The Fundamentals of Economic Dynamics and Policy Analyses: Learning through Numerical Examples. Part II. Dynamic General Equilibrium

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1. Introduction

The objective of this paper is to provide an introductory exposition of dynamic general equilibrium theory which is a fundamental tool for modern macroeconomic analyses.

The paper consists of 5 sections. After the brief introduction in section 1, we define the concept of dynamic general equilibrium in section 2. A dynamic macroeconomic model is constructed either as a discrete time model or as a continuous time model, depending on model builder's purpose. The motions of economic variables in the dynamic general equilibrium of discrete time model are described by a system of difference equations, while the motions of economic variables in the dynamic general equilibrium of continuous time model are described by a system of differential equations. In section 3, we provide an introduction to the theories of difference equation and differential equation. These theories are applied to Solow-Swan economic growth model. In section 4, we present a discrete time Ramsey model as a basic example for the dynamic general equilibrium analyses. In section 5, we present a continuous time Ramsey model. Readers will notice that the discrete time Ramsey model and the continuous time Ramsey model share similar properties with respect to the behaviors of economic variables in dynamic general equilibrium.

Although the materials presented in this paper are rudimentary, readers will find them useful to understand the analyses of intermediate and advanced macroeconomic theories.

2. Dynamic General Equilibrium

In this section, we provide the definition of dynamic general equilibrium. Intuitively, a dynamic general equilibrium consists of prices and quantities of goods and production factors such that (i) given the prices, the quantities of goods and production factors maximize every economic agent's payoff function, and (ii) the prices clear every market in every time period. In other words, the prices equate demand and supply in every market in every time period. The concept of dynamic general equilibrium is a natural extension of the concept of general equilibrium of static economic models to dynamic models. In the following, firstly, we explain the concept of general equilibrium of static economic models. Then, we proceed to explain the concept of dynamic general equilibrium of dynamic economic models that is shown to be a natural extension of the general equilibrium of static economic models.

2-1. General Equilibrium of Static Economic Models.

We start this subsection with a simple two -goods static economy. Two goods are labeled as A and B. There are $m = 1, 2, \ldots, M$ consumers, $k = 1, 2, \ldots, K$ producers who produce good A, and $l = 1, 2, \ldots, L$ producers who produce good B. A representative consumer m solves the following constrained optimization problem.

$$(2.1) \quad \max_{\{c^d_{A,m},\, c^d_{B,m}\}} u_m(c^d_{A,m}, c^d_{B,m})$$

subject to

$$(2.2) \quad p_A c^d_{A,m} + p_B c^d_{B,m} \le w_m$$

given { p_A, p_B }.

$c^d_{A,m}$ is the demand for good A by consumer m, $c^d_{B,m}$ is the demand for good B by consumer m, p_A is the price of good A , p_B is the price of good B, and w_m is the income of consumer m. $u_m\ (c^d_{A,m}\ ,\ c^d_{B,m})$ is the utility of consumer m as a function of $\{c^d_{A,m},\ c^d_{B,m}\}$. The utility function is assumed to satisfy conditions that guarantee the existence of well-behaved demand functions.[1] The solution to the above problem is a set of demand for good A and good B as functions of prices p_A and p_B, denoted as

(2.3) $c^d_{A,m}(p_A, p_B)$, $m = 1, 2, \dots, M$, and

(2.4) $c^d_{B,m}(p_A, p_B)$, $m = 1, 2, \dots, M$.

The aggregate demand function for good A and the aggregate demand function for good B are calculated as follows.

(2.5) $C^d_A(p_A, p_B) \equiv \sum_{m=1}^{M} c^d_{A,m}(p_A, p_B)$

(2.6) $C^d_B(p_A, p_B) \equiv \sum_{m=1}^{M} c^d_{B,m}(p_A, p_B)$

A representative producer k of good A, given the price p_A of good A, decides supply $c^s_{A,k}$ that maximizes profit $\pi_{A,k}$ which is defined as

(2.7) $\pi_{A,k} \equiv p_A c^s_{A,k} - h_{A,k}(c^s_{A,k})$.

$h_{A,k}\ (c^s_{A,k})$ is production cost as an increasing and convex function of $c^s_{A,k}$. The supply that maximizes the profit is a function of price p_A denoted as

(2.8) $c^s_{A,k}(p_A)$, $k = 1, 2, \dots, K$.

The aggregate supply function of good A is calculated as follows.

(2.9) $C^s_A(p_A) \equiv \sum_{k=1}^{K} c^s_{A,k}(p_A)$

A representative producer l of good B, given the price p_B of good B, decides supply $c^s_{B,l}$ that maximizes profit $\pi_{B,l}$ which is defined as

(2.10) $\pi_{B,l} \equiv p_B c^s_{B,l} - h_{B,l}(c^s_{B,l})$.

$h_{B,l}\ (c^s_{B,l})$ is production cost as an increasing and convex function of $c^s_{B,l}$. The individual supply function of good B and the aggregate supply function of good B are derived likewise. They are expressed as follows.

(2.11) $c^s_{B,l}(p_B)$, $l = 1, 2, \dots, L$.

(2.12) $C^s_B(p_B) \equiv \sum_{l=1}^{L} c^s_{B,l}(p_B)$

A general equilibrium of this two-goods economy is a set of prices and quantities $\{p_A{}^*, p_B{}^*, \{c^d_{A,m}{}^*, c^d_{B,m}{}^*\ ;\ m = 1, 2, \dots, M\}, \{c^s_{A,k}{}^*\ ;\ k = 1, 2, \dots, K\}, \{c^s_{B,l}{}^*\ ;\ l = 1, 2, \dots, L\}\}$ such that (i) given the prices $\{p_A{}^*, p_B{}^*\}$, the quantities $\{c^d_{A,m}{}^*, c^d_{B,m}{}^*\ ;\ m = 1, 2, \dots, M\}$ maximize every consumer's utility $u_m\ (c^d_{A,m}, c^d_{B,m})$ subject to budget constraint $p_A{}^*\, c^d_{A,m} + p_B{}^*\, c^d_{B,m} \leq w_m$, $m = 1, 2, \dots, M$, and (ii) the prices $\{p_A{}^*, p_B{}^*\}$ equate aggregate demand and aggregate supply in every market, i.e.,

(2.13) $\sum_{m=1}^{M} c^d_{A,m}{}^* = \sum_{k=1}^{K} c^s_{A,k}{}^*$, and

(2.14) $\sum_{m=1}^{M} c^d_{B,m}{}^* = \sum_{l=1}^{L} c^s_{B,k}{}^*$.

The above definition implies that the general equilibrium prices $\{p_A{}^*, p_B{}^*\}$ are solutions to the following simultaneous equations with respect to $\{p_A, p_B\}$.

(2.15) $C^d_A(p_A, p_B) = C^s_A(p_A)$

(2.16) $C^d_B(p_A, p_B) = C^s_B(p_A)$

Given the solution $\{p_A{}^*, p_B{}^*\}$ to $\{(2.15), (2.16)\}$, the general equilibrium quantities are obtained as follows.

(2.17) $c^d_{A,m}{}^* = c^d_{A,m}(p_A{}^*, p_B{}^*)$, $m = 1, 2, \dots, M$

(2.18) $c^d_{B,m}{}^* = c^d_{B,m}(p_A{}^*, p_B{}^*)$, $m = 1, 2, \dots, M$

(2.19) $c^s_{A,k}{}^* = c^s_{A,k}(p_A{}^*)$, $k = 1, 2, \dots, K$.

(2.20) $c^s_{B,l}{}^* = c^s_{B,l}(p_B{}^*)$, $l = 1, 2, \dots, L$.

The simultaneous equations (2.15) and (2.16), however, are not independent each other. On the distributive side of production factor markets, consumers' income consists of the factor payments from producers and profits. Therefore, the following identity must hold.

(2.21) $$\sum_{m=1}^{M} w_m = \sum_{k=1}^{K} [h_{A,k}(c^s_{A,k}) + \pi_{A,k}] + \sum_{l=1}^{L} [h_{B,l}(c^s_{B,l}) + \pi_{B,l}]$$

[1] $u_m\ (c^d_{A,m}, c^d_{B,m})$ is continuous, strictly increasing, and strongly quasi-concave with respect to $\{c^d_{A,m}, c^d_{B,m}\}$. See, for example, Varian (1992).

By (2.7) and (2.10), (2.21) is rewritten as

$$(2.22)\quad \sum_{m=1}^{M} w_m = \sum_{k=1}^{K} [p_A c^s_{A,k}] + \sum_{l=1}^{L} [p_B c^s_{B,l}]$$
$$= p_A C^s_A + p_B C^s_B$$

On the other hand, the aggregation of every consumer's budget constraint (2.2) gives the following identity.

$$(2.23)\quad \sum_{m=1}^{M} [p_A c^d_{A,m} + p_B c^d_{B,m}] = p_A C^d_A + p_B C^d_B = \sum_{m=1}^{M} w_m$$

By (2.22) and (2.23), the following identity must hold.

$$(2.24)\quad p_A(C^d_A - C^s_A) + p_B(C^d_B - C^s_B) = 0$$

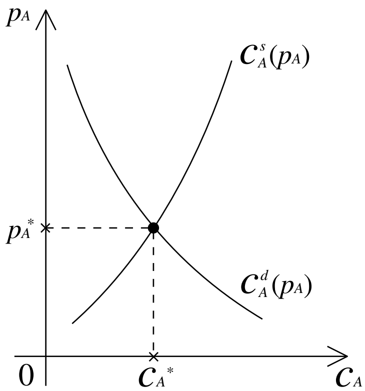
Although there are two markets, one for good A and the other for good B, (2.24) implies that if aggregate demand and aggregate supply are equal in one market, so are they in the other market. This observation is known as Walras' law. Walras' law in more general situation is described as follows. suppose there are $N \geq 2$ markets. If aggregate demand is equal to aggregate supply in each of N-1 markets, then so are they in the remaining Nth market. Walras' law (2.24) implies that (2.15) and (2.16) are linearly dependent. One cannot obtain a unique solution $\{p_A, p_B\}$ for (2.15) and (2.16). One way to calculate the general equilibrium prices $\{p_A, p_B\}$ is to use an additional condition. For example, one may use good B as a measure of accounting. This implies

$$(2.25)\quad p_B = 1.$$

In other words, good be B is chosen as a numeraire. Then, we can solve either (2.15) or (2.16) for the general equilibrium price p_A*. In this case p_A* is the number of good B that is exchanged for one unit of good A.

Figure 2.1 is a graphical exposition of equation (2.15). The horizontal axis measures the amount of good A, and the vertical axis measures the price of good A. The downward-sloping curve labeled $C^d_A(p_A)$ is the graph of aggregate demand for good A as a decreasing function of p_A, and the upward-sloping curve labeled $C^s_A(p_A)$ is the graph of aggregate supply of good A as an increasing function of p_A.The intersection of the demand curve and the supply curve determines the equilibrium price p_A* where the

Figure 2.1 Good A market Eguilibrium



aggregate demand and the aggregate supply are equated at $C^d_A(p_A*) = C^s_A(p_A*)$.

A general equilibrium in more generalized model is defined as follows. In the model, there are $m = 1, 2, \ldots, M$ consumers and $k = 1, 2, \ldots, K$ producers. A representative consumer m consumes N types of goods, denoted as

$$(2.26)\quad \tilde{y}^d_m \equiv (y^d_{m,1}, y^d_{m,2}, \cdots, y^d_{m,n}, \cdots, y^d_{m,N})$$

which is a N-dimensional vector, and supplies L types of factors of production, denoted as

$$(2.27)\quad \tilde{x}^s_m \equiv (x^s_{m,1}, x^s_{m,2}, \cdots, x^s_{m,l}, \cdots, x^s_{m,L})$$

which is a L-dimensional vector. The consumer chooses $\{\tilde{x}^s_m, \tilde{y}^d_m\}$ to maximize utility

$$(2.28)\quad u_m(\tilde{x}^s_m, \tilde{y}^d_m)$$

subject to budget constraint

$$(2.29)\quad \tilde{p}_y \tilde{y}^d_m \leq \tilde{p}_x \tilde{x}^s_m .$$

The right-hand side of (2.29) is factor income and the left-hand side of (2.29) is consumption expenditure.

$$(2.30)\quad \tilde{p}_y \equiv (p_{y,1}, p_{y,2}, \cdots, p_{y,n}, \cdots, p_{y,N})$$

is the N-dimensional vector of factor prices, and

$$(2.31)\quad \tilde{p}_x \equiv (p_{x,1}, p_{x,2}, \cdots, p_{x,l}, \cdots, p_{x,L})$$

is the L-dimensional vector of goods prices. The budget constraint (2.29) is rewritten as

(2.32)

$$p_{y,1}\,y^d_{m,1} + p_{y,2}\,y^d_{m,2} + \dots + p_{y,n}\,y^d_{m,n} + \dots + p_{y,N}\,y^d_{m,N} = p_{x,1}\,x^s_{m,1} + p_{x,2}\,x^s_{m,2} + \dots + p_{x,l}\,x^s_{m,l} + \dots + p_{x,L}\,x^s_{m,L}\,.$$

The solution to this constrained optimization problem consists of a N-dimensional vector of demand for goods as a function of prices, denoted as

(2.33) $\tilde{y}^d_m(\tilde{p}_x, \tilde{p}_y)$,

and a L-dimensional vector of supply of production factors, denoted as

(2.34) $\tilde{x}^s_m(\tilde{p}_x, \tilde{p}_y)$.

The aggregate demand for goods, which is also a N-dimensional vector, is calculated by summing the individual demand (2.33) with respect to $m = 1, 2, \dots, M$ as follows.

(2.35) $\tilde{Y}^d(\tilde{p}_x, \tilde{p}_y) \equiv \sum_{m=1}^{M} \tilde{y}^d_m(\tilde{p}_x, \tilde{p}_y)$

Similarly, the aggregate supply of production factors, which is also a L-dimensional vector, is calculated by summing the individual factor supply (2.34) with respect to $m = 1, 2, \dots, M$ as follows.

(2.36) $\tilde{X}^s(\tilde{p}_x, \tilde{p}_y) \equiv \sum_{m=1}^{M} \tilde{x}^s_m(\tilde{p}_x, \tilde{p}_y)$

A representative producer k demands L types of factors of production, denoted as

(2.37) $\tilde{x}^d_k \equiv (\,x^d_{k,1}\,,\ x^d_{k,2}\,, \dots,\ x^d_{k,l}\,, \dots,\ x^d_{k,L}\,)$

which is a L-dimensional vector, and supplies N types of goods denoted as

(2.38) $\tilde{y}^s_k \equiv (\,y^s_{k,1}\,,\ y^s_{k,2}\,, \dots,\ y^s_{k,n}\,, \dots,\ y^s_{k,N}\,)$

which is a N-dimensional vector. The producer chooses $\{\tilde{x}^d_k, \tilde{y}^s_k\}$ to maximize profit

(2.39) $\pi_k \equiv \tilde{p}_y \tilde{y}^s_k - \tilde{p}_x \tilde{x}^d_k$

subject to production technology constraint

(2.40) $\tilde{y}^s_k \in \Gamma(\tilde{x}^d_k)$

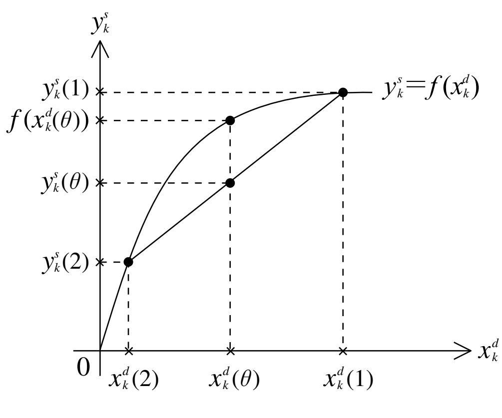
where $\Gamma(\tilde{x}^d_k)$ is a N-dimensional convex set defined on $\tilde{x}^d_k \in \mathfrak{R}^N_+$. $\Gamma(\tilde{x}^d_k)$ satisfies the following properties. (i) Choose two non-negative real vectors $\tilde{x}^d_k(1) \in \mathfrak{R}^N_+$ and $\tilde{x}^d_k(2) \in \mathfrak{R}^N_+$, and a scalar $\theta \in [0, 1]$. If $\tilde{x}^d_k(1) \geq \tilde{x}^d_k(2)$, then there exist $\tilde{y}^s_k(1) \in \Gamma(\tilde{x}^d_k(1))$ and $\tilde{y}^s_k(2) \in \Gamma(\tilde{x}^d_k(2))$ such that $\tilde{y}^s_k(1) \geq \tilde{y}^s_k(2)$. (ii) Define $\tilde{x}^d_k(\theta) \equiv \theta\, \tilde{x}^d_k(1) + (1-\theta)\, \tilde{x}^d_k(2)$ and $\tilde{y}^s_k(\theta) \equiv \theta\, \tilde{y}^s_k(1) + (1-\theta)\, \tilde{y}^s_k(2)$. Because $\Gamma(\tilde{x}^d_k)$ is a convex set defined on $\tilde{x}^d_k \in \mathfrak{R}^N_+$, $\tilde{y}^s_k(\theta) \in \Gamma(\tilde{x}^d_k(\theta))$. ($\tilde{x}^d_k(1) \geq \tilde{x}^d_k(2)$ implies $x^d_{k,n}(1) \geq x^d_{k,n}(2)$ for all $n = 1, 2, \dots, N$, and $x^d_{k,n}(1) > x^d_{k,n}(2)$ holds for at least one n.)

The constraint implies that the production technology exhibits diminishing returns. For example, if $N = 1$ and $L = 1$, then (2.40) is a familiar production function

(2.41) $y^s_k = f(x^d_k)$

where $f(x^d_k)$ is an increasing concave function. For two non-negative real numbers $x^d_k(1) \geq 0$ and $x^d_k(2) \geq 0$, if $x^d_k(1) > x^d_k(2)$, then $y^s_k(1) \equiv f(x^d_k(1)) > y^s_k(2) \equiv f(x^d_k(2))$. Define $x^d_k(\theta) \equiv \theta\, x^d_k(1) + (1-\theta)\, x^d_k(2)$ and $y^s_k(\theta) \equiv \theta\, y^s_k(1) + (1-\theta)\, y^s_k(2)$, where $\theta \in [0, 1]$. Then $y^s_k(\theta) \leq f(x^d_k(\theta))$. These properties of the production function is depicted by figure 2.2.

