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Weighted energy estimates for wave equations in exterior domains

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Abstract. Weighted energy estimates including the Keel, Smith and Sogge estimate is obtained for solutions of exterior problem of the wave equation in three or higher dimensional Euclidean spaces. For the solutions of the Cauchy problem, which is corresponding to the free system in scattering theory, the estimates are given by using the ideas introduced by Morawetz and summarized by Mochizuki for the Dirichlet problem in the outside of star shaped obstacles. From the estimates for the free system, the corresponding estimates for exterior domains are given if it is assumed that the local energy decays uniformly with respect to initial data, which depends on the structures of propagation of singularities.

Keywords. Wave equations, weighted energy estimates, local energy decay, Keel, Smith and Sogge estimate.

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1 Introduction

Let $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$ be an exterior domain of a bounded obstacle $\mathcal{O} = \mathbb{R}^n \setminus \Omega$. Assume that the boundary $\partial\Omega$ is C^∞ and compact, and Ω is connected. Consider the following mixed problem of the usual wave equation:

$$\begin{cases} (\partial_t^2 - \triangle)u(t,x) = f(t,x) & \text{in } \mathbb{R} \times \Omega, \\ \mathcal{B}u(t,x) = 0 & \text{on } \mathbb{R} \times \partial\Omega, \\ u(0,x) = f_1(x), \quad \partial_t u(0,x) = f_2(x) & \text{on } \Omega, \end{cases} \tag{1.1}$$

where the boundary operator $\mathcal{B}$ is given by $\mathcal{B}u(t,x) = u(t,x)$ (the Dirichlet condition) or $\mathcal{B}u(t,x) = \frac{\partial u}{\partial \nu}(t,x) = \sum_{j=1}^n \nu_j(x)\frac{\partial u}{\partial x_j}(t,x)$ (the Neumann condition). In the above, $\nu(x) = {}^t(\nu_1(x), \nu_2(x), \nu_3(x))$ is the unit outer normal vector of $\partial\Omega$ at $x \in \partial\Omega$ pointing into the outside of Ω. Since $\mathcal{O}$ is compact, we have $\mathcal{O} \subset B_{R_0}$ for some fixed constant $R_0 > 0$, where $B_{R_0} = B_{R_0}(0)$ and $B_{R_0}(a) = \{x \in \mathbb{R}^n \mid |x-a| < R_0\}$.

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The main purpose of this paper is to consider weighted energy estimates of L^2-type for solutions of problem (1.1). In [10], Keel, Smith and Sogge proposed a new approach to obtain existence theorems of non-linear wave equations. In this approach, a weighted energy estimate for critical case is essentially used. This estimate is called Keel, Smith and Sogge estimate (cf. Proposition 2.1 in Keel, Smith and Sogge [10], or for selfcontaindness let us refer the estimate of the case $l = 1$ in Theorem 1.1 or Corollary 2.2). In line with this new idea, various existence theorems of non-linear wave equations for the Cauchy problems and the Dirichlet boundary problems are investigated (see e.g. [3, 9, 15, 16] and the references therein).

In the case of the Dirichlet problem in exterior domains it can be expected to have existence theorems for solutions of non-linear wave equations for smooth initial data if the local energy of the solutions of the corresponding linearized wave equation decays sufficiently fast. These are also investigated by many authors ([9, 15, 16] and the references therein). In these works, one of the key estimates is weighted energy estimates of L^2-type in the space variable for the solution of linear equation in exterior domains.

In this paper, weighted energy estimates of L^2-type including the Keel, Smith and Sogge estimate in exterior domains in $\mathbb{R}^n$ ($n \geq 3$) are given if the local energy of the solutions of the corresponding linear equations decays fast uniformly. To describe them, let us introduce the function spaces $\dot{H}^m(\Omega) = \{v \in H^m_{\rm loc}(\Omega) \mid \partial_x^\alpha v \in L^2(\Omega)$ for $1 \leq |\alpha| \leq m, \lim_{r\to\infty} r^{-2}\int_{r\leq|x|\leq 2r}|v(x)|^2dx = 0\}$ ($m = 1, 2, \ldots$). Concerning the boundary conditions, we put $\dot{H}^m_{\mathscr{B}}(\Omega) = \{v \in \dot{H}^m(\Omega) \mid \mathscr{B}v = 0$ on $\partial\Omega\}$ for $m = 2$ or $m = 1$ and the case of $\mathscr{B}v = v$ on $\partial\Omega$, and $\dot{H}^1_{\mathscr{B}}(\Omega) = \dot{H}^1(\Omega)$ for the case of the Neumann boundary condition. For initial data $\{f_1, f_2\} \in \dot{H}^1_{\mathscr{B}}(\Omega) \times L^2(\Omega)$ and inhomogeneous data $f \in L^1_{\rm loc}(\mathbb{R}; L^2(\Omega))$, problem (1.1) has the unique solution $u(t,x) \in C(\mathbb{R}; \dot{H}^1(\Omega))$ with $\partial_t u(t,x) \in C(\mathbb{R}; L^2(\Omega))$. For any domain $D \subset \mathbb{R}^n$, we put

$$e(t,x;u) = \frac{1}{2}\left\{|\partial_t u(t,x)|^2 + |\nabla_x u(t,x)|^2\right\} \text{ and } E(u,D,t) = \int_D e(t,x;u)dx.$$

Let us introduce the following uniform decay rate $p_{m,R}(t)$ of the local energy of solutions $u(t,x)$ of (1.1) with zero inhomogeneous data $f(t,x) = 0$:

$$p_{m,R}(t) = \sup\left\{\frac{E(u,\Omega\cap B_R,t) + \|u(t,\cdot)\|^2_{L^2(\Omega\cap B_R)}}{\|\nabla_x f_1\|^2_{H^m(\Omega)} + \|f_2\|^2_{H^m(\Omega)}} \;\middle|\; 0 \neq f_1, f_2 \in C_0^\infty(\overline{\Omega}\cap B_R) \text{ such that } u \in C^\infty(\mathbb{R}\times\overline{\Omega})\right\},$$

where $R > 0$ is a constant satisfying $\mathcal{O} \subset B_R$, $m \geq 0$ is an integer. Note that $p_{0,R}(t)$ is the uniform decay rate in the sense of Morawetz [23].

In this paper, from now on, we say that a pair (f_1, f_2, f) of the data satisfies the compatibility condition of order m if and only if the unique solution u of (1.1) with $f_1 \in \dot{H}^{m+1}(\Omega) \cap \dot{H}^1_{\mathcal{B}}(\Omega)$, $f_2 \in H^m(\Omega)$, $f \in \bigcap_{j=0}^{m-1} W^{j+1,1}_{\mathrm{loc}}(\mathbb{R}; H^{m-1-j}(\Omega))$ belongs to $\bigcap_{j=0}^{m} C^j(\mathbb{R}; \dot{H}^{m+1-j}(\Omega))$ with $\partial_t u \in \bigcap_{j=0}^{m} C^j(\mathbb{R}; H^{m-j}(\Omega))$. In the definition of $p_{m,R}(t)$, we put the compatibility conditions of infinite order for the data (f_1, f_2, f) (with $f = 0$). But this is not necessary. We only need the one of order m.

For solutions $u(t,x)$ of (1.1), we define the weighted L^2-energy $W_l(t;u)$ by

$$W_l(t;u) = \int_0^t \int_\Omega (1+|x|)^{-l} e(s,x;u)\,dx\,ds + \int_0^t \int_\Omega (1+|x|)^{-(l+2)} |u(s,x)|^2\,dx\,ds.$$

The purpose of this paper is to give estimates of $W_l(t;u)$ for solutions of (1.1) if local energy decays sufficiently fast. For the uniform decay rate $p_{m,R}(t)$ of the local energy, we assume that

(E1) there exists an integer $m \geq 0$, a constant $\tilde{R}_0 > 0$ and a function $p_m \in C([0,\infty)) \cap L^1([0,\infty))$ such that $\sqrt{p_{m,R_0+\tilde{R}_0}(t)} \leq p_m(t)$ $(t \geq 0)$.

To state our main theorem we introduce the following notations:

$$q_l(t) = \begin{cases} \dfrac{(1+t)^{1-l}}{1-l}, & 0 \leq l < 1, \\ \log(1+t), & l = 1, \\ \max\{1, (l-1)^{-1}\}, & l > 1. \end{cases}$$

Theorem 1.1. *Assume that* $n \geq 3$ *and* (E1) *is satisfied. Then there exists a constant* $C > 0$ *such that*

$$W_l(t;u) \leq C q_l(t) \Bigg\{ \|\nabla_x f_1\|^2_{H^m(\Omega)} + \|f_2\|^2_{H^m(\Omega)} + \left(\int_0^t \|f(s,\cdot)\|_{L^2(\Omega)} ds \right)^2 + C_m \left(\sum_{k=0}^{m-1} \sum_{j=0}^{1} \int_0^t \|\partial_s^{j+k} f(s,\cdot)\|_{H^{m-1-k}(\Omega)} ds \right)^2 \Bigg\}$$

$(t \geq 0$, $l \geq 0$ *and* (f_1, f_2, f) *satisfies the compatibility condition of order* $m)$,

where C_m *is the constant defined by* $C_0 = 0$ *and* $C_m = 1$ *for* $m \geq 1$.

Note that the case of $l = 1$ in the estimate in Theorem 1.1 is the Keel, Smith and Sogge estimate.

Let us mention about the decay rate $p_{m,R}(t)$ before going to handle our concerned estimates in Theorem 1.1. Many authors investigated the decay rate $p_{m,R}(t)$ from the point of view in scattering theory. The case of $m = 0$ is different from the other cases. For the Dirichlet problem, Morawetz [22] shows $p_{0,R}(t) = O(t^{-1})$ as $t \to \infty$ if the obstacle $\mathcal{O}$ is star shaped. In this line, serial works of Ikehata makes the argument simpler and remove the restriction that the support of the initial data is compact (see e.g. Ikehata [6] and the references therein).

Morawetz [23] also gives an interesting result and argument that we can obtain $p_{0,R}(t) = O(e^{-\alpha t})$ for some $\alpha > 0$ if the space dimension $n \geq 3$ is odd and we a priori know $p_{0,R}(t) \to 0$ as $t \to \infty$. In the case of the Neumann boundary condition, this interesting result is also valid. Note that in this argument, Huygens' principle is essentially used. Hence the space dimension n should be odd. For even dimension, it is expected that we can get the estimate $p_{0,R}(t) = O(t^{-2(n-1)})$. For even $n \geq 4$, the estimate $p_{0,R}(t) = O(t^{-2(n-1)})$ is given in [8], and for even $n \geq 2$ the same estimate is obtained by Vodev [30]. In [8], the translation representation of the scattering theory of Lax and Phillips [11] are essentially used to decompose the waves. This idea is originally introduced by Melrose [12] to show the same estimate for the case of non-trapping obstacles as is in the next paragraph. Note that in [30], Vodev introduces a new approach via analyzing "cutoff resolvents", and also show the following uniform estimate: $E(u, \Omega \cap B_R, t) = O(t^{-2n})$.

