Large time asymptotic problems for optimal stochastic control with superlinear cost

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Abstract

The paper is concerned with stochastic control problems of finite time horizon whose running cost function is of superlinear growth with respect to the control variable. We prove that, as the time horizon tends to infinity, the value function converges to a function of variable separation type which is characterized by an ergodic stochastic control problem. Asymptotic problems of this type arise in utility maximization problems in mathematical finance. From the PDE viewpoint, our results concern the large time behavior of solutions to semilinear parabolic equations with superlinear nonlinearity in gradients.

1 Introduction

In this paper we deal with optimal stochastic control problems, or stochastic calculus of variations, having some specific cost functions. As a typical model, we consider for given $m^* > 1$ and $\beta > 0$ the following minimizing problem of finite time horizon:

$$\text{Minimize} \qquad E^x\Big[\int_0^T \Big(\frac{1}{m^*}|\xi_t|^{m^*} + |X_t^\xi|^\beta\Big)\,dt\Big], \tag{1.1}$$

$$\text{subject to} \qquad X_t^\xi = X_0 - \int_0^t \xi_s\,ds + W_t, \quad t \geq 0, \tag{1.2}$$

where $\xi = (\xi_t)_{0\leq t\leq T}$ denotes a control process taking its values in $\mathbb{R}^N$, and $W = (W_t)_{0\leq t\leq T}$ stands for an N-dimensional standard Brownian motion on some probability space (see [8, 10] for general information on optimal stochastic control).

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2010 MSC: Primary 93E20; Secondary 60J60, 49L20.

Keywords: Stochastic control, large time behavior, Hamilton-Jacobi-Bellman equation, ergodic control.

Short title: Asymptotic problems for optimal stochastic control.

The objective of this paper is to investigate the asymptotic behavior of the value function $u_V(T,x)$ for (1.1)-(1.2) as time horizon T tends to infinity. It turns out that u_V behaves as

$$u_V(T,\,\cdot\,) - \lambda T - \phi(\,\cdot\,) \longrightarrow 0 \quad \text{in} \;\; C(\mathbb{R}^N) \;\; \text{as} \;\; T \to \infty, \tag{1.3}$$

for some real constant λ and function ϕ on $\mathbb{R}^N$ that are characterized by an ergodic stochastic control problem. More specifically, λ is represented as

$$\lambda = \inf_{\xi} \liminf_{T\to\infty} \frac{1}{T} E^0\Big[\int_0^T \Big(\frac{1}{m^*}|\xi_t|^{m^*} + |X_t^\xi|^\beta\Big)\,dt\Big], \tag{1.4}$$

and the function $\underline{\xi}(x) := |D\phi(x)|^{(2-m^*)/(m^*-1)} D\phi(x)$, where $D\phi$ denotes the gradient of ϕ, gives an optimal Markov control policy for (1.4). The precise formulation will be given in the next section. We refer to [3] and the references therein for ergodic stochastic control in $\mathbb{R}^N$. Remark that (1.3) implies

$$\frac{u_V(T,\,\cdot\,)}{T} \longrightarrow \lambda \quad \text{in} \;\; C(\mathbb{R}^N) \;\; \text{as} \;\; T \to \infty. \tag{1.5}$$

Although properties (1.3) and (1.5) are natural, proving their validity is not obvious even in this simple model. The major difficulty comes from the fact that the control region for $\xi = (\xi_t)_{0\le t\le T}$ is not compact and the running cost function in (1.1) is unbounded with respect to both control and space variables.

The analytical counterpart of the above problem can be described as follows. Let $m > 1$ be the conjugate number of m^*, i.e., $m := m^*/(m^*-1)$. Then, u_V is a solution to the Cauchy problem for Hamilton-Jacobi-Bellman (or viscous Hamilton-Jacobi) equation

$$\begin{cases} \partial_t u - \dfrac{1}{2}\Delta u + \dfrac{1}{m}|Du|^m = |x|^\beta & \text{in} \;\; (0,+\infty)\times\mathbb{R}^N, \\ u(0,\,\cdot\,) = 0 & \text{in} \;\; \mathbb{R}^N, \end{cases} \tag{1.6}$$

where $\partial_t := \partial/\partial t$ and $\Delta := \sum_{i=1}^N \partial^2/\partial x_i^2$, while (λ,ϕ) in (1.3) is a solution to the associated ergodic type Hamilton-Jacobi-Bellman equation

$$\lambda - \frac{1}{2}\Delta\phi + \frac{1}{m}|D\phi|^m = |x|^\beta \quad \text{in} \;\; \mathbb{R}^N. \tag{1.7}$$

Thus, from the PDE point of view, our study concerns the convergence as $T \to \infty$ of solutions of (1.6) to that of (1.7). Asymptotics of type (1.3) for solutions of viscous Hamilton-Jacobi equations have been studied in [1, 2, 11, 24, 26] by purely analytical methods. See [1] for results under the periodic setting, [2, 24, 26] under Dirichlet boundary conditions, and [11, 24] for equations in the whole space. Compared to these earlier works, the principal novelty of this paper lies in the unbounded nature of the problem. In our setting, the superlinear nonlinearity in gradients for (1.6)-(1.7) is essential since it naturally happens that $|Du_V| \to \infty$ as $|x| \to \infty$. This makes a substantial contrast to [11, 24] where Du_V remains bounded on the whole space.

The large time behavior of solutions to Hamilton-Jacobi-Bellman equations has also been studied in the context of risk-sensitive stochastic control (see [7, 9, 13, 21, 22]). In connection with utility maximization problems in mathematical finance, Hata, Nagai and Sheu [13] and Nagai [22] discuss down-side risk minimization problems in which the convergence of type (1.5) arises on the dual side of the large deviations control. In those papers, they derived a family of Hamilton-Jacobi-Bellman equations with quadratic nonlinearity in gradients, and it turns out that establishing (1.5) for solutions of such equations is the key to solving the original problem.

In this paper, we focus on the analysis of (1.5), as well as (1.3), to develop a general theory available for Hamilton-Jacobi-Bellman equations not necessarily quadratic in gradients. Although cost functions of type (1.1) are natural and typical in the classical stochastic control theory, the analysis becomes more involved when $m^* > 2$. In such superquadratic cases, it is crucial to specify the growth order of $\phi(x)$ in (1.3) as $|x| \to \infty$, whereas this kind of estimates are unnecessary for $1 < m^* \leq 2$ (cf. [16]).

Another point to be mentioned is that we show not only (1.5) but also the refined convergence (1.3). Notice here that (1.3) is not an easy corollary. Indeed, the function ϕ in (1.3) is sensitive to the terminal cost while λ in (1.5) is not. That is, if the payoff (1.1) contains a terminal cost, say $g(X_T^\xi)$, in addition to the running cost, then ϕ may vary according to the choice of g. See Section 2 for the precise statement. We remark finally that the convergence (1.3) has an interpretation in terms of indifference pricing for volatility derivatives in incomplete markets. We refer, for instance, to [12] and the references therein for more information in this direction. Applications of our results to this topic will be discussed in a future work.

This paper is organized as follows. In the next section, we state our assumptions and main results precisely. Our framework admits slightly general cost functions than (1.1). In Section 3, we study the dynamic programming equation for value function u_V. Section 4 is concerned with the dynamic programming equation associated with ergodic stochastic control (1.4). Asymptotic behaviors (1.3) and (1.5) are studied in Section 5. Appendices are devoted to some technical estimates needed in this paper.

2 Preliminaries and Main results

Let $(\Omega, \mathcal{F}, P; (\mathcal{F}_t)_{t\geq 0})$ be a filtered probability space on which is defined an $(\mathcal{F}_t)$-adapted standard Brownian motion $W = (W_t)_{t\geq 0}$ in $\mathbb{R}^N$. For a given $\mathbb{R}^N$-valued $(\mathcal{F}_t)$-progressively measurable control process $\xi = (\xi_t)_{t\geq 0}$, we denote by $X^\xi = (X_t^\xi)_{t\geq 0}$

the controlled process governed by (1.2). Let us define the cost functional of finite time horizon $T > 0$ by

$$J_T(x;\xi) := E^x\Big[\int_0^T (l(X_t^\xi,\xi_t) + f(X_t^\xi))\,dt + g(X_T^\xi)\Big], \quad x \in \mathbb{R}^N, \tag{2.1}$$

and that of long-run average by

$$J_\infty(\xi) := \liminf_{T\to\infty} \frac{1}{T} E^0\Big[\int_0^T (l(X_t^\xi,\xi_t) + f(X_t^\xi))\,dt + g(X_T^\xi)\Big], \tag{2.2}$$

where $E^x[\,\cdot\,]$ denotes the expectation conditioning $X_0 = x$ in (1.2). Throughout the paper, functions l, f and g are assumed to satisfy the following conditions (H1)-(H3):

(H1) $l \in C^2(\mathbb{R}^N \times (\mathbb{R}^N \setminus \{0\}))$, $\xi \mapsto l(x,\xi)$ is strictly convex for all $x \in \mathbb{R}^N$, and there exist some $l_0 > 0$ and $m^* > 1$ such that

$$l_0|\xi|^{m^*} \le l(x,\xi) \le l_0^{-1}|\xi|^{m^*}, \quad |D_x l(x,\xi)| \le l_0^{-1}(1 + |\xi|^{m^*}), \quad (x,\xi) \in \mathbb{R}^{2N},$$

where $D_x l(x,\xi)$ is the partial derivative of $l(x,\xi)$ with respect to x.

(H2) $f \in C^2(\mathbb{R}^N)$, and there exist constants $f_0 > 0$ and $\beta > 0$ such that

$$f_0|x|^\beta - f_0^{-1} \le f(x) \le f_0^{-1}(1 + |x|^\beta), \quad |Df(x)| \le f_0^{-1}(1 + |x|^{\beta-1}), \quad x \in \mathbb{R}^N.$$

(H3) $g \in \Phi_0 := \{v \in C_p(\mathbb{R}^N) \mid \inf_{\mathbb{R}^N} v > -\infty\}$.

Here $C_p(\mathbb{R}^N)$ denotes the totality of continuous functions on $\mathbb{R}^N$ that are at most polynomially growing, i.e., $|v(x)| \le C(1 + |x|^q)$ in $\mathbb{R}^N$ for some $C > 0$ and $q > 0$.

Let $h = h(x,p)$ be the Fenchel-Legendre transform of $l(x,\xi)$ with respect to ξ, i.e.,

$$h(x,p) := \sup_{\xi \in \mathbb{R}^N} (p \cdot \xi - l(x,\xi)), \quad (x,p) \in \mathbb{R}^{2N}. \tag{2.3}$$

In view of the duality between l and h, we see that (H1) is equivalent to (H1)$'$ below:

(H1)$'$ $h \in C^2(\mathbb{R}^N \times (\mathbb{R}^N \setminus \{0\}))$, $p \mapsto h(x,p)$ is strictly convex for all $x \in \mathbb{R}^N$, and there exist some $h_0 > 0$ and $m > 1$ such that

$$h_0|p|^m \le h(x,p) \le h_0^{-1}|p|^m, \quad |D_x h(x,p)| \le h_0^{-1}(1 + |p|^m), \quad (x,p) \in \mathbb{R}^{2N}.$$

Notice here that $1/m^* + 1/m = 1$. The equivalence between (H1) and (H1)$'$ can be seen from Theorem 3.4 in the next section.

We now define the set of admissible control processes. For $T > 0$, a control process $\xi = (\xi_t)_{0 \le t \le T}$ is called admissible if

$$E^x\Big[\int_0^T (|\xi_t|^{m^*} + |X_t^\xi|^\beta)\,dt\Big] < \infty, \quad x \in \mathbb{R}^N. \tag{2.4}$$

We denote by $\mathcal{A}_T$ the totality of admissible control processes. As far as the ergodic stochastic control for (2.2) is concerned, we use the notation $\mathcal{A}_\infty$ to represent the set of control processes $\xi = (\xi_t)_{t\geq 0}$ satisfying (2.4) for all $T > 0$.

Let us consider the minimizing problem for (2.1), and denote its value function by

$$u_V(T, x) := \inf_{\xi\in\mathcal{A}_T} J_T(x; \xi). \tag{2.5}$$

In Section 3, we prove that u_V is a solution to the Cauchy problem

$$\begin{cases} \partial_t u - \dfrac{1}{2}\Delta u + h(x, Du) = f & \text{in } \ Q, \\ u(0, \,\cdot\,) = g & \text{on } \ \partial_p Q, \end{cases} \tag{CP}$$

where $Q := (0, \infty) \times \mathbb{R}^N$ and $\partial_p Q := \{0\} \times \mathbb{R}^N$. In the present paper, any solution is understood in the classical sense, namely, we call a function $u : \overline{Q} \longrightarrow \mathbb{R}$ solution (resp. subsolution, supersolution) of (CP) if $u \in C^{1,2}(Q) \cap C_p(\overline{Q})$ and

$$\frac{\partial u}{\partial t}(t, x) - \frac{1}{2}\Delta u(t, x) + h(x, Du(t, x)) = f(x) \qquad (\text{resp. } \leq f(x), \quad \geq f(x))$$

for all $(t, x) \in Q$, and $u(0, x) = g(x)$ (resp. $\leq g(x)$, $\geq g(x)$) for all $x \in \mathbb{R}^N$. Here $C_p(\overline{Q})$ stands for the set of continuous functions v on $\overline{Q}$ such that, for each $T > 0$, $|v(t, x)| \leq C(1 + |x|^q)$ in $[0, T] \times \mathbb{R}^N$ for some $C > 0$ and $q > 0$.