Figure 2.2 Production Function



The solution to the producer's profit maximization problem consists of a L-dimensional vector of demand for production factors as a function of prices denoted as

(2.42) $\tilde{x}^d_k(\tilde{p}_x, \tilde{p}_y)$,

and a N-dimensional vector of supply of goods denoted as

(2.43) $\tilde{y}^s_k(\tilde{p}_x, \tilde{p}_y)$.

The aggregate demand for production factors, which is also a L-dimensional vector, is calculated by summing the individual demand (2.42) with respect to $k = 1, 2, \ldots, K$ as follows.

$$(2.44)\quad \tilde{X}^d(\tilde{p}_x, \tilde{p}_y) \equiv \sum_{k=1}^{K} \tilde{x}_k^d(\tilde{p}_x, \tilde{p}_y) \ .$$

Similarly, the aggregate supply of goods, which is also a N-dimensional vector, is calculated by summing the individual supply (2.43) with respect to $k = 1, 2, \ldots, K$ as follows.

$$(2.45)\quad \tilde{Y}^s(\tilde{p}_x, \tilde{p}_y) \equiv \sum_{k=1}^{K} \tilde{y}_k^s(\tilde{p}_x, \tilde{p}_y) \ .$$

A general equilibrium of this economy is a set of prices and quantities $\{\tilde{p}_x{}^*, \tilde{p}_y{}^*, \{\tilde{x}_m^s{}^*, \tilde{y}_m^d{}^*\ ; m = 1, 2, \ldots, M\}, \{\tilde{x}_k^d{}^*, \tilde{y}_k^s{}^*\ ; k = 1, 2, \ldots, K\}\}$ such that (i) given the prices $\{\tilde{p}_x{}^*, \tilde{p}_y{}^*\}$, the quantities maximize every consumer's utility at $u_m(\tilde{x}_m^s{}^*, \tilde{y}_m^d{}^*)$ subject to budget constraint $\tilde{p}_y{}^* \tilde{y}_m^d \leq \tilde{p}_x{}^* \tilde{x}_m^s{}^*$ as well as every producer's profit at $\pi_k{}^* = \tilde{p}_y{}^* \tilde{y}_k^s - \tilde{p}_x{}^* \tilde{x}_k^d{}^*$ subject to technology constraint $\tilde{y}_k^s \in \Gamma(\tilde{x}_k^d)$, and (ii) the prices $\{\tilde{p}_x{}^*, \tilde{p}_y{}^*\}$ equate aggregate demand and aggregate supply in every market, i.e.,

$$(2.46)\quad \sum_{m=1}^{M} \tilde{y}_m^d{}^* = \sum_{k=1}^{K} \tilde{y}_k^k{}^* \ , \text{ and}$$

$$(2.47)\quad \sum_{k=1}^{K} \tilde{x}_k^d{}^* = \sum_{m=1}^{M} \tilde{x}_m^s{}^* \ .$$

The above definition implies that the general equilibrium prices $\{\tilde{p}_x{}^*, \tilde{p}_y{}^*\}$ are solutions to the following simultaneous equations system with respect to $\{\tilde{p}_x, \tilde{p}_y\}$.

$$(2.48)\quad \tilde{Y}^d(\tilde{p}_x, \tilde{p}_y) = \tilde{Y}^s(\tilde{p}_x, \tilde{p}_y)$$

$$(2.49)\quad \tilde{X}^d(\tilde{p}_x, \tilde{p}_y) = \tilde{X}^s(\tilde{p}_x, \tilde{p}_y)$$

(2.48) and (2.49) are a system of L + N equations with respect to L + N unknowns $\{\tilde{p}_x, \tilde{p}_y\}$. Like the two goods economy example, however, these L + N equations are linearly dependent because of Walras' law. Therefore, an additional condition must be imposed to calculate the equilibrium prices $\{\tilde{p}_x{}^*, \tilde{p}_y{}^*\}$. An example of the additional condition is to set

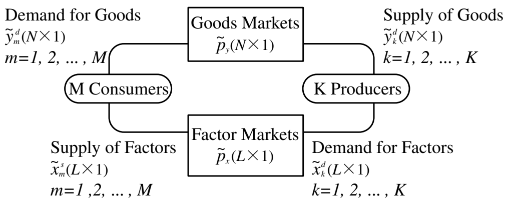
$$(2.50)\quad p_{y,1} = 1,$$

which implies that the pr ice of the first production factor is chosen as a numeraire.

Figure 2.3 depicts how the markets work in this model economy. In the goods market, N types of goods are traded. Given the prices $\tilde{p}_y$, each consumer $m = 1, 2, \ldots, M$ posts the demand for goods $\tilde{y}_m^d$, and each producer $k = 1, 2, \ldots, K$ posts the supply of goods $\tilde{y}_k^s$. In the production factors markets, L types of factors are traded. Given the factor prices $\tilde{p}_x$, each producer $k = 1, 2, \ldots$, K posts the demand for factors $\tilde{x}_k^d$, and each consumer $m = 1, 2, \ldots, M$ posts the supply of factors $\tilde{x}_m^s$. In the general equilibrium, prices are set at the level such that the aggregate demand and the aggregate supply are equated in every L + N markets.

In the following sections, we will analyze dynamic general equilibrium of dynamic economic models. It will be understood through specific examples that the dynamic general equilibrium is a natural extension of the general equilibrium of static economic models to models in which the decision makings of economic agents involve not only the types of goods and factors in one point of time but also across time.

Figure 2.3



3. The Analyses of Difference Equations and Differential Equations.

It will be shown that the analyses of dynamic general equilibrium are conducted through the analyses of difference equations in discrete time models, and through the analyses of differential equations in continuous models. In this section, we make a brief review of the analyses of difference equations and differential equations.

3-1. The Analyses of Difference Equations.

A difference equation describes the motions of variables across discrete time periods. For an expositional purpose, consider the following difference equation with respect to x_t.

$$(3.1)\quad f(x_{t+1}, x_t) = x_{t+1} - \frac{1}{2}x_t - 1 = 0$$

The subscript t expresses discrete time periods $t \in (-\infty, \ldots, -2, -1, 0, 1, 2, \ldots, \infty)$. Specifically, (3.1) is said to be a first-order linear non-homogenous difference equation with respect to x_t. A solution to the difference equation (3.1) is a function of time $x_t = g(t)$ that satisfies (3.1), i.e.,

$$(3.2)\quad x_{t+1} - \frac{1}{2}x_t - 1 = g(t+1) - \frac{1}{2}g(t) - 1 = 0$$

for all $t \in (-\infty, \ldots, -2, -1, 0, 1, 2, \ldots, \infty)$. For example, we may try if

$$(3.3)\quad x_t = g(t) = 2t$$

satisfies (3.1) for all t. The answer is no because

$$(3.4)\quad x_{t+1} - \frac{1}{2}x_t - 1 = [2(t+1)] - \frac{1}{2}(2t) - 1 = t + 1.$$

Therefore, $g(t) = 2t$ does not satisfy (3.1) unless $t = -1$. How about the following candidate function of time?

$$(3.5)\quad x_t = g(t) = 2 - \left(\frac{1}{2}\right)^t$$

We can verify that (3.5) is a solution to (3.1) because

$$(3.6)\quad x_{t+1} - \frac{1}{2}x_t - 1 = \left[2 - \left(\frac{1}{2}\right)^{t+1}\right] - \frac{1}{2}\left[2 - \left(\frac{1}{2}\right)^t\right] - 1$$
$$= 0.$$

(3.5) is not the only solution to (3.1). Readers can verify that

$$(3.7)\quad x_t = g(t) = 2 + \left(\frac{1}{2}\right)^t$$

also satisfies (3.1). In fact, there are continuums of solutions for (3.1) which is obtained by the following recursive substitution method. Given x_0 at time $t = 0$, x_1 is calculated by using (3.1) as follows.

$$(3.8)\quad x_1 = \frac{1}{2}x_0 + 1$$

Then, at $t = 1$, x_2 is calculated again by using (3.1) as follows.

$$(3.9)\quad x_2 = \frac{1}{2}x_1 + 1$$

By eliminating x_1 from (3.8) and (3.9), x_2 is expressed as a function of initial value x_0 as follows.

$$(3.10)\quad x_2 = \frac{1}{2}x_1 + 1 = \frac{1}{2}\left[\frac{1}{2}x_0 + 1\right] + 1$$
$$= \left(\frac{1}{2}\right)^2 x_0 + 1 + \frac{1}{2}$$

By repetition, at an arbitrary time t, x_t is expressed as a function of x_0 and t as follows.

$$(3.11)\quad x_t = \left(\frac{1}{2}\right)^t x_0 + \sum_{s=0}^{t-1}\left(\frac{1}{2}\right)^s$$

Readers can verify that (3.11) satisfies (3.1) for any $x_0 \in \Re$ and any t. (3.11) is called a general solution of (3.1). On the other hand, if we specify the initial value x_0, (3.11) is called a specific solution of (3.1). For example, if $x_0 = 1$, then (3.11) is written as

$$(3.12)\quad x_t = \left(\frac{1}{2}\right)^t \times 1 + \sum_{s=0}^{t-1}\left(\frac{1}{2}\right)^s = 2 - \left(\frac{1}{2}\right)^t.$$

This implies that (3.5) is a specific solution of (3.1) with the initial value $x_0 = 1$. On the other hand, if $x_0 = 3$, then (3.11) is rewritten as

$$(3.13)\quad x_t = \left(\frac{1}{2}\right)^t \times 3 + \sum_{s=0}^{t-1}\left(\frac{1}{2}\right)^s = 2 + \left(\frac{1}{2}\right)^t.$$

This implies that (3.7) is a specific solution of (3.1) with the initial value $x_0 = 3$.

For any initial value $x_0 \in \Re$, the general solution (3.11) implies that x_t converges to 2 as t becomes larger, i.e.,

$$(3.14)\quad \lim_{t\to\infty} x_t = 2, \forall x_0 \in \Re.$$

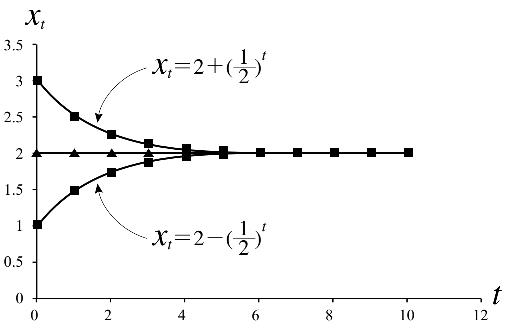
In fact, $x_t = 2$ is said to be a steady state of the difference equation (3.1). A steady state of the difference equation (3.1) is defined as a value of x_t such that x_t remains at the steady state as time t passes. By this definition, a steady state x^* of the difference equation (3.1) satisfies

$$(3.15)\quad f(x^*, x^*) = x^* - \frac{1}{2}x^* - 1 = 0.$$

From this, we obtain $x^* = 2$. Then, (3.14) implies that the steady state is globally stable. Figure 3.1 depicts the trajectory of the two specific solution of (3.1), one is (3.5) with initial value $x_0 = 1$, and the other is (3.7) with initial value $x_0 = 3$. Both solutions converge to the steady state $x^* = 2$. If the initial value x_0 is smaller than the steady state $x^* = 2$, then the solution is a monotonically increasing sequence $x_0 < x_1 < \ldots <$

$x_{t-1} < x_t < x_{t+1} <$... converging to the steady state. On the other hand, if the initial value x_0 is larger than the steady state $x^* = 2$, then the solution is a monotonically decreasing sequence $x_0 > x_1 >$ ... $> x_{t-1} > x_t > x_{t+1} >$... converging to the steady state.

Figure 3.1



The global stability of the steady state is depicted by using a phase-diagram. See figure 3.2. The horizontal axis measures x_t and the vertical axis measures x_{t+1}. On the (x_t, x_{t+1}) plane, there are two lines, one is a 45-degree line on which $x_{t+1} = x_t$ holds, and the other is the graph of the difference equation (3.1). The intersection of these two lines corresponds to the steady state $x^* = 2$ because it satisfies both $x_{t+1} = x_t$ and the difference equation. By using figure 3.2, the global stability of the steady state is demonstrated by the following steps.

Step 0. Pick an arbitrary initial value x_0. Then x_1 is found on the graph of (3.1).

Step 1. x_1 , measured on the vertical axis is projected on the horizontal axis by using 45-degree line.

Step 2. Then, x_2 is found on the graph of (3.1).

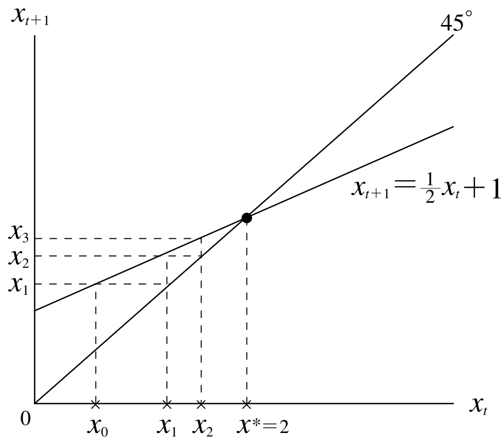
Step 3. x_2, measured on the vertical axis is projected on the horizontal axis by using 45-degree line.

Repeat these steps to generate the sequence $\{x_0, x_1, x_2, ...\}$. It is understood that for any initial value $x_0 \in \Re$, the solution converges to the steady state $x^* = 2$.

Example 1. Solow-Swan Economic Growth Model in Discrete Time.

A discrete time Solow-Swan model consists of the following equations. For $t = 0, 1, 2, ...$,

Figure 3.2



(3.16) $Y_t = AK_t^{\alpha} L_t^{1-\alpha}$, $0 < \alpha < 1$

(3.17) $Y_t = C_t + I_t$

(3.18) $L_{t+1} = (1 + n) L_t$

(3.19) $K_{t+1} = (1 - d) K_t + I_t$, $0 \leq d \leq 1$

(3.20) $I_t = sY_t$

(3.16) is a production function. Y_t is output, K_t is capital, L_t is labor. The constant parameter A in (3.16) expresses the level of technology. Given the factor inputs K_t and L_t , higher technology level A enables larger output Y_t. The constant parameter $\alpha \in (0, 1)$ is the share of capital in the production function. (3.16) is said to be a Cobb-Douglas production function. (3.17) implies that the output Y_t is divided between consumption C_t and investment I_t. (3.18) implies that the growth rate of labor L_t is n. (3.19) implies the capital accumulation process. By the end of time period t, $d \times 100$ % of capital K_t depreciates. The capital at the beginning of time period $t + 1$, K_{t+1}, is the leftover capital $(1 - d)$ K_t plus investment I_t. (3.20) is a investment function, implying that $s \times 100$ % of output Y_t is used for investment. s is the investment rate. In a closed economy where there is no lending to or borrowing from other economies, s is also the saving rate. s is called as well the propensity to invest or the propensity to save. By (3.17) and (3.20), the consumption C_t is $(1 - s) \times 100$ % of output, i.e.,

(3.21) $C_t = (1 - s) Y_t$.

1 − s is called the consumption rate or the propensity to consume.

These equations (3.16) ~ (3.21) are meant to express the economic activities at the national level. Therefore, the variables in (3.16) ~ (3.21) are understood as aggregate variables.

The analyses of the Solow-Swan model is conducted in 3 steps. In step 1, we derive a difference equation with respect to capital-labor ratio. In step 2, we calculate the steady state of the difference equation. In step 3, we analyze the stability of the steady state.

Step 1. Divide both side of (3.19) by L_{t+1}. By (3.16), (3.18), (3.19), and (3.20), we have

(3.22) $$\frac{K_{t+1}}{L_{t+1}} = \frac{(1-d)K_t}{(1+n)+L_t} + \frac{sAK_t^{\alpha}L_t^{1-\alpha}}{(1+n)L_t}.$$

(3.22) is rewritten as

(3.23) $$k_{t+1} = \left(\frac{1-d}{1+n}\right)k_t + \left(\frac{sA}{1+n}\right)k_t^{\alpha}$$

where k_t is capital-labor ratio defined as

(3.24) $k_t \equiv K_t / L_t$.

(3.23) is a nonlinear first-order difference equation with respect to k_t.

Step 2. At the steady state of (3.23), k_{t+1} and k_t are equal and constant at k_s. Therefore,

(3.25) $$k_s = \left(\frac{1-d}{1+n}\right)k_s + \left(\frac{sA}{1+n}\right)k_s^{\alpha}$$

holds. (3.25) is solved for the steady state k_s as follows.

(3.26) $$k_s = \left(\frac{sA}{n+d}\right)^{\frac{1}{1-\alpha}}$$

We have the following theorem with respect to the steady state (3.26).

Theorem 3.1. The steady state capital-labor ratio (3.26) is increasing in the technology level A and the investment rate s, while decreasing in the growth rate n of labor and the capital depreciation rate d.

Theorem 3.1 is expressed as follows.

(3.27) $k_s(\overset{+}{s}, \overset{+}{A}, \overset{-}{n}, \overset{-}{d})$

The sign (+ or −) above each parameter in the parenthesis of (3.27) implies the effect of an increase of the parameter on the steady state capital-labor ration k_s.

Step 3. The stability of steady state capital-labor ratio k_s is summarized by the following theorem.

Theorem 3.2. Given the initial capital labor ratio $k_0 \equiv K_0 / L_0$, (i) if $k_0 < k_s$, then the difference equation (3.23) generates a monotonically increasing sequence $k_0 < k_1 < ... < k_t < k_{t+1} < ...$ that converges to the steady state k_s, or (ii) if $k_0 > k_s$, then the difference equation (3.23) generates a monotonically decreasing sequence $k_0 > k_1 > ... > k_t > k_{t+1} > ...$ that converges to the steady state k_s

The convergence of capital-labor ratio to the steady state is expressed as

(3.28) $\lim_{t\to\infty} k_t = k_s, \forall k_0 \in \Re_{++}$.

Figure 3.3 depicts a graph of (3.23) where the horizontal axis measures k_t and the vertical axis measures k_{t+1}. Define the right-hand side of (3.23) as a function of k_t as follows.

