.e., an oriented connected closed surface embedded in $\mathbb{R}^4$. Let $f : F \to \mathbb{R}^3$ be the restriction to F of the projection map $\mathbb{R}^4 \to \mathbb{R}^3; (x, y, z, t) \mapsto (x, y, z)$. Assume that f is a generic map. Let U_- be an open regular neighborhood of the lower decker curves in F. Then the image $f(F \setminus U_-)$ in $\mathbb{R}^3$ is called a *surface knot diagram* of F. Refer to [3] for the definition of decker curves and details. Let U be an open regular neighborhood of all decker curves in F. Then the image $f(F \setminus U)$ is a compact and oriented surface in $\mathbb{R}^3$. We call each connected component of this surface a *semi-sheet* of the surface knot diagram.

Definition 2.4. Let D be a surface knot diagram. The *fundamental biquandle* $BQ(D)$ of D is the biquandle defined by the following presentation.

1. Let $a_1, \dots, a_n$ be the semi-sheets of D. Then the generating set is $\{a_1, \dots, a_n\}$.

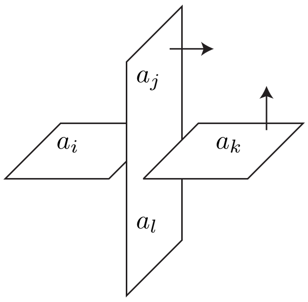
2. For a double point curve d of D, we associate d with two relations

$$r_1(d) : a_k = {a_i}^{a_j} \quad \text{and} \quad r_2(d) : a_l = {a_j}_{a_i},$$

where a_i, a_j, a_k, a_l are the semi-sheets around d as illustrated in Fig. 1. Let $d_1, \dots, d_m$ be the double point curves of D. The set of relations is $\{r_1(d_1), r_2(d_1), \dots, r_1(d_m), r_2(d_m)\}$.

A presentation of a biquandle will be explained in Sec. 4.

Carrell [2] showed that if two surface knot diagrams D and D' present the same surface knot, then the fundamental biquandles $BQ(D)$ and $BQ(D')$ are isomorphic. The *fundamental biquandle* $BQ(F)$ of a surface knot F is defined by the fundamental biquandle $BQ(D)$, where D is a surface knot diagram of F.



$$r_1(d) : a_k = a_i{}^{a_j}, r_2(d) : a_l = a_{j\,a_i}.$$

Figure 1: A neighborhood of the double point curve d

3 Virtual arc/knot presentation of a ribbon 2-knot/torus-knot

Let D be a virtual arc diagram or a virtual knot diagram. (It is an oriented arc or circle generically immersed in $\mathbb{R}^2$ such that each double point is given information of a type of positive, negative or virtual illustrated in Fig. 2 (cf. [9, 12]).)

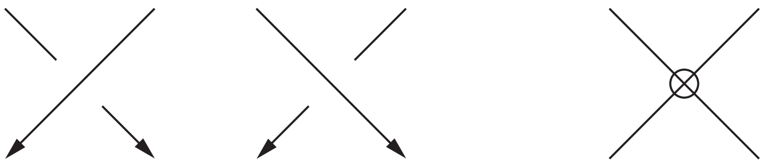


Figure 2: Positive, negative and virtual crossings.

Let tube(D) denote a surface knot diagram in $\mathbb{R}^3$ associated with D in the sense of Satoh [12]: It is obtained from D by placing a thin tube wherever we see an edge in the diagram D such that (1) at a classical crossing, the undergoing path of D corresponds to the tube which goes through the other tube, (2) at a virtual crossing, the tubes pass over/under each other, and (3) at an endpoint, we cap the tube. We orient the surface so that the normal vector in surface knot diagram points outward. See Fig. 3. Let Tube(D) denote a surface knot presented by the diagram tube(D). Note that Tube(D) is a ribbon 2-sphere if D is a virtual arc diagram or a ribbon torus-knot if D is a virtual knot diagram.

An *arc* of a virtual arc/knot diagram proceeds from one classical undercrossing or endpoint to another classical undercrossing or endpoint.

Definition 3.1. Let D be a virtual arc/knot diagram. The *fundamental quandle* $Q(D)$ of D is the quandle defined by the following presentation.

1. Let $a_1, \dots, a_n$ be the arcs of D. Then the generating set is $\{a_1, \dots, a_n\}$.

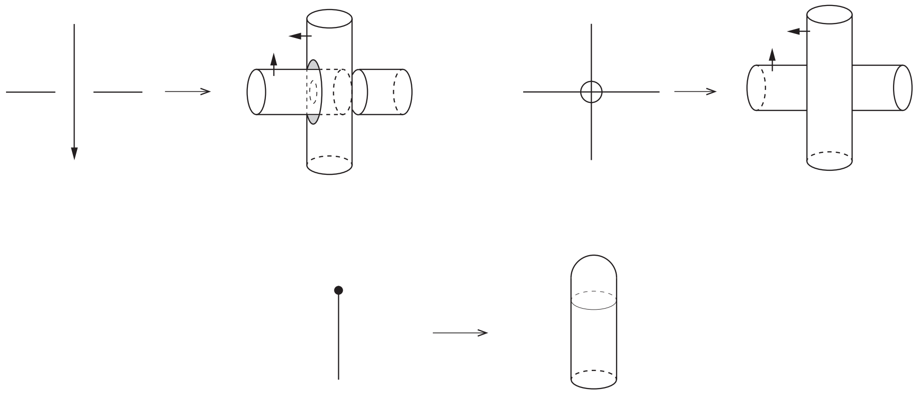
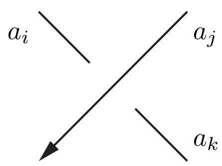


Figure 3: Correspondence at crossings and endpoints.

2. For a classical crossing c of D, we associate c with a quandle relation

$$r(c) : a_k = {a_i}^{a_j},$$

where a_i, a_j and a_k are the arcs adjacent to c as illustrated in Fig. 4. Let $c_1, \ldots, c_m$ be the classical crossings of D. The set of relations is $\{r(c_1), r(c_2), \ldots, r(c_m)\}$.



$$r(c) : a_k = {a_i}^{a_j}.$$

Figure 4: A neighborhood of the classical crossing c

A presentation of a quandle will be explained in Sec. 4.