For later use, we set $Q_T := (0, T) \times \mathbb{R}^N$ and

$$\Phi := \{u \in C^{1,2}(Q) \cap C_p(\overline{Q}) \mid \inf_{Q_T} u > -\infty \ \text{ for all } \ T > 0\}.$$

Theorem 2.1. *Assume (H1)-(H3). Let u_V be the value function defined by (2.5). Then u_V belongs to Φ and is the minimal solution of (CP) in the sense that $u_V \leq v$ for any solution v of (CP) such that $v \in \Phi$. Moreover, if $1 < m^* \leq 2$ in (H1), then it is the unique solution of (CP) in the class Φ.*

Let us consider the stationary equation

$$\lambda - \frac{1}{2}\Delta\phi + h(x, D\phi) = f \quad \text{in } \ \mathbb{R}^N, \quad \phi(0) = 0, \tag{EP}$$

where unknown is a pair $(\lambda, \phi) \in \mathbb{R} \times C^2(\mathbb{R}^N)$. The constraint $\phi(0) = 0$ is imposed to avoid the ambiguity of additive constant with respect to ϕ. In section 4, we study the solvability of (EP). For a given $\gamma > 0$, we set

$$\Phi_\gamma := \{v \in C^2(\mathbb{R}^N) \cap C_p(\mathbb{R}^N) \mid \liminf_{|x|\to\infty} \frac{v(x)}{|x|^\gamma} > 0\,\}.$$

Clearly, $\Phi_\gamma \subset \Phi_{\gamma'} \subset \Phi_0$ for all $\gamma \geq \gamma' > 0$, where Φ_0 is defined by (H3).

Theorem 2.2. *Assume (H1) and (H2). Then, there exists a unique solution* (λ, φ) *of (EP) such that* $\varphi \in \Phi_0$*. Moreover,* φ *belongs to* $\Phi_{(\beta/m)+1}$*, where* $m := m^*/(m^* - 1)$ *and* $m^* > 1$*,* $\beta > 0$ *are the constants in (H1) and (H2), respectively.*

We are now in a position to state our main results. Let us consider the minimizing problem for (2.2), and set

$$\lambda_\infty := \inf_{\xi \in \mathcal{A}_\infty} J_\infty(\xi). \tag{2.6}$$

In Section 5, we prove the following.

Theorem 2.3. *Assume (H1)-(H3). Let* u_V *be the value function defined by (2.5), and let* (λ, φ) *be the unique solution of (EP) such that* $\varphi \in \Phi_0$*. Then,*

$$\frac{u_V(T, \cdot\,)}{T} \longrightarrow \lambda \quad \text{in} \;\; C(\mathbb{R}^N) \quad \text{as} \;\; T \to \infty. \tag{2.7}$$

Moreover, $\lambda = \lambda_\infty$*, and* $\underline{\xi}(x) := D_p h(x, D\varphi(x))$ *gives an optimal Markov control policy for ergodic stochastic control (2.6).*

Theorem 2.4. *In addition to the hypothesis of Theorem 2.3, we assume that* $\beta \geq m^*$*, where* $m^* > 1$ *and* $\beta > 0$ *are the constants in (H1) and (H2), respectively. Then, there exists a real constant* c *such that*

$$u_V(T, \cdot\,) - (\varphi(\,\cdot\,) + \lambda T) \longrightarrow c \quad \text{in} \;\; C(\mathbb{R}^N) \quad \text{as} \;\; T \to \infty.$$

3 Proof of Theorem 2.1

This section is devoted to the proof of Theorem 2.1, namely, we show that u_V defined by (2.5) is the minimal solution of (CP). The proof is divided into two parts. In the first half, we construct a particular solution of (CP), denoted by $\bar{u}$, such that $\bar{u} \leq u_V$ (see Theorem 3.3). In the second half, we verify the identity $\bar{u} = u_V$ by establishing a comparison theorem (Proposition 3.8) for solutions of (CP). Minimality of u_V is also derived from the same comparison principle.

Throughout the paper, $m, m^* > 1$ and $\beta > 0$ denote the constants in (H1), (H1)$'$, and (H2), respectively. Recall that $1/m + 1/m^* = 1$. We also use the notation $B_R := \{x \in \mathbb{R}^N \mid |x| < R\}$ for $R > 0$.

3.1 Existence of a solution.

Let us consider Cauchy problem (CP). We construct a solution of (CP) by a suitable approximation procedure. Let $\{f_n\} \subset C_b^\infty(\mathbb{R}^N)$ be a sequence of functions such that $\inf_{\mathbb{R}^N} f \leq f_n \leq f \wedge n$, $|Df_n| \leq |Df|$ in $\mathbb{R}^N$ for all n, and $f_n \to f$ in $C(\mathbb{R}^N)$ as

$n \to \infty$. Loosely speaking, f_n is a regularization of $f \wedge n$. Similarly, we fix a sequence $\{g_n\} \subset C_b^\infty(\mathbb{R}^N)$ such that $\inf_{\mathbb{R}^N} g \le g_n \le g \wedge n$ in $\mathbb{R}^N$ for all n and $g_n \to g$ in $C(\mathbb{R}^N)$ as $n \to \infty$.

For each n, we define the cost functional $J_T^{(n)}$ by

$$J_T^{(n)}(x;\xi) := E^x\Big[\int_0^T (l(X_t^\xi,\xi_t) + f_n(X_t^\xi))\,dt + g_n(X_T^\xi)\Big], \tag{3.1}$$

and its value function $u_V^{(n)}$ by

$$u_V^{(n)}(T,x) := \inf_{\xi\in\mathcal{A}_T} J_T^{(n)}(x;\xi), \quad (T,x) \in Q. \tag{3.2}$$

Theorem 3.1. $u_V^{(n)}$ *is the unique solution of*

$$\begin{cases} \partial_t u - \dfrac{1}{2}\Delta u + h(x,Du) = f_n & \text{in } Q, \\ u(0,\,\cdot\,) = g_n & \text{on } \partial_p Q, \end{cases} \tag{CP$_n$}$$

such that $\sup_{Q_T}(|u| + |Du|) < \infty$ *for all* $T > 0$.

Proof. The assertion of this theorem has been proved in [10, Theorem IV.11.1, Remark IV.11.2], so that we omit to reproduce the proof. □

The following theorem gives a gradient estimate for solutions of (CP)

Theorem 3.2. *Let u be a solution of (CP). Then, for any $\varepsilon \in (0,1)$, $r > 0$ and $\delta \in (0,1)$, there exists a constant $K > 0$ not depending on u and f such that*

$$\sup_{(\delta,T]\times B_r} |Du| \le K\{1 + \sup_{B_{r+1}} |f| + \sup_{B_{r+1}} |Df| + \sup_{(\delta/2,T]\times B_{r+1}} |u|\}^{1+\varepsilon}.$$

Proof. We prove this theorem in Appendix A (see Theorem A.1). □

Theorem 3.3. *There exists a solution $\bar{u} \in \Phi$ of (CP) such that $\bar{u} \le u_V$ in Q.*

Proof. Define $u_-, u_+ : \overline{Q} \longrightarrow \mathbb{R}$ by

$$u_-(T,x) := T \inf_{\mathbb{R}^N} f + \inf_{\mathbb{R}^N} g, \qquad u_+(T,x) := E^x\Big[\int_0^T f(W_t)\,dt + g(W_T)\Big].$$

Remark that u_- and u_+ are sub- and supersolutions of (CP). Let $u_V^{(n)}$ be the solution of (CP$_n$) given by (3.2). By the definition of $u_\pm$ and $u_V^{(n)}$, we see that $u_- \le u_V^{(n)} \le u_+$ in Q for all n. Since $|f_n| \le |f|$ and $|Df_n| \le |Df|$ in $\mathbb{R}^N$, we see, in view of Theorem 3.2 with $u = u_V^{(n)}$ and $f = f_n$, that $\sup_{Q'} |Du_V^{(n)}|$ is bounded by a constant not depending on n for any $Q' \subset\subset Q$. Taking into account the classical regularity theory for quasilinear parabolic equation (e.g., [19, Theorem V.3.1]), there exists a $\theta \in (0,1)$ such that $D_i u_V^{(n)}$ belongs to Hölder space $C^{\frac{\theta}{2},\theta}(Q)$ for all $i = 1 \dots N$.

We now set $F_n(t,x) := f_n(x) - h(x, Du_V^{(n)}(t,x))$ and regard $u_V^{(n)}$ as a solution of the linear parabolic equation

$$\partial_t u - \frac{1}{2}\Delta u = F_n(t,x) \quad \text{in} \quad Q.$$

Then, it follows from Schauder's theory that the Hölder norm of $u_V^{(n)}$ in the space $C^{1+\frac{\theta}{2},2+\theta}(Q')$ is bounded by a constant not depending on n for any $Q' \subset\subset Q$. Hence, there exist a subsequence $\{n_j\}_j$ and a function $\bar{u} \in C^{1,2}(Q)$ such that, as $n \to \infty$, $u_V^{(n)}$, $\partial_t u_V^{(n)}/\partial t$, $Du_V^{(n)}$ and $D^2 u_V^{(n)}$ converge, respectively, to $\bar{u}$, $\partial_t \bar{u}/\partial t$, $D\bar{u}$ and $D^2\bar{u}$ uniformly on compacts. In particular, $\bar{u}$ satisfies (CP). It is also obvious from the definition of $u_V^{(n)}$ that $\bar{u} \in \Phi$ and $\bar{u} \le u_V$ in Q. Hence, the proof is complete. □

3.2 Minimality and uniqueness.

We establish in this subsection a couple of comparison theorems for sub- and supersolutions of (CP). We begin with recalling the duality between l and h.

Theorem 3.4. *Let $l = l(x,\xi)$ satisfy (H1), and let $h = h(x,p)$ be the function defined by (2.3). Then, the following (a)-(e) hold.*

(a) $h \in C^2(\mathbb{R}^N \times (\mathbb{R}^N \setminus \{0\}))$, and $p \mapsto h(x,p)$ is strictly convex for all $x \in \mathbb{R}^N$.

(b) $h(x,p) + l(x,\xi) \ge \xi \cdot p$ for any $x, p, \xi \in \mathbb{R}^N$. Moreover,

$$h(x,p) + l(x,\xi) = \xi \cdot p \iff \xi = D_p h(x,p) \iff p = D_\xi l(x,\xi).$$

(c) There exists a constant $h_0 > 0$ such that $h_0|p|^m \le h(x,p) \le h_0^{-1}|p|^m$ for all $x, p \in \mathbb{R}^N$, where $m := m^*/(m^*-1)$.

(d) There exist constants $h_1, l_1 > 0$ such that, for any $x, p, \xi \in \mathbb{R}^N$,

$$h_1|p|^{m-1} \le |D_p h(x,p)| \le h_1^{-1}|p|^{m-1}, \quad l_1|\xi|^{m^*-1} \le |D_\xi l(x,\xi)| \le l_1^{-1}|\xi|^{m^*-1}.$$

(e) There exists an $h_2 > 0$ such that $|D_x h(x,p)| \le h_2(1+|p|^m)$ for all $x, p \in \mathbb{R}^N$.

Proof. Assertions (a) and (b) can be verified in view of [5, Theorem A.2.5] with minor changes, so that we omit the proof. Verifying (c) is also easy from the very definition of h.

To check (d), we observe from (b) and (H1) that

$$l_0|D_p h(x,p)|^{m^*} \le l(x, D_p h(x,p)) = p \cdot D_p h(x,p) - h(x,p) \le |p||D_p h(x,p)|$$

for all $x, p \in \mathbb{R}^N$. Noting the relation $1/m^* + 1/m = 1$, we obtain

$$|D_p h(x,p)| \le (l_0^{-1}|p|)^{1/(m^*-1)} = l_0^{1-m}|p|^{m-1}.$$

On the other hand, we see by Young's inequality that

$$\begin{aligned}0 \le l_0|D_p h(x,p)|^{m^*} &\le |p||D_p h(x,p)| - h_0|p|^m \\ &\le \Big(\frac{h_0}{2}\Big)^{1-m^*}|D_p h(x,p)|^{m^*} - \frac{h_0}{2}|p|^m.\end{aligned}$$

In particular, $(h_0/2)|p|^{m-1} \le |D_p h(x,p)|$. Therefore, the first inequality is proved. The second inequality can be verified similarly.

We finally show (e). Observe first that $h(x,p) = p\cdot D_p h(x,p) - l(x, D_p h(x,p))$ for all $x, p \in \mathbb{R}^N$. Differentiating both sides by x and noting $p = D_\xi l(x, D_p h(x,p))$, we have

$$\begin{aligned}D_x h(x,p) &= D_{xp}h(x,p)p - D_x l(x, D_p h(x,p)) - D_{xp}h(x,p)D_\xi l(x, D_p h(x,p)) \\ &= -D_x l(x, D_p h(x,p)).\end{aligned}$$

In particular, using (d) and $(m-1)m^* = m$,

$$|D_x h(x,p)| = |D_x l(x, D_p h(x,p))| \le l_0^{-1}(1 + |D_p h(x,p)|^{m^*}) \le l_0^{-1}(1 + h_1^{-m^*}|p|^m).$$

Hence, the proof is complete. □

Now, we set $\alpha := (\beta/m) + 1$. This number will be frequently referred to in later discussions. Note that $\beta \ge m^*$ if and only if $\beta \ge \alpha$. Given a control process $\xi = (\xi_t)_{0\le t\le T}$, we denote by $X^\xi = (X^\xi_t)_{0\le t\le T}$ the controlled process governed by (1.2). Set $\tau_R := \inf\{t > 0 \,|\, X^\xi_t \notin B_R\}$ for $R > 0$. In what follows, unless otherwise specified, C denotes various positive constants that may take different values from line to line.

Lemma 3.5. *Suppose that $\xi \in \mathcal{A}_T$. Then $E^x\Big[\sup_{0\le t\le T}|X^\xi_t|^\alpha\Big] < \infty$ for all $x \in \mathbb{R}^N$.*

Proof. This lemma is easily verified by the standard argument. The proof is given in Appendix C for the convenience of the reader. □

The following result will be used in Sections 4 and 5.

Proposition 3.6. *Let u be a subsolution of (CP) for some $g \in C_p(\mathbb{R}^N)$ (not necessarily belonging to Φ_0), and suppose that $\sup_{Q_T}(|u|/(1+|x|^\alpha)) < \infty$ for all $T > 0$. Then, for any $x \in \mathbb{R}^N$ and $T, S \ge 0$,*

$$u(S+T,x) \le \inf_{\xi\in\mathcal{A}_T} E^x\Big[u(S, X^\xi_T) + \int_0^T (l(X^\xi_t, \xi_t) + f(X^\xi_t))\,dt\Big]. \tag{3.3}$$

In particular, $u \le u_V$ in Q.