(3.29) $$\psi(k_t) = \left(\frac{1-d}{1+n}\right)k_t + \left(\frac{sA}{1+n}\right)k_t^{\alpha}$$

$\Psi(k_t)$ has the following properties. $\Psi(0) = 0$. The slope of $\Psi(k_t)$ is

(3.30) $$\psi'(k_t) = \left(\frac{1-d}{1+n}\right) + \left(\frac{sA}{1+n}\right)\alpha k_t^{\alpha-1} > 0.$$

When k_t is small (large), $k_t^{\alpha-1}$ is large (small) because $\alpha \in (0, 1)$. Specifically, we have the followings.

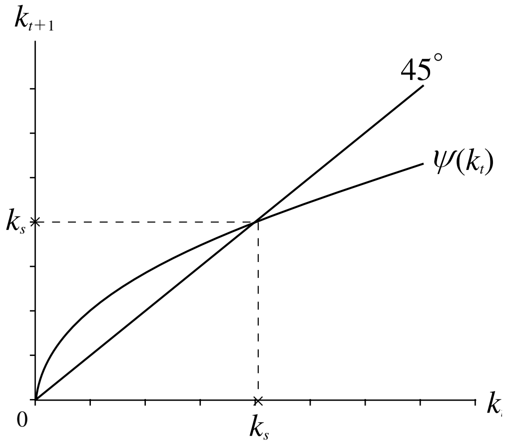
(3.31) $$\lim_{k_t\to 0}\psi'(k_t) = \infty \text{ and } \lim_{k_t\to\infty}\psi'(k_t) = \frac{1-d}{1+n} < 1.$$

In addition, $\Psi(k_t)$ is concave because

(3.32) $$\psi''(k_t) = \left(\frac{sA}{1+n}\right)\alpha(1-\alpha)k_t^{\alpha-2} < 0.$$

From these observations, we summarize that $\Psi(k_t)$ is increasing and concave function of k_t, and the graph of $\Psi(k_t)$ has a unique inter section $k_{t+1} = k_t \equiv k_s$ with a 45-degree line.

Figure 3.3



Given the initial capital-labor ratio k_0, we can trace the transition of k_t toward the steady state k_s. Figure 3.4 is a phase-diagram for a sequence $\{k_t\ ;\ t = 0, 1, 2, ...\}$ generated by the difference equation given an arbitrary initial capital-labor ratio k_0 which is assumed to be smaller than the steady state k_s. Given k_0, k_1 is found on the vertical axis at $k_1 = \Psi(k_0)$. By using the 45-degree line, we can project k_1 on the horizontal axis. Then, k_2 is found on the vertical axis at $k_2 = \Psi(k_1)$. By using the 45-degree line again, we can project k_2 on the horizontal axis. Then, k_3 is found on the vertical axis at $k_3 = \Psi(k_2)$. These steps are repeated to prove the first statement of theorem 3.2. Readers may prove the second statement of theorem 3.2 by starting with an arbitrary initial capital-labor ratio k_0 which is larger than the steady state k_s.

Figure 3.4

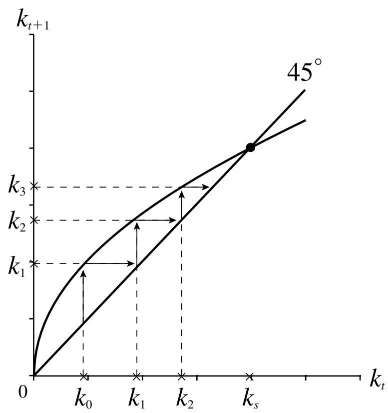
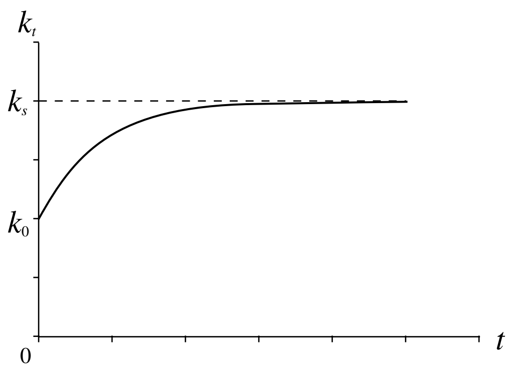


Figure 3.5 is a trajectory of capital-labor ratio $\{k_t\ ;\ t = 0, 1, 2, ...\}$ generated by the difference equation (3.23) with an initial capital-labor ratio k_0 which is smaller than the steady state k_s. In the figure, the horizontal axis measures time $t = 0, 1, ...$, and the vertical axis measures $\{k_t\ ;\ t = 0, 1, 2, ...\}$. The figure shows that the sequence $\{k_t\ ;\ t = 0, 1, 2, ...\}$ generated by the difference equation (3.23) is a monotonically increasing sequence converging to the steady state k_s.

Figure 3.5



Once the motion of capital-labor ratio k_t is characterized, the motions of other variables are characterized as well because they are related to the capital-labor ratio as follows. By definition, capital is calculated by

$$(3.33)\quad K_t = k_t L_t .$$

At the steady state, the capital becomes $K_t = k_s L_t$. Therefore, the growth rate of capital is equal to the growth rate n of labor. By rewriting the production function (3.16), output is calculated by

$$(3.34)\quad Y_t = AK_t^{\alpha} L_t^{1-\alpha} = A(K_t/L_t)^{\alpha} L_t = Ak_t^{\alpha} L_t .$$

At the steady state, $Y_t = Ak_s^{\alpha} L_t$. Therefore, the growth rate of output is also equal to the growth rate n of labor. Define the output per labor by

$$(3.35)\quad y_t \equiv Y_t/L_t .$$

(3.35) is also called the labor productivity. By (3.34) and (3.35), output per labor is calculated by

$$(3.36)\quad y_t = Y_t/L_t = Ak_t^{\alpha} .$$

At the steady state, output per labor is constant at

$$(3.37)\quad y_t = y_s = A k_s^{\alpha} .$$

Therefore, on the steady state of the basic Solow-Swan model, the growth rates of aggregate (macro) variables such as $\{K_t, Y_t\}$ are equal to the growth rate of labor, and the variables per worker such as $\{k_t, y_t\}$ become constants.

Example 2. Solow-Swan Economic Growth Model with Technological Progress.

The properties of the basic Solow-Swan model is insufficient to describe the economic growth of industrialized countries because the labor productivities of these countries seem to grow over extended periods of time. The reason for such a sustained growth in real world could be explained by technological progress. We can modify the basic Solow-Swan model to incorporate technological progress as follows. For $t = 0, 1, 2, \ldots$,

$$(3.38)\quad Y_t = A_t K_t^{\alpha} L_t^{1-\alpha} ,\quad 0 < \alpha < 1$$

$$(3.39)\quad A_{t+1} = (1+g)A_t ,\quad g \geq 0$$

$$(3.40)\quad Y_t = C_t + I_t$$

$$(3.41)\quad L_{t+1} = (1+n)L_t$$

$$(3.42)\quad K_{t+1} = (1-d)K_t + I_t ,\quad 0 \leq d \leq 1$$

$$(3.43)\quad I_t = sY_t$$

In the production function (3.38), the technology level A_t is not constant. As implied by (3.39), A_t grows at the growth rate g. The other equations (3.40) ~ (3.43) are same as the basic Solow-Swan model. If $g = 0$, then A_t is constant, and the model becomes the basic Solow-Swan model.

The analyses of the modified Solow-Swan model is also conducted in 3 steps that we followed before in the analyses of the basic model. In the analysis of the modified model, we need to define the variables so that they become constant at the steady state. Define the efficiency unit of labor by

$$(3.44)\quad \tilde{L}_t \equiv L_t A_t^{\frac{1}{1-\alpha}} .$$

(3.44) implies that labor is augmented by technology. Then, the production function (3.38) is rewritten as

$$(3.45)\quad Y_t = A_t K_t^{\alpha} L_t^{1-\alpha} = K_t^{\alpha}[L_t A_t^{\frac{1}{1-\alpha}}]^{1-\alpha} = K_t^{\alpha}\tilde{L}_t^{1-\alpha} .$$

(3.45) is called a production function with labor-augmenting technological progress. Define the capital-labor ratio in efficiency unit of labor by

$$(3.46)\quad \tilde{k}_t \equiv K_t/\tilde{L}_t = K_t/[L_t A_t^{\frac{1}{1-\alpha}}] .$$

Step 1. Divide both sides of (3.42) by $\tilde{L}_{t+1}$. By (3.38), (3.39), (3.41), (3.42), and (3.43), we have

$$(3.47)\quad \frac{K_{t+1}}{L_{t+1}A_{t+1}^{\frac{1}{1-\alpha}}} = \frac{(1-d)K_t}{(1+n)L_t[(1+g)A_t]^{\frac{1}{1-\alpha}}} + \frac{sA_t K_t^{\alpha} L_t^{1-\alpha}}{(1+n)L_t[(1+g)A_t]^{\frac{1}{1-\alpha}}} .$$

(3.47) is written in terms of $\tilde{k}_t$ as

$$(3.48)\quad \tilde{k}_{t+1} = \left[\frac{1-d}{(1+n)(1+g)^{\frac{1}{1-\alpha}}}\right]\tilde{k}_t + \left[\frac{s}{(1+n)(1+g)^{\frac{1}{1-\alpha}}}\right]\tilde{k}_t^{\alpha} .$$

Again, (3.48) is a nonlinear first-order difference equation with respect to $\tilde{k}_t$.

Step 2. At the steady state of (3.48), $\tilde{k}_{t+1}$ and $\tilde{k}_t$ are equal and constant at $\tilde{k}_s$. Therefore, we have

$$(3.49)\quad \tilde{k}_s = \left[\frac{1-d}{(1+n)(1+g)^{\frac{1}{1-\alpha}}}\right]\tilde{k}_s + \left[\frac{s}{(1+n)(1+g)^{\frac{1}{1-\alpha}}}\right]\tilde{k}_s^{\alpha} .$$

(3.49) is solved for the steady state $\tilde{k}_s$ as follows.

$$(3.50)\quad \tilde{k}_s = \left[\frac{s}{(1+n)(1+g)^{\frac{1}{1-\alpha}} + d - 1}\right]^{\frac{1}{1-\alpha}}$$

(3.50) implies the following theorem.

Theorem 3.3. The steady state capital-ratio in efficiency unit (3.50) is increasing in the investment rate s, while decreasing in the growth rate n of labor, the capital depreciation rate d, and the growth rate g of technological progress.

Theorem 3.3 is expressed as follows.

$$(3.51)\quad \tilde{k}_s(\overset{+}{s}, \overset{-}{n}, \overset{-}{d}, \overset{-}{g})$$

In addition, like theorem 3.2 of the basic Solow-Swan model, we have the following theorem with respect to the global stability of the steady state $\tilde{k}_s$.

Theorem 3.4. Given the initial capital-labor ratio in efficiency unit $\tilde{k}_0 \equiv K_0/[L_0 A_0^{\frac{1}{1-\alpha}}]$, (i) if $\tilde{k}_0 < \tilde{k}_s$, then the difference equation (3.48) generates a monotonically increasing sequence $\tilde{k}_0 < \tilde{k}_1 < \ldots < \tilde{k}_t < \tilde{k}_{t+1} < \ldots$ that converges to the steady state $\tilde{k}_s$, or (ii) if $\tilde{k}_0 > \tilde{k}_s$, then the difference equation (3.48) generates a monotonically decreasing sequence $\tilde{k}_0 > \tilde{k}_1 > \ldots > \tilde{k}_t > \tilde{k}_{t+1} > \ldots$ that converges to the steady state $\tilde{k}_s$

The convergence of capital-labor ratio to the steady state is expressed as

$$(3.52)\quad \lim_{t\to\infty} \tilde{k}_t = \tilde{k}_s, \forall \tilde{k}_0 \in \Re_{++}.$$

Like the proof of theorem 3.2, the proof of theorem 3.4 is obtained through the analysis of the difference equation (3.48). The right-hand side of (3.48) is increasing and concave with respect to $\tilde{k}_t$ with the slope approaching to infinite as $\tilde{k}_t$ approaching to zero, and the slope approaching to $(1 - d)/[(1 + n)(1 + g)^{1/(1-\alpha)}] < 1$ as $\tilde{k}_t$ approaching to infinity.

Step 3. Once the motion of capital-labor ratio in efficiency unit $\tilde{k}_t$ is characterized, the motion of the other variables are characterized as well because, like those in the basic Solow-Swan model, they are related to $\tilde{k}_t$ as follows. By definition, the capital is calculated by

$$(3.53)\quad K_t = \tilde{k}_t \tilde{L}_t = \tilde{k}_t [L_t A_t^{\frac{1}{1-\alpha}}] .$$

Given L_0 and A_0, by the repeated substitutions of (3.39) and (3.41), we have the following expressions for A_t and L_t.

$$(3.54)\quad A_t = (1+g)^t A_0$$

$$(3.55)\quad L_t = (1+n)^t L_0$$

Then, the labor in efficiency unit is expressed as follows.

$$(3.56)\quad \tilde{L}_t = L_t A_t^{\frac{1}{1-\alpha}} = (1+n)^t L_0 \times [(1+g)^t A_0]^{\frac{1}{1-\alpha}}$$
$$= [(1+n)(1+g)^{\frac{1}{1-\alpha}}]^t L_0 A_0^{\frac{1}{1-\alpha}} = \Gamma^t L_0 A_0^{\frac{1}{1-\alpha}}$$

In (3.56),

$$(3.57)\quad \Gamma \equiv (1+n)(1+g)^{\frac{1}{1-\alpha}}$$

is the growth rate of the labor in efficiency unit. By (3.53) and (3.56),

$$(3.58)\quad K_t = \tilde{k}_t \Gamma^t L_0 A_0^{\frac{1}{1-\alpha}} .$$

At the steady state, $\tilde{k}_t$ is constant at $\tilde{k}_s$. Then (3.58) implies that the growth rate of capital at the steady state is Γ. By (3.38), (3.46), and (3.56), the production function is rewritten as follows.

$$(3.59)\quad Y_t = K_t^\alpha \tilde{L}_t^{1-\alpha} = (K_t/\tilde{L}_t)^\alpha \tilde{L}_t = \tilde{k}_t^\alpha \tilde{L}_t$$
$$= \tilde{k}_t^\alpha \Gamma^t L_0 A_0^{\frac{1}{1-\alpha}}.$$

At the steady state,

$$(3.60)\quad Y_t = \tilde{k}_s^\alpha \Gamma^t L_0 A_0^{\frac{1}{1-\alpha}}.$$

Therefore, the growth rate of output in the steady state also is Γ. By definition, the capital-labor ratio is expressed as follows.

$$(3.61)\quad k_t = K_t/L_t = \tilde{k}_t \tilde{L}_t / L_t$$
$$= \tilde{k}_t A_t^{\frac{1}{1-\alpha}} = \tilde{k}_t [(1+g)^t A_0]^{\frac{1}{1-\alpha}}$$
$$= \tilde{k}_t [(1+g)^{\frac{1}{1-\alpha}}]^t A_0^{\frac{1}{1-\alpha}} = \gamma^t \tilde{k}_t A_0^{\frac{1}{1-\alpha}}$$

In (3.61),

$$(3.62)\quad \gamma \equiv (1+g)^{\frac{1}{1-\alpha}}.$$

At the steady state, $\tilde{k}_t$ is constant at $\tilde{k}_s$. Then (3.61) implies that the growth rate of capital-labor ratio at the steady state is γ. By (3.44), (3.54), and (3.59), the output per labor (labor productivity) is calculated by

$$(3.63)\quad y_t = Y_t/L_t = \tilde{k}_t^\alpha \tilde{L}_t / L_t = \tilde{k}_t^\alpha A_t^{\frac{1}{1-\alpha}} = \gamma^t \tilde{k}_t^\alpha A_0^{\frac{1}{1-\alpha}}.$$

At the steady state,

$$(3.64)\quad y_t = \gamma^t \tilde{k}_s^\alpha A_0^{\frac{1}{1-\alpha}} .$$

Therefore, the growth rate of output per labor at the steady state also is γ.

We can summarize that, at the steady state of the Solow-Swan model with technological progress, the growth rates of the aggregate (macro) variables such as $\{K_t, Y_t\}$ are $\Gamma = (1 + n)(1 + g)^{1/(1-\alpha)}$, and the growth rates of the variables per labor such as $\{k_t, y_t\}$ are $\gamma = (1 + g)^{1/(1-\alpha)}$. Unlike the basic Solow-Swan model, the variables per labor grow at the steady state because of technological progress $g > 0$.

Numerical Simulation of the Basic Solow-Swan Model.

Readers can numerically simulate the Solow-Swan models. As an example, we present a numerical simulation of the basic Solow-Swan model. The simulation is conducted through the following 3 steps.

Step 1. Specify the values of parameters and initial variables of the model. In this simulation example, we specify these values as follows; $\{A = 10, s = 0.2, \alpha = 0.3, n = 0.02, d = 0.1\}$ and $\{K_0 = 150, L_0 = 100\}$. Therefore, the initial capital-labor ratio is $k_0 = K_0 / L_0 = 1.5$. By (3.26), the steady state capital-labor ratio is $k_s = 55.65$ which is larger than the initial capital-labor ratio $k_0 = 1.5$. Therefore, the difference equation (3.23) will generate an monotonically increasing sequence of capital-labor ratio $\{k_t\ ;\ t = 0, 1, 2, \ldots\}$ which converges to k_s. Given $L_0 = 100$, the sequence of labor $\{L_t\ ;\ t = 0, 1, 2, \ldots\}$ is also calculated by (3.18).

Step 2. By (3.16), (3.23), (3.33), and (3.35), the values of variables at $t = 1$ are calculated as follows.

$$(3.65)\quad k_1 = \left(\frac{1-d}{1+n}\right)k_0 + \left(\frac{sA}{1+n}\right)k_0^{\alpha}$$

$$(3.66)\quad K_1 = k_1 L_1$$

$$(3.67)\quad Y_1 = A K_1^{\alpha} L_1^{1-\alpha}$$

$$(3.68)\quad y_1 = Y_1 / L_1$$

Step 3. The values of variables for $t = 2, 3, 4, \ldots$ are calculated by repetitions of step 2. Table 3.1 shows the values of variables $\{L_t, K_t, k_t, Y_t, y_t\}$ for $t = 0, 1, 2, \ldots$, the growth rate of k_t (labeled as Dk(t)/k(t)), the growth rate of Y_t (labeled as DY(t)/Y(t)), and the growth rate of y_t (labeled as Dy(t)/y(t)).