Theorem 3.2 (Satoh [12])**.** *For any ribbon 2-knot/torus-knot F, there exists a virtual arc/knot diagram D such that F is ambient isotopic to* $\mathrm{Tube}(D)$*. Moreover, $Q(F) \cong Q(D)$.*

4 Presentations of a quandle and a biquandle

Let S be a set, and let $\overline{S} = \{\bar{x} \mid x \in S\}$ be the set of symbols $\bar{x}$ for $x \in S$. Here we assume $S \cap \overline{S} = \emptyset$. By a word in S, we mean a (possibly empty) finite sequence $x_1 x_2 \cdots x_n$ ($x_i \in S \cup \overline{S}, n \in \mathbb{N} \cup \{0\}$). We denote the empty word by 1. Then the set, $W(S)$, of words in S is a monoid whose product is the word concatenation with identity element 1. The free group, $FG(S)$, on S is the quotient of $W(S)$ by the relation on $W(S)$ generated by $ux\bar{x}v \sim u\bar{x}xv \sim uv$ $(u, v \in W(S), x \in S)$.

The *free rack* $FR(S)$ on S is a rack with underlying set $S \times FG(S)$ and with operations $(a, w)^{(b,u)} = (a, w\bar{u}bu)$ and $(a, w)^{\overline{(b,u)}} = (a, w\bar{u}\bar{b}u)$. We write the element $(a, w) \in FR(S)$ as a^w, so $FR(S) = \{a^w \mid a \in S, w \in FG(S)\}$. Then $(a^w)^{(b^u)} = a^{w\bar{u}bu}$ and $(a^w)^{\overline{(b^u)}} = a^{w\bar{u}\bar{b}u}$. Let $\sim_q$ be the equivalence relation on $FR(S)$ generated by the relation $a^w \sim a^{aw}(a \in S, w \in FG(S))$. The *free quandle* $FQ(S)$ on S is the quotient $FQ(S) = FR(S)/\sim_q$ with the quandle operations induced from the operations of $FR(S)$ (cf. [5, 8]).

Let S_∞ denote the set of all expressions obtained from S by using eight binary symbols $(a, b) \mapsto a^b, a_b, a^{\bar{b}}, a_{\bar{b}}, a^{b^{-1}}, a_{b^{-1}}, a^{\bar{b}^{-1}}$ and $a_{\bar{b}^{-1}}$: S_∞ is defined to be the union $\bigcup_{i=0}^{\infty} S_i$, where $\{S_i\}_{i=0}^{\infty}$ is the sequence of sets of expressions defined inductively as follows.

- $S_0 := S$.
- $S_{i+1} := S_i \cup \{a^b, a_b, a^{\bar{b}}, a_{\bar{b}}, a^{b^{-1}}, a_{b^{-1}}, a^{\bar{b}^{-1}}, a_{\bar{b}^{-1}} \mid a, b \in S_i\}$.

Eight binary operations $(a, b) \mapsto a^b, a_b, a^{\bar{b}}, a_{\bar{b}}, a^{b^{-1}}, a_{b^{-1}}, a^{\bar{b}^{-1}}$ and $a_{\bar{b}^{-1}}$ on S_∞ are naturally defined. Let $\sim$ be the equivalence relation on S_∞ generated by the identities in (B1′), (B2), (B3) and (B4). The *free biquandle* $FBQ(S)$ on S is $S_\infty/\sim$ with binary operations induced from those of S_∞. (Refer to [1] for free algebras.) For $w \in S_\infty$, the element of $FBQ(S)$ represented by w is simply written as w.

A *biquandle presentation* is a pair of a set S and a subset R of $FBQ(S) \times FBQ(S)$, denoted by $\langle S \mid R\rangle_{bq}$. It determines a biquandle that is the quotient of the free biquandle $FBQ(S)$ by the congruence relation generated by R (cf. [5, 8]). Elements (x, y) of $R \subset FBQ(S) \times FBQ(S)$ are called *biquandle relations* and usually denoted by $x = y$ or $(x = y) \in R$, etc.

A *quandle presentation* $\langle S \mid R\rangle_q$ is similarly defined by using the free quandle $FQ(S)$ (see. [5, 8]).

5 The biquandle presentation associated with a quandle presentation

In this section, we associate with each quandle presentation $\langle S \mid R\rangle_q$ a biquandle presentation $J_0(\langle S \mid R\rangle_q)$.

Define a map $\phi_0 : S \times W(S) \to FBQ(S)$ by the following inductive rule with respect to the length of the word in $W(S)$.

- $\phi_0(a, 1) = a$, $(a \in S$ and $1 \in W(S))$.
- $\phi_0(a, wb) = (\phi_0(a, w)_{\bar{b}^{-1}})^b$ and $\phi_0(a, w\bar{b}) = (\phi_0(a, w)^{b^{-1}})_{\bar{b}}$, $(a \in S, w \in W(S)$ and $b \in S)$.

Proposition 5.1. *The map $\phi_0 : S \times W(S) \to FBQ(S)$ induces a well-defined map $\phi : FQ(S) \to FBQ(S)$.*

Proof. First we show that the map $\phi_0 : S \times W(S) \to FBQ(S)$ factors through $S \times FG(S)$, i.e., we show that $\phi_0(a, ub\bar{b}v) = \phi_0(a, uv)$ and $\phi_0(a, u\bar{b}bv) = \phi_0(a, uv)$ for $a, b \in S$ and $u, v \in W(S)$. Let $l(v)$ denote the length of v, and we use induction on the length $l(v)$.

When $l(v) = 0$, we have

$$\begin{aligned} \phi_0(a, ub\bar{b}) &= ((\phi_0(a,u)_{\bar{b}^{-1}})^{bb^{-1}})_{\bar{b}} \\ &= \phi_0(a,u) \end{aligned}$$

and

$$\begin{aligned} \phi_0(a, u\bar{b}b) &= ((\phi_0(a,u)^{b^{-1}})_{\bar{b}\bar{b}^{-1}})^{b} \\ &= \phi_0(a,u) \end{aligned}$$

for $a, b \in S$ and $u \in W(S)$.