In the case of $m = 0$, decay estimates of $p_{0,R}(t)$ are closely connected with the non-trapping property of singularities of the solutions of problem (1.1). If we have a trapping ray of geometrical optics, we have no decay property of $p_{0,R}(t)$. This is shown in Ralston [26]. On the other hand, as is in Vainberg [29], Morawetz, Ralston and Strauss [24], Melrose [12] and Ralston [27], if all singularities near the obstacle escape far away within fixed finite time, we have the estimates of $p_{0,R}(t)$ stated above. Melrose and Sjöstrand [13, 14] show that all singularities propagate along the generalized broken rays introduced in [13, 14]. Thus Melrose and Sjöstrand reduced analytical conditions about non-trapping obstacles to geometrical conditions.

On the other hand, in the case of $m > 0$, Walker [31] shows that $p_{m,R}(t) \to 0$ as $t \to \infty$ if $m > 0$. Hence the problem is how fast it decays. About this, let us introduce the work of Ikawa [4, 5] for the case of the Dirichlet boundary condition. In [4], Ikawa shows that for $m \geq 5$, $p_{m,R}(t) = O(e^{-\alpha t})$ with some fixed constant $\alpha > 0$ if $n = 3$ and $\mathcal{O}$ consists of two strictly convex bodies. Further in Ikawa [5], the case that $n = 3$ and $\mathcal{O}$ consists of finitely many convex bodies is considered.

If the convex hull of each two bodies does not intersect with the other bodies and some additional condition holds, then Ikawa [5] obtains $p_{m,R}(t) = O(e^{-\alpha t})$ for $m \geq 2$. Note that if all these bodies are balls and they separate each other well, the additional condition is satisfied.

Even for the transmisson problem, Cardoso, Popov and Vodev [2] give the same estimate as that of the non-trapping case stated above if the phase speed of the inside medium is greater than that of the outside one. Hence, if this is the case, thanks to the work of Cardoso, Popov and Vodev [2], we can obtain the same result as Theorem 1.1 by the similar argument in Section 5. On the contrary, in the case that the phase speed of the inside medium is less than that of the outside one, Popov and Vodev [25] show the existence of the sequence of the resonances approaching to the real axis. Thus we cannot expect to have such decay estimates, nor have any polynomial bound for $p_{m,R}(t)$. Finally, let us mention an interesting result of Burq [1] that for every obstacle, the upper bound $p_{m,R}(t) = O((\log(1+t)^{-m})$ is given. Even in the cases of no polynomial bound for $p_{m,R}(t)$, the method of Burq [1] gives the above logarithmic bound. Unfortunately, this bound seems to be too weak to obtain the weighted energy estimates in Theorem 1.1.

In Sections 2 and 3, we consider the weighted energy estimates for the solutions of the case of the Cauchy problem or that of the Dirichlet boundary condition in the exterior of star shaped obstacles $\mathcal{O}$. These are considered as free systems in scattering theory. In Section 2, we state the estimates for free space case. In our approach, as is in Theorem 1.1, we need to see how the coefficients of the estimates depend on l ($l \neq 1$) explicitly. Using this information we show the estimate for $l = 1$, that is the Keel, Smith and Sogge estimate. This argument is given in Section 2.

For the Keel, Smith and Sogge estimate, in the whole Euclidean space $\mathbb{R}^n$, Hidano and Yokoyama [3] give precise arguments and investigations about the estimate itself and the scaling invariant version of the estimate. For star shaped obstacles, the estimate is also obtained in Metcalfe and Sogge [16] by using the argument of Morawetz [22] for estimating decay of the local energy. Thus some parts of these cases have already been shown. Still in Section 3, we give a proof of these estimates since we can see how the argument developed by Morawetz and Mochizuki explains well why the estimates hold. As another reason, to obtain Theorem 1.1, we need to have L^2-type estimates in time integral for inhomogeneous data $f(t,x)$ in (1.1) (cf. Theorem 4.1). For the purpose, we use the argument of Morawetz and Mochizuki mentioned above.

In the context of scattering theory, Mochizuki develops the idea of Morawetz [22] to show various energy decay estimates (cf. Mochizuki [18, 19] and Mochizuki and Nakazawa [20, 21]). These correspond to the case of $l > 1$ in the estimate of Theorem 1.1 for star shaped obstacles. In the case of the Cauchy problem, as is in Mochizuki [17], the estimates for $l > 1$ correspond to the smooth operator estimate introduced by Kato [7]. Thus the estimates for $l > 1$ are implicitly well known in scattering theory. We can also show the estimate for $l = 1$ if we choose the multiplier in [19] and [20] in a proper way. But we do not use this approach since for the Cauchy problem, it seems to be difficult to perform integration by parts to obtain necessary identities. To obtain the estimates for the Cauchy problems (cf. Section 3.1), we use rather simple multiplier with parameters studied in Sugimoto [28]. One of the advantages of our choice of the multipliers is to see why the case $l = 1$ is critical explicitly.

The estimates given in Theorem 1.1 are of L^1-type in time variable t. To handle the perturbed system, we have to control L^2-type integrals in time variable. This means that for the solutions of the free systems, we need to have L^2-type estimates in time variable. In Section 4, these estimates are given (cf. Theorem 4.1). In these estimates, since we take L^2-type integrals in time, we have to put some weight to the space variables. Hence the arguments to obtaining these estimates are more complicated than those in Section 3 since we need the weights for the space variables x. Last in Section 5, using the estimates for free systems, we show Theorem 1.1. As we can see in the proof, these estimates for perturbed systems can be obtained if we have the estimates for free systems and the uniform decay estimates about local energy for perturbed systems.

2 Weighted energy estimates for free systems

Let us consider the case of the Dirichlet boundary condition or the Cauchy problem (i.e. the non-obstacle case). In these cases, the Morawetz identity is used to obtain weighted energy estimates. As is in Morawetz [22], we consider the following problem:

$$\begin{cases} (\partial_t^2 - \triangle)u(t,x) = f(t,x) & \text{in } \mathbb{R}\times\Omega, \\ u(t,x) = 0 & \text{on } \mathbb{R}\times\partial\Omega, \\ u(0,x) = f_1(x), \quad \partial_t u(0,x) = f_2(x) & \text{on } \Omega. \end{cases} \tag{2.1}$$

Here we assume that the obstacle $\mathcal{O} = \mathbb{R}^n \setminus \Omega$ satisfies one of the following conditions:

(H) the obstacle $\mathcal{O}$ is star shaped with respect to a point $x_0 \in \mathcal{O}$

or

(C) the obstacle $\mathcal{O}$ is empty.

Note that condition (C) means the case of the Cauchy problem for the whole Euclidean space $\mathbb{R}^n$.

We shall state weighted energy estimates separately as considering cases of the weights though they can be given simultaneously as is in Theorem 1.1.

Theorem 2.1. *Assume that $n \geq 3$ and the obstacle $\mathcal{O}$ satisfies* (H) *or* (C). *Then there exists a constant $C > 0$ such that for any $l \geq 0$, $l \neq 1$,*

$$\int_0^t \int_\Omega (1+|x|)^{-l} e(s,x;u)\,dx\,ds + \int_0^t \int_\Omega (1+|x|)^{-(l+2)} |u(s,x)|^2\,dx\,ds \leq C q_l(t) \left\{ E(u,\Omega,0) + \left(\int_0^t \|f(s,\cdot)\|_{L^2(\Omega)} ds \right)^2 \right\}.$$

From Theorem 2.1, we have the weighted energy estimate with the weight $(1+|x|)^{-1}$, that is the Keel, Smith and Sogge estimate.

Corollary 2.2. *Assume that $n \geq 3$ and the obstacle $\mathcal{O}$ satisfies* (H) *or* (C). *Then there exists a constant $C > 0$ such that for any $t \geq 0$*

$$\int_0^t \int_\Omega (1+|x|)^{-1} e(s,x;u)\,dx\,ds + \int_0^t \int_\Omega (1+|x|)^{-3} |u(s,x)|^2\,dx\,ds \leq C \log(1+t) \left\{ E(u,\Omega,0) + \left(\int_0^t \|f(s,\cdot)\|_{L^2(\Omega)} ds \right)^2 \right\}.$$

The proof of Theorem 2.1 is given in the following section. In the rest of this section, using Theorems 2.1, we show Corollary 2.2.

Proof of Corollary 2.2. It suffices to show the estimate in the case of $t \geq e^2 - 1$. For fixed $t \geq e^2 - 1$, choose l as $l = 1 - (\log(1+t))^{-1}$. Since $t \geq e^2 - 1$ we have $1/2 < l < 1$ and $(1+t)^{1-l} = e$. Hence from the fact $(1+|x|)^{-1} \leq (1+|x|)^{-l}$, Theorem 2.1 implies that

$$\int_0^t \int_\Omega (1+|x|)^{-1} e(s,x;u)\,dx\,ds \leq C \log(1+t) I(t), \tag{2.2}$$

where

$$I(t) = \left\{ E(u,\Omega,0) + \left(\int_0^t \|f(s,\cdot)\|_{L^2(\Omega)} ds \right)^2 \right\}. \tag{2.3}$$

Since it follows that $(1+|x|)^{-(l+2)} \geq (1+|x|)^{-3}$ $(0 \leq l < 1, x \in \mathbb{R}^n)$, from Theorem 2.1, the argument for (2.2) implies

$$\int_0^t \int_\Omega (1+|x|)^{-3} |u(s,x)|^2\,dx\,ds \leq C \log(1+t) I(t).$$

This completes the proof of Corollary 2.2. $\square$

3 Weighted energy estimates from the Morawetz identity

In this section, we give a proof of Theorem 2.1. If condition (H) holds, by translation, we can assume that x_0 in condition (H) is just the origin, i.e.

$$\nu(x) \cdot x \leq 0 \quad (x \in \partial\Omega). \tag{3.1}$$

3.1 The Morawetz identity and basic estimates

We begin with stating the Morawetz identity.