Figure 3.6 depicts the trajectory of capital-labor ratio k_t. The figure shows that, starting with $k_0 = 1.5$, the capital-labor ratio k_t is a monotonically increasing sequence converging to the steady state $k_s = 55.65$.

Table 3.1 Simulation of the Basic Solow-Swan Model.

t	L (t)	K (t)	Y (t)	C (t)	I (t)	y (t)	r (t)	w (t)	G [Y (t)]	g [y (t)]
0	100.00	150.00	1129.35	903.48	225.87	11.29	2.26	7.91		
1	102.00	360.87	1490.13	1192.11	298.03	14.61	1.24	10.23	0.319	0.294
2	104.04	622.81	1779.70	1423.76	355.94	17.11	0.86	11.97	0.194	0.171
3	106.12	916.47	2026.26	1621.01	405.25	19.09	0.66	13.37	0.139	0.116
4	108.24	1230.07	2244.19	1795.35	448.84	20.73	0.55	14.51	0.108	0.086
5	110.41	1555.90	2441.72	1953.37	488.34	22.12	0.47	15.48	0.088	0.067
6	112.62	1888.66	2624.02	2099.21	524.80	23.30	0.42	16.31	0.075	0.054
7	114.87	2224.60	2794.58	2235.66	558.92	24.33	0.38	17.03	0.065	0.044
8	117.17	2561.05	2955.88	2364.70	591.18	25.23	0.35	17.66	0.058	0.037
9	119.51	2896.12	3109.75	2487.80	621.95	26.02	0.32	18.21	0.052	0.031
10	121.90	3228.46	3257.62	2606.09	651.52	26.72	0.30	18.71	0.048	0.027
11	124.34	3557.14	3400.57	2720.45	680.11	27.35	0.29	19.14	0.044	0.023
12	126.82	3881.54	3539.51	2831.60	707.90	27.91	0.27	19.54	0.041	0.020
13	129.36	4201.28	3675.16	2940.13	735.03	28.41	0.26	19.89	0.038	0.018
14	131.95	4516.19	3808.14	3046.51	761.63	28.86	0.25	20.20	0.036	0.016
15	134.59	4826.20	3938.98	3151.18	787.80	29.27	0.24	20.49	0.034	0.014

Benchmark Case: {A = 10, s = 0.2, alpha = 0.3, n = 0.02, d = 0.1, L (0)= 100, K (0)= 150}

Figure 3.6

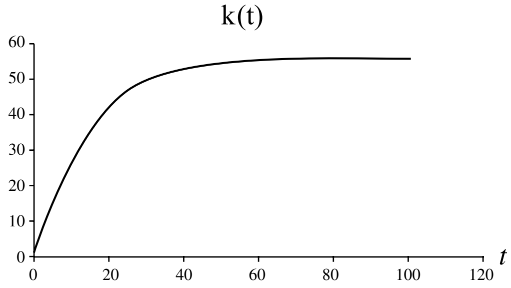


Figure 3.7 depicts the trajectory of output Y_t, and figure 3.8 depicts the growth rate of output DY(t)/Y(t). Along the transition path toward the steady state, the growth rate of Y_t is positive but decreasing toward the steady state growth rate which is equal to the growth rate $n = 0.02$ of labor.

Figure 3.7

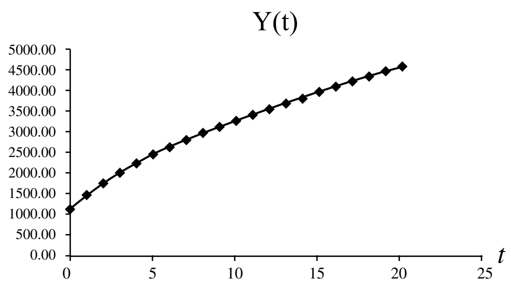


Figure 3.8

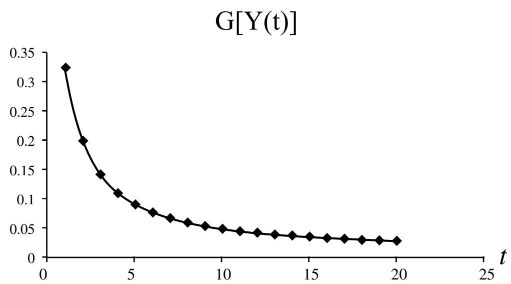


Figure 3.9 depicts the trajectory of output per labor (labor productivity) y_t. Because $y_t = Ak_t^{\alpha}$, the shape of the trajectory of labor productivity reflects the motion of capital-labor ratio k_t. $\{y_t \,;\, t = 0, 1, 2, \ldots\}$ is a monotonically increasing sequence converging to the steady state $y_s = Ak_s^{\alpha}$. Figure 3.10 depicts the growth rate of output per labor Dy(t)/y(t). Along the transition path toward the steady state, the growth rate of y_t is positive but decreasing toward zero because y_t becomes constant y_s at the steady state.

Figure 3.9

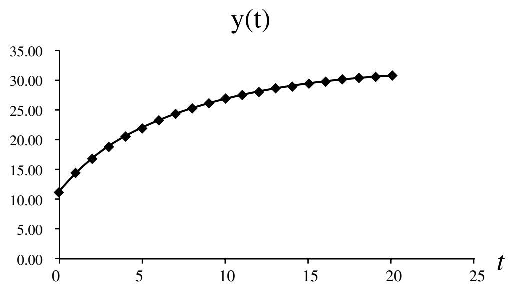
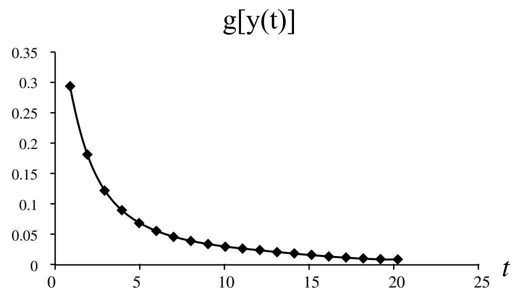


Figure 3.10



3-2. The Analyses of Differential Equations.

A differential equation, like a difference equation, describes the motions of variables. However, time flows continuously $t \in \Re = (-\infty, \infty)$. Like we did for an expositional purpose in the analyses of difference equations in 3-1, we start with a specific example of differential equation which is given as follows.

$$(3.69) \quad f(\dot{x}, x) = \dot{x} + x - 1 = 0$$

In (3.69), x is a function of time, and $\dot{x} \equiv dx/dt$. $\dot{x} > 0$ implies that x increases as time t increases, $\dot{x} < 0$ implies that x decreases as time t increases, and $\dot{x} = 0$ implies that x is constant. (3.69) is said to be a first-order linear non-homogenous differential equation with respect to x. A solution to the differential equation(3.69) is a function of time $x = h(t)$ that satisfies (3.69). We may ask if a candidate function

$h(t)$ satisfies (3.69), i.e.,

$$(3.70)\quad \dot{x} + x - 1 = \frac{dh(t)}{dt} + h(t) - 1 = 0$$

holds for all $t \in \Re$. For example, we may try if

$$(3.71)\quad x = h(t) = \frac{1}{2}t^2 + 2$$

satisfies (3.69) for all $t \in \Re$. The answer is no because

$$(3.72)\quad \dot{x} + x - 1 = t + [\frac{1}{2}t^2 + 2] - 1 = \frac{1}{2}t^2 + t + 1$$

is not zero for all $t \in \Re$. How about the following candidate?

$$(3.73)\quad x = h(t) = e^{-t} + 1$$

We can verify that (3.73) is a solution to (3.69) because

$$(3.74)\quad \dot{x} + x - 1 = [-e^{-t}] + [e^{-t} + 1] - 1 = 0.$$

This holds for all $t \in \Re$. (3.73) is not the only solution to (3.69). Readers can verify that

$$(3.75)\quad x = h(t) = -e^{-t} + 1$$

also satisfies (3.69). In fact, like what we saw in the analyses of difference equations, there are continuums of solutions for (3.69). A general solution of (3.69) is given by

$$(3.76)\quad x(t) = (x(0) - 1)e^{-t} + 1$$

where $x(0)$ is the initial value of $x(t)$. Readers man verify that (3.76) indeed satisfies (3.69) for any $x(0) \in \Re$ and any $t \in \Re$. On the other hand, if we specify the initial value $x(0)$, (3.76) is called a specific solution of (3.69). For example, if $x(0) = 2$, then (3.76) becomes (3.73). This implies that (3.73) is a specific solution of (3.69) with the initial value $x(0) = 2$. Furthermore, if $x(0) = 0$, then (3.76) becomes (3.75). This implies that (3.75) is a specific solution of (3.69) with the initial value $x(0) = 0$.

A steady state of a difference equation is defined as the value of $x(t)$ where it becomes constant $x(t) = x^*$. Because $x(t)$ becomes constant at the steady state, $\dot{x} = 0$. At the steady state, hence, the differential equation (3.69) satisfies

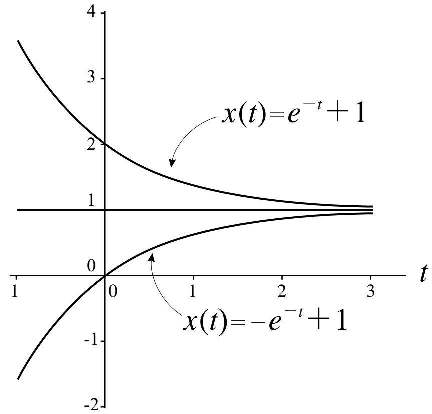
$$(3.77)\quad x^* = 1.$$

For any initial value $x(0) \in \Re$, the general solution (3.76) implies that $x(t)$ converges to the steady state $x^* = 1$ as t gets larger, i.e.,

$$(3.78)\quad \lim_{t\to\infty} x(t) = 1, \forall x(0) \in \Re.$$

These observations imply that the steady state x^* is globally stable. Figure 3.11 depicts the trajectories of two specific solutions of (3.69), one with the initial value $x(0) = 2$, and the other with $x(0) = 0$.

Figure 3.11



We can also use a phase-diagram to analyze the global stability of the steady state of the differential equation (3.69). In this example, the phase-diagram of (3.69) is a real line $x(t) \in \Re$. See Figure 3.12. The differential equation (3.69) implies that

$$(3.79)\quad x(t) \gtreqless 1 \Leftrightarrow \dot{x} \lesseqgtr 0.$$

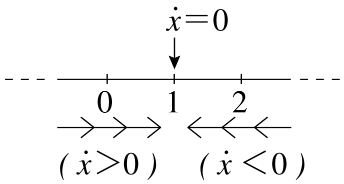
In figure 3.12, there are two arrows. The one that is labeled $\dot{x} > 0$ implies that when $x(t)$ is larger than the steady state $x^* = 1$, it decreases monotonically toward the steady state. The other arrow that is labeled $\dot{x} < 0$ implies that when $x(t)$ is smaller than the steady state $x^* = 1$, it increases monotonically toward the steady state. $x(t)$ becomes constant only if it is on the steady state $x^* = 1$.

Example 3. Solow-Swan Economic Growth Model in Continuous Time.

As an example for the analyses of differential equation applied to economic model, we use a Solow-

Figure 3.12

Phase-Diagram of $\dot{x}+x-1=0$



Swan economic growth model in continuous time. A continuous time Solow-Swan model consists of the following equations. For $t \geq 0$,

(3.80) $Y(t) = AK(t)^{\alpha} L(t)^{1-\alpha}$, $0 < \alpha < 1$

(3.81) $Y(t) = C(t) + I(t)$

(3.82) $\dot{L}(t)/L(t) = n$

(3.83) $\dot{K}(t) = I(t) - dK(t)$, $d \geq 0$

(3.84) $I(t) = sY(t)$, $0 \leq s \leq 1$

(3.80) ~ (3.84) are almost identical to (3.16) ~ (3.20) of the discrete time Solow-Swan model. In (3.80) ~ (3.84), all the variables {$Y(t)$, $K(t)$, $L(t)$, $C(t)$, $I(t)$} are continuous functions of time t. (3.82) implies that the growth rate of labor is n. If $\dot{L}(t)$ is replaces with $\dot{L}(t) \cong L(t+1) - L(t)$, then (3.82) becomes identical to (3.18). (3.83) implies that the increment of capital $\dot{K}(t)$ consists of gross investment $I(t)$ minus depreciation $dK(t)$. If $\dot{K}(t)$ is replaced with $\dot{K}(t) \cong K(t+1) - K(t)$, then (3.83) becomes identical to (3.19). In (3.80), we assume that the level of production technology A is constant. The model, however, is easily modified to incorporate a technological progress by replacing A with $\dot{A}(t)/A(t) = g$ which corresponds to (3.39).

The analyses of the continuous time Solow-Swan model is also conducted in 3 steps. In step 1, we derive a differential equation with respect to capital-labor ratio. In step 2, we calculate the steady state of the difference equation. In step 3, we analyze the stability of the steady state.

Step 1. Divide both sides of (3.83) with labor L. By (3.80), (3.83), and (3.84), we have

(3.85) $\dfrac{\dot{K}}{L} = \dfrac{sAK^{\alpha}L^{1-\alpha}}{L} - \dfrac{dK}{L}$

Define capital-labor ratio by

(3.86) $k = K/L$.

Differentiate both sides of (3.86) with respect to time. Then, we have the following.

(3.87) $\dot{k} = \dfrac{\dot{K}}{L} - \dfrac{K}{L^2}\dot{L}$

By (3.82) and (3.86), (3.87) is rewritten as follows.

(3.88) $\dfrac{\dot{K}}{L} = \dot{k} + nk$

By (3.85), (3.86), and (3.88), we have

(3.89) $\dot{k} = sAk^{\alpha} - (n+d)k$.

(3.89) is a first-order nonlinear differential equation with respect to k.

Step 2. At the steady state of (3.89), k is constant at k_s so that $\dot{k} = 0$. Therefore, at the steady state, (3.89) becomes

(3.90) $0 = sAk_s^{\alpha} - (n+d)k_s$.

(3.90) is solved for the steady state k_s as follows.

(3.91) $k_s = \left(\dfrac{sA}{n+d}\right)^{\frac{1}{1-\alpha}}$

(3.91) is identical to (3.26) of the discrete time Solow-Swan model. Therefore, theorem 3.1 also holds for the continuous time Solow-Swan model that is summarized as follows.

(3.92) $k_s(\overset{+}{s}, \overset{+}{A}, \overset{-}{n}, \overset{-}{d})$

Step 3. With respect to the stability of the steady state, we have the following theorem.

Theorem 3.5. The steady state capital-labor ratio k_s of differential equation (3.89) is globally stable.

We can prove the theorem by using a phase-diagram as follows. In figure 3.13, where the horizontal axis measures capital-labor ratio k, we draw the graph of sAk^{α} and the graph of $(n+d)k$. The graph of sAk^{α} is an increasing concave curve. At $k = 0$, $sAk^{\alpha} = 0$. The slope of the graph is $sA\alpha k^{\alpha-1}$. When k is small, the slope is large. When k is large, the slope is small. Specifically, we have the followings.

(3.93) $\lim_{k\to 0} sA\alpha k^{\alpha-1} = \infty$, and $\lim_{k\to\infty} sA\alpha k^{\alpha-1} = 0$.

The graph of $(n+d)k$ is a straight line with slope $n+d$. Therefore, in figure 3.13, the graph of sAk^{α} and the graph of $(n+d)k$ have a unique intersection. (3.89) implies that $\dot{k}=0$ at the intersection because $sAk^{\alpha}=(n+d)k$ at the intersection. Therefore, the intersection in figure 3.13 implies the steady state k_s of the differential equation (3.89). Furthermore, figure 3.13 implies the following relationship.

(3.94)

$$k(t) \gtrless k_s \Leftrightarrow sAk(t)^{\alpha} \gtrless (n+d)k(t) \Leftrightarrow \dot{k}(t) \gtrless 0$$

On the horizontal axis of figure 3.13, we label two arrows indicating the motions of capital-labor ratio stated by (3.94). For example, if $k(t) < k_s$ at some time $t \in \Re$, then $\dot{k}(t) > 0$ because $sAk(t)^{\alpha} > (n+d)k(t)$. Theorem 3.5 implies that if the initial capital-labor ratio $k(0)$ is smaller than the steady state k_s, then the differential equation generates a sequence $\{k(t)\,;\, t \geq 0\}$ that is monotonically increasing and converging to the steady state k_s. On the other hand, if the initial capital-labor ratio $k(0)$ is larger than the steady state k_s, then the differential equation generates a sequence $\{k(t)\,;\, t \geq 0\}$ that is monotonically decreasing and converging to the steady state k_s.

Figure 3.13

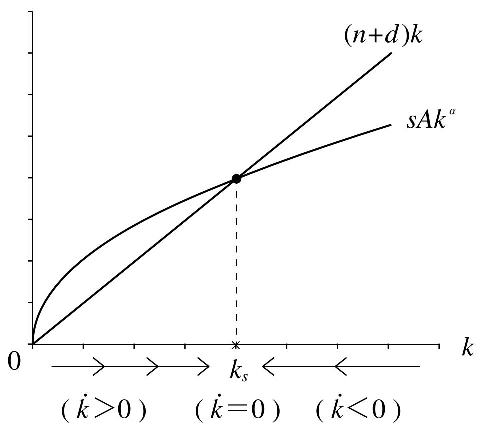
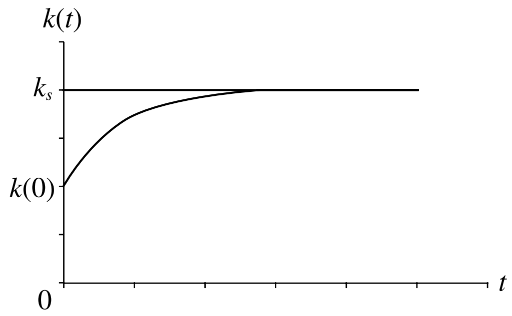


Figure 3.14 depicts a trajectory of capital-labor ratio $k(t)$ with the initial capital-labor ratio $k(0)$ smaller than the steady state k_s.

Figure 3.14



Once the motion of capital-labor ratio $k(t)$ is characterized, the motions of other variables are characterized as well because, like those in the discrete time Solow-Swan model, they are related to the capital-labor ratio as follows. The capital is calculated by

(3.95) $K(t) = k(t)L(t)$.