Our inductive assumption is: Assume when $l(v) = n$, the identities $\phi_0(a, ub\bar{b}v) = \phi_0(a, uv)$ and $\phi_0(a, u\bar{b}bv) = \phi_0(a, uv)$ hold. Then, for $a, b, c \in S$ and $u, v \in W(S)$ with $l(v) = n$, we have

$$\begin{aligned} \phi_0(a, ub\bar{b}vc) &= (\phi_0(a, ub\bar{b}v)_{\bar{c}^{-1}})^{c} \\ &= (\phi_0(a, uv)_{\bar{c}^{-1}})^{c} \\ &= \phi_0(a, uvc), \end{aligned}$$

$$\begin{aligned} \phi_0(a, ub\bar{b}v\bar{c}) &= (\phi_0(a, ub\bar{b}v)^{c^{-1}})_{\bar{c}} \\ &= (\phi_0(a, uv)^{c^{-1}})_{\bar{c}} \\ &= \phi_0(a, uv\bar{c}), \end{aligned}$$

$$\begin{aligned} \phi_0(a, u\bar{b}bvc) &= (\phi_0(a, u\bar{b}bv)_{\bar{c}^{-1}})^{c} \\ &= (\phi_0(a, uv)_{\bar{c}^{-1}})^{c} \\ &= \phi_0(a, uvc) \end{aligned}$$

and

$$\begin{aligned} \phi_0(a, u\bar{b}bv\bar{c}) &= (\phi_0(a, u\bar{b}bv)^{c^{-1}})_{\bar{c}} \\ &= (\phi_0(a, uv)^{c^{-1}})_{\bar{c}} \\ &= \phi_0(a, uv\bar{c}). \end{aligned}$$

This completes the inductive step. Hence we see that ϕ_0 induces a well-defined map from $S \times FG(S)$ to $FBQ(S)$.

Next we show that $\phi_0(a, a) = \phi_0(a, 1)$ for $a \in S$. By the second identity of (B3), we see that $x = x_{x^{x^{-1}}\overline{x^{x^{-1}x}}}$ for any $x \in FBQ(S)$. Thus we have $x = x_{x^{x^{-1}}\overline{x}}$, and hence $x_{\overline{x}^{-1}} = x_{x^{x^{-1}}}$. Then we have

$$\begin{aligned}
\phi_0(a, a) &= (a_{\overline{a}^{-1}})^a \\
&= (a_{a^{a^{-1}}})^a \\
&= a^{a^{-1}a} \quad \text{(by the second identity of (B2))} \\
&= a \\
&= \phi_0(a, 1).
\end{aligned}$$

This implies that $\phi_0(a, aw) = \phi_0(a, w)$ for $a \in S$ and $w \in FG(S)$. Since the equivalence relation $\sim_q$ is generated by $a^{aw} \sim a^w$, we see that the map $\phi : FQ(S) \to FBQ(S)$ is well-defined. $\square$

For a set R of quandle relations, let $\Phi(R)$ denote the set of biquandle relations $\{\phi(a^w) = \phi(b^z) \mid (a^w = b^z) \in R\}$. For a quandle presentation $\langle S \mid R\rangle_q$, we define $J_0(\langle S \mid R\rangle_q)$ to be the biquandle presentation $\langle S \mid \Phi(R)\rangle_{bq}$.

Lemma 5.2. *Let B be a biquandle. Define two binary operations $*$ and $\overline{*}$ by*

$$a * b := (a_{\overline{b}^{-1}})^b, \quad a \,\overline{*}\, b := (a^{b^{-1}})_{\overline{b}}$$

for $a, b \in B$. Then B with the operations $*$ and $\overline{*}$ is a quandle.

In the above lemma, the symbols $a * b$ and $a \,\overline{*}\, b$ denote quandle operations.

Proof. By the second identity of (B3), we see that $a = a_{a^{a^{-1}}\overline{a^{a^{-1}a}}}$ for any $a \in B$. Thus we have $a = a_{a^{a^{-1}}\overline{a}}$, and hence $a_{\overline{a}^{-1}} = a_{a^{a^{-1}}}$. Then we have

$$\begin{aligned}
a * a &= (a_{\overline{a}^{-1}})^a \\
&= (a_{a^{a^{-1}}})^a \\
&= a^{a^{-1}a} \quad \text{(by the second identity of (B2))} \\
&= a.
\end{aligned}$$

Thus $*$ satisfies (Q1).

Next we show that $*$ and $\overline{*}$ satisfy (Q2). By the definition of $*$ and $\overline{*}$, we have $(a * b) \,\overline{*}\, b = ((a_{\overline{b}^{-1}})^{bb^{-1}})_{\overline{b}} = a$ and $(a \,\overline{*}\, b) * b = ((a^{b^{-1}})_{\overline{b}^{-1}\overline{b}})^b = a$ for $a, b \in B$.

Finally we show that $*$ satisfies (Q3). From the second identity of (B3), we have $b = b_{a^{b^{-1}}\overline{a^{b^{-1}b}}}$ for $a, b \in B$. Thus we have $b = b_{a^{b^{-1}}\overline{a}}$, and hence

$$b_{a^{b^{-1}}} = b_{\overline{a}^{-1}}. \tag{1}$$

We have

$$\begin{aligned}
a^{c^{-1}b^{-1}} &= a^{c^{-1}b^{-1}c_b b^c (b^c)^{-1}(c_b)^{-1}} \\
&= a^{c^{-1}b^{-1}bc(b^c)^{-1}(c_b)^{-1}} \quad \text{(by the first identity of (B4))} \\
&= a^{(b^c)^{-1}(c_b)^{-1}}
\end{aligned}$$

for $a, b, c \in B$. Thus we have

$$a^{c^{-1}b^{-1}} = a^{(b^c)^{-1}(c_b)^{-1}}. \tag{2}$$

We have

$$\begin{aligned}
(b^c)_{\overline{a}^{-1}} &= (b^c)_{a^{(b^c)^{-1}}} \quad \text{(by the identity (1))} \\
&= (b^c)_{a^{(b^c)^{-1}(c_b)^{-1}c_b}} \\
&= (b^c)_{a^{c^{-1}b^{-1}c_b}} \quad \text{(by the identity (2))} \\
&= (b_{a^{c^{-1}b^{-1}}})^{c_{a^{c^{-1}b^{-1}b}}} \quad \text{(by the third identity of (B4))} \\
&= (b_{a^{c^{-1}b^{-1}}})^{c_{a^{c^{-1}}}} \\
&= (b_{\overline{a^{c^{-1}}}^{-1}})^{c_{\overline{a}^{-1}}} \quad \text{(by the identity (1))}
\end{aligned}$$

for $a, b, c \in B$. Thus we have

$$(b^c)_{\overline{a}^{-1}} = (b_{\overline{a^{c^{-1}}}^{-1}})^{c_{\overline{a}^{-1}}}. \tag{3}$$