Proposition 3.1. *For any $v \in H^2_{\mathrm{loc}}(\mathbb{R} \times \Omega)$ and a scalar valued function $F \in C^2(\mathbb{R}^n \setminus \{0\})$, we have the following identity:*

$$\begin{aligned} \mathrm{Re}\Big[F\Big(x \cdot \nabla_x \overline{v} + \frac{n-1}{2}\overline{v}\Big)(\partial_t^2 - \triangle)v\Big] \\ &= \partial_t(X(t,x;v)) + \mathrm{div}(Y(t,x;v)) + Z(t,x;v) \\ &\quad + \Big(2^{-1}\mathrm{div}(Fx) - \frac{n-1}{2}F\Big)\big(|\partial_t v|^2 - |\nabla_x v|^2\big) + F|\nabla_x v|^2 \\ &\quad + \mathrm{Re}[(\nabla_x F \cdot \nabla_x v)x \cdot \nabla_x \overline{v}], \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} X(t,x;v) &= \mathrm{Re}\Big[F(x)\partial_t v(t,x)\Big(x \cdot \nabla_x \overline{v(t,x)} + \frac{n-1}{2}\overline{v(t,x)}\Big)\Big], \\ Y(t,x;v) &= 2^{-1}\big(|\nabla_x v|^2 - |\partial_t v|^2\big)Fx \\ &\qquad - \mathrm{Re}\Big[F(x \cdot \nabla_x \overline{v})\nabla_x v + \frac{n-1}{2}Fv\nabla_x \overline{v} - \frac{n-1}{4}|v|^2\nabla_x F\Big], \\ Z(t,x;v) &= -\frac{n-1}{4}(\triangle F)(x)|v(t,x)|^2. \end{aligned}$$

In (3.2), the case that $F = 1$ is the original Morawetz identity given in [22] to show the decay estimate $E(u, \Omega \cap B_R, t) = O(t^{-1})$. This argument is summarized by Mochizuki (cf. Mochizuki [19], Mochizuki and Nakazawa [21], and the references therein). In these works, basically the multiplier F is chosen as $F(x) = |x|^{-1}\phi(|x|)$ with an appropriate function ϕ satisfying $\phi(0) > 0$.

As is in [19] for example, this choice of the multiplier makes the identity simple, however, it seems to be difficult to handle the case of the Cauchy problem. Hence in what follows, for $0 < \delta \leq 1$, $0 < \beta$ and $0 \leq l$ we choose F in (3.2) as $F(x) = (\delta + |x|^{2\beta})^{-l/2}$.

Lemma 3.2. *For $F(x) = (\delta + |x|^{2\beta})^{-l/2}$ ($0 < \delta \leq 1$, $0 < \beta$, $0 \leq l$), we have*

(1) $2^{-1}\mathrm{div}(Fx) - \frac{n-1}{2}F = -\frac{l\beta}{2}F(x)(\delta + |x|^{2\beta})^{-1}|x|^{2\beta} + \frac{1}{2}F(x)$,

(2) $(\nabla_x F \cdot \nabla_x v) x \cdot \nabla_x \overline{v} = -l\beta F(x)(\delta + |x|^{2\beta})^{-1}|x|^{2\beta} \left||x|^{-1} x \cdot \nabla_x v\right|^2$,

(3) $\triangle F(x) = -l\beta F(x)(\delta + |x|^{2\beta})^{-1}|x|^{2(\beta-1)}\{n - l\beta - 2 + \delta\beta(l+2)(\delta + |x|^{2\beta})^{-1}\}$.

Proof. Since $\partial_{x_j} F(x) = -l\beta x_j F(x)(\delta + |x|^{2\beta})^{-1}|x|^{2(\beta-1)}$ we have $\mathrm{div}(Fx) = x \cdot \nabla_x F + nF = -l\beta F(x)(\delta + |x|^{2\beta})^{-1}|x|^{2\beta} + nF$. This implies (1). Statement (2) is obvious.

We show (3). From the form of $\partial_{x_j} F(x)$, it follows that

$$|x|^2 F^{-1} \nabla_x F = -l\beta(\delta + |x|^{2\beta})^{-1}|x|^{2\beta} x, \tag{3.3}$$

which implies $\triangle F = |x|^{-2} F \mathrm{div}\big(-l\beta(\delta + |x|^{2\beta})^{-1}|x|^{2\beta} x\big) - 2|x|^{-2} x \cdot \nabla_x F + F^{-1}(\nabla_x F)^2$. Note that from (3.3) it follows that $F^{-1}(\nabla_x F)^2 = (l\beta)^2 F(\delta + |x|^{2\beta})^{-2}|x|^{2\beta}|x|^{2(\beta-1)}$ and $|x|^{-2} x \cdot \nabla_x F = -l\beta F(\delta + |x|^{2\beta})^{-1}|x|^{2(\beta-1)}$. Combining them with the equality

$$\begin{aligned}&\mathrm{div}\big(-l\beta(\delta|x|^{2\beta})^{-1}|x|^{2\beta} x\big)\\ &\quad = -l\beta|x|^2(\delta + |x|^{2\beta})^{-1}|x|^{2(\beta-1)}\{n + 2\delta\beta(\delta + |x|^{2\beta})^{-1}\},\end{aligned}$$

we obtain (3) of Lemma 3.2. This completes the proof of Lemma 3.2. □

From (3.2) and Lemma 3.2, it follows that

$$\begin{aligned}&\mathrm{Re}\Big[F\Big(x \cdot \nabla_x \overline{v} + \frac{n-1}{2}\overline{v}\Big)(\partial_t^2 - \triangle)v\Big]\\ &\quad = \partial_t X(t,x;v) + \mathrm{div} Y + Z(t,x;v)\\ &\qquad + F(x)\big(1 - l\beta + \delta l\beta(\delta + |x|^{2\beta})^{-1}\big)e(t,x;v)\\ &\qquad + l\beta F(x)(\delta + |x|^{2\beta})^{-1}|x|^{2\beta}\{|\nabla_x v|^2 - \left||x|^{-1} x \cdot \nabla_x v\right|^2\}.\end{aligned} \tag{3.4}$$

We have the following basic estimate from identity (3.4):

Proposition 3.3. *Assume that $n \geq 3$. For star shaped obstacles with respect to the origin, every solution $u \in C^0(\mathbb{R}; \dot{H}^1(\Omega))$, $\partial_t u \in C^0(\mathbb{R}; L^2(\Omega))$ of equation* (2.1) *satisfies the following estimate:*

$$\begin{aligned}&\int_0^t \int_\Omega F(x)\big(1 - l\beta + \delta l\beta(\delta + |x|^{2\beta})^{-1}\big)e(s,x;u)\,dx\,ds + \int_0^t \int_\Omega Z(s,x;u)\,dx\,ds\\ &\qquad \leq I_{l,\beta}(t;u,\delta) - \int_\Omega X(t,x;u)\,dx + \int_\Omega X(0,x;u)\,dx\\ &\qquad\qquad \textit{for any } t \geq 0,\ 1 \geq \delta > 0,\ \beta > 0 \textit{ and } l \geq 0,\end{aligned}$$

where $F(x) = (\delta + |x|^{2\beta})^{-l/2}$ *and*

$$K_{l,\beta}(t;u,\delta) = \int_0^t \int_\Omega F(x)\big(1 - l\beta + \delta l\beta(\delta + |x|^{2\beta})^{-1}\big)e(s,x;u)dxds, \quad (3.5)$$

$$I_{l,\beta}(t;u,\delta) = \mathrm{Re}\int_0^t \int_\Omega F(x)\Big(x\cdot\nabla_x\overline{u(s,x)} + \frac{n-1}{2}\overline{u(s,x)}\Big) f(s,x)dxds.$$

If $u \in C^0(\mathbb{R};\dot{H}^1(\mathbb{R}^n))$, $\partial_t u \in C^1(\mathbb{R};L^2(\mathbb{R}^n))$ *is a solution of the Cauchy problem of the wave equation in the whole space* $\mathbb{R}^n$*, we also have the estimate given by replacing* Ω *with the whole space* $\mathbb{R}^n$ *in the above estimate.*

Proof. First we consider the case of the Dirichlet problem. Putting $v = u$ in (3.4), and integrating (3.4) over $[0,t]\times\Omega$, we obtain

$$\begin{aligned} I_{l,\beta}(t;u,\delta) = &\int_\Omega X(t,x;u)dx - \int_\Omega X(0,x;u)dx + \int_0^t\int_\Omega \mathrm{div}Ydxds \\ &+ \int_0^t\int_\Omega Z(s,x;u)dxds + K_{l,\beta}(t;u,\delta) \\ &+ l\beta\int_0^t\int_\Omega \Big(F(x)(\delta+|x|^{2\beta})^{-1}|x|^{2\beta} \\ &\qquad\qquad \times\{|\nabla_x u|^2 - \big||x|^{-1}x\cdot\nabla_x u\big|^2\}\Big)dxds. \end{aligned}$$

In the case of the Dirichlet condition, it follows that $\nabla_x u(t,x) = \nu(x)\partial_\nu u(t,x)$ on $\mathbb{R}\times\partial\Omega$. Hence as is in Morawetz [22], integrating by parts and using (3.1) we have

$$\begin{aligned} \int_0^t\int_\Omega \mathrm{div}Ydxds &= \int_0^t\int_{\partial\Omega} \nu(x)\cdot YdS_xds \\ &= -\frac{1}{2}\int_0^t\int_{\partial\Omega} F\nu\cdot x|\partial_\nu u|^2dS_xds \\ &\geq 0. \end{aligned}$$

From this estimate and the fact that $|\nabla_x v|^2 - \big||x|^{-1}x\cdot\nabla_x v\big|^2 \geq 0$, we obtain Proposition 3.3 in the case of the Dirichlet condition.