In the steady, $k(t)$ is constant at k_s. Therefore, at the steady state, (3.95) becomes

(3.96) $K(t) = k_s L(t)$.

(3.96) implies that the growth rate of capital $K(t)$ at the steady state is equal to the growth rate n of labor. The output $Y(t)$ is also expressed as a function of capital-labor ratio $k(t)$. By (3.80), we have

(3.97) $Y(t) = AK(t)^{\alpha} L(t)^{1-\alpha} = Ak(t)^{\alpha} L(t)$.

At the steady state (3.97) becomes

(3.98) $Y(t) = Ak_s^{\alpha} L(t)$.

(3.98) implies that the growth rate of output $Y(t)$ at the steady state is also equal to the growth rate n of labor. These observations are summarized as follows.

(3.99) $\dfrac{\dot{K}(t)}{K(t)} = \dfrac{\dot{L}(t)}{L(t)} = n$

By (3.97), the output per labor (labor productivity) is

(3.100) $y(t) = \dfrac{Y(t)}{L(t)} = Ak(t)^{\alpha}$.

In the steady, $k(t)$ is constant at k_s. Therefore, at the steady state, (3.100) becomes

(3.101) $y(t) = Ak_s^{\alpha}$.

(3.101) implies that the labor productivity is also constant at the steady state.

Readers may understand that the continuous time Solow-Swan model and the discrete time Solow-Swan model share the same properties stated by theorem 3.1 ~ theorem 3.5.

4. Discrete Time Ramsey Model

In this section, we analyze a discrete time Ramsey model. The model was first proposed by Ramsey (1928). It is the most basic and fundamental tool to analyze dynamic general equilibrium. The simplest bare-bones Ramsey model consists of many households and many firms whose behaviors will be described in the following.

4-1. Households

There are N identical households. N is assumed to be constant, although we can extend the basic model to incorporate population growth $N_{t+1} = (1 + n) N_t$. In each period $t = 0, 1, 2, \ldots$, a representative household faces the following budget constraint.

$$(4.1) \quad c_t + i_t = r_t k_t^s + w_t l_t^s$$

The right-hand side of (4.1) is income. k_t^s is the household's asset, and r_t is the interest rate. l_t^s is the household's labor supply, and w_t is the wage rate. Hence, $r_t k_t^s$ is asset income, and $w_t l_t^s$ is labor income. k_t^s is also regarded as the supply of capital because it will be used as capital input by firms. In the following, for analytical simplicity, we assume that the household supplies a fixed amount of labor in each period. Specifically, we assume

$$(4.2) \quad l_t^s = 1, \quad \text{for all } t = 0, 1, 2, \ldots .$$

The left-hand side of (4.1) implies that the income is used for consumption c_t and investment i_t. The household's asset changes through investment as follows.

$$(4.3) \quad k_{t+1}^s = k_t^s + i_t$$

For analytical simplicity, we assume that the asset does not depreciate, although we can extend the basic model to incorporate capital depreciation $k_{t+1}^s = (1 - \delta) k_t^s + i_t$, $0 \leq \delta \leq 1$. The household's utility from consumption c_t is

$$(4.4) \quad u(c_t) = \frac{c_t^{1-\sigma} - 1}{1-\sigma}, \quad \sigma \neq 1, \quad \sigma > 0 .$$

Notice $u(c_t)$ is an increasing and concave function of c_t because $u'(c_t) = c_t^{-\sigma} > 0$ and $u''(c_t) = -\sigma\ c_t^{-\sigma-1} < 0$. σ is a parameter that measures the curvature of $u(c_t)$. Define the curvature of $u(c_t)$ by $-u''(c_t) \times c_t / u'(c_t)$. Then, by (4.4),

$$(4.5) \quad -u''(c_t) \times c_t / u'(c_t) = \sigma .$$

When σ is very small, $u(c_t)$ is approximately a linear function of c_t. On the other hand, when σ is large, $u(c_t)$ is highly "bending" function of c_t. (The relative price of consumption in $t+1$ and consumption in t is $1 + r_{t+1}$. σ is shown to be the inverse of the price elasticity of substitution between c_{t+1} and c_t. See Blanchard and Fischer (1989) for a reference.) When $\sigma = 1$, (4.4) can not be defined. In this case, we replace (4.4) with

$$(4.6) \quad u(c_t) = \ln c_t .$$

In fact, when $\sigma = 1$, (4.6) is obtained by applying L'Hopital's rule to (4.4). The household's lifetime utility function is defined as follows.

$$(4.7) \quad \sum_{t=0}^{T} \rho^t u(c_t), \quad 0 < \rho < 1$$

In (4.7), T is the last period of the household. $\rho \in (0, 1)$ is a discount factor which measures the relative importance of future utilities. Because ρ^t decreases as t increases, the utility of future consumption is less important than the utility from current consumption. The household is "impatient". In the following, we do not specify T. Instead, we analyze a case with $T = \infty$ which approximates the behavior of a long-life household. The representative household's constrained optimization problem is summarized as follows. Given the initial asset k_0^s, the household chooses a sequence of consumption and asset $\{c_t, k_{t+1}^s ; t = 0, 1, 2, \ldots\}$ to maximize

$$(4.8) \quad \sum_{t=0}^{\infty} \rho^t u(c_t)$$

subject to

$$(4.9) \quad c_t + k_{t+1}^s = (1 + r_t) k_t^s + w_t, \quad t = 0, 1, 2, \ldots .$$

The household takes the sequence of interest rate and wage rate $\{r_t, w_t \,;\, t = 0, 1, 2, ...\}$ as givens as a price-taker. We also impose the following "transversality condition" on the household's constrained optimization problem.

$$(4.10)\quad \lim_{T\to\infty}\left[\frac{k^s_{T+1}}{(1+r_1)(1+r_2)\dots(1+r_T)}\right]=0$$

The transversality condition is derived by the recursive substitution of (4.9) as follows. At $t = 0$, and $t = 1$, (4.9) is expressed as

$$(4.11)\quad c_0 + k^s_1 = (1+r_0)k^s_0 + w_0 \text{ , and}$$

$$(4.12)\quad c_1 + k^s_2 = (1+r_1)k^s_1 + w_1 \text{ .}$$

We eliminate k^s_1 from (4.11) and (4.12), and arrange the result to have

$$(4.13)\quad c_0+\frac{c_1}{1+r_1}+\frac{k^s_2}{1+r_1} = (1+r_0)k^s_0+w_0+\frac{w_1}{1+r_1}\text{ .}$$

At $t = 2$, (4.9) is

$$(4.14)\quad c_2 + k^s_3 = (1+r_2)k^s_2 + w_2 \text{ .}$$

We eliminate k^s_2 from (4.13) and (4.14), and arrange the result to have

$$(4.15)\quad c_0+\frac{c_1}{1+r_1}+\frac{c_2}{(1+r_1)(1+r_2)}+\frac{k^s_3}{(1+r_1)(1+r_2)} = (1+r_0)k^s_0+w_0+\frac{w_1}{1+r_1}+\frac{w_2}{(1+r_1)(1+r_2)}\text{ .}$$

By repeating these steps for $t = 0, 1, 2, ... , T$, we have the following.

(4.16)

$$c_0+\frac{c_1}{1+r_1}+\frac{c_2}{(1+r_1)(1+r_2)}+\dots+\frac{c_T}{(1+r_1)(1+r_2)\dots(1+r_T)}+\frac{k^s_{T+1}}{(1+r_1)(1+r_2)\dots(1+r_T)}$$
$$= (1+r_0)k^s_0+w_0+\frac{w_1}{1+r_1}+\frac{w_2}{(1+r_1)(1+r_2)}+\dots+\frac{w_T}{(1+r_1)(1+r_2)\dots(1+r_T)}$$

The transversality condition (4.10) is obtained by letting $T\to\infty$ in (4.16) and setting the last term of the left-hand side of (4.16) to be zero.

Remark 1. If the transversality condition (4.10) holds, then (4.16) becomes

$$(4.17)\quad \sum_{t=0}^{\infty}\left\{\left[\prod_{s=0}^{t}\left(\frac{1}{1+r_s}\right)\right]c_t\right\} = (1+r_0)k^s_0 + \sum_{t=0}^{\infty}\left\{\left[\prod_{s=0}^{t}\left(\frac{1}{1+r_s}\right)\right]w_t\right\}\text{ .}$$

Remark 2. If the interest rate r_t is constant at r, then the transversality condition becomes

$$(4.18)\quad \lim_{T\to\infty}\left[\frac{k^s_{T+1}}{(1+r)^T}\right]=0\text{ ,}$$

and (4.16) becomes

$$(4.19)\quad \sum_{t=0}^{\infty}\left[\frac{c_t}{(1+r)^t}\right] = (1+r_0)k^s_0 + \sum_{t=0}^{\infty}\left[\frac{w_t}{(1+r)^t}\right]\text{ .}$$

The transversality condition (4.10) follows from two logical reasons. First, because all the households are identical, no one can borrow from other households. This implies k^s_{T+1} can not be negative. Second, if (4.10) is positive, then the household is not optimizing. In fact, if the last term of the left-hand side of (4.16) is positive, then the household can increase consumption (hence utility) by reducing the capital of indefinite future.[2]

The solution to the household's constrained optimization problem is obtained as follows. Define the Lagrangean as a function of the sequence of consumption and asset $\{c_t, k^s_{t+1} \,;\, t = 0, 1, 2, ...\}$, and the sequence of Lagrangean multiplier $\{\lambda_t \,;\, t = 0, 1, 2, ...\}$ as follows.

$$(4.20)\quad \pounds(c_t,\ k^s_{t+1},\ \lambda_t\ ;\ t=0,1,2,\dots) \equiv \sum_{t=0}^{\infty}\left\{\rho^t u(c_t)+\lambda_t[(1+r_t)k^s_t+w_t-c_t-k^s_{t+1}]\right\}$$

The optimal solution $\{c_t, k^s_{t+1} \,;\, t = 0, 1, 2, ...\}$, together with $\{\lambda_t \,;\, t = 0, 1, 2, ...\}$ must satisfy the following first-order conditions.

$$(4.21)\quad \frac{\partial\pounds}{\partial c_t}=\rho^t c_t^{-\sigma}-\lambda_t=0\text{ ,}\qquad \text{for all } t = 0, 1, 2, \dots .$$

$$(4.22)\quad \frac{\partial\pounds}{\partial k^s_{t+1}}=\lambda_{t+1}(1+r_{t+1})-\lambda_t=0\text{ ,}\qquad \text{for all } t = 0, 1, 2, \dots .$$

[2] (4.10) is also called the non-Ponzi game condition. The precise definition of a transversality condition is the value of capital measured in terms of utility units converges to zero in indefinite future. See Leonard and Long (1992) for rigorous treatment of transversality conditions.

$$(4.23)\quad \frac{\partial £}{\partial \lambda_t} = (1+r_t)k_t^s + w_t - c_t - k_{t+1}^s = 0,$$

for all t = 0, 1, 2, ...

4-2. Firms

Although the number of firms is inessential in the Ramsey model because of liner-homogeneity of production function, we assume that it is equal to the number of households N for analytical simplicity. By this assumption, one firm employs one labor in labor market equilibrium.

In each period $t = 0, 1, 2, \ldots$, a representative firm chooses the demand for capital k_t^d and the demand for labor l_t^d to maximize profit π_t which is defined as

$$(4.24)\quad \pi_t = y_t - r_t k_t^d - w_t l_t^d$$

subject to production function

$$(4.25)\quad y_t = A(k_t^d)^{\beta}(l_t^d)^{1-\beta}, \quad 0 < \beta < 1.$$

In (4.25), A is a constant parameter that measures the level of technology, and β is the weight of capital in the Cobb-Douglas production function. The optimal capital and labor must satisfy the following first-order conditions.

$$(4.26)\quad \frac{\partial \pi_t}{\partial k_t^d} = \frac{\partial y_t}{\partial k_t^d} - r_t = 0, \quad \text{for all } t = 0, 1, 2, \ldots .$$

$$(4.27)\quad \frac{\partial \pi_t}{\partial l_t^d} = \frac{\partial y_t}{\partial l_t^d} - w_t = 0, \quad \text{for all t} = 0, 1, 2, \ldots .$$

(4.26) implies that the marginal product of capital is equal to the interest rate, and (4.27) implies that the marginal product of labor is equal to the wage rate. By (4.25), (4.26) and (4.27) are rewritten as follows.

$$(4.28)\quad r_t = \beta A(k_t^d)^{\beta-1}(l_t^d)^{1-\beta}, \quad \text{for all } t = 0, 1, 2, \ldots .$$

$$(4.29)\quad w_t = (1-\beta)A(k_t^d)^{\beta}(l_t^d)^{-\beta},$$

for all t = 0, 1, 2, ...

The maximized profit, however, is zero because of the linear-homogeneity of production function with respect to capital and labor. This claim is confirmed as follows.

$$\begin{aligned}(4.30)\quad \pi_t &= y_t - r_t k_t^d - w_t l_t^d \\ &= A(k_t^d)^{\beta}(l_t^d)^{1-\beta} - [\beta A(k_t^d)^{\beta-1}(l_t^d)^{1-\beta}]\times k_t^d \\ &\quad - [(1-\beta)A(k_t^d)^{\beta}(l_t^d)^{-\beta}]\times l_t^d \\ &= 0\end{aligned}$$

(4.30) also implies

$$(4.31)\quad y_t = A(k_t^d)^{\beta}(l_t^d)^{1-\beta} = r_t k_t^d + w_t l_t^d .$$

4-3. Dynamic General Equilibrium

A dynamic general equilibrium of the discrete time Ramsey model is a sequence of prices and quantities $\{\{r_t, w_t ; t = 0, 1, 2, \ldots\}, \{k_{t+1}^d, k_{t+1}^s, l_t^d, l_t^s, c_t, i_t, y_t ; t = 0, 1, 2, \ldots\}\}$ such that (i) given the prices $\{r_t, w_t ; t = 0, 1, 2, \ldots\}$ the quantities maximize every household's lifetime utility (4.8) subject to budget constraint (4.9) as well as every firm's profit (4.24) subject to production function (4.25), and (ii) the prices equate aggregate demand and aggregate supply in every market in every time period $t = 0, 1, 2, \ldots$.

There are three markets in this model; capital market, labor market, and output market. In each period $t = 0, 1, 2, \ldots$, the aggregate demand for capital by firms is $N \times k_t^d$, and the aggregate supply of capital by households is $N \times k_t^s$. Therefore the capital market equilibrium is described as

$$(4.32)\quad k_t^d = k_t^s, \quad t = 0, 1, 2, \ldots .$$

In the following, we drop the superscripts “d ” and “s”, and express the amount of capital in equilibrium by k_t. At the initial time period $t = 0$, the aggregate supply of capital $N \times k_0^s$ is given. The interest rate r_0 and the wage rate w_0 adjust to equate the aggregate demand for capital $N \times k_0^d$ to the given aggregate supply of capital.

In each period $t = 0, 1, 2, \ldots$, the aggregate demand for labor by firms is $N \times l_t^d$, and the aggregate supply of capital by households is $N \times l_t^s = N$. Therefore the labor market equilibrium is described as

$$(4.33)\quad l_t^d = l_t^s = 1, \quad t = 0, 1, 2, \ldots .$$

In each period $t = 0, 1, 2, \ldots$, the aggregate demand for output by households is $N \times c_t + N \times i_t$, and the aggregate supply of output by firms is $N \times y_t$. Therefore, the output market equilibrium is described by

$$(4.34)\quad y_t = c_t + i_t, \quad t = 0, 1, 2, \ldots .$$

Although there are three markets in each period $t = 0, 1, 2, \ldots$, because of Walras' law, if two of the three markets are in equilibrium, so is the third market. For example, if the capital market and the labor market are in equilibrium, the interest rate, that is the price in capital market, is equal to the marginal product of capital (4.28), and the wage rate, that is the price in labor market, is equal to the marginal product of labor (4.29). Then, a representative consumer's budget constraint (4.1) is

$$\begin{aligned}(4.35)\quad c_t + i_t &= r_t k_t^s + w_t l_t^s \\ &= [\beta A(k_t^d)^{\beta-1}(l_t^d)^{1-\beta}]\times k_t^d \\ &\quad + [(1-\beta)A(k_t^d)^{\beta}(l_t^d)^{-\beta}]\times l_t^d \\ &= \beta A(k_t^d)^{\beta}(l_t^d)^{1-\beta} \\ &\quad + (1-\beta)A(k_t^d)^{\beta}(l_t^d)^{1-\beta} \\ &= y_t \ .\end{aligned}$$

In (4.35), the third equality is obtained by the capital market equilibrium condition $k_t^d = k_t^s$ and the labor market equilibrium condition $l_t^d = l_t^s = 1$. We could have added the price of output, say p_t, $t = 0, 1, 2, \ldots$, to the model. Instead, We set $p_t = 1$ for all $t = 0, 1, 2, \ldots$. In other words, we choose the output pr ice as a numeraire. We are able to do this because of Walras' law.

In the dynamic general equilibrium, given the initial capital k_0, the sequence of capital and consumption $\{k_{t+1}, c_t\ ;\ t = 0, 1, 2, \ldots\}$ satisfies the following equations. By (4.23), (4.28), and (4.29),

$$(4.36)\quad k_{t+1} = k_t + Ak_t^{\beta} - c_t \ ,$$

and by (4.21), (4.22), and (4.28),

$$(4.37)\quad c_{t+1} = [\rho(1+\beta Ak_{t+1}^{\beta-1})]^{1/\sigma} c_t \ .$$

(4.36) and (4.37) form a system of first-order nonlinear difference equations with respect to $\{k_{t+1}, c_t\ ;\ t = 0, 1, 2, \ldots\}$. Although our objective is to find solutions to (4.36) and (4.37), nonlinear equations are difficult to solve. Therefore, in the following, we approximate (4.36) and (4.37) by a system of first-order linear difference equations, and find solutions of the approximated system. This analysis consists of 4 steps.