From the third identity of (B3), we have $a = a^{\overline{b_{\overline{a}^{-1}}} b_{\overline{a}^{-1}\overline{a}}}$ for $a, b \in B$. Thus we have $a = a^{\overline{b_{\overline{a}^{-1}}} b}$, and hence

$$a^{b^{-1}} = a^{\overline{b_{\overline{a}^{-1}}}}. \tag{4}$$

It is known that

$$x_{\overline{b}\overline{a}} = x_{\overline{a^{\overline{b}}}\,\overline{b_{\overline{a}}}} \tag{5}$$

holds for $a, b, x \in B$. This identity corresponds to Reidemeister move of type III with all negative crossings and is obtained from the axioms of a biquandle (see [2]). Put $x = c_{\overline{a}^{-1}\overline{b}^{-1}}$ in the identity (5), and we see that

$$c_{\overline{a}^{-1}\overline{b}^{-1}\overline{a^{\overline{b}}}\,\overline{b_{\overline{a}}}} = c. \tag{6}$$

Thus we have

$$\begin{aligned}
c_{\overline{a}^{-1}\overline{b}^{-1}} &= c_{\overline{a}^{-1}\overline{b}^{-1}\overline{a^{\overline{b}}}\,\overline{b_{\overline{a}}}\,\overline{b_{\overline{a}}}^{-1}\overline{a^{\overline{b}}}^{-1}} \\
&= c_{\overline{b_{\overline{a}}}^{-1}\overline{a^{\overline{b}}}^{-1}} \quad \text{(by the identity (6))}
\end{aligned}$$

for $a, b, c \in B$. Thus we have

$$c_{\overline{a}^{-1}\overline{b}^{-1}} = c_{\overline{b_{\overline{a}}}^{-1}\overline{a^{\overline{b}}}^{-1}}. \tag{7}$$

Then we have

$$\begin{aligned}
c_{\overline{a}^{-1}\overline{b_{\overline{a}^{-1}}}^{-1}} &= c_{\overline{b_{\overline{a}^{-1}\overline{a}}}^{-1}\overline{a^{\overline{b_{\overline{a}^{-1}}}}}^{-1}} \quad \text{(by the identity (7))} \\
&= c_{\overline{b}^{-1}\overline{a^{\overline{b_{\overline{a}^{-1}}}}}^{-1}} \\
&= c_{\overline{b}^{-1}\overline{a^{b^{-1}}}^{-1}} \quad \text{(by the identity (4))}
\end{aligned}$$

for $a, b, c \in B$. Thus we have

$$c_{\overline{a}^{-1}\overline{b_{\overline{a}^{-1}}}^{-1}} = c_{\overline{b}^{-1}\overline{a^{b^{-1}}}^{-1}}. \tag{8}$$

From the second identity of (B3), we have $b = b_{a_{\overline{b}^{-1}}\overline{(a_{\overline{b}^{-1}})^{b}}}$, for $a, b \in B$. Thus we have

$$b_{a_{\overline{b}^{-1}}} = b_{\overline{(a_{\overline{b}^{-1}})^{b}}^{-1}}. \tag{9}$$

We have

$$\begin{aligned}
(b^c)_{\overline{(a_{\overline{c}^{-1}})^{c}}^{-1}} &= (b_{\overline{(a_{\overline{c}^{-1}})^{cc^{-1}}}^{-1}})^{c_{\overline{(a_{\overline{c}^{-1}})^{c}}^{-1}}} \quad \text{(by the identity (3))}\\
&= (b_{\overline{a_{\overline{c}^{-1}}}^{-1}})^{c_{\overline{(a_{\overline{c}^{-1}})^{c}}^{-1}}}\\
&= (b_{\overline{a_{\overline{c}^{-1}}}^{-1}})^{c_{a_{\overline{c}^{-1}}}} \quad \text{(by the identity (9))}
\end{aligned}$$

for $a, b, c \in B$. Thus we have

$$(b^c)_{\overline{(a_{\overline{c}^{-1}})^{c}}^{-1}} = (b_{\overline{a_{\overline{c}^{-1}}}^{-1}})^{c_{a_{\overline{c}^{-1}}}}. \tag{10}$$

Thus we have

$$\begin{aligned}
(a * b) * c &= (((a_{\overline{b}^{-1}})^{b})_{\overline{c}^{-1}})^{c} \quad \text{(by the definition of } *)\\
&= (a_{\overline{b}^{-1}\overline{c^{b^{-1}}}^{-1}})^{b_{\overline{c}^{-1}}c} \quad \text{(by the identity (3))}\\
&= (a_{\overline{c}^{-1}\overline{b_{\overline{c}^{-1}}}^{-1}})^{b_{\overline{c}^{-1}}c} \quad \text{(by the identity (8))}\\
&= (a_{\overline{c}^{-1}\overline{b_{\overline{c}^{-1}}}^{-1}})^{c_{b_{\overline{c}^{-1}}}(b_{\overline{c}^{-1}})^{c}} \quad \text{(by the first identity of (B4))}\\
&= (((a_{\overline{c}^{-1}})^{c})_{\overline{(b_{\overline{c}^{-1}})^{c}}^{-1}})^{(b_{\overline{c}^{-1}})^{c}} \quad \text{(by the identity (10))}\\
&= (a * c) * (b * c), \quad \text{(by the definition of } *)
\end{aligned}$$

for $a, b, c \in B$. Hence $*$ satisfies (Q3). $\square$

Then we have the following Corollary.