Next consider the case of the Cauchy problem. Let $B_\varepsilon(0) = \{x\in\mathbb{R}^n \mid |x| < \varepsilon\}$. If this is the case, replacing Ω with $\mathbb{R}^n\setminus B_\varepsilon(0)$, we follow the same argument as in the case of the Dirichlet condition, and take the limit $\varepsilon\to 0$. When we take the limit, every term not containing derivatives of F converges to the integral over $[0,t]\times\mathbb{R}^n$. Thus it suffices to show the following limits for the terms containing derivatives of F:

$$\int_0^t \int_{\mathbb{R}^n \setminus B_\varepsilon(0)} \operatorname{div} Y dx ds = -\int_0^t \int_{\partial B_\varepsilon(0)} \frac{x}{|x|} \cdot Y dS_x ds \to 0 \quad (\varepsilon \to 0), \tag{3.6}$$

$$\int_0^t \int_{\mathbb{R}^n \setminus B_\varepsilon(0)} Z(s, x; v) dx ds \to \int_0^t \int_{\mathbb{R}^n} Z(s, x; v) dx ds \qquad (\varepsilon \to 0). \tag{3.7}$$

We give a proof of (3.6). From the definition of Y and the estimate of the derivatives of F, it is enough to show

$$\begin{aligned} &\int_0^t \int_{\partial B_\varepsilon(0)} |\nabla_x F||u(s,x)|^2 dS_x ds \\ &\qquad \le C \int_0^t \int_{\partial B_\varepsilon(0)} |x|^{2\beta-1} |u(s,x)|^2 dS_x ds \\ &\qquad \le C\varepsilon^{2\beta-1} \int_0^t \int_{\partial B_\varepsilon(0)} |u(s,x)|^2 dS_x ds \to 0 \quad (\varepsilon \to 0). \end{aligned} \tag{3.8}$$

To show (3.8), we use the fact that $f_1 \in H^1_{\mathrm{loc}}(\mathbb{R}^n)$ for any $f_1 \in \dot{H}^1(\mathbb{R}^n)$ and Hardy's inequality

$$\int_{\tilde{\Omega}} |x|^{-2} |f_1(x)|^2 dx \le C \int_{\tilde{\Omega}} |\nabla_x f_1(x)|^2 dx, \tag{3.9}$$

where $\tilde{\Omega} = \Omega$ or $\tilde{\Omega} = \mathbb{R}^n$. Note that (3.9) holds only in the case that $n \ge 3$. For $f_1 \in \dot{H}^1(\mathbb{R}^n)$, the trace theorem implies that $f_1 \in L^2(\partial B_\varepsilon(0))$, and there exists a $C > 0$ such that for any $\varepsilon > 0$

$$\begin{aligned} \int_{\partial B_\varepsilon(0)} |f_1(x)|^2 dS_x &= \varepsilon^{n-1} \int_{\partial B_1(0)} |f_1(\varepsilon\omega)|^2 dS_\omega \\ &\le C\varepsilon^{n-1} \int_{B_1(0)} \{\varepsilon^2 |(\nabla_x f_1)(\varepsilon x)|^2 + |f_1(\varepsilon x)|^2\} dx. \end{aligned}$$

Changing the variables in the above estimate, we have

$$\int_{\partial B_\varepsilon(0)} |f_1(x)|^2 dS_x \le C\varepsilon^{-1} \int_{B_\varepsilon(0)} \{\varepsilon^2 |(\nabla_x f_1)(x)|^2 + |f_1(x)|^2\} dx. \tag{3.10}$$

From Hardy's inequality (3.9), it follows that

$$\int_{B_\varepsilon(0)} |f_1(x)|^2 dx \le C\varepsilon^2 \int_{B_\varepsilon(0)} |x|^{-2} |f_1(x)|^2 dx \le C\varepsilon^2 \int_{\mathbb{R}^n} |\nabla_x f_1(x)|^2 dx.$$

Combining the above estimate, (3.10) and noting that $\nabla_x u \in C^0(\mathbb{R}; L^2(\mathbb{R}^2))$ we obtain (3.8) in the following way:

$$\begin{aligned}\varepsilon^{2\beta-1} &\int_0^t \int_{\partial B_\varepsilon(0)} |u(s,x)|^2 dS_x ds \\ &\le C\varepsilon^{2\beta} \int_0^t \int_{\mathbb{R}^n} |\nabla_x u(s,x)|^2 dx ds \to 0 \quad (\varepsilon \to 0).\end{aligned}$$

Hence we have proved (3.6).

For (3.7), from the definition of Z and the estimate of the derivatives of F, it suffices to show that

$$\int_0^t \int_{B_\varepsilon(0)} |x|^{2\beta-2} |u(t,x)|^2 dx ds \to 0 \quad (\varepsilon \to 0).$$

This limit can be obtained by the same argument as in the proof of (3.8). Hence we have (3.7). Thus in the case of the Cauchy problem, we have shown Proposition 3.3. □

We need estimates of the weight $F(x) = (\delta + |x|^{2\beta})^{-l/2}$ used in Proposition 3.3.

Lemma 3.4. *For any* $1 \ge \delta > 0$, $\beta > 0$ *and* $l \ge 0$, *we have the following estimate:*

$$(\delta + |x|^{2\beta})^{-l/2} \ge C_{l,\beta}(\delta)(\delta + |x|)^{-l\beta},$$

where $C_{l,\beta}(\delta) = \min\{1, (1+\delta)^{\frac{l}{2}(2\beta-1)}, \delta^{\frac{l}{2}(2\beta-1)}\}$.

Proof. We put $\varphi(r) = (\delta + r)^{l\beta}(\delta + r^{2\beta})^{-l/2}$ $(r \ge 0)$. Since

$$\varphi'(r) = l\beta(\delta + r)^{l\beta-1}(\delta + r^{2\beta})^{-l/2-1}\delta\{1 - r^{2\beta-1}\},$$

for $0 < \beta < 1/2$, it follows that

$$\varphi(r) \ge \varphi(1) = (1+\delta)^{l\beta}(1+\delta)^{-l/2} = (1+\delta)^{\frac{l}{2}(2\beta-1)}.$$

For $\beta = 1/2$, we have $\varphi(r) = 1$ $(r \ge 0)$. For $\beta > 1/2$, noting that

$$\varphi(r) = \frac{(\delta + r)^{l\beta}}{(\delta + r^{2\beta})^{l/2}} = \frac{(r^{-1}\delta + 1)^{l\beta}}{(r^{-2\beta}\delta + 1)^{l/2}} \to 1 \quad (r \to \infty)$$

and $\varphi(0) = \delta^{l\beta}\delta^{-l/2} = \delta^{\frac{l}{2}(2\beta-1)}$, we have proved Lemma 3.4. □

We introduce the following notations:

$$K_{l,\beta}^{(1)}(t;u,\delta) = \int_0^t \int_\Omega (\delta + |x|)^{-l\beta} e(s,x;u)dxds,$$

$$K_{l,\beta}^{(2)}(t;u,\delta) = \int_0^t \int_\Omega (\delta + |x|)^{-(l+2)\beta} e(s,x;u)dxds, \tag{3.11}$$

$$Z_{l,\beta}^{(1)}(t;u,\delta) = \int_0^t \int_\Omega (\delta + |x|)^{-(l+2)\beta} |x|^{2(\beta-1)}|u(s,x)|^2 dxds, \tag{3.12}$$

$$Z_{l,\beta}^{(2)}(t;u,\delta) = \int_0^t \int_\Omega (\delta + |x|)^{-(l+4)\beta} |x|^{2(\beta-1)}|u(s,x)|^2 dxds. \tag{3.13}$$

From (3.5) and Lemma 3.4, it follows that for any $t \geq 0$, $l, \beta > 0$ and $0 < \delta \leq 1$

$$K_{l,\beta}(t;u,\delta) \geq C_{l,\beta}(\delta)(1 - l\beta)K_{l,\beta}^{(1)}(t;u,\delta) + C_{l+2,\beta}(\delta)\delta l\beta K_{l,\beta}^{(2)}(t;u,\delta).$$

In the estimate of Proposition 3.3, the weighted L^2-norms of $u(t,x)$ are contained in the term consisting of the integral of Z. To pick up these terms defined by (3.12) and (3.13), we essentially need $n \geq 3$.

Lemma 3.5. *Assume that $n \geq 3$. There exists a constant $C > 0$ such that for any $1 \geq \delta > 0$, $\beta > 0$ and $l > 0$ with $l\beta \leq 1$*

$$\int_0^t \int_\Omega Z(s,x;u)dxds \geq \frac{(n-1)l\beta}{4}\Big\{C_{l+2,\beta}(\delta)(1 - l\beta)Z_{l,\beta}^{(1)}(t;u,\delta) + \delta C_{l+4,\beta}(\delta)\beta(l+2)Z_{l,\beta}^{(2)}(t;u,\delta)\Big\}.$$

From this estimate, we also have

$$\int_0^t \int_\Omega Z(s,x;u)dxds \geq 0.$$

Proof. Since $n \geq 3$, we have $n - l\beta - 2 \geq 1 - l\beta \geq 0$. Hence by the definition of $Z(t,x;u)$ and (3) of Lemma 3.2, we have

$$Z(t,x;u) \geq \frac{(n-1)l\beta}{4}\Big\{(1 - l\beta)F(x)(\delta + |x|^{2\beta})^{-1}|x|^{2(\beta-1)}|u(t,x)|^2 + \delta\beta(l+2)F(x)(\delta + |x|^{2\beta})^{-2}|x|^{2(\beta-1)}|u(t,x)|^2\Big\}.$$

From Lemma 3.4, it follows that

$$Z(t,x;u) \geq \frac{(n-1)l\beta}{4}\Big\{\min\Big\{1, (1+\delta)^{\frac{l+2}{2}(2\beta-1)}, \delta^{\frac{l+2}{2}(2\beta-1)}\Big\} \times (1-l\beta)\frac{|x|^{2(\beta-1)}|u(t,x)|^2}{(1+|x|)^{(l+2)\beta}} + \min\Big\{1, (1+\delta)^{\frac{l+4}{2}(2\beta-1)}, \delta^{\frac{l+4}{2}(2\beta-1)}\Big\} \times \delta\beta(l+2)\frac{|x|^{2(\beta-1)}|u(t,x)|^2}{(1+|x|)^{(l+4)\beta}}\Big\}.$$

Integrating the above inequality over $[0,t]\times\Omega$, we obtain Lemma 3.5. □

Summing up the above arguments, we obtain the following proposition:

Proposition 3.6. *Assume that $n \geq 3$. For star shaped obstacles with respect to the origin, every solution $u \in C^0(\mathbb{R};\dot{H}^1(\Omega))$, $\partial_t u \in C^0(\mathbb{R};L^2(\Omega))$ of equation* (2.1) *satisfies the following estimate:*

$$\begin{aligned}(1-l\beta)\Big\{C_{l,\beta}(\delta)K^{(1)}_{l,\beta}(t;u,\delta) + \frac{n-1}{4}l\beta C_{l+2,\beta}(\delta)Z^{(1)}_{l,\beta}(t;u,\delta)\Big\} \\ + \delta l\beta\Big\{C_{l+2,\beta}(\delta)K^{(2)}_{l,\beta}(t;u,\delta) + \frac{n-1}{4}(l+2)\beta C_{l+4,\beta}(\delta)Z^{(2)}_{l,\beta}(t;u,\delta)\Big\} \\ \leq I_{l,\beta}(t;u,\delta) - \int_\Omega X(t,x;u)dx + \int_\Omega X(0,x;u)dx \\ \textit{for any } t \geq 0,\ 1 \geq \delta > 0,\ \beta > 0 \textit{ and } l \geq 0 \textit{ with } l\beta \leq 1.\end{aligned}$$

If $u \in C^0(\mathbb{R};\dot{H}^1(\mathbb{R}^n))$, $\partial_t u \in C^1(\mathbb{R};L^2(\mathbb{R}^n))$ is a solution of the Cauchy problem of the wave equation in the whole space $\mathbb{R}^n$, we also have the estimate given by replacing Ω with the whole space $\mathbb{R}^n$ in the above estimate.