Step 1. The steady state capital and consumption $\{k^*, c^*\}$ of the difference equations (4.36) and (4.37) satisfy the followings.

$$(4.38)\quad k^* = k^* + A(k^*)^{\beta} - c^*$$

$$(4.39)\quad c^* = [\rho(1+\beta A(k^*)^{\beta-1})]^{1/\sigma} c^* \ .$$

By (4.39), the steady state capital k^* is

$$(4.40)\quad k^* = \left(\frac{\rho\beta A}{1-\rho}\right)^{\frac{1}{1-\beta}} \ .$$

Then, by (4.38), steady state consumption c^* is

$$(4.41)\quad c^* = A(k^*)^{\beta} \ .$$

Step 2. Rewrite the difference equations (4.36) and (4.37) as follows.

$$\begin{aligned}(4.42)\quad k_{t+1} - k^* &= k_t + Ak_t^{\beta} - c_t - k^* \\ &\equiv F_1(k_t, c_t)\end{aligned}$$

$$\begin{aligned}(4.43)\quad c_{t+1} - c^* &= [\rho(1+\beta Ak_{t+1}^{\beta-1})]^{1/\sigma} c_t - c^* \\ &\equiv F_2(k_{t+1}, c_t)\end{aligned}$$

By definition, the followings hold at the steady state $\{k^*, c^*\}$.

$$(4.44)\quad F_1(k^*, c^*) = 0$$

$$(4.45)\quad F_2(k^*, c^*) = 0$$

We apply first-order linear approximations to the right-hand sides of (4.42) and (4.43) evaluated at the steady state $\{k^*, c^*\}$. The results are given as follows.

$$\begin{aligned}(4.46)\quad k_{t+1} - k^* &\cong F_1(k^*, c^*) \\ &\quad + (k_t - k^*)\frac{\partial}{\partial k_t}F_1(k^*, c^*) \\ &\quad + (c_t - c^*)\frac{\partial}{\partial c_t}F_1(k^*, c^*)\end{aligned}$$

$$\begin{aligned}(4.47)\quad c_{t+1} - c^* &\cong F_2(k^*, c^*) \\ &\quad + (k_{t+1} - k^*)\frac{\partial}{\partial k_{t+1}}F_2(k^*, c^*) \\ &\quad + (c_t - c^*)\frac{\partial}{\partial c_t}F_2(k^*, c^*)\end{aligned}$$

In (4.46), $\partial F_1(k^*, c^*)/\partial k_t$ implies the partial differentiation of $F_1(k_t, c_t)$ with respect to k_t evaluated at the steady state $\{k^*, c^*\}$. This term is calculated as follows.

$$(4.48)\quad \frac{\partial}{\partial k_t}F_1(k_t, c_t) = 1 + A\beta k_t^{\beta-1}$$

If we evaluate (4.48) at the steady state $\{k^*, c^*\}$, (4.48) becomes

$$(4.49)\quad \frac{\partial}{\partial k_t}F_1(k^*, c^*) = 1 + A\beta\left[\left(\frac{\rho\beta A}{1-\rho}\right)^{\frac{1}{1-\beta}}\right]^{\beta-1} = 1/\rho.$$

The other terms in (4.46) and (4.47) become as follows.

$$(4.50)\quad \frac{\partial}{\partial c_t}F_1(k^*, c^*) = -1$$

$$(4.51)\quad \frac{\partial}{\partial k_{t+1}}F_2(k^*, c^*) = -\frac{(1-\rho)^2(1-\beta)}{\sigma\rho\beta} = -M,$$

$$M \equiv \frac{(1-\rho)^2(1-\beta)}{\sigma\rho\beta} > 0.$$

$$(4.52)\quad \frac{\partial}{\partial c_t}F_2(k^*, c^*) = 1$$

Therefore, (4.46) and (4.47) are approximated by the following first-order linear difference equations.

$$(4.53)\quad k_{t+1} - k^* = \frac{1}{\rho}(k_t - k^*) - (c_t - c^*)$$

$$(4.54)\quad c_{t+1} - c^* = -M(k_{t+1} - k^*) + (c_t - c^*)$$

Step 3. Define the deviations of capital and consumption from the steady sate as follows.

$$(4.55)\quad \hat{k}_t \equiv k_t - k^*$$

$$(4.56)\quad \hat{c}_t \equiv c_t - c^*$$

Then, (4.53) and (4.54) are rewritten as follows.

$$(4.57)\quad \hat{k}_{t+1} = \frac{1}{\rho}\hat{k}_t - \hat{c}_t$$

$$(4.58)\quad \hat{c}_{t+1} = -M\hat{k}_{t+1} + \hat{c}_t$$

By (4.57), $\hat{c}_t = (1/\rho)\hat{k}_t - \hat{k}_{t+1}$ and $\hat{c}_{t+1} = (1/\rho)\hat{k}_{t+1} - \hat{k}_{t+2}$. We put these into (4.58) to eliminate $\hat{c}_t$ and $\hat{c}_{t+1}$. The result is the following second-order linear difference equation with respect to $\hat{k}_t$.

$$(4.59)\quad \hat{k}_{t+2} - (1 + \frac{1}{\rho} + M)\hat{k}_{t+1} + \frac{1}{\rho}\hat{k}_t = 0$$

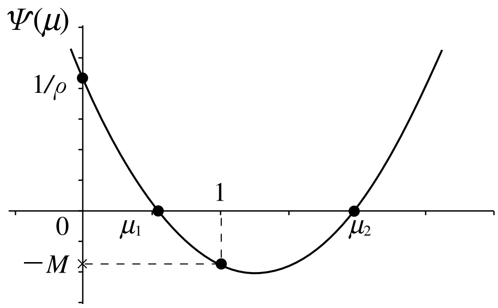
Step 4. A general solution to (4.59) is calculated as follows. Define the "characteristic equation" of (4.59) as

$$(4.60)\quad \psi(\mu) \equiv \mu^2 - (1 + \frac{1}{\rho} + M)\mu + \frac{1}{\rho} = 0$$

Because (4.60) is a quadratic equation with respect to μ, it has two solutions denoted as $\{\mu_1, \mu_2\}$. They are called the characteristic roots of (4.60). Figure 4.1 depicts the graph of $\Psi(\mu)$. Because $\Psi(0) = 1/\rho > 0$ and $\Psi(1) = -M < 0$, one root of (4.60) is between zero and one, and the other root is larger than one, i.e.,

$$(4.61)\quad 0 < \mu_1 < 1 \text{ and } \mu_2 > 1.$$

Figure 4.1



The general solution of (4.59) is expressed as follows.

$$(4.62)\quad \hat{k}_t = H_1\mu_1^t + H_2\mu_2^t$$

In (4.62), H_1 and H_2 are arbitrary real constants. We can verify that (4.62) satisfies (4.59) in any time period t and any real constants $\{H_1, H_2\}$ as follows. (4.62) implies

$$(4.63)\quad \hat{k}_{t+1} = H_1\mu_1^{t+1} + H_2\mu_2^{t+1}, \text{ and}$$

$$(4.64)\quad \hat{k}_{t+2} = H_1\mu_1^{t+2} + H_2\mu_2^{t+2}.$$

Put these expressions into the left-hand side of (4.59) to verify the claim, i.e.,

$$(4.65)\quad \hat{k}_{t+2} - (1 + \frac{1}{\rho} + M)\hat{k}_{t+1} + \frac{1}{\rho}\hat{k}_t$$

$$= (H_1\mu_1^{t+2} + H_2\mu_2^{t+2}) - (1 + \frac{1}{\rho} + M)$$

$$\cdot (H_1\mu_1^{t+1} + H_2\mu_2^{t+1}) + \frac{1}{\rho}(H_1\mu_1^t + H_2\mu_2^t)$$

$$= H_1\mu_1^t[\mu_1^2 - (1 + \frac{1}{\rho} + M)\mu_1 + \frac{1}{\rho}]$$

$$+ H_2\mu_2^t[\mu_2^2 - (1 + \frac{1}{\rho} + M)\mu_2 + \frac{1}{\rho}]$$

$$= 0.$$

Although there are continuums of solution (4.62), we can find a unique specific solution that satisfies the initial condition k_0 and the transversality condition.

First, notice if $H_2 \neq 0$, then $\hat{k}_t = k_t - k^*$ will explode either to $+\infty$ or to $-\infty$ because $\mu_2 > 1$. Therefore, in order for the sequence of capital $\{\hat{k}_{t+1}$; 0, 1, 2, ...$\}$ generated by (4.62) satisfy the transversality condition (4.10),

$$(4.66)\quad H_2 = 0$$

must hold. Then (4.62) becomes

$$(4.67)\quad \hat{k}_t = k_t - k^* = H_1 \mu_1^t \ .$$

At the initial time period $t = 0$, k_0 is given. Then, (4.67) implies

$$(4.68)\quad H_1 = k_0 - k^* \ .$$

By (4.67) and (4.68), we have

$$(4.69)\quad \hat{k}_t = (k_0 - k^*)\mu_1^t$$

(4.69) is rewritten as

$$(4.70)\quad k_t = k^* + (k_0 - k^*)\mu_1^t \ .$$

(4.70) is a unique specific solution of the difference equation (4.59) that satisfies the initial condition k_0 and the transversality condition (4.10). Because $0 < \mu_1 < 1$, the sequence of capital $\{k_{t+1}$; 0, 1, 2, ...$\}$ converges to the steady state k^* given any initial capital k_0, i.e.,

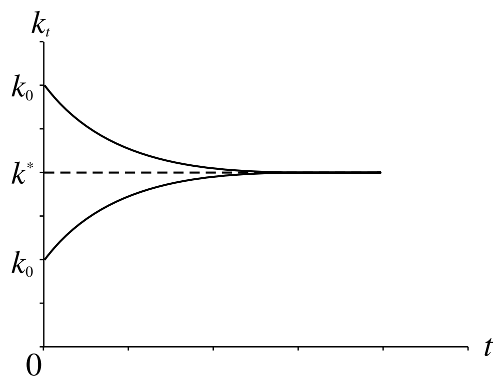
$$(4.71)\quad \lim_{t\to\infty} k_t = k^* \ .$$

Furthermore, if the initial capital k_0 is smaller than the steady state k^*, then the second term of the right-hand side of (4.70) is a positive sequence that monotonically decreasing to zero. On the other hand, if the initial capital k_0 is larger than the steady state k^*, then the second term of the right-hand side of (4.70) is a negative sequence that monotonically increasing to zero. We summarize these observations as the following theorem.

Theorem 4.1. The dynamic general equilibrium sequence of capital $\{k_{t+1}$; 0, 1, 2, ...$\}$ satisfies (i) if the initial capital k_0 is smaller than the steady state k^*, then $\{k_{t+1}$; 0, 1, 2, ...$\}$ is a monotonically increasing sequence $k_0 < k_1 < ... < k_t < k_{t+1} < ...$ converging to k^*, and (ii) if the initial capital k_0 is larger than the steady state k^* , then $\{k_{t+1}$; 0, 1, 2, ...$\}$ is a monotonically decreasing sequence $k_0 > k_1 > ... > k_t > k_{t+1} < ...$ converging to k^*.

Figure 4.2 depicts the statements (i) and (ii) of theorem 4.1.

Figure 4.2



Once the dynamic general equilibrium sequence of capital $\{k_{t+1}$; 0, 1, 2, ...$\}$ is specified, the motions of other variables of the Ramsey model are specified as well as follows. (4.70) implies that the growth rate of $\hat{k}_t$ is μ_1 because

$$(4.72)\quad \hat{k}_{t+1} = (k_0 - k^*)\mu_1^{t+1} = \mu_1(k_0 - k^*)\mu_1^t = \mu_1 \hat{k}_t \ .$$

By (4.57) and (4.72), we have

$$(4.73)\quad \hat{c}_t = \frac{1}{\rho}\hat{k}_t - \hat{k}_{t+1} = (\frac{1}{\rho} - \mu_1)\hat{k}_t \ .$$

(4.73) is rewritten as

$$(4.74)\quad c_t = c^* + (\frac{1}{\rho} - \mu_1)(k_t - k^*) \ .$$

From the inspection of (4.74), we summarize the properties of the dynamic general equilibrium sequence of consumption $\{c_t$; 0, 1, 2, ...$\}$ by the following theorem.

Theorem 4.2. The dynamic general equilibrium sequence of consumption $\{c_t$; 0, 1, 2, ...$\}$ satisfies the following properties; (i) Given the initial capital k_0 at $t = 0$, there is a unique initial consumption c_0 that is determined by

$$(4.75)\quad c_0 = c^* + (\frac{1}{\rho} - \mu_1)(k_0 - k^*) \ .$$

(ii) By theorem 4.1, the dynamic general equilibrium sequence of capital $\{k_t \,;\, t = 0, 1, 2, ...\}$ converges to the steady state k^*. Therefore, (4.74) implies that the dynamic general equilibrium sequence of consumption $\{c_t \,;\, 0, 1, 2, ...\}$ also converges to the steady state c^*, i.e.,

$$(4.76)\quad \lim_{t\to\infty} c_t = c^* .$$

Because $(1/\rho) > \mu_1 \in (0, 1)$, if the initial capital k_0 is smaller than the steady state k^*, then (4.75) implies that the initial consumption c_0 is also smaller than the steady state c^*. In addition, by theorem 4.1, the dynamic general equilibrium sequence of consumption $\{c_t \,;\, 0, 1, 2, ...\}$ is a monotonically increasing sequence $c_0 < c_1 < ... < c_t < c_{t+1} < ...$ converging to the steady state c^*. (iv) The statement in (iii) also implies that if the initial capital k_0 is larger than the steady state k^*, then the initial consumption c_0 is larger than the steady state c^*. In addition, the dynamic general equilibrium sequence of consumption $\{c_t \,;\, 0, 1, 2, ...\}$ is a monotonically decreasing sequence $c_0 > c_1 > ... > c_t > c_{t+1} > ...$ converging to the steady state c^*.

In the discrete time Ramsey model, when the characteristic roots of (4.60) satisfies $\{0 < \mu_1 < 1$ and $\mu_2 > 1\}$, the steady state $\{k^*, c^*\}$ of the dynamic general equilibrium is said to be a "saddle point" of the difference equations (4.53) and (4.54).

In the dynamic general equilibrium, by (4.25), the output per labor (labor productivity) is

$$(4.77)\quad y_t = Ak_t^{\beta} \;,\; t = 0, 1, 2, ...$$

By (4.28) and (4.29), the interest rate and the wage rate are

$$(4.78)\quad r_t = \beta A k_t^{\beta-1} \;,\; t = 0, 1, 2, ... \text{, and}$$

$$(4.79)\quad w_t = (1-\beta) A k_t^{\beta} \;,\; t = 0, 1, 2, ...$$

The aggregate variables are obtained by multiplying per-household variables with the number of households and by multiplying per-firm variables with the number of firms. (In this example, the number of households and the number of firms are assumed to be same.)

4-4. Numerical Example

In the following, a numerical simulation example of the discrete time Ramsey model will be presented. Readers may replicate the result, and conduct simulation experiments by changing parameter values and initial conditions. The simulation is conducted in 2 steps.

Step 1. The parameter values and the initial capital at $t = 0$ are set as follows.

$$(4.80)\quad \{\beta = 0.3,\ A = 10,\ \rho = 0.9,\ \sigma = 2\} \text{ and } k_0 = 50.$$

By (4.40) and (4.41), these values imply that the steady state capital and consumption are calculated as

$$(4.81)\quad k^* = 110.87 \quad \text{and} \quad c^* = 41.06 .$$

In addition, by (4.60), the characteristic roots are calculated as

$$(4.82)\quad \mu_1 = 0.932 \quad \text{and} \quad \mu_2 = 1.192 .$$

Step 2. The dynamic general equilibrium sequences of capital, consumption, output per labor, interest rate, and wage rate $\{k_{t+1}, c_t, y_t, r_t, w_t \,;\, t = 0, 1, 2, ...\}$ are calculated by using (4.70), (4.74), (4.77), (4.78), and (4.79). The results are summarized in table 4.1.

Figure 4.3 depicts the dynamic general equilibrium trajectory of capital $\{k_t \,;\, t = 0, 1, 2, ...\}$. As predicted by theorem 4.1, because the initial capital $k_0 = 50$ is smaller than the steady state $k^* = 110.87$, figure 4.3 shows that the dynamic general equilibrium sequence of capital $\{k_t \,;\, t = 0, 1, 2, ...\}$ is monotonically increasing and converging to the steady state. Figure 4.4 depicts the dynamic general equilibrium trajectory of consumption $\{c_t \,;\, t = 0, 1, 2, ...\}$. Given the initial capital $k_0 = 50$ at $t = 0$, a unique value of consumption $c_0 = 30.18$ is calculated from (4.75). Starting from $\{k_0 = 50,\ c_0 = 30.18\}$, the difference equations $\{(4.53), (4.54)\}$ generates the dynamic general equilibrium sequence of capital and consumption $\{k_{t+1}, c_t \,;\, t = 0, 1, 2, ...\}$ which converges to the steady state $\{k^* = 110.87,\ c^* = 41.06\}$. As predicted by theorem 4.2, the dynamic general equilibrium sequence of consumption $\{c_t \,;\, t = 0, 1, 2, ...\}$ is also monotonically increasing and converging to the steady state c^*.