Corollary 5.3. *Let B be a biquandle. Then the following identities hold for any elements $a, b, c \in B$.*

$$\begin{aligned}
(a_{\overline{(b_{\overline{c}^{-1}})^{c}}^{-1}})^{(b_{\overline{c}^{-1}})^{c}} &= ((((a^{c^{-1}})_{\overline{c}\overline{b}^{-1}})^{b})_{\overline{c}^{-1}})^{c},\\
(a_{\overline{(b^{c^{-1}})_{\overline{c}}}^{-1}})^{(b^{c^{-1}})_{\overline{c}}} &= ((((a_{\overline{c}^{-1}})^{c})_{\overline{b}^{-1}})^{bc^{-1}})_{\overline{c}},\\
(a^{((b_{\overline{c}^{-1}})^{c})^{-1}})_{\overline{(b_{\overline{c}^{-1}})^{c}}} &= ((((a^{c^{-1}})_{\overline{c}})^{b^{-1}})_{\overline{b}\overline{c}^{-1}})^{c},\\
(a^{((b^{c^{-1}})_{\overline{c}})^{-1}})_{\overline{(b^{c^{-1}})_{\overline{c}}}} &= ((((a_{\overline{c}^{-1}})^{cb^{-1}})_{\overline{b}})^{c^{-1}})_{\overline{c}}.
\end{aligned}$$

Proof. Since the binary operations $*$ and $\overline{*}$ on B defined in Lemma 5.2 satisfy (Q2) and (Q3), the following identities hold (cf. [5]).

$$\begin{aligned}
a * (b * c) &= ((a \,\overline{*}\, c) * b) * c,\\
a * (b \,\overline{*}\, c) &= ((a * c) * b) \,\overline{*}\, c,\\
a \,\overline{*}\, (b * c) &= ((a \,\overline{*}\, c) \,\overline{*}\, b) * c,\\
a \,\overline{*}\, (b \,\overline{*}\, c) &= ((a * c) \,\overline{*}\, b) \,\overline{*}\, c.
\end{aligned}$$

The desired identities follow from these identities. $\square$

Lemma 5.4. *For a set S and $FG(S)$, the following identities hold for $a, b \in S$ and $u, w \in FG(S)$.*

$$\begin{aligned}
(\phi(a^u)_{\overline{\phi(b^w)}^{-1}})^{\phi(b^w)} &= \phi(a^{u\bar{w}bw}),\\
(\phi(a^u)^{\phi(b^w)^{-1}})_{\overline{\phi(b^w)}} &= \phi(a^{u\bar{w}\bar{b}w}).
\end{aligned}$$

Proof. First, we show the first identity by induction on $l(w)$ on the length of w as an element of $W(S)$.

When $l(w) = 0$, we have $(\phi(a^u)_{\overline{\phi(b^1)}^{-1}})^{\phi(b^1)} = (\phi(a^u)_{\bar{b}^{-1}})^b = \phi(a^{ub})$ for $a, b \in S, u \in FG(S)$.

Our inductive assumption is: $(\phi(a^u)_{\overline{\phi(b^w)}^{-1}})^{\phi(b^w)} = \phi(a^{u\bar{w}bw})$ for any $w \in FG(S)$ with $l(w) = n$. We show that $(\phi(a^u)_{\overline{\phi(b^w)}^{-1}})^{\phi(b^w)} = \phi(a^{u\bar{w}bw})$ for any $w \in FG(S)$ with $l(w) = n+1$. Put $w = vd$ or $v\bar{d}$ where $v \in W(S)$ with $l(v) = n$ and $d \in S$.

When $w = vd$, we have

$$\begin{aligned}
(\phi(a^u)_{\overline{\phi(b^w)}^{-1}})^{\phi(b^w)} &= (\phi(a^u)_{\overline{\phi(b^{vd})}^{-1}})^{\phi(b^{vd})} && (11)\\
&= (\phi(a^u)_{\overline{(\phi(b^v)_{\bar{d}^{-1}})^d}^{-1}})^{(\phi(b^v)_{\bar{d}^{-1}})^d} && (12)\\
&= ((((\phi(a^u)^{d^{-1}})_{\bar{d}\,\overline{\phi(b^v)}^{-1}})^{\phi(b^v)})_{\bar{d}^{-1}})^d && (13)\\
&= (((\phi(a^{u\bar{d}})_{\overline{\phi(b^v)}^{-1}})^{\phi(b^v)})_{\bar{d}^{-1}})^d && (14)\\
&= (\phi(a^{u\bar{d}\bar{v}bv})_{\bar{d}^{-1}})^d && (15)\\
&= \phi(a^{u\overline{vd}bvd}) && (16)\\
&= \phi(a^{u\bar{w}bw}) && (17)
\end{aligned}$$

for $a, b, d \in S, u \in FG(S)$. Here (11) is a replacement of w by vd, (12) follows from the definition of ϕ, (13) follows from the first identity of Corollary 5.3, (14) follows from the definition of ϕ, (15) is the inductive assumption, (16) follows from the definition of ϕ, and (17) is a replacement of vd by w.

When $w = v\bar{d}$, we have

$$
\begin{aligned}
(\phi(a^u)_{\overline{\phi(b^w)}^{-1}})^{\phi(b^w)} &= (\phi(a^u)_{\overline{\phi(b^{v\bar{d}})}^{-1}})^{\phi(b^{v\bar{d}})} &(18)\\
&= (\phi(a^u)_{\overline{(\phi(b^v)^{d^{-1}})_{\overline{d}}}^{-1}})^{(\phi(b^v)^{d^{-1}})_{\overline{d}}} &(19)\\
&= ((((\phi(a^u)_{\overline{d}^{-1}})^d)_{\overline{\phi(b^v)}^{-1}})^{\phi(b^v)d^{-1}})_{\overline{d}} &(20)\\
&= ((\phi(a^{ud})_{\overline{\phi(b^v)}^{-1}})^{\phi(b^v)d^{-1}})_{\overline{d}} &(21)\\
&= (\phi(a^{ud\bar{v}bv})^{d^{-1}})_{\overline{d}} &(22)\\
&= \phi(a^{u\overline{v\bar{d}}bv\bar{d}}) &(23)\\
&= \phi(a^{u\bar{w}bw}) &(24)
\end{aligned}
$$

for $a, b, d \in S, u \in FG(S)$. Here (18) is a replacement of w by $v\bar{d}$, (19) follows from the definition of ϕ, (20) follows from the second identity of Corollary 5.3, (21) follows from the definition of ϕ, (22) is the inductive assumption, (23) follows from the definition of ϕ, and (24) is a replacement of $v\bar{d}$ by w. This completes the proof of the first identity.

Next, we show the second identity by induction on $l(w)$ on the length of w as an element of $W(S)$.

When $l(w) = 0$, we have $(\phi(a^u)^{\phi(b^1)^{-1}})_{\overline{\phi(b^1)}} = (\phi(a^u)^{b^{-1}})_{\overline{b}} = \phi(a^{u\bar{b}})$ for $a, b \in S, u \in FG(S)$.