Proof. Fix any $\xi \in \mathcal{A}_T$, and apply Ito's formula to $u(S+T-t, X_t^\xi)$. Then, noting Theorem 3.4 (b), as well as the subsolution property for u, we see that

$$u(S+T,x) \le E^x\Big[u(S+T-T\wedge\tau_R, X^\xi_{T\wedge\tau_R}) + \int_0^{T\wedge\tau_R}(l(X_t^\xi,\xi_t)+f(X_t^\xi))\,dt\Big].$$

We now send $R\to\infty$. Since $|u(t,x)| \le C(1+|x|^\alpha)$ in Q_{S+T} for some $C>0$, and l, f are bounded below, we conclude in view of Lemma 3.5 that

$$u(S+T,x) \le E^x\Big[u(S, X_T^\xi) + \int_0^T (l(X_t^\xi,\xi_t)+f(X_t^\xi))\,dt\Big].$$

Taking the inf over $\xi \in \mathcal{A}_T$, we obtain (3.3). □

Proposition 3.7. *Suppose that $1 < m^* \le 2$ in (H1), or equivalently $m \ge 2$ in (H1)′. Then, (3.3) is valid for any subsolution u of (CP) such that $u \in \Phi$. In particular, $u \le u_V$ in Q.*

Proof. Observe from $m \ge 2$ that, for any $\varepsilon > 0$, there exists a $\kappa_\varepsilon > 0$ such that

$$h(x,p) - h(x,q) - D_p h(x,q)\cdot(p-q) \ge \frac{\kappa_\varepsilon}{2}|p-q|^2 - \varepsilon, \quad x,p,q\in\mathbb{R}^N.$$

We can also see in view of Theorem 3.4 (b) that $\xi = D_p h(x, D_\xi l(x,\xi))$ for all $(x,\xi) \in \mathbb{R}^{2N}$, and that $h(x,q)+l(x,\xi) = \xi\cdot q$ if and only if $q = D_\xi l(x,\xi)$. Thus,

$$\begin{aligned} &h(x,p) + l(x,\xi) - \xi\cdot p \\ &\quad = h(x,p) - h(x, D_\xi l(x,\xi)) - D_p h(x, D_\xi l(x,\xi))\cdot(p - D_\xi l(x,\xi)) \\ &\quad \ge \frac{\kappa_\varepsilon}{2}|p - D_\xi l(x,\xi)|^2 - \varepsilon, \quad x,p,\xi\in\mathbb{R}^N. \end{aligned}$$

Let u be a subsolution of (CP) such that $u \in \Phi$, and fix any $\xi \in \mathcal{A}_T$. Then, by the previous estimate, we have

$$\begin{aligned} &u(S, X_T^\xi) - u(S+T,x) \\ &\quad = \int_0^T (h(X_t^\xi, Du) - \xi_t\cdot Du - f(X_t^\xi))\,dt + \int_0^T Du\,dW_t \\ &\quad \ge \int_0^T (-l(X_t^\xi,\xi_t) - f(X_t^\xi) + \frac{\kappa_\varepsilon}{2}|Du - q_t|^2 - \varepsilon)\,dt + \int_0^T Du\,dW_t, \end{aligned}$$

where we have set $Du = Du(S+T-t, X_t^\xi)$ and $q_t := D_\xi l(X_t^\xi, \xi_t)$. In particular,

$$\begin{aligned} &u(S, X_T^\xi) + \int_0^T (l(X_t^\xi,\xi_t) + f(X_t^\xi))\,dt - \int_0^T q_t\,dW_t \\ &\quad \ge u(S+T,x) - \varepsilon T + \frac{\kappa_\varepsilon}{2}\int_0^T |Du - q_t|^2\,dt + \int_0^T (Du - q_t)\,dW_t. \end{aligned}$$

In view of Theorem 3.4 (d), $|D_\xi l(x,\xi)|^m \le (l_1^{-1}|\xi|^{m^*-1})^m = l_1^{-m}|\xi|^{m^*}$ for all $(x,\xi) \in \mathbb{R}^{2N}$. This infers that $E^x[\int_0^T |q_t|^m\,dt] < \infty$. Hence, $\int_0^T q_t dW_t$ is an $(\mathcal{F}_t)$-martingale. Using Jensen's inequality, we have

$$\begin{aligned}
&E^x\Big[u(S,X_T^\xi) + \int_0^T (l(X_t^\xi,\xi_t) + f(X_t^\xi))\,dt\Big] \\
&\quad \ge E^x\Big[u(S+T,x) - \varepsilon T + \frac{\kappa_\varepsilon}{2}\int_0^T |Du - q_t|^2\,dt + \int_0^T (Du - q_t)\,dW_t\Big] \\
&\quad \ge -\frac{1}{\kappa_\varepsilon}\log E^x\Big[e^{-\kappa_\varepsilon(u(S+T,x)-\varepsilon T)-(\kappa_\varepsilon^2/2)\int_0^T |Du-q_t|^2\,dt - \kappa_\varepsilon\int_0^T (Du-q_t)\,dW_t}\Big] \\
&\quad \ge u(S+T,x) - \varepsilon T.
\end{aligned}$$

Sending $\varepsilon \to 0$, we conclude that (3.3) holds. $\square$

Proposition 3.8. *Let v be a supersolution of (CP) such that $v \in \Phi$. Then, for any $x \in \mathbb{R}^N$ and $T, S \ge 0$,*

$$v(S+T,x) \ge \inf_{\xi\in\mathcal{A}_T} E^x\Big[v(S,X_T^\xi) + \int_0^T (l(X_t^\xi,\xi_t) + f(X_t^\xi))\,dt\Big].$$

In particular, $v \ge u_V$ in Q.

Proof. Let $X^* = (X_t^*)$ be the diffusion process governed by

$$dX_t^* = -D_p h(X_t^*, Dv(T-t,X_t^*))\,dt + dW_t, \qquad 0 \le t < T\wedge\tau_\infty,$$

where $\tau_\infty := \lim_{R\to\infty}\tau_R$. We set $\xi_t^* := D_p h(X_t^*, Dv(T-t,X_t^*))$ for $0 \le t < T\wedge\tau_\infty$. Then, we observe that

$$l(X_t^*,\xi_t^*) + h(X_t^*,Dv) = \xi_t^*\cdot Dv, \quad Dv := Dv(T-t,X_t^*).$$

Applying Ito's formula to $v(S+T-t,X_t^*)$ and noting the supersolution property for v, we see that

$$\begin{aligned}
&v(S+T-T\wedge\tau_R, X^*_{T\wedge\tau_R}) + \int_0^{T\wedge\tau_R} (l(X_t^*,\xi_t^*) + f(X_t^*))\,dt \\
&\le v(S+T,x) + \int_0^{T\wedge\tau_R} (l(X_t^*,\xi_t^*) + h(X_t^*,Dv) - \xi_t^*\cdot Dv)\,dt + \int_0^{T\wedge\tau_R} Dv\,dW_t \\
&= v(S+T,x) + \int_0^{T\wedge\tau_R} Dv\,dW_t.
\end{aligned}$$

Taking expectation, we obtain

$$v(S+T,x) \ge E^x\Big[v(S+T-T\wedge\tau_R, X^*_{T\wedge\tau_R}) + \int_0^{T\wedge\tau_R} (l(X_t^*,\xi_t^*) + f(X_t^*))\,dt\Big].$$

Since l, f and v are bounded below on $\mathbb{R}^{2N}$, $\mathbb{R}^N$ and Q_{S+T}, respectively, we can apply Fatou's lemma to deduce that

$$v(S+T,x) \geq E^x\Big[v(S+T-T\wedge\tau_\infty, X^*_{T\wedge\tau_\infty}) + \int_0^{T\wedge\tau_\infty} (l(X^*_t,\xi^*_t) + f(X^*_t))\,dt\Big].$$

Notice here that $P^x(\tau_\infty < T) = 0$. Otherwise, $E^x[\int_0^{T\wedge\tau_\infty} f(X^*_t)\,dt] = \infty$, which does not agree with the last inequality. Thus, $P^x(T\wedge\tau_\infty = T) = 1$ and

$$v(S+T,x) \geq E^x\Big[v(S,X^*_T) + \int_0^T (l(X^*_t,\xi^*_t) + f(X^*_t))\,dt\Big].$$

Since $\xi^* \in \mathcal{A}_T$ in view of (H1) and (H2), we obtain the required estimate. □

Gathering the results of this section, we can prove Theorem 2.1.

Proof of Theorem 2.1. Let $\bar{u} \in \Phi$ be the solution of (CP) given in Theorem 3.3. Then $\bar{u} \leq u_V$ in Q. By Proposition 3.8, we also see that $\bar{u} \geq u_V$ in Q. Hence, $u_V = \bar{u}$ in Q. Furthermore, Proposition 3.8 implies that $u_V \leq v$ in Q for any solution v of (CP) such that $v \in \Phi$. Thus, u_V is the minimal solution of (CP) in the class Φ. Uniqueness under $1 < m^* \leq 2$ is a direct consequence of Proposition 3.7 in combination with Proposition 3.8. Hence, the proof of Theorem 2.1 is complete. □

Remark 3.9. *Let Φ' be the totality of $u \in \Phi$ such that $\sup_{Q_T}(|u|/(1+|x|^\alpha)) < \infty$ for all $T > 0$. Then, the uniqueness of solutions to (CP) in the class Φ' is valid as a direct consequence of Propositions 3.6 and 3.8. However, we do not know, in general, whether a solution of (CP) belongs to Φ' without assuming any upper bound for g. This is the reason why the uniqueness in the class Φ is not guaranteed for $m^* > 2$.*

4 Proof of Theorem 2.2

The proof of Theorem 2.2 is divided into two parts. We first construct a suitable solution of (EP) by a standard analytical approximation procedure. We then establish a uniqueness result using some probabilistic arguments.

4.1 Existence.

We begin with the following gradient estimate for solutions of (EP).

Theorem 4.1. *For any $r > 0$, there exists a constant $K > 0$ depending only on r, N, and the constants in (H1)$'$ such that for any solution (λ,ϕ) of (EP),*

$$\sup_{B_r}|D\phi| \leq K(1 + \sup_{B_{r+1}}|f-\lambda|^{1/m} + \sup_{B_{r+1}}|Df|^{1/(2m-1)}). \tag{4.1}$$

Proof. The proof of this theorem will be given in Appendix B (see Theorem B.1). $\square$

Proposition 4.2. *Let (λ, ϕ) be a solution of (EP). Then, there exists a $K > 0$ such that*

$$|D\phi(x)| \leq K(1 + |x|^{\alpha-1}), \quad |\phi(x)| \leq K(1 + |x|^{\alpha}), \quad x \in \mathbb{R}^N,$$

where $\alpha = (\beta/m) + 1$.

Proof. Fix any $r > 0$. Since $\beta/m = \alpha - 1$ and $(\beta - 1)/(2m - 1) < \alpha - 1$, we see by virtue of Theorem 4.1 that

$$\sup_{B_r} |D\phi| \leq C(1 + \sup_{B_{r+1}} |f - \lambda|^{1/m} + \sup_{B_{r+1}} |Df|^{1/(2m-1)}) \leq C + C(r + 1)^{\alpha-1}.$$

This yields the first estimate of this proposition. The second estimate is easily deduced from the first one. Hence, we have completed the proof. $\square$

In what follows, we use the notation

$$F[\psi](x) := -\frac{1}{2}\Delta\psi(x) + h(x, D\psi(x)) - f(x), \quad x \in \mathbb{R}^N, \quad \psi \in C^2(\mathbb{R}^N). \tag{4.2}$$

Lemma 4.3. *There exist constants $\nu_0 > 0$ and $\rho_0 \in (0, 1)$ such that, for any $\rho \in [-\rho_0, \rho_0]$ and $\gamma \in [0, \alpha]$, function $\phi_0(x) := \rho(1 + |x|^2)^{\gamma/2}$ satisfies*

$$F[\phi_0](x) \leq -\nu_0|x|^{\beta} + \nu_0^{-1}, \quad x \in \mathbb{R}^N. \tag{4.3}$$

Proof. Let $\rho \in [-1, 1]$ and $\gamma \in [0, \alpha]$. Observe that

$$\begin{aligned} D\phi_0(x) &= \gamma\rho(1 + |x|^2)^{(\gamma-2)/2}x, \\ \Delta\phi_0(x) &= \gamma\rho\{(\gamma + N - 2)|x|^2 + N\}(1 + |x|^2)^{(\gamma-4)/2}. \end{aligned}$$

Since $\gamma \leq \alpha$ implies $m(\gamma - 1) \leq \beta$, we see, in view of (H1)$'$, (H2) and $|\rho| \leq 1$, that

$$F[\phi_0](x) \leq C(1 + |x|^{\gamma-2} + |\rho|^m|x|^{m(\gamma-1)}) - f_0|x|^{\beta} \leq (|\rho|C - f_0)|x|^{\beta} + C$$

for some $C > 0$ independent of ρ and γ. Choosing $\rho_0 \in (0, 1)$ so small that $\rho_0 < C^{-1}f_0$ and setting $\nu_0 := \min\{f_0 - \rho_0 C, C^{-1}\}$, we obtain (4.3). $\square$

Lemma 4.4. *There exist a constant $\rho_1 > 1$ such that function $\psi_0(x) := \rho_1(1 + |x|^2)^{\alpha/2}$ satisfies $F[\psi_0](x) \geq -K_1$ in $\mathbb{R}^N$ for some $K_1 > 0$.*

Proof. Similarly as in the previous lemma, we easily see, in view of (H1)$'$, (H2) and $m(\alpha - 1) = \beta$, that

$$\begin{aligned} F[\psi_0](x) &\geq -\rho_1 C(1 + |x|^{\alpha-2}) + h_0\rho_1^m|x|^{m(\alpha-1)} - f_0^{-1}(1 + |x|^{\beta}) \\ &\geq (h_0\rho_1^m - \rho_1 C - f_0^{-1})|x|^{\beta} - C(1 + \rho_1) \end{aligned}$$

for some $C > 0$ not depending on ρ_1. Choosing ρ_1 so large that $h_0\rho_1^m - \rho_1 C - f_0^{-1} \geq 0$ and setting $K_1 := C(1 + \rho_1)$, we obtain the required estimate. $\square$

We now construct a solution (λ, φ) of (EP) such that $\varphi \in \Phi_\alpha$. For this purpose, fix any $\phi_0(x) := \rho_0(1+|x|^2)^{\gamma/2}$ satisfying (4.3) for some $\rho_0 \in (0,1)$ and $\gamma \in [\alpha \wedge \beta, \alpha]$. For $\varepsilon \in (0,1)$, let us consider the elliptic equation

$$F[v_\varepsilon] + \varepsilon v_\varepsilon = \varepsilon \phi_0 \quad \text{in } \mathbb{R}^N. \tag{4.4}$$

Proposition 4.5. *For any ε, there exists a solution $v_\varepsilon \in C^2(\mathbb{R}^N)$ of (4.4) such that $\varepsilon v_\varepsilon(0)$ is bounded uniformly in $\varepsilon \in (0,1)$.*

Proof. Let ψ_0 be the function given in Lemma 4.4. Fix any ε. By the definitions of ϕ_0 and ψ_0, we see that $\phi_0 \le \psi_0$ in $\mathbb{R}^N$. Moreover, $\phi_0 - 1/(\varepsilon\nu_0)$ and $\psi_0 + K_1/\varepsilon$ are sub- and supersolutions of (4.4), respectively.