3.2 Proof of Theorem 2.1

We choose a cutoff function $\psi \in C_0^\infty(\mathbb{R}^n)$ with $\psi(x) = 1$ (for $|x| \leq 1$), $\psi(x) = 0$ (for $|x| \geq 2$), and put $\psi_R(x) = \psi(R^{-1}x)$. We consider the solutions $u_R^{(1)}(t,x)$ and $u_R^{(2)}(t,x)$ of the following equations respectively:

$$\begin{cases}(\partial_t^2 - \triangle)u_R^{(1)}(t,x) = \psi_R(x)f(t,x) & \text{in } \mathbb{R}\times\Omega, \\ u_R^{(1)}(t,x) = 0 & \text{on } \mathbb{R}\times\partial\Omega, \\ u_R^{(1)}(0,x) = \psi_R(x)f_1(x), \quad \partial_t u_R^{(1)}(0,x) = \psi_R(x)f_2(x) & \text{on } \Omega\end{cases} \tag{3.14}$$

and

$$\begin{cases} (\partial_t^2 - \triangle)u_R^{(2)}(t,x) = (1-\psi_R(x))f(t,x) & \text{in } \mathbb{R}\times\Omega, \\ u_R^{(2)}(t,x) = 0 & \text{on } \mathbb{R}\times\partial\Omega, \\ u_R^{(2)}(0,x) = (1-\psi_R(x))f_1(x) & \text{on } \Omega, \\ \partial_t u_R^{(2)}(0,x) = (1-\psi_R(x))f_2(x) & \text{on } \Omega. \end{cases} \tag{3.15}$$

Note that from the property of the finite propagation speed, we have

$$\operatorname{supp} u_R^{(1)}(t,\cdot) \subset \{x \in \mathbb{R}^n \mid |x| \le 2R + t\}, \tag{3.16}$$

$$\operatorname{supp} u_R^{(2)}(t,\cdot) \subset \{x \in \mathbb{R}^n \mid R - t \le |x|\} \tag{3.17}$$

since $\operatorname{supp}\psi_R \subset \{x \in \mathbb{R}^n \mid |x| \le 2R\}$ and $\operatorname{supp}(1-\psi_R) \subset \{x \in \mathbb{R}^n \mid R \le |x|\}$. We also note that there exists a constant $C > 0$ such that

$$E(u_R^{(j)}, \Omega, 0) \le CE(u, \Omega, 0) \quad (R \ge 0,\ j = 1, 2). \tag{3.18}$$

To show this, we use the fact that $R \le |x| \le 2R$ if $\nabla_x\psi(R^{-1}x) \ne 0$. From this, we have

$$E(u_R^{(j)}, \Omega, 0) \le 2E(u, \Omega, 0) + 4 \sup_{x\in\mathbb{R}^n} |\nabla_x\psi(x)|^2 \int_\Omega |x|^{-2}|f_1(x)|^2 dx.$$

This estimate and Hardy's inequality (3.9) imply (3.18).

First we give estimates of $u_R^{(1)}(t,x)$. For this, we use Proposition 3.6. We begin with considering the term of $X(t,x;u_R^{(1)})$.

Lemma 3.7. *There exists a constant $C > 0$ such that for any $\beta \ge 0$ and $l \ge 0$ with $l\beta \le 1$*

$$\left| \int_\Omega X(t,x;u_R^{(1)})dx \right| \le C(2R+t)^{1-l\beta} I(t) \quad (t \ge 0, R > 0),$$

where $I(t)$ is introduced in (2.3).

Proof. Since for $1 \ge \delta > 0$, β, $l \ge 0$, it follows that $|x|F(x) = |x|(\delta + |x|^{2\beta})^{-l/2} \le |x|(|x|^{2\beta})^{-l/2} = |x|^{1-l\beta}$, we have

$$\begin{aligned} |X(t,x;u_R^{(1)})| &\le |x|F(x)|\partial_t u_R^{(1)}(t,x)|\Big(|\nabla_x u_R^{(1)}(t,x)| + \frac{n-1}{2}|x|^{-1}|u_R^{(1)}(t,x)|\Big) \\ &\le |x|^{1-l\beta}|\partial_t u_R^{(1)}(t,x)|\Big(|\nabla_x u_R^{(1)}(t,x)| + \frac{n-1}{2}|x|^{-1}|u_R^{(1)}(t,x)|\Big). \end{aligned} \tag{3.19}$$

Hence noting (3.16), (3.9) and $l\beta \le 1$, we obtain

$$\left| \int_\Omega X(t, x; u_R^{(1)}) dx \right| \le C(2R + t)^{1-l\beta} E(u_R^{(1)}, \Omega, t). \tag{3.20}$$

Since the usual energy estimate implies

$$E(u_R^{(1)}, t, \Omega) \le E(u_R^{(1)}, \Omega, 0) + C \left(\int_0^t \| f(s, \cdot) \|_{L^2(\Omega)} ds \right)^2 \quad (t \ge 0, R \ge 0), \tag{3.21}$$

combining (3.21), (3.20) with (3.18), we obtain Lemma 3.7. □

For the term $I_{l,\beta}(t; u_R^{(1)}, \delta)$, using the argument for (3.19), we have

$$\begin{aligned} I_{l,\beta}(t; u_R^{(1)}, \delta) \le \int_0^t \int_\Omega &|x|^{1-l\beta} |\psi_R(x) f(s, x)| \Big(|\nabla_x u_R^{(1)}(s, x)| \\ &+ \frac{n-1}{2} |x|^{-1} |u_R^{(1)}(s, x)| \Big) dx ds \\ \le (2R + t)^{1-l\beta} &\int_0^t \int_\Omega |f(s, x)| \Big(|\nabla_x u_R^{(1)}(s, x)| \\ &+ \frac{n-1}{2} |x|^{-1} |u_R^{(1)}(s, x)| \Big) dx ds. \end{aligned}$$

Hence Hardy's inequality and (3.21) yield

$$I_{l,\beta}(t; u_R^{(1)}, \delta) \le C(2R + t)^{1-l\beta} \left\{ E(u, \Omega, 0) + \left(\int_0^t \| f(s, \cdot) \|_{L^2(\Omega)} ds \right)^2 \right\}. \tag{3.22}$$

From Lemma 3.7, (3.22) and Proposition 3.6, there exists a constant $C > 0$ such that

$$\begin{aligned} &(1 - l\beta) \Big\{ C_{l,\beta}(\delta) K_{l,\beta}^{(1)}(t; u_R^{(1)}, \delta) + l\beta C_{l+2,\beta}(\delta) Z_{l,\beta}^{(1)}(t; u_R^{(1)}, \delta) \Big\} \\ &\quad + \delta l\beta \Big\{ C_{l+2,\beta}(\delta) K_{l,\beta}^{(2)}(t; u_R^{(1)}, \delta) + \beta(l + 2) C_{l+4,\beta}(\delta) Z_{l,\beta}^{(2)}(t; u_R^{(1)}, \delta) \Big\} \\ &\qquad \le C(2R + t)^{1-l\beta} I(t) \end{aligned} \tag{3.23}$$

for any $t, R \ge 0$, $1 \ge \delta > 0$, $\beta \ge 0$ and $l \ge 0$ with $l\beta \le 1$.

For $u_R^{(2)}(t,x)$ we have the following estimate:

Proposition 3.8. *There exists a constant* $C > 0$ *such that*

$$\int_0^t\int_\Omega \frac{e(s,x;u_R^{(2)})}{(\delta+|x|)^\alpha}dxds + \alpha\int_0^t\int_\Omega \frac{|x|^{2(\beta-1)}|u_R^{(2)}(s,x)|^2}{(\delta+|x|)^{\alpha+2\beta}}dxds$$
$$\leq Ct(R-t)^{-\alpha}I(t) \quad (0\leq\alpha,\beta,\ 0<\delta\leq 1, 0\leq t\leq R).$$

Proof. For t with $0\leq t\leq R$, from (3.17), $u^{(2)}(t,x)=0$ for $|x|\leq R-t$. Since $(\delta+|x|)^{-\alpha}\leq(R-t)^{-\alpha}$ $(0\leq\alpha,\ |x|\geq R-t)$ we have

$$\begin{aligned}
&\int_0^t\int_\Omega(\delta+|x|)^{-\alpha}e(s,x;u_R^{(2)})dxds\\
&\quad=\int_0^t\int_{\Omega\setminus B_{R-t}(0)}(\delta+|x|)^{-\alpha}e(s,x;u_R^{(2)})dxds\\
&\quad\leq(R-t)^{-\alpha}\int_0^t E(u_R^{(2)},\Omega,s)ds.
\end{aligned}\tag{3.24}$$

From (3.21) and (3.18), it follows that

$$\begin{aligned}
E(u_R^{(2)},\Omega,s)&\leq E(u_R^{(2)},\Omega,0)+\left(\int_0^s\|(1-\psi_R(\cdot))f(s,\cdot)\|_{L^2(\Omega)}ds\right)^2\\
&\leq CI(s)\quad(0\leq s\leq t),
\end{aligned}$$

which yields

$$\begin{aligned}
\int_0^t\int_\Omega\frac{e(s,x;u_R^{(2)})}{(\delta+|x|)^\alpha}dxds&\leq C(R-t)^{-\alpha}\int_0^t E(s,u_R^{(2)},\Omega)ds\\
&\leq Ct(R-t)^{-\alpha}I(t).
\end{aligned}\tag{3.25}$$

Since we also have $(\delta+|x|)^{-(\alpha+2\beta)}|x|^{2\beta}\leq(\delta+|x|)^{-\alpha}\leq(R-t)^{-\alpha}$ $(\alpha\geq 0)$ for $|x|\geq R-t$, noting Hardy's inequality (3.9) and using the argument for (3.25), we obtain

$$\int_0^t\int_\Omega(\delta+|x|)^{-(\alpha+2\beta)}|x|^{2(\beta-1)}|u_R^{(2)}(s,x)|^2dxds\leq\frac{Ct}{(R-t)^\alpha}I(t).\tag{3.26}$$