Our inductive assumption is: $(\phi(a^u)^{\phi(b^w)^{-1}})_{\overline{\phi(b^w)}} = \phi(a^{u\bar{w}\bar{b}w})$ for any $w \in FG(S)$ with $l(w) = n$. We show that $(\phi(a^u)^{\phi(b^w)^{-1}})_{\overline{\phi(b^w)}} = \phi(a^{u\bar{w}\bar{b}w})$ for any $w \in FG(S)$ with $l(w) = n+1$. Put $w = vd$ or $v\bar{d}$ where $v \in W(S)$ with $l(v) = n$ and $d \in S$.

When $w = vd$, we have

$$
\begin{aligned}
(\phi(a^u)^{\phi(b^w)^{-1}})_{\overline{\phi(b^w)}} &= (\phi(a^u)^{\phi(b^{vd})^{-1}})_{\overline{\phi(b^{vd})}} &(25)\\
&= (\phi(a^u)^{((\phi(b^v)_{\overline{d}^{-1}})^d)^{-1}})_{\overline{(\phi(b^v)_{\overline{d}^{-1}})^d}} &(26)\\
&= ((((\phi(a^u)^{d^{-1}})_{\overline{d}})^{\phi(b^v)^{-1}})_{\overline{\phi(b^v)}\,\overline{d}^{-1}})^d &(27)\\
&= ((\phi(a^{u\bar{d}})^{\phi(b^v)^{-1}})_{\overline{\phi(b^v)}\,\overline{d}^{-1}})^d &(28)\\
&= (\phi(a^{u\bar{d}\bar{v}\bar{b}v})_{\overline{d}^{-1}})^d &(29)\\
&= \phi(a^{u\overline{vd}\,\bar{b}vd}) &(30)\\
&= \phi(a^{u\bar{w}\bar{b}w}) &(31)
\end{aligned}
$$

for $a, b, d \in S, u \in FG(S)$. Here (25) is a replacement of w by vd, (26) follows from the definition of ϕ, (27) follows from the third identity of Corollary 5.3, (28) follows from the definition of ϕ, (29) is the inductive assumption, (30) follows from the definition of ϕ, and (31) is a replacement of vd by w.

When $w = v\bar{d}$, we have

$$
\begin{aligned}
(\phi(a^u)^{\phi(b^w)^{-1}})_{\overline{\phi(b^w)}} &= (\phi(a^u)^{\phi(b^{v\bar{d}})^{-1}})_{\overline{\phi(b^{v\bar{d}})}} && (32)\\
&= (\phi(a^u)^{((\phi(b^v)^{d^{-1}})_{\overline{d}})^{-1}})_{\overline{(\phi(b^v)^{d^{-1}})_{\overline{d}}}} && (33)\\
&= ((((\phi(a^u)_{\overline{d}^{-1}})^{d\phi(b^v)^{-1}})_{\overline{\phi(b^v)}})^{d^{-1}})_{\overline{d}} && (34)\\
&= (((\phi(a^{ud})^{\phi(b^v)^{-1}})_{\overline{\phi(b^v)}})^{d^{-1}})_{\overline{d}} && (35)\\
&= (\phi(a^{ud\bar{v}\bar{b}v})^{d^{-1}})_{\overline{d}} && (36)\\
&= \phi(a^{u\overline{v\bar{d}}\bar{b}v\bar{d}}) && (37)\\
&= \phi(a^{u\bar{w}\bar{b}w}) && (38)
\end{aligned}
$$

for $a, b, d \in S, u \in FG(S)$. Here (32) is a replacement of w by $v\bar{d}$, (33) follows from the definition of ϕ, (34) follows from the fourth identity of Corollary 5.3, (35) follows from the definition of ϕ, (36) is the inductive assumption, (37) follows from the definition of ϕ, and (38) is a replacement of $v\bar{d}$ by w. This completes the proof of the second identity. $\square$

For a set R of biquandle relations, let $\langle\langle R\rangle\rangle_{bq}$ denote the set of biquandle consequences of R.

Lemma 5.5. *If $(\phi(a^u) = \phi(b^v)) \in R$ for some $a, b \in S$ and $u, v \in FG(S)$, then $(\phi(a^{uw}) = \phi(b^{vw})) \in \langle\langle R\rangle\rangle_{bq}$ for any $w \in FG(S)$. Hence, $\Phi(a^{uw} = b^{vw}) \in \langle\langle \Phi(a^u = b^v)\rangle\rangle_{bq}$.*

Proof. Assuming $\phi(a^u) = \phi(b^v)$, we prove that $\phi(a^{uw}) = \phi(b^{vw})$ for any $w \in FG(S)$ by induction on the length $l(w)$ of w as an element of $W(S)$.

The case of $l(w) = 0$ is obvious.

Our inductive assumption is: $\phi(a^{uw}) = \phi(b^{vw})$ for any $w \in FG(S)$ with $l(w) = n$. We show that $\phi(a^{uw}) = \phi(b^{vw})$ for any $w \in FG(S)$ with $l(w) = n+1$. Put $w = zd$ or $z\overline{d}$ where $z \in W(S)$ with $l(z) = n$ and $d \in S$.

When $w = zd$, $\phi(a^{uw}) = \phi(a^{uzd}) = (\phi(a^{uz})_{\overline{d}^{-1}})^d = (\phi(b^{vz})_{\overline{d}^{-1}})^d = \phi(b^{vzd}) = \phi(b^{vw})$.

When $w = z\overline{d}$, $\phi(a^{uw}) = \phi(a^{uz\overline{d}}) = (\phi(a^{uz})^{d^{-1}})_{\overline{d}} = (\phi(b^{vz})^{d^{-1}})_{\overline{d}} = \phi(b^{vz\overline{d}}) = \phi(b^{vw})$.