For each $R > 0$, we consider the Dirichlet problem

$$F[v] + \varepsilon v = \varepsilon \phi_0 \quad \text{in } B_R, \qquad v = \phi_0 \quad \text{on } \partial B_R. \tag{4.5}$$

It is well known (e.g., [18, Theorem 4.8.3]) that (4.5) has a solution $v = v_R \in C^2(\overline{B}_R)$. We also see by the standard comparison theorem that $\phi_0 - 1/(\varepsilon\nu_0) \le v_R \le \psi_0 + K_1/\varepsilon$ in $\overline{B}_R$. Moreover, for any $r > 0$, there exists a $K > 0$ such that $\sup_{B_r} |Dv_R| \le K$ for all $R > r$ (see Theorem B.1 in Appendix B). These facts, together with the classical regularity theory for quasilinear elliptic equations (e.g., [18, Theorem 4.6.1]), imply that the Hölder norm $|Dv_R|_{\theta;B_r}$ for some $\theta \in (0,1)$ is bounded by a constant not depending on $R > r$. Applying Schauder's theory for linear elliptic equations, we also see that the Hölder norm $|v_R|_{2+\theta;B_r}$ is bounded by a constant not depending on $R > r$. In particular, the family $\{v_R\}_{R>r}$ is pre-compact in $C^2(\mathbb{R}^N)$, namely, there exist a sequence $\{R_j\}_j$ with $R_j \to \infty$ as $j \to \infty$, and a function $v \in C^2(\mathbb{R}^N)$ such that v_{R_j}, Dv_{R_j}, and $D^2 v_{R_j}$ converge, respectively, to v, Dv, and $D^2 v$ in $C(\mathbb{R}^N)$ as $j \to \infty$. Thus, we conclude that v is a solution of (4.4) satisfying

$$\phi_0(x) - \frac{1}{\varepsilon\nu_0} \le v(x) \le \psi_0(x) + \frac{K_1}{\varepsilon}, \quad x \in \mathbb{R}^N. \tag{4.6}$$

This implies also that $\varepsilon v(0)$ is bounded by a constant not depending on ε. Hence, we have completed the proof. □

The following lemma will be needed in Section 5.

Lemma 4.6. *Let $\phi_0(x) := \rho_0(1+|x|^2)^{(\alpha\wedge\beta)/2}$ satisfy (4.3) for some $\rho_0 \in (0,1)$. Then, for each $\varepsilon \in (0,1)$, there exists a supersolution ψ_ε of (4.4) such that*

$$\phi_0(x) \le \psi_\varepsilon(x) \le K_\varepsilon(1+|x|^2)^{(\alpha\wedge\beta)/2}, \quad x \in \mathbb{R}^N, \tag{4.7}$$

for some $K_\varepsilon > 1$.

Proof. Fix any $\varepsilon \in (0,1)$ and set $\psi(x) := \rho(1+|x|^2)^{(\alpha\wedge\beta)/2}$, where $\rho > 1$ will be determined later. Then, we observe that

$$\begin{aligned}
&F[\psi](x) + \varepsilon\psi(x) - \varepsilon\phi_0(x) \\
&\quad \geq -\rho C(1+|x|^{\alpha\wedge\beta-2}) + C^{-1}\rho^m|x|^{m(\alpha\wedge\beta-1)} - C(1+|x|^\beta) \\
&\qquad\quad + \varepsilon(\rho-1)(1+|x|^2)^{(\alpha\wedge\beta)/2} \\
&\quad \geq (C^{-1}\rho^m - C\rho)|x|^{m(\alpha\wedge\beta-1)} - C|x|^\beta + \varepsilon(\rho-1)(1+|x|^2)^{(\alpha\wedge\beta)/2} - C(1+\rho).
\end{aligned}$$

Here and in what follows, $C > 0$ denotes various constants not depending on ρ and ε.

We first consider the case where $\alpha\wedge\beta = \alpha$. Then $m(\alpha\wedge\beta-1) = m(\alpha-1) = \beta$. Choosing ρ so that $C^{-1}\rho^m - C\rho \geq C$ and setting $\psi_\varepsilon(x) := \psi(x) + C(1+\rho)/\varepsilon$, we see that ψ_ε is a supersolution of (4.4). Suppose next that $\alpha\wedge\beta = \beta$. In this case, we choose $\rho = \rho_\varepsilon$ so large that $C^{-1}\rho^m - C\rho \geq 0$ and $C|x|^\beta \leq \varepsilon(\rho-1)(1+|x|^2)^{\beta/2}$ in $\mathbb{R}^N$. Then $\psi_\varepsilon(x) := \psi(x) + C(1+\rho)/\varepsilon$ is a supersolution of (4.4). Estimate (4.7) can be verified in both cases by the definition of ψ_ε. Hence, the proof is complete. □

Proposition 4.7. *Let $\phi_0(x) := \rho_0(1+|x|^2)^{\gamma/2}$ be any function satisfying (4.3) for some $\rho_0 \in (0,1)$ and $\gamma \in [\alpha\wedge\beta, \alpha]$, and let v_ε be the solution of (4.4) constructed in Proposition 4.5. Set $\varphi_\varepsilon(x) := v_\varepsilon(x) - v_\varepsilon(0)$. Then, the family $\{\varphi_\varepsilon\}_{\varepsilon\in(0,1)}$ is pre-compact in $C^2(\mathbb{R}^N)$. Moreover, there exists a constant $M > 0$ such that $\varphi_\varepsilon \geq \phi_0 - M$ in $\mathbb{R}^N$ for all ε.*

Proof. Set $\lambda_\varepsilon := \varepsilon v_\varepsilon(0)$. Then $C_1 := \sup_{\varepsilon\in(0,1)}|\lambda_\varepsilon| < \infty$ and φ_ε is a solution of

$$\lambda_\varepsilon + F[\varphi_\varepsilon] + \varepsilon\varphi_\varepsilon = \varepsilon\phi_0 \quad \text{in } \mathbb{R}^N, \qquad \varphi_\varepsilon(0) = 0. \tag{4.8}$$

In view of Theorem B.1 in Appendix B and $\varphi_\varepsilon(0) = 0$, we observe that, for any $R > 0$, $\sup_{B_R}|\varphi_\varepsilon|$ and $\sup_{B_R}|D\varphi_\varepsilon|$ are bounded by a constant not depending on ε. In particular, by the same argument as in the proof of Proposition 4.5, we see that Hölder norm $|\varphi_\varepsilon|_{2+\theta;B_R}$ for some $\theta \in (0,1)$ is bounded uniformly in ε. Hence, $\{\varphi_\varepsilon\}_\varepsilon$ is pre-compact in $C^2(\mathbb{R}^N)$.

We next prove the latter claim. By the convexity of $F[\,\cdot\,]$ and Lemma 4.3, we see that, for any $\delta \in (1/2,1)$,

$$F[\delta\phi_0](x) \leq \delta F[\phi_0] + (1-\delta)F[0](x) \leq \nu_0^{-1} - \frac{f_0}{2}|x|^\beta + f_0^{-1}, \quad x \in \mathbb{R}^N,$$

where f_0 and ν_0 are the constants in (H2) and (4.3), respectively. Taking into account this estimate, we can choose an $R > 0$ such that $F[\delta\phi_0](x) \leq -C_1$ for all $|x| \geq R$ and $\delta \in (1/2,1)$, and then find an $M > 0$ such that $\sup_{0<\varepsilon<1}\sup_{B_R}(|\phi_0| + |\varphi_\varepsilon|) \leq M$. Notice that M is finite since $\sup_{B_R}|\varphi_\varepsilon|$ is bounded by a constant not depending on ε.

We now claim that $\varphi_\varepsilon \geq \delta\phi_0 - M$ in $\mathbb{R}^N$ for all $\delta \in (1/2, 1)$. To prove this, we first observe that $\varphi_\varepsilon(x) - \delta\phi_0(x) \geq -\sup_{B_R}(|\varphi_\varepsilon| + |\phi_0|) = -M$ for all $|x| \leq R$. On the other hand, since $\inf_{\mathbb{R}^N}(\varphi_\varepsilon - \phi_0) > -\infty$ by virtue of (4.6), and

$$\varphi_\varepsilon(x) - (\delta\phi_0(x) - M) = (\varphi_\varepsilon - \phi_0)(x) + (1-\delta)\phi_0(x) + M \longrightarrow \infty$$

as $|x| \to \infty$, we can find an $R_{\varepsilon,\delta} > R$ such that $\varphi_\varepsilon(x) \geq \delta\phi_0(x) - M$ for all $|x| \geq R_{\varepsilon,\delta}$.

Set $D := \{x \in \mathbb{R}^N \mid R < |x| < R_{\varepsilon,\delta}\}$. Then, for any $x \in D$, we have $F[\delta\phi_0 - M](x) + \varepsilon(\delta\phi_0(x) - M) \leq \varepsilon\phi_0(x) - C_1$ and $F[\varphi_\varepsilon](x) + \varepsilon\varphi_\varepsilon(x) \geq \varepsilon\phi_0 - C_1$. Therefore, $\delta\phi_0 - M$ and φ_ε are, respectively, sub- and supersolutions of

$$F[v] + \varepsilon v = \varepsilon\phi_0 - C_1 \quad \text{in} \quad D,$$

and satisfy $\delta\phi_0 - M \leq \varphi_\varepsilon$ on ∂D. Applying the standard comparison theorem, we obtain $\delta\phi_0 - M \leq \varphi_\varepsilon$ in D. Hence, $\delta\phi_0 - M \leq \varphi_\varepsilon$ in $\mathbb{R}^N$ for all $\delta \in (1/2, 1)$. Letting $\delta \to 1$, we conclude that $\phi_0 - M \leq \varphi_\varepsilon$ in $\mathbb{R}^N$. □

Theorem 4.8. *Let $\phi_0(x) := \rho_0(1 + |x|^2)^{\gamma/2}$ be any function satisfying (4.3) for some $\rho_0 \in (0, 1)$ and $\gamma \in [\alpha \wedge \beta, \alpha]$. Then there exists a solution (λ, φ) of (EP) such that $\inf_{\mathbb{R}^N}(\varphi - \phi_0) > -\infty$.*

Proof. Let v_ε be the solution of (4.4) given in Proposition 4.5. Set $\varphi_\varepsilon(x) := v_\varepsilon(x) - v_\varepsilon(0)$ and $\lambda_\varepsilon := \varepsilon v_\varepsilon(0)$. Then, by virtue of Proposition 4.7 and the fact that $\sup_\varepsilon |\lambda_\varepsilon| < \infty$, there exist a sequence $\{\varepsilon_n\}$ with $\varepsilon_n \to 0$ as $n \to \infty$, a real constant λ and a function $\varphi \in C^2(\mathbb{R}^N)$ such that $\lambda_{\varepsilon_n} \to \lambda$ and $\varphi_\varepsilon \to \varphi$ in $C^2(\mathbb{R}^N)$ as $n \to \infty$. Since $(\lambda_{\varepsilon_n}, \varphi_{\varepsilon_n})$ is a solution of (4.8) with $\varepsilon = \varepsilon_n$, we conclude by sending $n \to \infty$ that (λ, φ) is a solution of (EP). We can also see that $\inf_{\mathbb{R}^N}(\varphi - \phi_0) > -\infty$ in view of the latter claim of Proposition 4.7. Hence, we have completed the proof. □

Corollary 4.9. *There exists a solution (λ, φ) of (EP) such that $\varphi \in \Phi_\alpha$.*

Proof. This corollary is obvious from Theorem 4.8. Indeed, it suffices to set $\gamma = \alpha$ and choose a $\rho_0 \in (0, 1)$ so that $\phi_0(x) = \rho_0(1 + |x|^2)^{\alpha/2}$ satisfies (4.3). □

Proposition 4.10. *Let (λ, φ) be a solution of (EP) such that $\varphi \in \Phi_0$. Then,*

$$\varphi(x) + \lambda T = \inf_{\xi \in \mathcal{A}_T} E^x\Big[\int_0^T (l(X_t^\xi, \xi_t) + f(X_t^\xi))\,dt + \varphi(X_T^\xi)\Big], \quad T > 0. \tag{4.9}$$

Moreover, the optimal Markov control policy for the right-hand side of (4.9) is given by $\underline{\xi}(x) := D_p h(x, D\varphi(x))$.