Table 4.1 Simulation of Discrete Time Ramsey Model

t	k (t)	c (t)	y (t)	r (t)	w (t)
0	50.000	30.184	32.336	0.194	22.635
1	54.115	30.920	33.113	0.184	23.179
2	57.951	31.605	33.800	0.175	23.660
3	61.528	32.245	34.413	0.168	24.089
4	64.864	32.841	34.962	0.162	24.474
5	67.974	33.396	35.457	0.156	24.820
6	70.873	33.915	35.904	0.152	25.133
7	73.577	34.398	36.310	0.148	25.417
8	76.098	34.848	36.679	0.145	25.675
9	78.448	35.268	37.015	0.142	25.910
10	80.640	35.660	37.322	0.139	26.125
11	82.683	36.025	37.603	0.136	26.322
12	84.589	36.366	37.861	0.134	26.503
13	86.365	36.683	38.098	0.132	26.699
14	88.021	36.979	38.316	0.131	26.821
15	89.566	37.255	38.516	0.129	26.961
16	91.006	37.513	38.701	0.128	27.091
17	92.349	37.752	38.871	0.126	27.210
18	93.600	37.976	39.029	0.125	27.320
19	94.768	38.185	39.174	0.124	27.422
20	95.856	38.379	39.308	0.123	27.516
21	96.871	38.561	39.433	0.122	27.603
22	97.817	38.730	39.548	0.121	27.684
23	98.699	38.887	39.655	0.121	27.758
24	99.522	39.034	39.754	0.120	27.827
25	100.289	39.172	39.845	0.119	27.892

Parameters: beta = 0.3, A = 10, rho = 0.9, sigma = 2

Figure 4.3

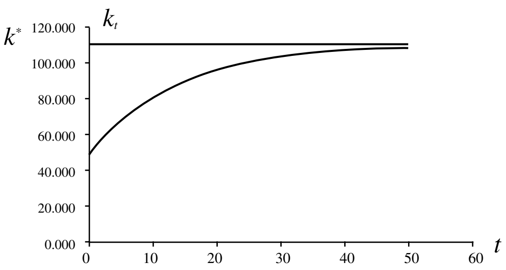
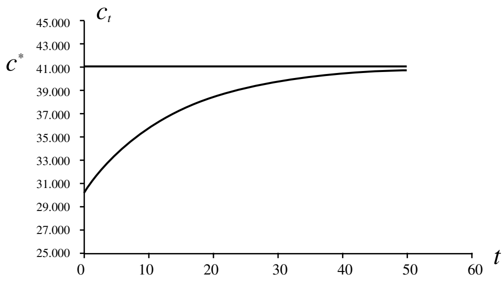


Figure 4.4



5. Continuous Time Ramsey Model

In section 3, we saw that the discrete time Solow-Swan model and the continuous time Solow-Swan model share similar properties. In addition, we saw that while the analysis of discrete time Solow-Swan model is conducted through the analysis of difference equation, the analysis of continuous time Solow-Swan model is conducted through the analysis of differential equation. Likewise, in this section, we are going to see that the discrete time Ramsey model and the continuous time Ramsey model share similar properties. In addition, while the analysis of discrete time Ramsey model is conducted through the analysis of difference equations, the analysis of continuous time Ramsey model is conducted through the analysis of differential equations.

The simplest bare-bones continuous time Ramsey model consists of many households and many firms whose behaviors, that are similar to those of the discrete time Ramsey model, will be presented in the following.

5-1. Households

There are N identical households. Although N is assumed to be constant, the model can be extended to handle cases with population growth described by $\dot{N}/N = n$. At time $t \geq 0$, a representative household faces the following budget constraint.

$$(5.1)\quad c(t) + i(t) = r(t)k^s(t) + w(t)l^s(t)$$

In (5.1), $c(t)$ is consumption, $i(t)$ is investment, $k^s(t)$ is capital supply, $r(t)$ is interest rate, $l^s(t)$ is labor supply, and $w(t)$ is wage rate. Each variable is a continuous function of time. The right-hand side of (5.1) is the household's income consisting of capital income $r(t)$ $k^s(t)$ and labor income $w(t)\ l^s(t)$. The left-hand side of (5.1) implies that the income is used for consumption $c(t)$ and investment $i(t)$. The stock of capital is accumulate through investment as follows.

(5.2) $\dot{k}^s(t) = i(t)$

where $\dot{k}^s(t) = dk^s(t)/dt$ is the increment of capital between time t and $t + \Delta t$ where Δt is very small. If $\dot{k}^s(t)$ is approximated by $\dot{k}^s(t) \cong k^s(t+1) - k^s(t)$, then (5.2) is same as (4.3) of the discrete time Ramsey model. In the following, we assume

(5.3) $l^s(t) = 1$, for all $t \geq 0$.

The household's utility from consumption $c(t)$ is

(5.4) $u(c(t)) = \dfrac{c(t)^{1-\sigma}-1}{1-\sigma}$, $\sigma \neq 1$, $\sigma > 0$.

If $\sigma = 1$, then (5.4) is replaced with log-utility function

(5.5) $u(c(t)) = \ln c(t)$.

The household's lifetime utility function is defined by

(5.6) $\int_0^\infty u(c(t))e^{-\theta t}\,dt$.

In (5.6), $\theta > 0$ is a discount factor which measures the relative importance of future utilities. Because $e^{-\theta t}$ decreases as t increases, the utility from future consumptions is less important than the utility from current consumption. Given the initial capital supply $k^s(0)$, the household chooses a sequence of capital supply and consumption $\{k^s(t), c(t)\ ;\ t \geq 0\}$ to maximize the lifetime utility (5.6) subject to budget constraint (5.1). The household takes the sequence of interest rate and wage rate $\{r(t), w(t)\ ;\ t \geq 0\}$ as givens. We also impose the transversality condition on the household's constrained optimization problem. The transversality condition of the continuous time Ramsey model is given as follows.

(5.7) $\lim\limits_{T\to\infty}\left\{\exp\left[-\int_0^T r(s)ds\right]k^s(T)\right\} = 0$

Readers may notice the equivalence implied by the transversality condition (4.10) of discrete time Ramsey model and the transversality condition (5.7) of continuous time Ramsey model. In fact, (5.7) is obtained by integrating the household's budget constraint, a procedure which is equivalent to the procedure of deriving the transversality condition (4.10) of discrete time Ramsey model. (5.1) is rewritten as follows.

(5.8) $-r(t)k^s(t) + \dot{k}^s(t) = w(t) - c(t)$.

Multiply $\exp[\int_t^T r(s)ds]$ on the both sides of (5.8), and integrate with respect to t between 0 and T. Then, the left-hand side of (5.8) becomes

(5.9) $\int_0^T \exp\left[\int_t^T r(s)ds\right][-r(t)k^s(t)+\dot{k}^s(t)]dt$

$$= \exp[\int_t^T r(s)ds]\times k^s(t)\ \Big|_0^T$$

$$= k^s(T) - \exp[\int_0^T r(s)ds]\times k^s(0)$$

The second equality holds because of the following integration by parts.

(5.10) $\dfrac{d}{dt}\{\exp[\int_t^T r(s)ds]\times k^s(t)\}$

$$= \exp[\int_t^T r(s)ds]\times[-r(t)]\times k^s(t) + \exp[\int_t^T r(s)ds]\times \dot{k}^s(t)$$

$$= \exp[\int_t^T r(s)ds]\times[-r(t)k^s(t)+\dot{k}^s(t)]$$

The right-hand side of (5.8) becomes

(5.11) $\int_0^T \exp\left[\int_t^T r(s)ds\right][w(t)-c(t)]dt$

Next, multiply $\exp[-\int_0^T r(s)ds]$ on the both sides to obtain

(5.12) $\exp[-\int_0^T r(s)ds]\times k^s(T) - k^s(0)$

$$= \int_0^T \exp\left[-\int_0^t r(s)ds\right][w(t)-c(t)]dt$$

(5.12) is rewritten as

(5.13) $\int_0^T \exp\left[-\int_0^t r(s)ds\right]c(t)dt$

$$+ \exp[-\int_0^T r(s)ds]\times k^s(T)$$

$$= \int_0^T \exp\left[-\int_0^t r(s)ds\right]w(t)dt + k^s(0)$$.

The transversality condition (5.7) is obtained by letting $T\to\infty$ in (5.13) and setting the second term of the left-hand side of (5.13) to be zero.

Remark 1. If the transversality condition (5.7) holds, then (5.13) becomes

$$(5.14)\quad \int_0^\infty \exp\left[-\int_0^t r(s)ds\right]c(t)dt = \int_0^\infty \exp\left[-\int_0^t r(s)ds\right]w(t)dt + k^s(0) \ .$$

Remark 2. If the interest rate $r(t)$ is constant at r, then $\exp[-\int_0^T r(s)ds] = e^{-rT}$. In this case, the transversality condition (5.7) becomes

$$(5.15)\quad \lim_{T\to\infty}\{e^{-rT}\ k^s(T)\} = 0 \ ,$$

and (5.13) becomes

$$(5.16)\quad \int_0^\infty e^{-rt}c(t)dt = \int_0^\infty e^{-rt}w(t)dt + k^s(0) \ .$$

The economic implications of the transversality condition (5.7) of continuous time Ramsey model are same as those of the transversality condition (4.10) of discrete time Ramsey model. First, $k^s(T)$ can not be negative because identical households can not borrow among themselves. Second, if (5.7) is positive, then the household is not optimizing. The house hold can increase consumption (hence utility) by reducing the capital of indefinite future.

The solution to the household's constrained optimization problem is obtained as follows. Define the Hamiltonian as a function of the sequence of capital supply and consumption $\{k^s(t), c(t)\ ;\ t \geq 0\}$ and the sequence of "shadow price" $\{\lambda(t)\ ;\ t \geq 0\}$ as follows.

$$(5.17)\quad H(k^s(t), c(t), \lambda(t)\,;t \geq 0) = u(c(t)) + \lambda(t)[r(t)k^s(t) + w(t) - c(t)]$$

The optimal solution $\{k^s(t), c(t)\ ;\ t \geq 0\}$ together with $\{\lambda(t)\ ;\ t \geq 0\}$ must satisfy the following first-order conditions.

$$(5.18)\quad \frac{\partial H}{\partial c(t)} = c(t)^{-\sigma} - \lambda(t) = 0$$

$$(5.19)\quad \dot{\lambda}(t) = \theta\lambda(t) - \frac{\partial H}{\partial k^s(t)}$$

$$(5.20)\quad \dot{k}^s(t) = \frac{\partial H}{\partial \lambda(t)}$$

(5.19) and (5.20) are rewritten as follows.

$$(5.21)\quad \dot{\lambda}(t) = \lambda(t)[\theta - r(t)]$$

$$(5.22)\quad \dot{k}^s(t) = r(t)k^s(t) + w(t) - c(t)$$

5-2. Firms

For analytical simplicity, we assume that the number of firms is N which is equal to the number of households. At time $t \geq 0$, a representative firm chooses the demand for capital $k^d(t)$ and the demand for labor $l^d(t)$ to maximize profit $\pi(t)$ which is defined as

$$(5.23)\quad \pi(t) = y(t) - r(t)k^d(t) - w(t)l^d(t)$$

subject to production function

$$(5.24)\quad y(t) = A(k^d(t))^\beta(l^d(t))^{1-\beta} \ .$$

The technology level A is assumed to be constant. The model, however, can be extended to handle cases with technological progress described as $\dot{A}/A = g$. The optimal capital and labor must satisfy the following first-order conditions.

$$(5.25)\quad \frac{\partial\pi(t)}{\partial k^d(t)} = \frac{\partial y(t)}{\partial k^d(t)} - r(t)\ , \quad \text{for all } t \geq 0.$$

$$(5.26)\quad \frac{\partial\pi(t)}{\partial l^d(t)} = \frac{\partial y(t)}{\partial l^d(t)} - w(t)\ , \quad \text{for all } t \geq 0.$$

By (5.25) and (5.26), the interest rate $r(t)$ and the wage rate $w(t)$ are expressed as

$$(5.27)\quad r(t) = \beta A(k^d(t))^{\beta-1}(l^d(t))^{1-\beta}\ , \quad \text{for all } t \geq 0.$$

$$(5.28)\quad w(t) = (1-\beta)A(k^d(t))^\beta(l^d(t))^{-\beta}\ , \quad \text{for all } t \geq 0.$$

By the linear homogeneity of production function, the maximized profit is zero. Therefore, the following equation holds.

$$(5.29)\quad y(t) = A(k^d(t))^\beta(l^d(t))^{1-\beta} = r(t)k^d(t) + w(t)l^d(t)$$

5-3. Dynamic General Equilibrium

A dynamic general equilibrium of the continuous time Ramsey model is a sequence of prices and quantities $\{\{r(t), w(t)\ ;\ t \geq 0\}, \{k^d(t), k^s(t), l^d(t), l^s(t), c(t), i(t), y(t)\ ;\ t \geq 0\}\}$ such that (i) given the prices $\{r(t), w(t)\ ;\ t \geq 0\}$, the quantities maximize every household's lifetime utility (5.6) subject to budget constraint (5.1) as well as every firm's profit (5.23) subject to production function (5.24), and (ii) the prices equates aggregate demand and aggregate supply in every market at all time $t \geq 0$.

There are three markets in this model; capital market, labor market, and output market. The equilibrium condition in capital market is given by

$$(5.30)\quad Nk^d(t) = Nk^s(t) \ , \quad \text{for all } t \geq 0.$$

The left-hand side of (5.30) is the aggregate demand for capital by firms, and the right-hand side of (5.30) is the aggregate supply of capital by households. (5.30) implies

$$(5.31)\quad k^d(t) = k^s(t) \ , \quad \text{for all } t \geq 0.$$

In the following, we drop the superscripts "d" and "s" from (5.31), and express the amount of capital in equilibrium by $k(t)$. At the initial time period $t = 0$, the aggregate supply of capital $Nk^s(0)$ is given. The interest rate $r(0)$ and the wage rate $w(0)$ adjust to equate the aggregate demand for capital $Nk^d(0)$ to the given aggregate supply of capital.

The equilibrium condition in labor market is given by

$$(5.32)\quad Nl^d(t) = Nl^s(t) \ , \quad \text{for all } t \geq 0.$$

The left-hand side of (5.32) is the aggregate demand for labor by firms, and the right-hand side of (5.32) is the aggregate supply of labor by households. (5.32) implies

$$(5.33)\quad l^d(t) = l^s(t) = 1 \ , \quad \text{for all } t \geq 0.$$

The equilibrium condition in output market is given by

$$(5.34)\quad Ny(t) = Nc(t) + Ni(t) \ .$$

The left-hand-side of (5.34) is the aggregate supply of output by firms, and the right-hand side is the aggregate demand for output by households consisting of aggregate consumption and aggregate investment. The condition however, is redundant because of Walras' law.

In the dynamic general equilibrium, given the initial capital $k(0)$, the sequence of capital and consumption $\{k(t), c(t) \, ; t \geq 0\}$ satisfies the following equations. By (5.22) and (5.29),

$$(5.35)\quad \dot{k}(t) = Ak(t)^{\beta} - c(t) \equiv F_1(k(t), c(t)) \ ,$$

and by (5.18), (5.21), and (5.27),

$$(5.36)\quad \dot{c}(t) = \frac{1}{\sigma}(\beta Ak(t)^{\beta-1} - \theta)c(t)$$
$$\equiv F_2(k(t), c(t)) \ .$$

(5.35) and (5.36) form a system of first-order nonlinear differential equations with respect to $\{k(t), c(t) \, ; t \geq 0\}$. Although our objective is to find solutions to (5.35) and (5.36), nonlinear equations are in general difficult to solve. Therefore, like we did for the analysis of discrete time Ramsey model, we approximate (5.35) and (5.36) by a system of first-order linear differential equations, and find solutions to the approximated system. The analysis consists of 4 steps.

Step 1. The steady state capital and consumption $\{k^*, c^*\}$ of the differential equations (5.35) and (5.36) satisfy the followings.

$$(5.37)\quad 0 = \dot{k} = A(k^*)^{\beta} - c^*$$

$$(5.38)\quad 0 = \dot{c} = \frac{1}{\sigma}(\beta A(k^*)^{\beta-1} - \theta)c^*$$

By (5.38), the steady state capital k^* is

$$(5.39)\quad k^* = (\beta A/\theta)^{\frac{1}{1-\beta}} \ .$$

Then, by (5.37), the steady state consumption c^* is

$$(5.40)\quad c^* = A(k^*)^{\beta} \ .$$

By definition, the followings hold at the steady state $\{k^*, c^*\}$.

$$(5.41)\quad F_1(k^*, c^*) = 0$$

$$(5.42)\quad F_2(k^*, c^*) = 0$$

We apply first-order linear approximations to the right-hand sides of (5.35) and (5.36) evaluated at the steady state $\{k^*, c^*\}$. The results are given as follows.

$$(5.43)\quad \dot{k}(t) \cong F_1(k^*, c^*) + (k(t) - k^*)\frac{\partial}{\partial k}F_1(k^*, c^*)$$
$$+ (c(t) - c^*)\frac{\partial}{\partial c}F_1(k^*, c^*)$$

(5.44) $\dot{c}(t) \cong F_2(k^*, c^*) + (k(t) - k^*)\frac{\partial}{\partial k}F_2(k^*, c^*)$

$$+ (c(t) - c^*)\frac{\partial}{\partial c}F_2(k^*, c^*)$$

In (5.43), $\partial F_1(k^*, c^*)/\partial k$ implies the partial differentiation of $F_1(k(t), c(t))$ with respect to $k(t)$ evaluated at the steady state $\{k^*, c^*\}$. This term is calculated as follows.

(5.45) $\frac{\partial}{\partial c}F_1(k(t), c(t)) = \beta A k(t)^{\beta-1}$

If we evaluate (5.45) at the steady state $\{k^*, c^*\}$, (5.45) becomes

(5.46) $\frac{\partial}{\partial c}F_1(k^*, c^*) = \beta A(k^*)^{\beta-1}$

$$= \beta A([\beta A/\theta]^{\frac{1}{1-\beta}})^{1-\beta} = \theta .$$

The other terms in (5.43) and (5.44) become as follows.

(5.47) $\frac{\partial}{\partial c}F_1(k^*, c^*) = -1$

(5.48) $\frac{\partial}{\partial k}F_2(k^*, c^*) = -\frac{(1-\beta)\theta^2}{\sigma\beta} = -M$,

$$M \equiv \frac{(1-\beta)\theta^2}{\sigma\beta} > 0.$$

(5.49) $\frac{\partial}{\partial c}F_2(k^*, c^*) = 0$

Therefore, (5.37) and (5.38) are approximated by the following first-order linear difference equations.

(5.50) $\dot{k}(t) = \theta(k(t) - k^*) - (c(t) - c^*)$

(5.51) $\dot{c}(t) = -M(k(t) - k^*)$

Step 3. Differentiate both sides of (5.50) with respect to time t to have

(5.52) $\ddot{k}(t) = \theta\dot{k}(t) - \dot{c}(t)$

where $\ddot{k}(t) = d\dot{k}(t)/dt = d[d\dot{k}(t)/dt]/dt$. Eliminate $\dot{c}(t)$ from (5.51) and (5.52). Then, we have

(5.53) $\ddot{k}(t) - \theta\dot{k}(t) - M(k(t) - k^*) = 0$.

Define the deviation of capital from its steady state by

(5.54) $\hat{k}(t) \equiv k(t) - k^*$.

Because $\dot{\hat{k}}(t) = \dot{k}(t)$ and $\ddot{\hat{k}}(t) = \ddot{k}(t)$, (5.53) is rewritten as follows.

(5.55) $\ddot{\hat{k}}(t) - \theta\dot{\hat{k}}(t) - M\hat{k}(t) = 0$

(5.55) is a second-order linear differential equation with respect to $\hat{k}(t)$.