From (3.25) and (3.26), we have Proposition 3.8. □

Now we show Theorem 2.1. Noting $C_{l,\beta}(\delta) \le 1$ ($l, \beta \ge 0$), from (3.23) and Proposition 3.8, we have

$$\begin{aligned}
&(1-l\beta)\Big\{C_{l,\beta}(\delta)K^{(1)}_{l,\beta}(t;u,\delta) + l\beta C_{l+2,\beta}(\delta)Z^{(1)}_{l,\beta}(t;u,\delta)\Big\}\\
&\quad + \delta l\beta\Big\{C_{l+2,\beta}(\delta)K^{(2)}_{l,\beta}(t;u,\delta) + \beta(l+2)C_{l+4,\beta}(\delta)Z^{(2)}_{l,\beta}(t;u,\delta)\Big\}\\
&\qquad \le CI(t)\{(2R+t)^{1-l\beta} + t(R-t)^{-l\beta}\}\\
&\qquad\qquad (0 \le \beta, l,\ 0 < \delta \le 1,\ l\beta \le 1,\ 0 \le t \le R).
\end{aligned}$$

In the above estimate, we choose R by $R = 2t + 1$. Since it follows that $(2R + t)^{1-l\beta} + t(R-t)^{-l\beta} = (2+5t)^{1-l\beta} + t(1+t)^{-l\beta} \le 6(1+t)^{1-l\beta}$, we have

$$\begin{aligned}
&(1-l\beta)\Big\{C_{l,\beta}(\delta)K^{(1)}_{l,\beta}(t;u,\delta) + l\beta C_{l+2,\beta}(\delta)Z^{(1)}_{l,\beta}(t;u,\delta)\Big\}\\
&\quad + \delta l\beta\Big\{C_{l+2,\beta}(\delta)K^{(2)}_{l,\beta}(t;u,\delta) + \beta(l+2)C_{l+4,\beta}(\delta)Z^{(2)}_{l,\beta}(t;u,\delta)\Big\}\\
&\qquad \le CI(t)(1+t)^{1-l\beta} \quad (0 \le \beta, l, 0 < \delta \le 1, l\beta \le 1, 0 \le t).
\end{aligned} \tag{3.27}$$

We choose an arbitrary $1 < l_0 \le 2$. For this l_0, let $\beta = (l_0 - 1)/2$ ($\le 1/2$), $l = \beta^{-1}$ and $\delta = \beta$ in (3.27). In this case, noting $C_{l',\beta}(\beta) = (1+\beta)^{l'(\beta-1/2)}$ ($l' > 0$), $|x|^{2(\beta-1)}(\beta + |x|)^{-(1+4\beta)} \ge (1+|x|)^{-(3+2\beta)} = (1+|x|)^{-(l_0+2)}$ and $(1+|x|)^{-(1+2\beta)} \le (\beta+|x|)^{-(1+2\beta)}$, we have

$$\int_0^t\int_\Omega \frac{e(s,x;u)}{(1+|x|)^{l_0}}dxds + \int_0^t\int_\Omega \frac{|u(s,x)|^2}{(1+|x|)^{l_0+2}}dxds \le \frac{C}{\beta}(1+\beta)^{\frac{1}{2\beta}(1-2\beta)}I(t).$$

Note that the constant $C > 0$ in the above estimate is independent of β with $0 < \beta \le 1/2$. Since $(1+\beta)^{\frac{1}{2}(\frac{1}{\beta}-2)} = (1+\beta)^{-1}\{(1+\beta)^{\frac{1}{\beta}}\}^{\frac{1}{2}} \to e^{\frac{1}{2}}$ ($\beta \to 0$), it follows that $\sup_{0<\beta\le 1/2}(1+\beta)^{\frac{1}{2}(\frac{1}{\beta}-2)} < \infty$. Hence we have

$$\begin{aligned}
&\int_0^t\int_\Omega \frac{e(s,x;u)}{(1+|x|)^{l_0}}dxds + \int_0^t\int_\Omega \frac{|u(s,x)|^2}{(1+|x|)^{l_0+2}}dxds\\
&\qquad \le \frac{C}{l_0-1}I(t) \quad (1 < l_0 \le 2).
\end{aligned} \tag{3.28}$$

For $l_0 \ge 2$, noting $(1+|x|)^{-l_0} \le (1+|x|)^{-2}$ and using (3.28), we have a constant $C > 0$ such that

$$\int_0^t\int_\Omega \frac{e(s,x;u)}{(1+|x|)^{l_0}}dxds + \int_0^t\int_\Omega \frac{|u(s,x)|^2}{(1+|x|)^{l_0+2}}dxds \le CI(t) \quad (l_0 \ge 2). \tag{3.29}$$

If $0 \le l_0 < 1$, we use (3.27) as $\beta > 0$, $l = l_0/\beta > 0$ and $\delta = \min\{1, \beta\}$. In this case, since $C_{l,\beta}(\delta) \ge 1/2$ for $\beta \ge 1/2$, $C_{l,\beta}(\delta) \ge \{(1+\beta)^{-1/\beta}\}^{l_0/2}$ for $0 < \beta \le 1/2$, we have a constant $C > 0$ such that $C_{l,\beta}(\delta) \ge C$ for any $0 < \beta$ and $0 \le l_0 < 1$ with $l = l_0/\beta$. Thus, there exists a constant $C > 0$ such that for any $\beta > 0$

$$\begin{aligned} &\int_0^t \int_\Omega \frac{e(s,x;u)}{(1+|x|)^{l_0}} dxds + l_0 \int_0^t \int_\Omega \frac{|x|^{2(\beta-1)}|u(s,x)|^2}{(1+|x|)^{l_0+2\beta}} dxds \\ &\qquad \le \frac{C(1+t)^{1-l_0}}{1-l_0} I(t). \end{aligned} \tag{3.30}$$

From (3.29), (3.28) and the case of $\beta = 1$ in (3.30), to finish the proof of Theorem 2.1, it suffices to show that there exist constants $0 < \alpha_0 < 1$ and $C > 0$ such that

$$\int_0^t \int_\Omega \frac{|x|^{2(\beta-1)}|u(s,x)|^2}{(1+|x|)^{l_0+2\beta}} dxds \le \frac{C(1+t)^{1-l_0}}{1-l_0} I(t) \tag{3.31}$$

for $t \ge 0$, $0 \le l_0 \le \alpha_0$, $0 < \beta$. Note that Hardy's inequality (3.9) yields

$$\begin{aligned} &\int_0^t \int_\Omega \frac{|x|^{-2}|u(s,x)|^2}{(1+|x|)^{l_0}} dxds \\ &\quad \le C \int_0^t \int_\Omega \left|\nabla_x\big((1+|x|)^{-l_0/2} u(s,x)\big)\right|^2 dxds \\ &\quad \le C \left\{ \int_0^t \int_\Omega \frac{e(s,x;u)}{(1+|x|)^{l_0}} dxds \right. \\ &\qquad\qquad \left. + \int_0^t \int_\Omega \left|\nabla_x\big((1+|x|)^{-l_0/2}\big)\right|^2 |u(s,x)|^2 dxds \right\}. \end{aligned}$$

This estimate, (3.30) and the estimate $|x|^{2\beta}(1+|x|)^{-2\beta} \le 1$ imply (3.31) since we have

$$\left|\nabla_x\big((1+|x|)^{-l_0/2}\big)\right|^2 \le \left| -(l_0/2)(1+|x|)^{-l_0/2-1} \frac{x}{|x|} \right|^2 \le \frac{l_0^2}{4} \frac{|x|^{-2}}{(1+|x|)^{l_0}}.$$

Thus we have Theorem 2.1.

4 L^2-type estimates for inhomogeneous data

The estimates given in Theorem 2.1 and Corollary 2.2 are estimates of L^1-type in t for the inhomogeneous data $f(t,x)$. For removing assumption (H) (or (C)) introduced in Section 2, we need estimates of L^2-type in t for the inhomogeneous data $f(t,x)$. In this section, we prepare these estimates to obtain the main theorem.

Theorem 4.1. *Assume that $n \geq 3$ and the obstacle $\mathcal{O}$ satisfies* (H) *or* (C). *Then there exists a constant $C > 0$ such that for any $l \geq 0$ and $t \geq 0$*

$$\begin{aligned}
&\int_0^t \int_\Omega (1+|x|)^{-l} e(s,x;u)\,dxds \\
&\quad + \int_0^t \int_\Omega (1+|x|)^{-(l+2\beta)} |x|^{2(\beta-1)} |u(s,x)|^2 dxds \\
&\qquad \leq C q_l(t) \left\{ E(u,\Omega,0) + \int_0^t \int_\Omega (1+|x|)^2 |f(s,x)|^2 dxds \right\}.
\end{aligned}$$

Thus we can also obtain L^2-type estimates in t. Instead of that, however, we need a weight in x.

In the rest of this section, we show Theorem 4.1. Note that the case $l = 1$ is given by the argument in the proof of Corollary 2.2. Hence we show the case that $l \geq 0, l \neq 1$.