Proof. Since $v(T, x) := \varphi(x) + \lambda T$ is a solution of (CP) with $g = \varphi \in \Phi_0$ and $v \in \Phi$, the $\geq$ part is deduced from Proposition 3.8. We can also obtain the opposite inequality in view of Propositions 3.6 and 4.2. The optimality of $\underline{\xi}$ is verified similarly as in the proof of Proposition 3.8. □

4.2 Uniqueness.

In this subsection, we establish a uniqueness result for (EP). Let (λ, φ) be any solution of (EP), and let $X = X^{\varphi}$ be the associated diffusion process governed by

$$dX_t = -D_p h(X_t, D\varphi(X_t))\, dt + dW_t, \quad t \geq 0. \tag{4.10}$$

The key to proving uniqueness lies in the ergodicity of X^{φ}. More precisely, we prove that X^{φ} is ergodic provided $\varphi \in \Phi_0$. The ergodicity of X^{φ} is also crucial in Section 5. We recall here the definition of ergodicity. Let $X = (X_t)_{t \geq 0}$ be a diffusion process in $\mathbb{R}^N$ with infinitesimal generator $A = (1/2)\Delta + b(x)D$ for some $b \in C(\mathbb{R}^N; \mathbb{R}^N)$. We say that X is ergodic if there exists a unique probability measure μ on $\mathbb{R}^N$ such that

$$\mu(B) = \int_{\mathbb{R}^N} P^x(X_t \in B)\, \mu(dx) \quad \text{for all} \quad t > 0, \quad B \in \mathcal{B}(\mathbb{R}^N).$$

The above μ is called the invariant probability measure for X. It is well known (see for instance [6, Theorem 4]) that, if X is ergodic, then

$$E^x[\psi(X_T)] \longrightarrow \int_{\mathbb{R}^N} \psi(y)\mu(dy) \quad \text{as} \quad T \to \infty \tag{4.11}$$

for any $\psi \in L^{\infty}(\mathbb{R}^N)$ and $x \in \mathbb{R}^N$.

The following two theorems on the ergodicity of diffusion processes are fundamental and will be frequently used in the rest of this paper. The first theorem gives a criterion for the ergodicity of a diffusion process (cf. [15, 16]).

Theorem 4.11. *Let X be a diffusion process in $\mathbb{R}^N$ with infinitesimal generator A. Suppose that there exist constants $r, \varepsilon > 0$ and a function $u \in C^2(\mathbb{R}^N \setminus B_r)$ such that $u(x) \to \infty$ as $|x| \to \infty$ and $Au \leq -\varepsilon$ in $\mathbb{R}^N \setminus B_r$. Then, X is ergodic.*

Proof. Observe first that X is ergodic if and only if it is positive recurrent in the sense that $E^x[\sigma_{y,\varepsilon}] < \infty$ for all $x, y \in \mathbb{R}^N$ and $\varepsilon > 0$, where $\sigma_{y,\varepsilon} := \inf\{t > 0 \,|\, |X_t - y| < \varepsilon\}$. See for instance [23, Theorem 4.9.6] for the proof of this fact.

It thus suffices to prove that X is positive recurrent. But it is known that the assumptions of this theorem imply the positive recurrence of X. See for instance [23, Theorem 4.6.3] or [15, Theorems 4.1, 5.5] for a complete proof. □

The second theorem claims that (4.11) is still valid for ψ not necessarily bounded but integrable with respect to μ, and that the convergence is uniform on compacts as a function of x.

Theorem 4.12. *Let X be a diffusion process in $\mathbb{R}^N$, and suppose that X is ergodic with invariant probability measure μ. Then,*

$$E^x[\psi(X_T)] \longrightarrow \int_{\mathbb{R}^N} \psi(y)\mu(dy) \quad \text{in} \quad C(\mathbb{R}^N) \quad \text{as} \quad T \to \infty$$

for all $\psi \in C(\mathbb{R}^N)$ satisfying $\int_{\mathbb{R}^N} \psi(y)\mu(dy) < \infty$.

Proof. This theorem has been proved in [16, Proposition 2.7] (cf. [17, Lemma 7.5]). $\square$

We now study the ergodicity of X^φ given in (4.10).

Proposition 4.13. *Let (λ, φ) be a solution of (EP) such that $\varphi \in \Phi_0$, and let X^φ be the associated diffusion process governed by (4.10). Then X^φ is ergodic. Moreover, let μ be the invariant probability measure for X^φ. Then, for any $(T, x) \in Q$ and $q > 1$,*

$$\sup_{R>0} E^x[|X^\varphi_{T\wedge\tau_R}|^q] < \infty, \qquad \int_{\mathbb{R}^N} |y|^q \mu(dy) < \infty. \tag{4.12}$$

Proof. Fix any $\rho_0 \in (0,1)$ such that $\phi_0(x) := -\rho_0(1 + |x|^2)^{\alpha/2}$ satisfies (4.3) for some $\nu_0 > 0$. Set $u := \varphi - \inf_{\mathbb{R}^N} \varphi - \phi_0$. Let A^φ be the infinitesimal generator for X^φ, that is,

$$A^\varphi v := \frac{1}{2}\Delta v - D_p h(x, D\varphi(x))Dv, \quad v \in C^2(\mathbb{R}^N). \tag{4.13}$$

Then, by the convexity of $h(x, p)$ in p, we see that

$$\begin{aligned}(A^\varphi u)(x) &= \frac{1}{2}(\Delta\varphi(x) - \Delta\phi_0(x)) - D_p h(x, D\varphi(x))(D\varphi(x) - D\phi_0(x)) \\ &\le F[\phi_0](x) - F[\varphi](x) \le -\nu_0|x|^\beta + \nu_0^{-1} + \lambda \longrightarrow -\infty\end{aligned}$$

as $|x| \to \infty$, where $F[\,\cdot\,]$ is defined by (4.2). Since $u(x) \to \infty$ as $|x| \to \infty$, we conclude in view of Theorem 4.11 that X^φ is ergodic.

To show the latter claim, let $q > 1$ be any number and apply Ito's formula to $u(X^\varphi_t)^q$. Then,

$$\begin{aligned}u(X^\varphi_{T\wedge\tau_R})^q - u(X^\varphi_0)^q = &\int_0^{T\wedge\tau_R} qu(X^\varphi_t)^{q-1}\Big(A^\varphi u(X^\varphi_t) + \frac{q-1}{2}\frac{|Du(X^\varphi_t)|^2}{u(X^\varphi_t)}\Big)\,dt \\ &+ \int_0^{T\wedge\tau_R} qu(X^\varphi_t)^{q-1} Du(X^\varphi_t)\,dW_t.\end{aligned} \tag{4.14}$$

Noting Proposition 4.2 and the fact that $u \ge -\phi_0 = \rho_0(1 + |x|^2)^{\alpha/2}$ in $\mathbb{R}^N$, we obtain

$$\begin{aligned}A^\varphi u(x) + \frac{q-1}{2}\frac{|Du(x)|^2}{u(x)} &\le F[\phi_0](x) - F[\varphi](x) + \frac{C(1 + |x|^{\alpha-1})^2}{\rho_0(1 + |x|^2)^{\alpha/2}} \\ &\le -\nu_0|x|^\beta + \nu_0^{-1} + \lambda + C(1 + |x|^2)^{(\alpha-2)/2}.\end{aligned}$$

Since $\alpha - 2 < \beta$, there exists a $\nu > 0$ such that

$$A^\varphi u(x) + \frac{q-1}{2}\frac{|Du(x)|^2}{u(x)} \le -\nu|x|^\beta + \nu^{-1} =: -k(x), \quad x \in \mathbb{R}^N.$$

Remark here that $k(x) \to \infty$ as $|x| \to \infty$. Plugging the last estimate into (4.14), taking expectation, and noting the fact that $M := \max_{x\in\mathbb{R}^N} qu(x)^{q-1}k_-(x) < \infty$, where $k_\pm(x) := \max\{0, \pm k(x)\}$, we have

$$E^x[u(X^\varphi_{T\wedge\tau_R})^q] + E^x\Big[\int_0^{T\wedge\tau_R} qu(X_t^\varphi)^{q-1}k_+(X_t^\varphi)\,dt\Big]$$
$$\le u(x)^q + E^x\Big[\int_0^{T\wedge\tau_R} qu(X_t^\varphi)^{q-1}k_-(X_t^\varphi)\,dt\Big] \le \varphi(x)^q + MT.$$

Since q is arbitrary and $u \ge -\phi_0 \ge \rho_0|x|^\alpha$ in $\mathbb{R}^N$, we obtain the first estimate in (4.12).

To establish the second estimate, we send $R \to \infty$ in the above inequality and divide both sides by T. Then,

$$\frac{1}{T}E^x\Big[\int_0^T qu(X_t^\varphi)^{q-1}k_+(X_t^\varphi)\,dt\Big] \le \frac{u(x)^q}{T} + M.$$

Letting $T \to \infty$ and taking into account Birkhoff's individual ergodic theorem, we have

$$\int_{\mathbb{R}^N} qu(y)^{q-1}k_+(y)\mu(dy) = \lim_{T\to\infty} E^x\Big[\frac{1}{T}\int_0^T qu(X_t^\varphi)^{q-1}k_+(X_t^\varphi)\,dt\Big] \le M.$$

Since q is arbitrary and $u(x)^{q-1}k_+(x) \ge |x|^{(q-1)\alpha}$ in $\mathbb{R}^N \setminus B_R$ for some $R > 0$, we obtain the second estimate in (4.12). □

We are now in position to establish a uniqueness for (EP).

Theorem 4.14. *Let (λ, φ) and (ν, ϕ) be two solutions of (EP) such that $\varphi, \phi \in \Phi_0$. Then $\lambda = \nu$ and $\varphi = \phi$.*

Proof. We first show that $\lambda = \nu$. Let X^φ be the diffusion associated with (λ, φ) and set $\xi_t^\varphi := D_p h(X_t^\varphi, D\varphi(X_t^\varphi))$. Note that $\xi^\varphi \in \mathcal{A}_T$ in view of Proposition 4.13. Set $u(T, x) := \phi(x) + \nu T$. Observe in view of Proposition 4.2 that $\sup_{Q_T}(|u|/(1+|x|^\alpha)) < \infty$ for all $T > 0$. Then, applying Proposition 3.6 to the above u and using Proposition 4.10, we see that, for any $(T, x) \in Q$,

$$\begin{aligned}\phi(x) + \nu T &\le \inf_{\xi\in\mathcal{A}_T} E^x\Big[\int_0^T (l(X_t^\xi, \xi_t) + f(X_t^\xi))\,dt + \phi(X_T^\xi)\Big]\\ &\le E^x\Big[\int_0^T (l(X_t^\varphi, \xi_t^\varphi) + f(X_t^\varphi))\,dt + \phi(X_T^\varphi)\Big]\\ &= \varphi(x) + \lambda T + E^x[(\phi - \varphi)(X_T^\varphi)].\end{aligned}$$

In particular,

$$(\phi - \varphi)(x) + (\nu - \lambda)T \le E^x[(\phi - \varphi)(X_T^\varphi)], \quad (T, x) \in Q. \tag{4.15}$$

Since $E^x[(\phi - \varphi)(X_T^\varphi)] \to \int_{\mathbb{R}^N}(\phi - \varphi)(y)\mu(dy) < \infty$ as $T \to \infty$ by virtue of Theorem 4.12, we have $\nu \le \lambda$. Changing the role of (λ, φ) and (ν, ϕ) in the above argument, we also see that $\lambda \le \nu$. Hence, $\lambda = \nu$.

To obtain the equality $\phi = \varphi$ in $\mathbb{R}^N$, we set $\lambda = \nu$ in (4.15) and send $T \to \infty$. Then, $(\phi - \varphi)(x) \le \int_{\mathbb{R}^N} (\phi - \varphi)(y)\, \mu(dy)$ for all $x \in \mathbb{R}^N$. Taking the sup over $x \in \mathbb{R}^N$, we have

$$0 \le \int_{\mathbb{R}^N} \{(\phi - \varphi)(y) - \sup_{\mathbb{R}^N}(\phi - \varphi)\}\, \mu(dy) \le 0.$$

Since $\operatorname{supp} \mu = \mathbb{R}^N$, we obtain $\phi - \varphi = \sup_{\mathbb{R}^N}(\phi - \varphi)$ in $\mathbb{R}^N$. Noting $\phi(0) = \varphi(0) = 0$ by definition, we conclude that $\phi = \varphi$ in $\mathbb{R}^N$. $\square$

The proof of Theorem 2.2 is now obvious from Corollary 4.9 and Theorem 4.14. We remark that, contrary to Cauchy problem (CP), the uniqueness of solutions to (EP) is guaranteed for any $m > 1$, or equivalently, for any $m^* > 1$. This comes from the fact that any solution ϕ of (EP) satisfies $\sup_{\mathbb{R}^N}(|\phi|/(1+|x|^\alpha)) < \infty$ by virtue of Proposition 4.2.

We close this section by making a remark on the value of λ. We first observe the following result on the solvability of (EP).

Theorem 4.15 (Theorem 2.1 of [15]). *There exists a critical constant λ^* such that (EP) has a solution $\phi \in C^2(\mathbb{R}^N)$ if and only if $\lambda \le \lambda^*$.*

Proposition 4.16. *Let (λ, φ) be the unique solution of (EP) such that $\varphi \in \Phi_0$. Then, $\lambda = \lambda^*$.*

Proof. Let ϕ be a solution of (EP) for $\lambda = \lambda^*$. Then, similarly as in the proof of Theorem 4.14, we see that $\lambda^* \le \lambda$. Since $\lambda^* \ge \lambda$ by Theorem 4.15, we obtain $\lambda = \lambda^*$. $\square$

5 Proof of the main results

This section is devoted to the proofs of Theorem 2.3 and Theorem 2.4.

5.1 Proof of Theorem 2.3.

In this subsection, we establish convergence (1.5) under our standing assumptions (H1)-(H3).

Proposition 5.1. *Let (λ, φ) be the solution of (EP) such that $\varphi \in \Phi_0$, and let u_V be the value function defined by (2.5). Then, for any $R > 0$ and $\eta > 0$, there exists a $T_0 > 0$ such that*

$$-\eta \le \frac{u_V(T,x)}{T} - \lambda \le \eta, \quad \textit{for all } \ T \ge T_0,\ x \in B_R. \tag{5.1}$$

Proof. Let $\phi_0(x) := \rho_0(1+|x|^2)^{(\alpha\wedge\beta)/2}$ satisfy (4.3) for some $\rho_0 \in (0,1)$, and let v_ε be the solution of (4.4) given in Proposition 4.5. Set $\varphi_\varepsilon := v_\varepsilon - v_\varepsilon(0)$ and $\lambda_\varepsilon := \varepsilon v_\varepsilon(0)$. Then, $(\lambda_\varepsilon, \varphi_\varepsilon)$ satisfies (4.8). In view of Proposition 4.7, we observe that there exists an $M > 0$ such that $\varphi_\varepsilon \geq \phi_0 - M$ in $\mathbb{R}^N$ for all ε. Furthermore, by the pre-compactness of $\{\varphi_\varepsilon\}_\varepsilon$ in $C^2(\mathbb{R}^N)$ and the uniqueness result for (EP), we also see that $\varphi_\varepsilon \to \varphi$ in $C(\mathbb{R}^N)$ and $\lambda_\varepsilon \to \lambda$ as $\varepsilon \to 0$.