This completes the proof. $\square$

Lemma 5.6 (Fenn and Rourke [5]). *Two presentations of isomorphic quandles are related by a finite sequence of the following transformations and their inverse transformations, so-called Tietze moves:*

T_1 *Repeat a relation.*

T_2 *Conjugate a relation, i.e. replace $a^t = b^w$ by $a^{tz} = b^{wz}$.*

T_3 *If $(a = b^w) \in R$, then we can replace $c^z = a^t$ by $c^z = b^{wt}$ or $a^z = c^t$ by $b^{wz} = c^t$.*

T_4 *If* $(a = b^w) \in R$*, then we can replace* $c^{taq} = d^z$ *by* $c^{t\bar{w}bwq} = d^z$*,* $c^{t\bar{a}q} = d^z$ *by* $c^{t\bar{w}\bar{b}wq} = d^z$*,* $c^z = d^{taq}$ *by* $c^z = d^{t\bar{w}bwq}$ *or* $c^z = d^{t\bar{a}q}$ *by* $c^z = d^{t\bar{w}\bar{b}wq}$*.*

T_5 *Introduce a new generator* x *and a new relation* $x = a^w$ *(where* x *does not occur in* w*).*

In what follows, for quandle/biquandle presentations P and P', $P \cong P'$ means that the quandles/biquandles presented by P and $P'$are isomorphic.

Proof of Theorem 1.2. It is sufficient to show that if a quandle presentation $\langle S' \mid R' \rangle_q$ is obtained from a quandle presentation $\langle S \mid R \rangle_q$ by one of the moves $T_1, T_2, \ldots, T_5$, then $J_0(\langle S \mid R \rangle_q) \cong J_0(\langle S' \mid R' \rangle_q)$.

Case 1. T_1-move.

In this case $S' = S$, and it is clear that $\Phi(R)$ equals $\Phi(R')$ setwise. Thus we have $\langle S \mid \Phi(R) \rangle_{bq} \cong \langle S' \mid \Phi(R') \rangle_{bq}$, and hence $J_0(\langle S \mid R \rangle_q) \cong J_0(\langle S' \mid R' \rangle_q)$.

Case 2. T_2-move.

In this case $S' = S$ and $R' = (R \setminus \{a^w = b^t\}) \cup \{a^{wz} = b^{tz}\}$ where $a, b \in S$ and $t, w, z \in FG(S)$. By Lemma 5.5, $\langle\langle \Phi(a^w = b^t) \rangle\rangle_{bq} = \langle\langle \Phi(a^{wz} = b^{tz}) \rangle\rangle_{bq}$. Thus, $\langle\langle \Phi(R) \rangle\rangle_{bq} = \langle\langle \Phi(R') \rangle\rangle_{bq}$, and we have $\langle S \mid \Phi(R) \rangle_{bq} \cong \langle S' \mid \Phi(R') \rangle_{bq}$. Hence we have $J_0(\langle S \mid R \rangle_q) \cong J_0(\langle S' \mid R' \rangle_q)$.

Case 3. T_3-move.

First, we show that if $(a = b^w) \in R$, then we can replace $c^z = a^t$ by $c^z = b^{wt}$. In this case $S' = S$ and $R' = (R \setminus \{c^z = a^t\}) \cup \{c^z = b^{wt}\}$ where $a, b, c \in S$ and $t, w, z \in FG(S)$. By Lemma 5.5, $\Phi(a^t = b^{wt}) \in \langle\langle \Phi(a = b^w) \rangle\rangle_{bq} \subset \langle\langle \Phi(R) \rangle\rangle_{bq} \cap \langle\langle \Phi(R') \rangle\rangle_{bq}$. Since $\Phi(a^t = b^{wt})$ and $\Phi(c^z = a^t)$ belong to $\langle\langle \Phi(R) \rangle\rangle_{bq}$, we see that $\Phi(c^z = b^{wt}) \in \langle\langle \Phi(R) \rangle\rangle_{bq}$. On the other hand, since $\Phi(a^t = b^{wt})$ and $\Phi(c^z = b^{wt})$ belong to $\langle\langle \Phi(R') \rangle\rangle_{bq}$, we see that $\Phi(c^z = a^t) \in \langle\langle \Phi(R') \rangle\rangle_{bq}$. Thus, $\langle\langle \Phi(R) \rangle\rangle_{bq} = \langle\langle \Phi(R') \rangle\rangle_{bq}$, and we have that $J_0(\langle S \mid R \rangle_q) \cong J_0(\langle S' \mid R' \rangle_q)$.

The latter assertion is proved similarly.

Case 4. T_4-move.

First, we show that if $(a = b^w) \in R$, then we can replace $c^{taq} = d^z$ by $c^{t\bar{w}bwq} = d^z$. In this case $S' = S$, and it suffices to show that $\langle\langle \Phi(a = b^w), \Phi(c^{taq} = d^z) \rangle\rangle_{bq} = \langle\langle \Phi(a = b^w), \Phi(c^{t\bar{w}bwq} = d^z) \rangle\rangle_{bq}$, equivalently to show the following assertion by Lemma 5.5.

Assertion 1. $\Phi(c^{ta} = c^{t\bar{w}bw}) \in \langle\langle \Phi(a = b^w) \rangle\rangle_{bq}$, i.e., $\phi(c^{ta}) = \phi(c^{t\bar{w}bw}) \mod \langle\langle \Phi(a = b^w) \rangle\rangle_{bq}$.

Proof of Assertion 1. For $a, b, c \in S$ and $t, w \in FG(S)$,

$$\phi(c^{ta}) = (\phi(c^t)_{\overline{a}^{-1}})^a \tag{39}$$

$$= (\phi(c^t)_{\overline{\phi(b^w)}^{-1}})^{\phi(b^w)} \tag{40}$$

$$= \phi(c^{t\bar{w}bw}). \tag{41}$$

Here (39) follows from the definition of ϕ, (40) is a congruence mod $\langle\langle\Phi(a = b^w)\rangle\rangle_{bq}$, and (41) follows from the first identity of Lemma 5.4. This completes the proof of the assertion.

Next, we show that if $(a = b^w) \in R$, then we can replace $c^{t\bar{a}q} = d^z$ by $c^{t\bar{w}\bar{b}wq} = d^z$. In this case $S' = S$, and it suffices to show that $\langle\langle\Phi(a = b^w), \Phi(c^{t\bar{a}q} = d^z)\rangle\rangle_{bq} = \langle\langle\Phi(a = b^w), \Phi(c^{t\bar{w}\bar{b}wq} = d^z)\rangle\rangle_{bq}$, equivalently to show the following assertion by Lemma 5.5.

Assertion 2. $\Phi(c^{t\bar{a}} = c^{t\bar{w}\bar{b}w}) \in \langle\langle\Phi(a = b^w)\rangle\rangle_{bq}$, i.e., $\phi(c^{t\bar{a}}) = \phi(c^{t\bar{w}\bar{b}w}) \mod \langle\langle\Phi(a = b^w)\rangle\rangle_{bq}$.