We begin with showing the following estimate:

Lemma 4.2. *For $F_1(x) = (1+|x|^{2\beta_1})^{-l_1/2}$ with $\beta_1, l_1 \geq 0$ and $v \in C(\mathbb{R}; \dot{H}^1(\Omega))$, we have*

$$\begin{aligned}
&\int_0^t \int_\Omega F_1(x) \left| \frac{x}{|x|} \cdot \nabla_x v(s,x) + \frac{n-1}{2|x|} v(s,x) \right|^2 dxds \\
&\quad \leq \int_0^t \int_\Omega F_1(x) |\nabla_x v(s,x)|^2 dxds \\
&\qquad + \frac{(n-1) l_1 \beta_1}{2} \int_0^t \int_\Omega F_1(x) (1+|x|^{2\beta_1})^{-1} |x|^{2(\beta_1 - 1)} |v(s,x)|^2 dxds.
\end{aligned}$$

Proof. We follow the argument given on page 472 of Morawetz, Ralston and Strauss [24]. Calculating the term

$$\begin{aligned}
&2F_1(x)\,\mathrm{Re}\left(|x|^{-2}(x \cdot \nabla_x v)(s,x)\overline{v(s,x)}\right) \\
&\quad = F_1(x)|x|^{-2} x \cdot \nabla_x \left(|v(s,x)|^2\right) \\
&\quad = \mathrm{div}\left(|v(s,x)|^2 F_1(x)|x|^{-2}x\right) - |v(s,x)|^2 \mathrm{div}\left(F_1(x)|x|^{-2}x\right) \\
&\quad = \mathrm{div}\left(|v(s,x)|^2 F_1(x)|x|^{-2}x\right) \\
&\qquad + F_1(x)|x|^{-2}\left(l_1\beta_1(1+|x|^{2\beta_1})^{-1}|x|^{2\beta_1} - (n-2)\right)|v(s,x)|^2,
\end{aligned}$$

we obtain

$$
\begin{aligned}
&F_1(x)\Big|\frac{x}{|x|}\cdot\nabla_x v(s,x)+\frac{n-1}{2|x|}v(s,x)\Big|^2\\
&\quad= F_1(x)\left|\frac{x}{|x|}\cdot\nabla_x v(s,x)\right|^2+\frac{n-1}{2}\Big\{\operatorname{div}\big(|v(s,x)|^2F_1(x)|x|^{-2}x\big)\\
&\qquad+|v(s,x)|^2F(x)|x|^{-2}\{l_1\beta_1(1+|x|^{2\beta_1})^{-1}|x|^{2\beta_1}-(n-2)\}\\
&\qquad+\frac{n-1}{2}|v(s,x)|^2F_1(x)|x|^{-2}\Big\}.
\end{aligned}
$$

Since $(n-1)/2-(n-2)=-(n-3)/2\le 0$ for $n\ge 3$, it follows that

$$
\begin{aligned}
&F_1(x)\Big|\frac{x}{|x|}\cdot\nabla_x v(s,x)+\frac{n-1}{2|x|}v(s,x)\Big|^2\\
&\quad\le F_1(x)\big|\nabla_x v(s,x)\big|^2+\frac{n-1}{2}\operatorname{div}\big(|v(s,x)|^2F_1(x)|x|^{-2}x\big)\\
&\qquad+\frac{n-1}{2}|v(s,x)|^2F_1(x)l_1\beta_1(1+|x|^{2\beta_1})^{-1}|x|^{2(\beta_1-1)}\\
&\qquad\qquad(l_1,\beta_1\ge 0, n\ge 3).
\end{aligned}
\tag{4.1}
$$

Now we consider the case of star shaped obstacles. If this is the case, from (3.1) we have

$$
\begin{aligned}
&\int_0^t\int_\Omega \operatorname{div}\big(|v(s,x)|^2F_1(x)|x|^{-2}x\big)dx\,ds\\
&\quad=\int_0^t\int_{\partial\Omega}|v(s,x)|^2F_1(x)\frac{\nu(x)\cdot x}{|x|^2}dS_x\,ds\le 0.
\end{aligned}
\tag{4.2}
$$

Hence we obtain the estimate in Lemma 4.2 by integrating (4.1) over $[0,t]\times\Omega$.

In the case of the Cauchy problem, it suffices to show inequality (4.2) in the case that $\Omega=\mathbb{R}^n$. Choose $\varphi\in C^1([0,\infty))$ satisfying $\varphi(s)=0$ for $0\le s\le 1$, $\varphi(s)=1$ for $s\ge 2$ and $\varphi'(s)\ge 0$ $(0\le s)$. For any $\varepsilon>0$, integration by parts implies

$$
\begin{aligned}
&\int_0^t\int_{\mathbb{R}^n}\varphi(\varepsilon^{-1}|x|)\operatorname{div}\big(|v(s,x)|^2F_1(x)|x|^{-2}x\big)dx\,ds\\
&\quad=-\int_0^t\int_{\mathbb{R}^n}\varepsilon^{-1}|x|\varphi'(\varepsilon^{-1}|x|)|v(s,x)|^2F_1(x)|x|^{-2}dx\,ds\le 0\quad(\varepsilon>0).
\end{aligned}
$$

From Hardy's inequality (3.9), the integrated function $\operatorname{div}\big(|u(s,x)|^2 F_1(x)|x|^{-2}x\big)$ is integrable in $[0,t]\times\mathbb{R}^n$. Thus, taking the limit as $\varepsilon \to 0$, we have estimate (4.2). This completes the proof of Lemma 4.2. □

As is in the proof of Theorem 2.1, using the solutions $u_R^{(1)}(t,x)$ and $u_R^{(2)}(t,x)$ of (3.14) and (3.15) we decompose the solution $u(t,x)$ of (2.1) by $u = u_R^{(1)} + u_R^{(2)}$. Since we have Theorem 2.1, we can assume that the initial data f_1 and f_2 in (1.1) vanish.

Lemma 4.3. *We have the following estimate:*

$$\begin{aligned} I_{l,\beta}(t;u_R^{(1)},\delta) \le{}& (2R+t)^{1-l\beta}\sqrt{L(t)} \\ &\times \big\{K^{(2)}_{2,2^{-1}}(t;u_R^{(1)},1) + (n-1)Z^{(2)}_{2,2^{-1}}(t;u_R^{(1)},1)\big\}^{1/2} \\ &\qquad (t \ge 0,\, R>0,\, \beta, l \ge 0,\, l\beta \le 1,\, 1 \ge \delta > 0), \end{aligned}$$

where $L(t)$ is defined by

$$L(t) = \int_0^t \int_\Omega (1+|x|)^2 |f(s,x)|^2 dxds.$$

Proof. Since $|x|F(x) = |x|(\delta + |x|^{2\beta})^{-1} \le |x|(|x|^{2\beta})^{-l/2} = |x|^{1-l\beta}$, for any $1 \ge \delta > 0$, $\beta > 0$ and $l > 0$ with $l\beta \le 1$, (3.16) and Schwarz's inequality imply

$$\begin{aligned} I_{l,\beta}(t;u_R^{(1)},\delta) \le{}& \int_0^t \int_\Omega \Big((1+|x|)^{-1} \left| \frac{x}{|x|}\cdot\nabla_x u_R^{(1)}(s,x) + \frac{n-1}{2|x|} u_R^{(1)}(s,x) \right| \\ &\times |x|F(x)(1+|x|)|\psi_R f(s,x)| \Big) dxds \\ \le{}& (2R+t)^{1-l\beta}\sqrt{L(t)} \Big\{ \int_0^t \int_\Omega \Big((1+|x|)^{-2} \left| \frac{x}{|x|}\cdot\nabla_x u_R^{(1)}(s,x) \right. \\ &\left. + \frac{n-1}{2|x|} u_R^{(1)}(s,x) \right|^2 \Big) dxds \Big\}^{1/2}. \end{aligned}$$

We choose $\beta_1 = 1/2$, $l_1 = 4$ and $v = u_R^{(1)}$ in Lemma 4.2. In this case, we have $F_1(x) = (1+|x|)^{-2}$ and $F_1(x)(1+|x|^{2\beta_1})^{-1}|x|^{2(\beta_1-1)} = (1+|x|)^{-3}|x|^{-1}$. Hence Lemma 4.2 implies that

$$\begin{aligned} &\int_0^t \int_\Omega (1+|x|)^{-2} \left| \frac{x}{|x|}\cdot\nabla_x u(s,x) + \frac{n-1}{2|x|}u(s,x) \right|^2 dxds \\ &\quad \le \int_0^t \int_\Omega \frac{e(s,x,u)}{(1+|x|)^2} dxds + (n-1)\int_0^t \int_\Omega \frac{|x|^{-1}|u(s,x)|^2}{(1+|x|)^3} dxds. \end{aligned}$$

From (3.11) and (3.13), we have

$$
\begin{aligned}
K^{(2)}_{2,2^{-1}}(t;u,1) &= \int_0^t \int_\Omega \frac{e(s,x,u)}{(1+|x|)^2}dxds,\\
Z^{(2)}_{2,2^{-1}}(t;u,1) &= \int_0^t \int_\Omega \frac{|x|^{-1}|u(s,x)|^2}{(1+|x|)^3}dxds,
\end{aligned}
\tag{4.3}
$$

which completes the proof of Lemma 4.3. □

The argument for (3.20) and the usual energy identity for solution of (3.14) imply

$$
\left|\int_\Omega X(t,x;u^{(1)}_R)dx\right| \le C(2R+t)^{1-l\beta}\int_0^t\int_\Omega |\partial_t u^{(1)}_R(s,x)||\psi_R(x)f(s,x)|dxds.
$$

From the above estimate and (4.3), it follows that

$$
\begin{aligned}
\left|\int_\Omega X(t,x;u^{(1)}_R)dx\right| &\le C(2R+t)^{1-l\beta}\sqrt{L(t)}\left\{K^{(2)}_{2,2^{-1}}(t;u^{(1)}_R,1)\right\}^{1/2}\\
&\qquad (t\ge 0, R>0, \beta, l\ge 0, l\beta\le 1, 1\ge\delta>0).
\end{aligned}
\tag{4.4}
$$

Now we choose $\beta = 1/2$, $l = 2$ and $\delta = 1$ in the estimate in Proposition 3.6. Then noting $C_{l',\beta}(1) = 1$ ($l' > 0$), Lemma 4.3 and (4.4), we have

$$
\begin{aligned}
&K^{(2)}_{2,2^{-1}}(t;u^{(1)}_R,1) + Z^{(2)}_{2,2^{-1}}(t;u^{(1)}_R,1)\\
&\qquad \le C\sqrt{L(t)}\left\{K^{(2)}_{2,2^{-1}}(t;u^{(1)}_R,1) + Z^{(2)}_{2,2^{-1}}(t;u^{(1)}_R,1)\right\}^{1/2},
\end{aligned}
$$

which yields

$$
K^{(2)}_{2,2^{-1}}(t;u^{(1)}_R,1) + Z^{(2)}_{2,2^{-1}}(t;u^{(1)}_R,1) \le CL(t) \quad (t\ge 0).
$$

Combining the above estimate, Lemma 4.3, (4.4) with Proposition 3.6, we have the following estimate for $u^{(1)}_R(t,x)$:

Proposition 4.4. *Assume that $n \ge 3$ and the obstacle $\mathcal{O}$ satisfies* (H) *or* (C). *Then there exists a constant $C > 0$ such that for any $\beta, l \ge 0$ with $l\beta \le 1$, $1 \ge \delta > 0$, $R > 0$ and $t \ge 0$*

$$
\begin{aligned}
&(1-l\beta)\left\{C_{l,\beta}(\delta)K^{(1)}_{l,\beta}(t;u^{(1)}_R,\delta) + l\beta C_{l+2,\beta}(\delta)Z^{(1)}_{l,\beta}(t;u^{(1)}_R,\delta)\right\}\\
&\quad + \delta l\beta\left\{C_{l+2,\beta}(\delta)K^{(2)}_{l,\beta}(t;u^{(1)}_R,\delta) + (l+2)\beta C_{l+4,\beta}(\delta)Z^{(2)}_{l,\beta}(t;u^{(1)}_R,\delta)\right\}\\
&\qquad \le C(2R+t)^{1-l\beta}L(t).
\end{aligned}
$$