Let ψ_ε be the supersolution of (4.4) given in Lemma 4.6. Then, similarly as in the proof of Proposition 4.5, we can verify that v_ε satisfies $\phi_0 - 1/(\varepsilon\nu_0) \leq v_\varepsilon \leq \psi_\varepsilon$ in $\mathbb{R}^N$. In particular, for each ε, there exists a $C_\varepsilon > 1$ such that

$$\phi_0(x) - M \leq \varphi_\varepsilon \leq C_\varepsilon(1+|x|^{\alpha\wedge\beta}), \quad x \in \mathbb{R}^N. \tag{5.2}$$

Fix any $\eta > 0$. We first prove the lower bound of (5.1). Set

$$v(T,x) := (1-e^{-\delta T})\varphi_\varepsilon(x) + (\lambda - 2\eta)T + q(T), \quad (T,x) \in Q,$$

for some $\varepsilon, \delta \in (0,1)$ and $q \in C^1([0,\infty))$. We find suitable ε, δ and q so that v is a subsolution of (CP). By the convexity of $F[\,\cdot\,]$, we observe that

$$\begin{aligned}\frac{\partial v}{\partial t} + F[v] &\leq e^{-\delta T}\delta\varphi_\varepsilon + \lambda - 2\eta + q' + (1-e^{-\delta T})F[\varphi_\varepsilon] + e^{-\delta T}F[0] \\ &\leq e^{-\delta T}\delta\varphi_\varepsilon + \lambda - 2\eta + q' + (1-e^{-\delta T})\{\varepsilon(\phi_0 - \varphi_\varepsilon) - \lambda_\varepsilon\} \\ &\qquad + e^{-\delta T}(-f_0|x|^\beta + f_0^{-1}).\end{aligned}$$

Taking into account (5.2), we have

$$\begin{aligned}\frac{\partial v}{\partial t} + F[v] &\leq e^{-\delta T}(\delta C_\varepsilon - f_0)|x|^\beta + q' + e^{-\delta T}(2\delta C_\varepsilon + f_0^{-1} + |\lambda|) \\ &\qquad + \varepsilon M + |\lambda - \lambda_\varepsilon| - 2\eta.\end{aligned}$$

We now choose ε and δ so that $\varepsilon M + |\lambda - \lambda_\varepsilon| < 2\eta$ and $\delta C_\varepsilon - f_0 < 0$. Then,

$$\frac{\partial v}{\partial t} + F[v] \leq q'(T) + e^{-\delta T}(2\delta C_\varepsilon + f_0^{-1} + |\lambda|).$$

We next define q so that the right-hand side is zero and $q(0) = \inf_{\mathbb{R}^N} g$, namely,

$$q(T) := \inf_{\mathbb{R}^N} g - \frac{2\delta C_\varepsilon + f_0^{-1} + |\lambda|}{\delta}(1-e^{-\delta T}), \quad T \geq 0.$$

Since $v(0,\,\cdot\,) = q(0) \leq g$ in $\mathbb{R}^N$, we conclude that v is a subsolution of (CP) such that $\sup_{Q_T}(|v|/(1+|x|^{\alpha\wedge\beta})) < \infty$ for all $T > 0$. Applying Proposition 3.6, we obtain

$$\begin{aligned}v(T,x) &\leq \inf_{\xi\in\mathcal{A}_T} E^x\Big[v(0,X_T^\xi) + \int_0^T (l(X_t^\xi,\xi) + f(X_t^\xi))\,dt\Big] \\ &\leq \inf_{\xi\in\mathcal{A}_T} E^x\Big[g(X_T^\xi) + \int_0^T (l(X_t^\xi,\xi) + f(X_t^\xi))\,dt\Big] = u_V(T,x).\end{aligned}$$

In particular,

$$\lambda - 2\eta + \frac{q(T) - |\varphi_\varepsilon(x)|}{T} \le \frac{u_V(T,x)}{T}, \quad (T,x) \in Q.$$

Noting $\inf_{T>0} q(T) > -\infty$, we conclude that, for any $R > 0$, there exists a $T_0 > 0$ such that $\lambda - \eta \le u_V(T,x)/T$ for all $x \in B_R$ and $T \ge T_0$.

We next show the upper bound of (5.1). Let $X^\varphi = (X_t^\varphi)_{t\ge 0}$ be the diffusion governed by (4.10) and set $\xi_t^\varphi := D_p h(X_t^\varphi, D\varphi(X_t^\varphi))$ for $t \ge 0$. Then, by the definition of u_V and Proposition 4.10, we see that

$$\begin{aligned}\frac{u_V(T,x)}{T} &\le \frac{1}{T} E^x\Big[\int_0^T (l(X_t^\varphi, \xi_t^\varphi) + f(X_t^\varphi))\,dt + g(X_T^\varphi)\Big] \\ &= \lambda + \frac{\varphi(x) + E^x[(g-\varphi)(X_T^\varphi)]}{T}.\end{aligned}$$

Since $E^x[(g-\varphi)(X_T^\varphi)]$ converges to $\int_{\mathbb{R}^N} (g-\varphi)(y)\mu(dy)$ in $C(\mathbb{R}^N)$ as $T \to \infty$ by virtue of Theorem 4.12, we can see that, for any $R > 0$, there exists a $T_0 > 0$ such that $u_V(T,x)/T \le \lambda + \eta$ for all $x \in B_R$ and $T \ge T_0$. Hence, the proof is complete. □

Proposition 5.2. *Let (λ, φ) be the solution of (EP) such that $\varphi \in \Phi_0$, and let λ_∞ be the constant defined by (2.6). Then $\lambda = \lambda_\infty$. Moreover, function $\underline{\xi}(x) := D_p h(x, D\varphi(x))$ gives an optimal Markov control policy for (2.6).*

Proof. Let u_V be the value function given by (2.5). Then, for any $\xi \in \mathcal{A}_\infty$ and $T > 0$,

$$\frac{u_V(T,0)}{T} \le \frac{1}{T} E^0\Big[\int_0^T (l(X_t^\xi, \xi_t) + f(X_t^\xi))\,dt + g(X_T^\xi)\Big].$$

Since the left-hand side converges to λ as $T \to \infty$ by Proposition 5.1, we obtain $\lambda \le \lambda_\infty$.

Let $X^\varphi = (X_t^\varphi)_{t\ge 0}$ be the diffusion given in (4.10) and set $\xi_t^\varphi := D_p h(X_t^\varphi, D\varphi(X_t^\varphi))$ for $t \ge 0$. Since $\xi^\varphi \in \mathcal{A}_T$ for all $T > 0$, we see that $\xi^\varphi \in \mathcal{A}_\infty$. Using Proposition 4.10, we have

$$\frac{\varphi(0) + \lambda T}{T} = \frac{1}{T} E^0\Big[\int_0^T (l(X_t^\varphi, \xi_t^\varphi) + f(X_t^\varphi))\,dt + g(X_T^\varphi)\Big] + \frac{E^0[(\varphi - g)(X_T^\varphi)]}{T}$$

for all $T > 0$. In particular,

$$\lambda \ge \liminf_{T\to\infty} \frac{1}{T} E^0\Big[\int_0^T (l(X_t^\varphi, \xi_t^\varphi) + f(X_t^\varphi))\,dt + g(X_T^\varphi)\Big].$$

The last equality together with $\lambda \le \lambda_\infty$ imply that $\lambda = \lambda_\infty$, and that ξ^φ is an optimal control for (2.6). Hence, we have completed the proof. □

Remark 5.3. *Proposition 5.2 implies that the value λ_∞ does not depend on $g \in \Phi_0$.*

Theorem 2.3 is now easily deduced from Propositions 5.1 and 5.2, so that we omit the proof.

5.2 Proof of Theorem 2.4.

Let (λ, φ) be the solution of (EP) such that $\varphi \in \Phi_0$, and let u_V be the value function given by (2.5). We set $w(T,x) := u(T,x) - (\varphi(x) + \lambda T)$ for $(T,x) \in Q$ and prove that $w(T,\,\cdot\,)$ converges in $C(\mathbb{R}^N)$ to a constant as $T \to \infty$. Observe that w is a solution of

$$\begin{cases} \partial_t w - A^\varphi w + H_\varphi(x, Dw) = 0 & \text{in } \; Q, \\ w(0,\,\cdot\,) = g - \varphi & \text{on } \; \partial_p Q, \end{cases} \tag{5.3}$$

where A^φ is the differential operator given by (4.13), and $H_\varphi(x,p)$ is defined by

$$H_\varphi(x,p) := h(x, p + D\varphi(x)) - h(x, D\varphi(x)) - D_p h(x, D\varphi(x)) \cdot p \geq 0.$$

Lemma 5.4. *Let (λ, φ) be the solution of (EP) such that $\varphi \in \Phi_0$, and let $X^\varphi = (X_t^\varphi)_{t \geq 0}$ be the associated ergodic diffusion. Then,*

$$w(T+S, x) \leq E^x[w(T, X_S^\varphi)], \quad T, S \geq 0, \quad x \in \mathbb{R}^N.$$

Proof. We apply Ito's formula to $w(T+S-t, X_t^\varphi)$. Then,

$$\begin{aligned} & w(T+S-S\wedge\tau_R, X^\varphi_{S\wedge\tau_R}) - w(T+S, X_0^\varphi) \\ &= \int_0^{S\wedge\tau_R} (-\partial_t w + A^\varphi w)(T+S-t, X_t^\varphi)\,dt + \int_0^{S\wedge\tau_R} Dw(T+S-t, X_t^\varphi)\,dW_t \\ &\geq \int_0^{S\wedge\tau_R} Dw(T+S-t, X_t^\varphi)\,dW_t. \end{aligned}$$

Taking expectation, we have

$$w(T+S,x) \leq E^x[w(T+S-S\wedge\tau_R, X^\varphi_{S\wedge\tau_R})].$$

Since $|w(t,x)| \leq C(1+|x|^q)$ in Q_{T+S} for some $C, q > 1$, and $\{|X^\varphi_{S\wedge\tau_R}|^q\,;\, R > 1\}$ is uniformly integrable by Proposition 4.13, we obtain the desired estimate after sending $R \to \infty$. □

Proposition 5.5. *For any $R > 0$, the family $\{w(T,\,\cdot\,)\,|\,T > 1\}$ is uniformly bounded from above on B_R. Moreover, if $\beta \geq m^*$, then it is also uniformly bounded from below on B_R.*

Proof. Let $X^\varphi = (X_t^\varphi)_{t\geq 0}$ be the ergodic diffusion associated with (λ, φ). Then, in view of Lemma 5.4 and Theorem 4.12, we see that

$$w(T,x) \leq E^x[(g-\varphi)(X_T^\varphi)] \longrightarrow \int_{\mathbb{R}^N} (g-\varphi)(y)\mu(dy) < \infty \quad \text{as } \; T \to \infty$$

uniformly on B_R. In particular, $w(T,\,\cdot\,)$ is bounded above on B_R uniformly in $T > 1$.

To get a lower bound, we assume $\beta \geq m^*$. Recall that $\beta \geq m^*$ if and only if $\beta \geq \alpha$. Set $v(T,x) := (1-e^{-\delta T})\varphi(x) + \lambda T + q(T)$ for some $\delta > 0$ and $q \in C^1([0,\infty))$ that will be determined later. Then, noting $\varphi(x) \leq K(1+|x|^\alpha)$ in $\mathbb{R}^N$ for some $K > 0$ by virtue of Proposition 4.2 and observing $\beta \geq \alpha$ by assumption, we see that

$$\begin{aligned}\frac{\partial v}{\partial t} + F[v] &\leq e^{-\delta T}\delta\varphi + \lambda + q' + (1-e^{-\delta T})F[\varphi] + e^{-\delta T}F[0]\\ &\leq e^{-\delta T}(\delta K - f_0)|x|^\beta + q' + e^{-\delta T}(2\delta K + |\lambda| + f_0^{-1}).\end{aligned}$$

We now choose $\delta := f_0/K$ and $q(T) := \inf_{\mathbb{R}^N} g - \delta^{-1}(2\delta K + |\lambda| + f_0^{-1})(1-e^{-\delta T})$. Then, $\partial_t v + F[v] \leq 0$ in Q and $v(0,\,\cdot\,) \leq g$ in $\mathbb{R}^N$. In particular, v is a subsolution of (CP) such that $\sup_{Q_T}(|v|/(1+|x|^\alpha)) < \infty$ for all $T > 0$. Applying Proposition 3.6, we obtain $v \leq u_V$ in Q. This infers that $-e^{-\delta T}\varphi(x) + q(T) \leq w(T,x)$ for all $(T,x) \in Q$. Since $\inf_T q(T) > -\infty$, we conclude that $w(T,\,\cdot\,)$ is bounded below on B_R uniformly in $T > 1$. □

Let Γ be the totality of all ω-limits of $\{w(T,\,\cdot\,) \,|\, T > 1\}$ in $C(\mathbb{R}^N)$, namely,

$$\Gamma := \{w_\infty \in C(\mathbb{R}^N) \,|\, \lim_{j\to\infty} w(T_j,\,\cdot\,) = w_\infty \text{ in } C(\mathbb{R}^N) \text{ for some } \lim_{j\to\infty} T_j = \infty\,\}.$$

Since $\sup_{[1,\infty)\times B_R}|Dw| < \infty$ for all $R > 0$ by virtue of Theorem 3.2 and Proposition 5.5, we see that $\{w(T,\,\cdot\,) \,|\, T > 1\}$ is pre-compact in $C(\mathbb{R}^N)$. In particular, $\Gamma \neq \emptyset$.