Proof of Assertion 2. For $a, b, c \in S$ and $t, w \in FG(S)$,

$$\phi(c^{ta}) = (\phi(c^t)^{a^{-1}})_{\overline{a}} \tag{42}$$

$$= (\phi(c^t)^{\phi(b^w)^{-1}})_{\overline{\phi(b^w)}} \tag{43}$$

$$= \phi(c^{t\bar{w}\bar{b}w}). \tag{44}$$

Here (42) follows from the definition of ϕ, (43) is a congruence mod $\langle\langle\Phi(a = b^w)\rangle\rangle_{bq}$, and (44) follows from the second identity of Lemma 5.4. This completes the proof of the assertion.

The other two cases of T_4 are proved similarly.

Case 5. T_5-move.

Put $S' = S \cup \{x\}$ and $R' = R \cup \{x = a^w\}$ ($a \in S$ and $w \in FG(S)$), where x does not occur in w or in R. (We regard $FQ(S) \subset FQ(S')$ and $FBQ(S) \subset FBQ(S')$.) Then we have $\Phi(R') = \Phi(R) \cup \{x = \phi(a^w)\}$, and the letter x does not occur in $\phi(a^w)$ or in $\Phi(R)$. Thus we have $\langle S \mid \Phi(R)\rangle_{bq} \cong \langle S \cup \{x\} \mid \Phi(R) \cup \{x = \phi(a^w)\}\rangle_{bq}$ (cf. [1]). $\square$

By Theorem 1.2, we associate with each quandle Q a biquandle $J(Q)$.

6 The fundamental biquandle of a ribbon 2-knot/torus-knot

In this section, we give a method for obtaining a presentation of the fundamental biquandle of a ribbon 2-knot/torus-knot directly from its virtual arc/knot presentation.

Let F be a ribbon 2-knot/torus-knot and D a virtual arc/knot diagram which presents F, and let S be the set of arcs of D and R the set of quandle relations associated with the classical crossings of D. (When the arcs adjacent to a classical crossing are a, b, c as in the left of Fig. 5, then its associated quandle relation is $c = a^b$.) Then $\langle S \mid R\rangle_q$ is a presentation of $Q(D)$.

Theorem 6.1. *In the above situation, the biquandle presentation $J_0(\langle S \mid R\rangle_q)$ is a presentation of the fundamental biquandle $BQ(F)$ of F.*

Proof of Theorem 6.1. Let S' be the set of semi-sheets of tube(D) and R' the set of biquandle relations associated with the double point curves of tube(D). Then $\langle S' \mid R' \rangle_{bq}$ is a presentation of $BQ(\text{tube}(D))$. At a classical crossing of D illustrated in the left of Fig. 5, the quandle relation is $c = a^b$. On the other hand, for the corresponding part of tube(D) illustrated in the right of Fig. 5, there are two double point curves and their associated biquandle relations are

$$b = d^a, \quad e = a_d, \quad c = e^b \quad \text{and} \quad f = b_e. \tag{45}$$

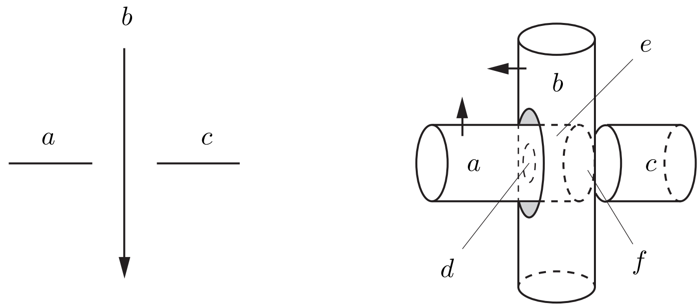


Figure 5:

From $b = d^a$, we have $d = b^{a^{-1}}$. Thus, $e = a_d$ becomes $e = a_{b^{a^{-1}}}$. By the second identity of (B3), we see $a = a_{x\overline{x^a}}$ for any x. Put $x = b^{a^{-1}}$ and we have $a = a_{b^{a^{-1}}\bar{b}}$. Thus, $a_{b^{a^{-1}}} = a_{\bar{b}^{-1}}$. Therefore, we have $e = a_{\bar{b}^{-1}}$, and hence $f = b_{a_{\bar{b}^{-1}}}$. Now we see that the relations in (45) can be replaced by

$$d = b^{a^{-1}}, \quad e = a_{\bar{b}^{-1}}, \quad c = (a_{\bar{b}^{-1}})^b \quad \text{and} \quad f = b_{a_{\bar{b}^{-1}}}.$$

Since none of other relations of R' contains d, e and f, we can delete generators d, e and f from S' and relations $d = b^{a^{-1}}, e = a_{\bar{b}^{-1}}$ and $f = b_{a_{\bar{b}^{-1}}}$ from R'. The remaining relation is $c = (a_{\bar{b}^{-1}})^b$, whose right hand side is $\phi(a^b)$. Applying this procedure to all classical crossings of D, we can transform $\langle S' \mid R' \rangle_{bq}$ into $J_0(\langle S \mid R \rangle_q)$. Since $BQ(F)$ is defined by $BQ(\text{tube}(D))$, $J_0(\langle S \mid R \rangle_q)$ is a presentation of $BQ(F)$. $\square$

Proof of Theorem 1.3. Let D be a virtual arc/knot diagram presenting F, i.e., Tube(D) is ambient isotopic to F. Then $Q(D) \cong Q(F)$ by Theorem 3.2. Let $\langle S \mid R \rangle_q$ be a presentation of $Q(D)$ as in Theorem 6.1. Then $J_0(\langle S \mid R \rangle_q)$ is a presentation of $BQ(\text{Tube}(D))$. Thus $J(Q(F)) \cong J(Q(D)) \cong BQ(\text{Tube}(D)) \cong BQ(F)$. $\square$

S. Kamada, S. Satoh and K. Tanaka suggested that the author should generalize the argument in this paper to ribbon surface knots of higher genera. We hope to discuss this problem elsewhere.

Acknowledgements

The author is deeply grateful to Professor Seiichi Kamada, Professor Makoto Sakuma and Professor Shin Satoh for helpful suggestions and support. This research was supported by Grant-in-Aid for JSPS Fellows 25·1512.

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