Step 4. A general solution of (5.55) is calculated as follows. The characteristic equation of (5.55) is given by

(5.56) $\varsigma(\mu) \equiv \mu^2 - \theta\mu - M = 0$.

Because (5.56) is a quadratic equation with respect to μ, (5.56) has two characteristic roots. These roots, denoted as $\{\mu_1, \mu_2\}$ satisfy

(5.57) $\mu_1 + \mu_2 = \theta > 0$ and $\mu_1\mu_2 = -M < 0$.

Therefore, one root must be positive while the other root must be negative. Let

(5.58) $\mu_1 < 0$ and $\mu_2 > 0$.

The general solution of (5.55) is expresses as follows.

(5.59) $\hat{k}(t) = H_1 e^{\mu_1 t} + H_2 e^{\mu_2 t}$.

In (5.59), H_1 and H_2 are arbitrary real constants. We can verify that (5.59) satisfies (5.55) in any time period t and any real constants $\{H_1, H_2\}$ as follows. (5.59) implies

(5.60) $\dot{\hat{k}}(t) = H_1\mu_1 e^{\mu_1 t} + H_2\mu_2 e^{\mu_2 t}$, and

(5.61) $\ddot{\hat{k}}(t) = H_1(\mu_1)^2 e^{\mu_1 t} + H_2(\mu_2)^2 e^{\mu_2 t}$.

Put these expressions into the left-hand side of (5.55) to verify the claim, i.e.,

(5.62) $\ddot{\hat{k}}(t) - \theta\dot{\hat{k}}(t) - M\hat{k}(t)$

$$= [H_1(\mu_1)^2 e^{\mu_1 t} + H_2(\mu_2)^2 e^{\mu_2 t}] - \theta[H_1\mu_1 e^{\mu_1 t} + H_2\mu_2 e^{\mu_2 t}] - M[H_1 e^{\mu_1 t} + H_2 e^{\mu_2 t}]$$

$$= H_1 e^{\mu_1 t}[(\mu_1)^2 - \theta(\mu_1) - M] + H_2 e^{\mu_2 t}(\mu_2)^2 - \theta(\mu_2) - M]$$

$$= 0 .$$

Although there are continuums of solution (5.59), we can find a unique specific solution that satisfies the initial condition $k(0)$ and the transversality condition (5.7). First, notice if $H_2 \neq 0$, then $\hat{k}(t) = k(t) - k^*$ will explode either to $+\infty$ or to $-\infty$ because $\mu_2 > 0$. Therefore, in order for the sequence of capital $\{k(t)\,;\, t \geq 0\}$ generated by (5.59) satisfy the transversality condition (5.7),

(5.63) $H_2 = 0$

must hold. Then (5.59) becomes

(5.64) $\hat{k}(t) = k(t) - k^* = H_1 e^{\mu_1 t}$.

At the initial time period $t = 0$, $k(0)$ is given. Then, (5.64) implies

(5.65) $H_1 = k(0) - k^*$.

By (5.59) and (5.65), we have

(5.66) $\hat{k}(t) = (k(0) - k^*)e^{\mu_1 t}$.

(5.66) is rewritten as

(5.67) $k(t) = k^* + (k(0) - k^*)e^{\mu_1 t}$.

(5.67) is a unique specific solution of the differential equation (5.53) that satisfies the initial condition $k(0)$ and the transversality condition (5.7). From the inspection of (5.67), we have the following theorem.

Theorem 5.1. The dynamic general equilibrium sequence of capital $\{k(t), t \geq 0\}$ satisfies that (i) if the initial capital $k(0)$ is smaller than the steady state k^*, then $\{k(t), t \geq 0\}$ is a monotonically increasing sequence converging to k^*, or (ii) if the initial capital $k(0)$ is larger than the steady state k^*, then $\{k(t), t \geq 0\}$ is a monotonically decreasing sequence converging to k^*.

The convergence of the general equilibrium sequence of capital $\{k(t), t \geq 0\}$ to the steady state k^* implies

(5.68) $\lim_{t\to\infty} k(t) = k^*$.

Theorem 4.1 and theorem 5.1 show that the dynamic general equilibrium sequence of capital $\{k_t ; t = 0, 1, 2, ...\}$ of discrete time Ramsey model and the dynamic general equilibrium sequence of capital $\{k(t), t \geq 0\}$ of continuous time Ramsey model have the same properties about monotonicity and convergence to unique steady state. Therefore, figure 4.2 is also used to depict the transition of the dynamic general equilibrium sequence of capital $\{k(t), t \geq 0\}$ to the steady state k^* for the continuous time Ramsey model.

Once the dynamic general equilibrium sequence of capital $\{k(t), t \geq 0\}$ is set, the motions of the other variables are set we well as follows. By (5.67),

(5.69) $\dot{k}(t) = \mu_1(k(0) - k^*)e^{\mu_1 t} = \mu_1(k(t) - k^*)$.

By (5.50) and (5.69), the dynamic general equilibrium sequence of consumption $\{c(t), t \geq 0\}$ is given as follows.

(5.70) $c(t) = c^* + (\theta - \mu_1)(k(t) - k^*)$

Because the dynamic general equilibrium sequence of capital $\{k(t), t \geq 0\}$ converges to the steady state k^*, (5.70) implies that the dynamic general equilibrium sequence of consumption $\{c(t), t \geq 0\}$ also converges to the steady state c^*. A further inspection of (5.70) enables us to characterize the properties of dynamic general equilibrium sequence of consumption $\{c(t), t \geq 0\}$ which are summarized by the following theorem.

Theorem 5.2. The dynamic general equilibrium sequence of consumption $\{c(t), t \geq 0\}$ satisfies the following properties; (i) Given the initial capital $k(t)$ at $t = 0$, there is a unique initial consumption $c(t)$ that is determined by

(5.71) $c(0) = c^* + (\theta - \mu_1)(k(0) - k^*)$.

(ii) The dynamic general equilibrium sequence of consumption $\{c(t), t \geq 0\}$ converges to the steady state c^*, i.e.,

(5.72) $\lim_{t\to\infty} c(t) = c^*$.

(iii) Because $\mu_1 < 0$, if the initial capital $k(0)$ is smaller than the steady state k^*, then (5.71) implies $c(0)$ is also smaller than the steady state c^*. In addition, by theorem 5.1, the dynamic general equilibrium sequence of consumption $\{c(t), t \geq 0\}$ is a monotonically increasing sequence converging to the steady state c^*. (iv) The statement in (iii) also implies that if the initial capital $k(0)$ is larger than the steady state k^*, then (5.71) implies $c(0)$ is larger than the steady state c^*. In addition, the dynamic general equilibrium sequence of consumption $\{c(t), t \geq 0\}$ is a monotonically decreasing sequence converging to the steady state c^*.

In the continuous time Ramsey model, when the characteristic roots of (5.56) satisfies $\{\mu_1 < 0$ and $\mu_2 > 0\}$, the steady state $\{k^*, c^*\}$ of the dynamic general equilibrium is said to be a "saddle point" of the differential equations (5.50) and (5.51).

By (5.24), (5.27), and (5.28), the output per labor (labor productivity), the interest rate, and the wage rate in the dynamic general equilibrium are given as follows.

(5.73) $y(t) = A(k(t))^{\beta}(l(t))^{1-\beta}$

(5.74) $r(t) = \beta A(k(t))^{\beta-1}(l(t))^{1-\beta}$

(5.75) $w(t) = (1-\beta)A(k(t))^{\beta}(l(t))^{-\beta}$

The aggregate variables are obtained by multiplying per-household variables with the number of households and by multiplying per-firm variables with the number of firms. (In this example, the number of households and the number of firms are assumed to be same.)

5-4. Phase-Diagram Analysis

We can characterize the dynamic general equilibrium sequence of capital and consumption $\{k(t), c(t)\ ;\ t \geq 0\}$ by the phase-diagram analysis of differential equations {(5.35), (5.36)} as follows. These equations are replicated here.

(5.76) $\dot{k}(t) = Ak(t)^{\beta} - c(t) \equiv F_1(k(t), c(t))$

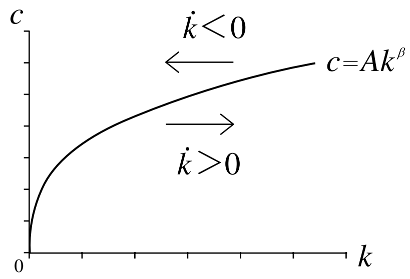
(5.77) $\dot{c}(t) = \dfrac{1}{\sigma}(\beta Ak(t)^{\beta-1} - \theta)c(t) \equiv F_2(k(t), c(t))$

(5.76) implies that $\dot{k}(t) = 0$ when capital $k(t)$ and consumption $c(t)$ satisfy the following relationship.

(5.78) $c(t) = Ak(t)^{\beta}$

Figure 5.1 depicts the locus of the combinations of capital and consumption that satisfy (5.78). In the figure, the horizontal axis measures capital k, and the vertical axis measures consumption c. (5.76) also implies that $\dot{k}(t) = Ak(t)^{\beta} - c(t) < 0$ holds at any point (k, c) above the locus, while $\dot{k}(t) = Ak(t)^{\beta} - c(t) > 0$ holds at any point (k, c) below the locus. Therefore, capital $k(t)$ is decreasing above the locus, while it is increasing below the locus. In figure 5.1, the left-pointing arrow above the locus indicates the decrease of capital, while the right-pointing arrow below the locus indicates the increase of capital.

Figure 5.1



(5.77) implies that $\dot{c}(t) = 0$ when capital $k(t)$ satisfies

(5.79) $\beta Ak(t)^{\beta-1} - \theta = 0$

regardless of consumption $c(t)$. In fact, (5.79) implies that $\dot{c}(t) = 0$ when capital $k(t)$ is at the steady state $k^* = (\beta A/\theta)^{1/(1-\beta)}$. (5.77) also implies that if $k(t) < k^*$, then $\dot{c} = (1/\sigma)(\beta Ak^{\beta-1} - \theta)\ c > 0$, or if $k(t) > k^*$, then $\dot{c} = (1/\sigma)(\beta Ak^{\beta-1} - \theta)\ c < 0$. The vertical line in figure 5.2 indicates $k(t) = k^*$. In figure 5.2, the horizontal axis measures capital and the vertical axis measures consumption. At any point (k, c) on the left of the vertical line, $k(t) < k^*$ holds. Therefore, consumption is increasing at the point, i.e., $\dot{c}(t) > 0$. The upward-pointing arrow at the left of the vertical line in figure 5.2 indicates that consumption is increasing. By the same reason, at any point (k, c) on the right of the vertical line, $k(t) > k^*$ holds. Therefore, consumption is decreasing at the point, i.e., $\dot{c}(t) < 0$. The downward-pointing arrow at the right of the vertical line in figure 5.2 indicates that consumption is decreasing.

By combining figure 5.1 and figure 5.2, the positive orthant of capital and consumption is divided into four phases as depicted by figure 5.3. These phases are labeled as phase Ⅰ, phase Ⅱ, phase Ⅲ, and phase Ⅳ.

At any point (k, c) in phase Ⅰ, capital $k(t)$ is decreasing because the point is above the $c = Ak^{\beta}$ locus, and consumption $c(t)$ is decreasing because the point is on the right of the vertical line indicating

Figure 5.2

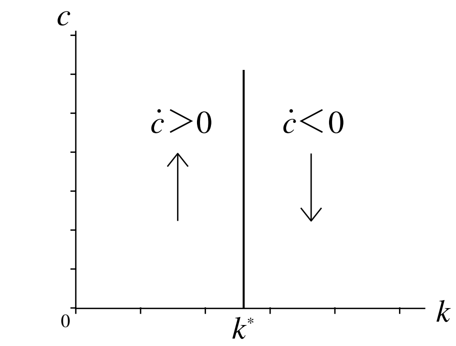
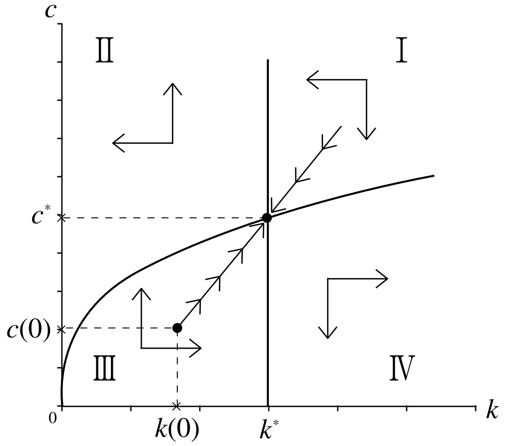


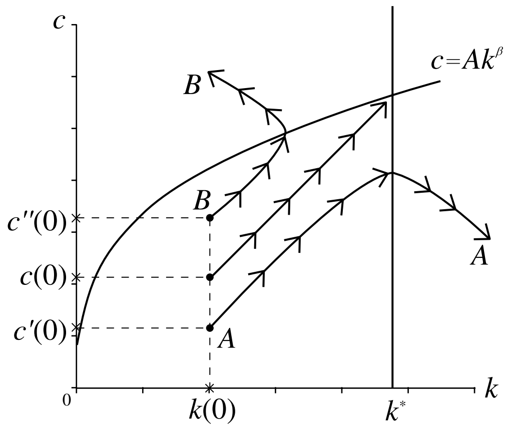
Figure 5.3



$k(t) = k^*$. By the same reason, $k(t)$ is decreasing and $c(t)$ is increasing in phase Ⅱ, $k(t)$ is increasing and $c(t)$ is increasing in phase Ⅲ, and $k(t)$ is increasing and $c(t)$ is decreasing in phase Ⅳ. From these observations, we can state that a sequence of capital and consumption $\{k(t), c(t) ; t \geq 0\}$ generated by the differential equations {(5.76), (5.77)} starting from $\{k(0), c(0)\}$ at $t = 0$ belonging to either phase Ⅱ or phase Ⅳ does not converge to the steady state $\{k^*, c^*\}$. In figure 5.3, the unique intersection of the locus of $c = Ak^{\beta}$ and the vertical line at $k(t) = k^*$ indicates the steady state $\{k^*, c^*\}$ of the differential equations {(5.76), (5.77)} because both $\dot{k} = 0$ and $\dot{c} = 0$ are satisfied. A dynamic general equilibrium sequence of capital and consumption $\{k(t), c(t) ; t \geq 0\}$ that satisfies the initial condition $k(0)$ and the transversality condition (5.7) must belong to either phase Ⅰ or phase Ⅲ. In figure 5.3, the downward-sloping dotted-arrow pointing to the steady state $\{k^*, c^*\}$ in phase Ⅰ and the upward-sloping dotted-arrow pointing to the steady state $\{k^*, c^*\}$ in phase Ⅲ indicate the possible dynamic general equilibrium sequence of capital and consumption. For example, consider an initial capital $k(0)$ at $t = 0$ which is smaller than the steady state capital k^*. As depicted in figure 5.3, there is a unique initial consumption $c(0)$ such that $\{k(0), c(0)\}$ is on the dynamic general equilibrium trajectory. As time t passes, the sequence of capital and consumption $\{k(t), c(t) ; t \geq 0\}$ generated by the differential equations {(5.76), (5.77)} stays on the dynamic general equilibrium trajectory converging to the steady state $\{k^*, c^*\}$. The dotted-arrow implies that along the process, both capital and consumption are monotonically increasing sequences converging to the steady state $\{k^*, c^*\}$.

The dynamic general equilibrium trajectory of capital and consumption is unique. Given the initial capital $k(0)$, consider an initial consumption $c'(0)$ that is below the dynamic general equilibrium trajectory, or an initial consumption $c''(0)$ that is above the dynamic general equilibrium trajectory. These two alternative initial consumptions are depicted in figure 5.4. In figure 5.4, the trajectory of capital and consumption generated by the difference equations {(5.76), (5.77)} starting with initial condition $\{k(0), c'(0)\}$ is labeled as AA. For awhile, capital and consumption increase because they are in phase Ⅲ. Compared to the capital and consumption on the dynamic general equilibrium trajectory, however, consumption is smaller and capital accumulation is faster. As a result, at some time t, capital and consumption enter phase Ⅳ. As time t proceeds further, either the non-negative constraint on consumption or the transversality condition for capital will be violated. On the other hand, in figure 5.4, the trajectory of capital and consumption generated by the difference equations {(5.76), (5.77)} starting with initial condition $\{k(0), c''(0)\}$ is labeled as BB. The capital and consumption increase for a while because they are in phase Ⅲ too. Compared to the capital and consumption on the dynamic general equilibrium trajectory, consumption is larger and capital accumulation is slower. As a result, at some time t, capital and consumption enter phase Ⅱ where the capital decreases while the consumption increases. as time t proceeds further, non-negativity constraint on capital will be violated.

Figure 5.4



If the initial capital $k(0)$ at $t=0$ is larger than the steady state k^*, then there is a unique initial consumption $c(0)$ such that $\{k(0), c(0)\}$ is on the dynamic general equilibrium trajectory in phase Ⅰ. Starting with $\{k(0), c(0)\}$, the differential equations $\{(5.76), (5.77)\}$ generate a monotonically decreasing sequence of capital and consumption $\{k(t), c(t)\ ;\ t \geq 0\}$ converging to the steady state.

These properties are summarized by the following theorem.

Theorem 5.3. Given the initial capital $k(0)$ at $t=0$, there is a unique consumption $c(0)$ such that (i) if $k(0)$ is smaller than the steady state k^*, then $c(0)$ is smaller than the steady state c^*, and the dynamic general equilibrium sequence of capital and consumption $\{k(t), c(t)\ ;\ t \geq 0\}$ generated by the differential equations $\{(5.76), (5.77)\}$ is a monotonically increasing sequence converging to the steady state $\{k^*, c^*\}$, or (ii) if $k(0)$ is larger than the steady state k^*, then $c(0)$ is larger than the steady state c^*, and the dynamic general equilibrium sequence of capital and consumption $\{k(t), c(t)\ ;\ t \geq 0\}$ generated by the differential equations $\{(5.76), (5.77)\}$ is a monotonically decreasing sequence converging to the steady state $\{k^*, c^*\}$.

Although these properties of the dynamic general equilibrium capital and consumption are similar to the properties stated by theorem 5.1 and theorem 5.2, theorem 5.3 holds as global properties, while theorem 5.1 and theorem 5.2 hold as local properties of the linearly approximated differential equations.

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