Proposition 5.6. *There exists a constant $c \in \mathbb{R}$ such that $\Gamma = \{c\}$.*

Proof. We first show that any element of Γ is constant. Let $w_\infty \in \Gamma$, i.e., $w(T_j,\,\cdot\,) \to w_\infty$ in $C(\mathbb{R}^N)$ as $j \to \infty$ for some diverging sequence $\{T_j\}$. By Lemma 5.4, we see that

$$w(T+S,x) \leq E^x[w(T, X_S^\varphi)], \quad T,S \geq 0, \quad x \in \mathbb{R}^N. \tag{5.4}$$

Take $S := T_j - T$ and send $j \to \infty$. Then, in view of Theorem 4.12, we have

$$w_\infty(x) \leq \int w(T,y)\mu(dy).$$

Since $\int |w_\infty(y)|\mu(dy) < \infty$ in view of Proposition 4.13, we deduce by choosing $T := T_j$ and letting $j \to \infty$ that

$$w_\infty(x) \leq \int w_\infty(y)\mu(dy).$$

In particular, w_∞ is bounded above on $\mathbb{R}^N$. Taking the sup over $x \in \mathbb{R}^N$, we obtain

$$0 \leq \int (w_\infty(y) - \sup_{\mathbb{R}^N} w_\infty)\mu(dy) \leq 0.$$

From the last estimate and the fact that $\operatorname{supp}\mu = \mathbb{R}^N$, we conclude that $w_\infty = \sup_{\mathbb{R}^N} w_\infty$ in $\mathbb{R}^N$. Hence, w_∞ is constant.

We next show that Γ consists of a single element. Suppose that there exist two diverging sequences $\{T_j\}$ and $\{S_j\}$ such that $w(T_j,\cdot\,) \to c_1$ and $w(S_j,\cdot\,) \to c_2$ in $C(\mathbb{R}^N)$ as $j \to \infty$ for some $c_1, c_2 \in \mathbb{R}$. We choose $S := S_j - T$ and $T := T_k$ in (5.4), and let $j \to \infty$ and $k \to \infty$ in this order. Then,

$$c_2 \le \lim_{k\to\infty} \int w(T_k, y)\mu(dy) = \int c_1\mu(dy) = c_1.$$

Thus, $c_2 \le c_1$. Changing the role of $\{T_j\}$ and $\{S_j\}$, we also have $c_1 \le c_2$. Hence, $c_1 = c_2$, and Γ consists of a single element which is constant. □

Theorem 2.4 is now easy to verify. We omit to reproduce the proof.

Remark 5.7. *In the statement of Theorem 2.4, u_V can be replaced by any solution u of (CP) such that $u \in \Phi$.*

We close this section by making a remark on our additional assumption $\beta \ge m^*$. This condition is needed only to obtain the lower bound of $w(T,x)$ in Proposition 5.5. Once we have proved it, Theorem 2.4 remains valid without assuming $\beta \ge m^*$. In particular, we have the following theorem.

Theorem 5.8. *The assertion of Theorem 2.4 remains valid if we assume (H1)-(H3) and $\inf_{\mathbb{R}^N}(g-\varphi) > -\infty$.*

Proof. Since $\inf_{\mathbb{R}^N}(g-\varphi) > -\infty$, there exists a $C > 0$ such that $g \ge \varphi - C$ in $\mathbb{R}^N$. Noting Proposition 4.10, we have

$$\begin{aligned} u_V(T,x) &\ge \inf_{\xi\in\mathcal{A}_T} E^x\Big[\int_0^T (l(X_t^\xi,\xi_t) + f(X_t^\xi))\,dt + \varphi(X_T^\xi)\Big] - C \\ &= \varphi(x) + \lambda T - C. \end{aligned}$$

This implies that $w(T,x) := u_V(T,x) - (\varphi(x)+\lambda T)$ is bounded below on B_R uniformly in $T > 1$ for all $R > 0$. Hence, the assertion of Theorem 2.4 is valid in view of Proposition 5.6. □

Appendix A: Gradient estimate for (CP)

Let Ω and Ω' be given bounded domains in $\mathbb{R}^N$ with C^3 boundary such that $\overline{\Omega}' \subset \Omega$. We set $Q_\delta := (\delta, T] \times \Omega$ and $Q'_\delta := (\delta, T] \times \Omega'$ for $\delta \ge 0$. Given a function $f \in C^2(\mathbb{R}^N)$, let us consider the parabolic equation

$$\partial_t u - \frac{1}{2}\Delta u + h(x, Du) = f \quad \text{in} \ \ Q_0, \tag{A.1}$$

where h is assumed to satisfy (H1)$'$.

Theorem A.1. *For any $\varepsilon, \delta \in (0,1)$, there exists a $K > 0$ depending only on ε, δ, the constants in (H1)$'$, and $d := \operatorname{dist}(\Omega', \partial\Omega)$ such that*

$$\sup_{Q'_\delta} |Du| \le K(1 + \sup_\Omega |f| + \sup_\Omega |Df| + \sup_{Q_{\delta/2}} |u|)^{1+\varepsilon}$$

for any smooth solution u of (A.1). Moreover, if $\sup_\Omega |Du(0,x)| < \infty$, then the above estimate holds with $\delta = 0$.

Proof. Let $\rho_0 \in C^2([0,\infty))$ be a cut-off function in time such that $\rho_0(t) = 0$ for $t \in [0, \delta/2]$ and $0 < \rho_0(t), \rho_0'(t) \le 1$ for $t \in (\delta, T]$. Let $\rho \in C^2(\mathbb{R}^N)$ be a cut-off function in space such that $\rho \equiv 1$ in Ω', $\operatorname{supp} \rho \subset \Omega$, and $0 \le \rho \le 1$ in Ω. Note that $\sup_\Omega |D\rho|$ and $\sup_\Omega |\Delta\rho|$ depend only on d.

Fix any number q such that $\max\{1/4, (3-m)/4\} < q < 1/2$ and $1/(2q) < 1 + \varepsilon$, and set $\eta(t,x) := \rho_0(t)^{m/(m-1)} \rho(x)^{2m/(1-2q)}$. We evaluate the function

$$z(t,x) := \eta(t,x)\{(1 + |Du(t,x)|^2)^q - u(t,x)\}$$

at its maximum point (t_0, x_0) on $\overline{Q}_{\delta/2}$. Note here that we have either $z(t_0,x_0) = 0$ or $z(t_0,x_0) > 0$. Suppose first that $z(t_0,x_0) = 0$. Then, for any $(t,x) \in (\delta, T] \times \Omega'$, we see that

$$\eta(t,x)(1 + |Du(t,x)|^2)^q = z(t,x) + \eta(t,x)u(t,x) \le z(t_0,x_0) + u(t,x) \le \sup_{Q_{\delta/2}} |u|.$$

Recalling $\rho(x) = 1$ and $\rho_0'(t) > 0$ for $t > \delta/2$, we have

$$\rho_0(\delta)^{m/(m-1)} |Du(t,x)|^{2q} \le \eta(t,x)(1 + |Du(t,x)|^2)^q \le \sup_{Q_{\delta/2}} |u|.$$

This implies that $\sup_{Q'_\delta} |Du| \le K(1 + \sup_{Q_{\delta/2}} |u|)^{1+\varepsilon}$ for some $K > 0$ depending only on ε, δ and m.

It remains to consider the case where $z(t_0,x_0) > 0$. Set $U(t,x) := 1 + |Du(t,x)|^2$ and $w(t,x) := U(t,x)^q - u(t,x)$, so that $z = \eta w$. Notice first that $(t_0,x_0) \in (\delta/2, T] \times \Omega$ since $\eta = 0$ in $(\{\delta/2\} \times \Omega) \cup ([\delta/2, T] \times \partial\Omega)$. This deduces that $z_t = w\eta_t + \eta w_t \ge 0$, $Dz = wD\eta + \eta Dw = 0$ and $\Delta z = w\Delta\eta + 2DwD\eta + \eta\Delta w \le 0$ at (t_0,x_0), where z_t, η_t and w_t denote the t-derivatives of z, η and w, respectively. In particular, at (t_0,x_0),

$$0 \le z_t - \frac{1}{2}\Delta z = \eta(w_t - \frac{1}{2}\Delta w) + w(\eta_t - \frac{1}{2}\Delta\eta + \eta^{-1}|D\eta|^2). \tag{A.2}$$

In what follows, since we evaluate the right-hand side of (A.2) only at (t_0,x_0), we omit the component (t_0,x_0) if there is no confusion.

We first estimate $w_t - (1/2)\Delta w$. By direct computation, we observe that $w_t = 2qU^{q-1}DuDu_t - u_t$, $Dw = qU^{q-1}DU - Du$, and

$$\begin{aligned}\Delta w &= q(q-1)U^{q-2}|DU|^2 + qU^{q-1}\Delta U - \Delta u \\ &= \frac{q-1}{q}U^{-q}|Dw + Du|^2 + 2qU^{q-1}\{\operatorname{tr}((D^2u)^2) + DuD(\Delta u)\} - \Delta u.\end{aligned}$$

Since $\mathrm{tr}((D^2u)^2) \geq 0$ and $u_t - (1/2)\Delta u = -h(x, Du) + f$, we have

$$\begin{aligned}
&w_t - \frac{1}{2}\Delta w \\
&\leq 2qU^{q-1}DuD(u_t - \frac{1}{2}\Delta u) - (u_t - \frac{1}{2}\Delta u) + \frac{1-q}{2q}U^{-q}|Dw + Du|^2 \\
&\leq -2qU^{q-1}Du(D_xh - Df + D^2uD_ph) + h - f + \frac{1-q}{q}U^{-q}(|Dw|^2 + |Du|^2).
\end{aligned}$$

Noting $1/4 < q < 1/2$, $2qU^{q-1}D^2uDu = Dw + Du$, and $|Du| \leq U^{1/2}$, we obtain

$$\begin{aligned}
&w_t - \frac{1}{2}\Delta w \\
&\leq U^{q-(1/2)}(|D_xh| + |Df|) - D_ph(Dw + Du) + h - f + 3(U^{-q}|Dw|^2 + U^{1-q}).
\end{aligned}$$

We now remind $|D_xh| \leq h_0^{-1}(1 + |p|^m)$, $|D_ph| \leq h_1^{-1}|p|^{m-1}$ and $1 - q < (m + 2q - 1)/2$ to deduce that

$$\begin{aligned}
w_t - \frac{1}{2}\Delta w &\leq |f| + |Df| + U^{q-(1/2)}h_0^{-1}(1 + |Du|^m) + h_1^{-1}|Du|^{m-1}|Dw| \\
&\quad + 3U^{-q}|Dw|^2 + 3U^{1-q} - D_phDu + h \\
&\leq |f| + |Df| + (3 + 2h_0^{-1})U^{(m+2q-1)/2} \\
&\quad + h_1^{-1}|Dw|U^{(m-1)/2} + 3U^{-q}|Dw|^2 - (D_phDu - h).
\end{aligned}$$

Since $D_ph \cdot p - h = l(x, D_ph) \geq l_0|D_ph|^{m^*} \geq l_0h_1^{m^*}|p|^m$ in view of (H1) and Theorem 3.4, there exists a constant $K_1 > 1$ such that

$$\begin{aligned}
w_t - \frac{1}{2}\Delta w &\leq 1 + |f| + |Df| - K_1^{-1}U^{m/2} \\
&\quad + K_1U^{(m+2q-1)/2}(1 + |Dw|U^{-q} + |Dw|^2U^{-2q}). \qquad \text{(A.3)}
\end{aligned}$$

We recall that $z(t_0, x_0) > 0$. This implies $w(t_0, x_0) > 0$, and therefore $u(t_0, x_0) < U(t_0, x_0)^q$. In particular, $w < U^q + u < 2U^q$ at (t_0, x_0). Noting this facts and plugging $|Dw| = w\eta^{-1}|D\eta| < 2U^q\eta^{-1}|D\eta|$ into (A.3),

$$\begin{aligned}
w_t - \frac{1}{2}\Delta w &\leq 1 + |f| + |Df| - K_1^{-1}U^{m/2} \\
&\quad + K_1U^{(m+2q-1)/2}(1 + 2\eta^{-1}|D\eta| + 4\eta^{-2}|D\eta|^2).
\end{aligned}$$

We set $\theta := m^{-1}(m + 2q - 1) \in (1/2, 1)$ and $V := \eta U^{m/2}$. Then, we have

$$\begin{aligned}
\eta(w_t - \frac{1}{2}\Delta w) &\leq 1 + |f| + |Df| - K_1^{-1}V + K_1V^\theta\eta^{1-\theta}(1 + 2\eta^{-1}|D\eta| + 4\eta^{-2}|D\eta|^2) \\
&\leq 1 + |f| + |Df| - K_1^{-1}V + K_1V^\theta(1 + 2\eta^{-\theta}|D\eta| + 4\eta^{-(1+\theta)}|D\eta|^2).
\end{aligned}$$

As to the second term of the right-hand side of (A.2), we see, in view of $w < 2U^q$ at (t_0, x_0) and $2q/m < (1/m) \wedge \theta$, that

$$
\begin{aligned}
w(\eta_t - \frac{1}{2}\Delta\eta + \eta^{-1}|D\eta|^2) &\le (\eta U^{m/2})^{2q/m}\eta^{-2q/m}(2\eta_t + |\Delta\eta| + 2\eta^{-1}|D\eta|^2) \\
&\le V^{2q/m}(2\eta^{-1/m}\eta_t + \eta^{-\theta}|\Delta\eta| + 2\eta^{-(1+\theta)}|D\eta|^2).
\end{aligned}
$$

Hence, plugging the last two estimates into (A.2), we conclude that

$$
V \le K_1(1 + |f| + |Df|) + K_2(1 \vee V^\theta)(1 + \eta^{-1/m}\eta_t + \eta^{-(1+\theta)}|D\eta|^2 + \eta^{-\theta}|\Delta\eta|)
$$

for some $K_2 > 0$.

We now set $\gamma := 2m/(1-2q) = 2/(1-\theta) > 4$. Then, we see that

$$
\begin{aligned}
\eta_t &= \frac{m}{m-1}\rho_0^{1/(m-1)}\rho_0'\rho^\gamma \le \frac{m}{m-1}(\rho_0^{m/(m-1)}\rho^\gamma)^{1/m} = \frac{m}{m-1}\eta^{1/m}, \\
|D\eta| &= \gamma\rho_0^{m/(m-1)}\rho^{\gamma-1}|D\rho| \le \gamma(\rho_0^{m/(m-1)}\rho^\gamma)^{(\gamma-1)/\gamma}|D\rho| = \gamma\eta^{(1+\theta)/2}|D\rho|,
\end{aligned}
$$

and

$$
\begin{aligned}
|\Delta\eta| &\le \gamma\rho_0^{m/(m-1)}\{\rho^{\gamma-1}|\Delta\rho| + (\gamma-1)\rho^{\gamma-2}|D\rho|^2\} \\
&\le \gamma(\rho_0^{m/(m-1)}\rho^\gamma)^{(\gamma-1)/2}|\Delta\rho| + \gamma(\gamma-1)(\rho_0^{m/(m-1)}\rho^\gamma)^{(\gamma-2)/2}|D\rho|^2 \\
&= \gamma\eta^{(1+\theta)/2}|\Delta\rho| + \gamma(\gamma-1)\eta^\theta|D\rho|^2.
\end{aligned}
$$

Thus, there exists a $K_3 > 0$ depending only on m, q and $d = \mathrm{dist}(\Omega', \partial\Omega)$ such that

$$
V \le K_1(1 + |f| + |Df|) + K_2K_3(1 \vee V^\theta).
$$

Since $\theta < 1$, we conclude in view of Young's inequality that

$$
V \le K_4(1 + |f| + |Df|) \tag{A.4}
$$

for some $K_4 > 0$ depending only on the constants in (H1)$'$, q and d. Thus, for any $(t, x) \in (\delta, T] \times \Omega'$,

$$
\begin{aligned}
\rho_0(t)^{m/(m-1)}w(t,x) = z(t,x) &\le z(t_0,x_0) = \eta(t_0,x_0)(U(t_0,x_0)^q - u(t_0,x_0)) \\
&\le V(t_0,x_0) + |u(t_0,x_0)| \le K_4(1 + |f| + |Df|) + \sup_{Q_{\delta/2}}|u|,
\end{aligned}
$$

which implies that

$$
|Du(t,x)|^{2q} \le \rho_0(\delta)^{-m/(m-1)}\{K_4(1 + |f| + |Df|) + 2\sup_{Q_{\delta/2}}|u|\}.
$$

The last inequality easily deduces the desired estimate.

The latter claim of this theorem can be seen by taking $\rho_0 \equiv 1$. Hence, we have completed the proof. □

Appendix B: Gradient estimate for (EP)

Let Ω and Ω' be bounded domains in $\mathbb{R}^N$ with C^3 boundary such that $\overline{\Omega}' \subset \Omega$. For given $\varepsilon \in [0, 1)$ and $f \in C^2(\mathbb{R}^N)$, we consider the elliptic equation

$$-\frac{1}{2}\Delta\phi + h(x, D\phi) + \varepsilon\phi = f \quad \text{in} \ \ \Omega, \tag{B.1}$$

where h is assumed to satisfy (H1)$'$.

Theorem B.1. *There exists a $K > 0$ depending only on N, $d := \operatorname{dist}(\Omega', \partial\Omega)$ and the constants in (H1)$'$ such that*

$$\sup_{\Omega'} |D\phi| \le K(1 + \sup_{\Omega}(\varepsilon\phi)_-^{1/m} + \sup_{\Omega} f_+^{1/m} + \sup_{\Omega} |Df|^{1/(2m-1)}) \tag{B.2}$$

for any solution $\phi \in C^3(\mathbb{R}^N)$ of (B.1), where $r_\pm := \max\{\pm r, 0\}$ for $r \in \mathbb{R}$.

Proof. Let $\rho \in C^2(\Omega)$ be a cut-off function such that $\rho \equiv 1$ in Ω', $\operatorname{supp}\rho \subset \Omega$, and $0 \le \rho \le 1$ in Ω. Set $\eta := \rho^{4m/(m-1)}$, $w := (1/2)|D\phi|^2$, and $z := \eta w$. Let x_0 be a maximum point of z on $\overline{\Omega}$. We may assume without loss of generality that $z(x_0) > 1$. Indeed, if $z(x_0) \le 1$, then for any $x \in \Omega'$, $(1/2)|D\phi(x)|^2 = z(x) \le z(x_0) \le 1$. Thus, $|D\phi| \le \sqrt{2}$ in Ω' and (B.2) is valid.

From the fact that $z(x_0) > 1$, we have $x_0 \in \Omega$. In particular, $Dz = \eta Dw + w D\eta = 0$ and $\Delta z \le 0$ at $x = x_0$. Noting $Dw = D^2\phi D\phi$ and $\Delta w = \operatorname{tr}((D^2\phi)^2) + D(\Delta\phi)D\phi = \operatorname{tr}((D^2\phi)^2) + 2((D_x h - Df)D\phi + D_p h Dw + \varepsilon w)$, we observe that, at $x = x_0$,

$$\begin{aligned}
0 \ge \Delta z &= \eta\Delta w + 2D\eta Dw + w\Delta\eta \\
&= \eta \operatorname{tr}((D^2\phi)^2) + 2\eta((D_x h - Df)D\phi + D_p h(-w\eta^{-1}D\eta) + 2\varepsilon w) \\
&\qquad + 2D\eta(-w\eta^{-1}D\eta) + w\Delta\eta \\
&\ge \eta \operatorname{tr}((D^2\phi)^2) - 2\eta((|D_x h| + |Df|)|D\phi| + w\eta^{-1}|D\eta||D_p h|) - w(2\eta^{-1}|D\eta|^2 + |\Delta\eta|).
\end{aligned}$$

From now on, since we evaluate values only at $x = x_0$, we omit the component x_0.

We now remind $|D_x h| \le h_0^{-1}(1 + |p|^m)$ and $|D_p h| \le h_1^{-1}|p|^{m-1}$, and observe that

$$N(\operatorname{tr}(D^2\phi)^2) \ge (\operatorname{tr}(D^2\phi))^2 = 4(h + \varepsilon\phi - f)^2 \ge 2h_0^2|D\phi|^{2m} - 16(\varepsilon\phi)_-^2 - 16f_+^2.$$

Then,

$$\begin{aligned}
\frac{2h_0^2}{N}\eta|D\phi|^{2m} &\le 16(\varepsilon\phi)_-^2 + 16f_+^2 + \eta \operatorname{tr}((D^2\phi)^2) \\
&\le 16(\varepsilon\phi)_-^2 + 16f_+^2 + 2\eta h_0^{-1}(1 + |D\phi|^m)|D\phi| + 2\eta|D\phi||Df| \\
&\qquad + h_1^{-1}|D\phi|^{m+1}|D\eta| + |D\phi|^2(\eta^{-1}|D\eta|^2 + \frac{1}{2}|\Delta\eta|).
\end{aligned}$$

Applying Young's inequality to $|D\phi||Df|$, we see that, for any $\delta > 0$, there exists a constant $C_\delta > 0$ such that $|D\phi||Df| \le \delta|D\phi|^{2m} + C_\delta|Df|^{2m/(2m-1)}$. Hence, there exists a $K_1 > 0$ depending only on N and the constants in (H1)$'$ such that

$$\begin{aligned}\eta|D\phi|^{2m} \le K_1\{1 + (\varepsilon\phi)_-^2 + f_+^2 + |Df|^{2m/(2m-1)} \\ + |D\phi|^{m+1}|D\eta| + |D\phi|^2(\eta^{-1}|D\eta|^2 + |\Delta\eta|)\}.\end{aligned}$$

We now set $V := \eta|D\phi|^{2m}$ and $\theta := (m+1)/2m \in (1/m, 1)$. Then,

$$\begin{aligned}V \le K_1\{1 + (\varepsilon\phi)_-^2 + f_+^2 + |Df|^{2m/(2m-1)} \\ + V^\theta\eta^{-\theta}|D\eta| + V^{1/m}(\eta^{-(m+1)/m}|D\eta|^2 + \eta^{-1/m}|\Delta\eta|)\}.\end{aligned}$$

Observing $1 < z < (\eta|D\phi|^2)^m \le V$ and $\theta > 1/m$, we have

$$\begin{aligned}V \le K_1(1 + (\varepsilon\phi)_-^2 + f_+^2 + |Df|^{2m/(2m-1)}) \\ + K_1V^\theta(\eta^{-\theta}|D\eta| + \eta^{-2\theta}|D\eta|^2 + \eta^{-\theta}|\Delta\eta|).\end{aligned}$$

We claim here that $\eta^{-\theta}|D\eta|$ and $\eta^{-\theta}|\Delta\eta|$ are bounded by a constant depending only on m and d. Indeed, recalling $\eta = \rho^\gamma$ with $\gamma := 4m/(m-1)$, we can verify that

$$\begin{aligned}&\eta^{-\theta}|D\eta| = \gamma\rho^{\gamma-1-\gamma\theta}|D\rho| = \gamma\rho|D\rho|, \\ &\eta^{-\theta}|\Delta\eta| \le \gamma\{\rho^{\gamma-1-\gamma\theta}|\Delta\rho| + (\gamma-1)\rho^{\gamma-2-\gamma\theta}|D\rho|^2\} = \gamma\{\rho|\Delta\rho| + (\gamma-1)|D\rho|^2\}.\end{aligned}$$

Hence, there exists a $K_2 > 0$ depending only on N, d and the constants in (H1)$'$ such that

$$V \le K_2(1 + (\varepsilon\phi)_-^2 + f_+^2 + |Df|^{2m/(2m-1)}),$$

from which we easily deduce (B.2). □

Appendix C: Moment estimate for controlled processes

Given a control $\xi = (\xi_t)_{0\le t\le T}$, let $X^\xi = (X_t^\xi)_{0\le t\le T}$ be the associated controlled process governed by (1.2).

Lemma C.1. *Let $\alpha := (\beta/m) + 1$. Then, there exists a constant $C > 0$ such that*

$$E^x\Big[\sup_{0\le t\le T}|X_t^\xi|^\alpha\Big] \le 2|x|^\alpha + CE^x\Big[\int_0^T(1 + |X_s^\xi|^\beta + |\xi_s|^{m^*})\,ds\Big]$$

for all $T > 0$, $x \in \mathbb{R}^N$, and $\xi \in \mathcal{A}_T$.

Proof. Fix any $R > 0$. By Ito's formula, Young's inequality, and $\beta = m(\alpha - 1) > \alpha - 2$, we see that

$$\begin{aligned}|X^{\xi}_{t\wedge\tau_R}|^{\alpha} - |X_0|^{\alpha} &= -\int_0^{t\wedge\tau_R} \alpha|X^{\xi}_s|^{\alpha-2}X^{\xi}_s \cdot \xi_s\, ds + \int_0^{t\wedge\tau_R} \alpha|X^{\xi}_s|^{\alpha-2}X^{\xi}_s\, dW_s \\ &\quad + \frac{\alpha(\alpha+N-2)}{2}\int_0^{t\wedge\tau_R} |X^{\xi}_s|^{\alpha-2}\, ds \\ &\le C\int_0^{t\wedge\tau_R} (1 + |X^{\xi}_s|^{\beta} + |\xi_s|^{m^*})\, ds + \int_0^{t\wedge\tau_R} \alpha|X^{\xi}_s|^{\alpha-2}X^{\xi}_s\, dW_s.\end{aligned}$$

Applying Burkholder's inequality, we have

$$\begin{aligned}E^x\Big[\sup_{0\le t\le T}|X^{\xi}_{t\wedge\tau_R}|^{\alpha}\Big] - |x|^{\alpha} &\le CE^x\Big[\sup_{0\le t\le T}\int_0^{t\wedge\tau_R}(1 + |X^{\xi}_s|^{\beta} + |\xi_s|^{m^*})\, ds\Big] \\ &\quad + \alpha E^x\Big[\sup_{0\le t\le T}\Big|\int_0^{t\wedge\tau_R}|X^{\xi}_s|^{\alpha-2}X^{\xi}_s\, dW_s\Big|\Big] \\ &\le CE^x\Big[\int_0^T (1 + |X^{\xi}_s|^{\beta} + |\xi_s|^{m^*})\, ds\Big] \\ &\quad + CE^x\Big[\Big(\int_0^{T\wedge\tau_R}|X^{\xi}_s|^{2(\alpha-1)}\, ds\Big)^{1/2}\Big].\end{aligned}$$

Since the last term can be estimated as

$$\begin{aligned}&CE^x\Big[\Big(\int_0^{T\wedge\tau_R}|X^{\xi}_s|^{2(\alpha-1)}\, ds\Big)^{1/2}\Big] \\ &\quad \le CE^x\Big[\Big(\sup_{0\le t\le T\wedge\tau_R}|X^{\xi}_t|^{\alpha-1}\int_0^{T\wedge\tau_R}|X^{\xi}_s|^{\alpha-1}\, ds\Big)^{1/2}\Big] \\ &\quad \le \frac{1}{2}E^x\Big[\sup_{0\le t\le T}|X^{\xi}_{t\wedge\tau_R}|^{\alpha-1}\Big] + CE^x\Big[\int_0^T |X^{\xi}_s|^{\alpha-1}\, ds\Big],\end{aligned}$$

we conclude that

$$E^x\Big[\sup_{0\le t\le T}|X^{\xi}_{t\wedge\tau_R}|^{\alpha}\Big] \le 2|x|^{\alpha} + CE^x\Big[\int_0^T (1 + |X^{\xi}_s|^{\beta} + |\xi_s|^{m^*})\, ds\Big] < \infty.$$

Sending $R \to \infty$, we obtain the desired estimate. □

Acknowledgements. The author would like to thank Professor Jun Sekine for his valuable comments on an earlier version of this paper. In particular, the link between Theorem 2.4 and the issue of indifference pricing was pointed out by him. The author also thanks two anonymous referees for their careful reading and several suggestions for a better organization of the paper.

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