Next we give an estimate of $u_R^{(2)}(t,x)$. Noting (3.21) with $E(u,\Omega,0)=0$ and $f(t,x)\to(1-\psi_R(x))f(t,x)$ and using Schwarz's inequality, we have

$$E(u_R^{(2)},\Omega,s)\le s\int_0^t\|(1-\psi_R(\cdot))f(\tau,\cdot)\|_{L^2(\Omega)}^2d\tau\quad(0\le s\le t).$$

This estimate and (3.24) yield

$$\int_0^t\int_\Omega\frac{e(s,x;u_R^{(2)})}{(\delta+|x|)^\alpha}dxds\le\frac{Ct^2}{(R-t)^\alpha}\int_0^t\|(1-\psi_R(\cdot))f(\tau,\cdot)\|_{L^2(\Omega)}^2d\tau$$
$$(0\le t\le R,0\le\alpha,0<\delta\le1).$$

Using the fact that $R\le|x|$ if $(1-\psi_R(x))\ne0$, we have

$$\|(1-\psi_R(\cdot))f(\tau,\cdot)\|_{L^2(\Omega)}^2\le R^{-2}\int_\Omega|x|^2|f(\tau,x)|^2dx.$$

From these estimates, there exists a constant $C>0$ such that

$$\int_0^t\int_\Omega\frac{e(s,x;u_R^{(2)})}{(\delta+|x|)^\alpha}dxds\le\frac{Ct^2}{(R-t)^\alpha R^2}L(t)\quad(0\le t\le R,0\le\alpha,0<\delta\le1).\tag{4.5}$$

Note that we also have

$$\int_0^t\int_\Omega\frac{|x|^{2(\beta-1)}|u_R^{(2)}(s,x)|^2}{(\delta+|x|)^{\alpha+2\beta}}dxds\le\frac{Ct^2}{(R-t)^\alpha R^2}L(t)\tag{4.6}$$
$$(0\le t\le R,0\le\alpha,\beta,0<\delta\le1)$$

by the argument for (4.5) since the estimate $(\delta+|x|)^{-(\alpha+2\beta)}|x|^{2\beta}\le(\delta+|x|)^{-\alpha}\le(R-t)^{-\alpha}$ $(\alpha\ge0)$ for $|x|\ge R-t$ and Hardy's inequality (3.9) imply

$$\int_0^t\int_\Omega\frac{|x|^{2(\beta-1)}|u_R^{(2)}(s,x)|^2}{(1+|x|)^{\alpha+2\beta}}dxds\le\frac{C}{(1+R-t)^\alpha}\int_0^tE(u_R^{(2)},\Omega,s)ds.$$

Now we show Theorem 4.1. As is in Section 3, noting $C_{l,\beta}(\delta)\le1$ $(\beta,l\ge0)$ and choosing $R=2t+1$ in Proposition 4.4, (4.5) and (4.6), we obtain

$$\begin{aligned}(1-l\beta)&\Big\{C_{l,\beta}(\delta)K_{l,\beta}^{(1)}(t;u,\delta)+l\beta C_{l+2,\beta}(\delta)Z_{l,\beta}^{(1)}(t;u,\delta)\Big\}\\&+\delta l\beta\Big\{C_{l+2,\beta}(\delta)K_{l,\beta}^{(2)}(t;u,\delta)+(l+2)\beta C_{l+4,\beta}(\delta)Z_{l,\beta}^{(2)}(t;u,\delta)\Big\}\\&\le C(1+t)^{1-l\beta}L(t)\quad(t\ge0,1\ge\delta>0,l,\beta\ge0).\end{aligned}\tag{4.7}$$

Hence noting that Theorem 2.1 is shown by using (3.27), from (4.7), we can obtain the estimates in Theorem 4.1 for $l\ne1$. This completes the proof of Theorem 4.1.

5 Proof of Theorem 1.1

In this section, for convenience we express the solutions of problem (1.1) in the energy space H. For initial data $\{f_1, f_2\}$ in problem (1.1), we set $\vec{f} = {}^t(f_1, f_2)$. The energy space H is defined by the completion of the set $\{\vec{f} \in C_0^\infty(\overline{\Omega}) \mid \mathscr{B} f_1 = 0 \text{ on } \partial\Omega\}$ by the norm

$$\|\vec{f}\|_H^2 = \frac{1}{2}\int_\Omega \{|\nabla_x f_1(x)|^2 + |f_2(x)|^2\}dx.$$

In the case that $n \geq 3$, H is given by $H = \dot{H}^1_{\mathscr{B}}(\Omega) \times L^2(\Omega)$. For initial data $\vec{f} = {}^t(f_1, f_2) \in H = \dot{H}^1_{\mathscr{B}}(\Omega) \times L^2(\Omega)$ and inhomogeneous data $f(t,x) = 0$, problem (1.1) has the unique solution $u \in C(\mathbb{R}; \dot{H}^1(\Omega))$ with $\partial_t u(t,x) \in C(\mathbb{R}; L^2(\Omega))$. For this solution $u(t,x)$, we define $U(t)$ by $U(t)\vec{f} = {}^t(u(t,\cdot), \partial_t u(t,\cdot))$. The energy conservation law implies that $\{U(t)\}_{t\in\mathbb{R}}$ is a one parameter family of unitary operators on H. The generator L of $\{U(t)\}_{t\in\mathbb{R}}$ is given by $L\vec{f} = {}^t(f_2, \triangle f_1)$ for $\vec{f} \in D(L) = \dot{H}^2_{\mathscr{B}}(\Omega) \times H^1(\Omega)$.

Note that for any initial data $\vec{f} = {}^t(f_1, f_2) \in H$ and inhomogeneous data $f \in L^1_{\text{loc}}(\mathbb{R}; L^2(\Omega))$, problem (1.1) has the unique solution $u \in C(\mathbb{R}; \dot{H}^1(\Omega))$ with $\partial_t u(t,x) \in C(\mathbb{R}; L^2(\Omega))$. For this solution $u(t,x)$, we set $V(t, \vec{f}, f) = {}^t(u(t,\cdot), \partial_t u\ (t,\cdot))$. Then we have

$$V(t, \vec{f}, f) = U(t)\vec{f} + \int_0^t U(t-s)F(s)ds \ \text{ in } H \quad (F(s) = {}^t(0, f(s,\cdot))). \tag{5.1}$$

For the Cauchy problem corresponding to (1.1), we introduce the energy space H_0, a one parameter family of unitary operators $\{U_0(t)\}_{t\in\mathbb{R}}$ and its generator L_0 corresponding to H, $\{U(t)\}_{t\in\mathbb{R}}$ and L respectively. Note that in this case, $H_0 = \dot{H}^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ and $D(L_0) = \dot{H}^2(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$. For the solution $u(t,x)$ of the Cauchy problem corresponding to (1.1), we put $V_0(t, \vec{f}, f) = {}^t(u(t,\cdot), \partial_t u(t,\cdot))$. We have

$$V_0(t, \vec{f}, f) = U_0(t)\vec{f} + \int_0^t U_0(t-s)F(s)ds \ \text{ in } H_0 \quad (F(s) = {}^t(0, f(s,\cdot))).$$

This is the same as in the case of (5.1).

We choose a cutoff function $\psi \in C^\infty(\mathbb{R}^n)$ with $\psi(x) = 0$ for $|x| \leq R_0 + \tilde{R}_0/3$, $\psi(x) = 1$ for $|x| \geq R_0 + 2\tilde{R}_0/3$. We put $\psi U(t)\vec{f} = {}^t(\psi u(t,\cdot), \psi\partial_t u(t,\cdot))$, where $u(t,x)$ is the solution of (1.1) for initial data $\vec{f} \in H$ and inhomogeneous data $f(t,x) = 0$.

From Hardy's inequality (3.9), Duhamel's principle and the fact that $\psi = 0$ near $\partial\Omega$, we have the following lemma.

Lemma 5.1. *We have the following properties:*

(1) *We have* $\psi U(t)\vec{f} \in C(\mathbb{R}; H_0)$ *for* $\vec{f} \in H$, *and* $\psi U(t)\vec{f} \in C^1(\mathbb{R}; H_0) \cap C^0(\mathbb{R}; D(L_0))$ *for* $\vec{f} \in D(L)$. *Further, there holds*

$$\psi U(t)\vec{f} = U_0(t)(\psi \vec{f}) + \int_0^t U_0(t-s)Q_\psi U(s)\vec{f}\,ds \ \text{ in } H_0 \quad (\vec{f} \in H),$$

where $Q_\psi \in B(H, H_0) \cap B(D(L), D(L_0))$ *is defined by satisfying the identity* $Q_\psi \vec{g} = {}^t(0, -[\triangle, \psi]g_1)$.

(2) *We have* $\psi U_0(t)\vec{f}_0 \in C(\mathbb{R}; H)$ *for* $\vec{f}_0 \in H_0$ *and* $\psi U_0(t)\vec{f}_0 \in C^1(\mathbb{R}; H) \cap C^0(\mathbb{R}; D(L))$ *for* $\vec{f}_0 \in D(L_0)$. *Further, there holds*

$$\psi U_0(t)\vec{f}_0 = U(t)(\psi \vec{f}_0) + \int_0^t U(t-s)Q_\psi U_0(s)\vec{f}_0\,ds \ \text{ in } H \quad (\vec{f}_0 \in H_0),$$

where $Q_\psi \in B(H_0, H) \cap B(D(L_0), D(L))$ *is defined by satisfying the identity* $Q_\psi \vec{g} = {}^t(0, -[\triangle, \psi]g_1)$.

From now on, for $l \geq 0$, $\vec{f} = {}^t(f_1, f_2)$, $m \in \mathbb{N} \cup \{0\}$ and $D \subset \mathbb{R}^n$, we define $\|\vec{f}\|_{H_{m,l}(D)}$ (≥ 0) and $\|\vec{f}\|_{H_m(D)}$ by

$$\|\vec{f}\|^2_{H_{m,l}(D)} = \sum_{1 \leq j+|\alpha| \leq m+1} \int_D \frac{|\partial_x^\alpha f_{j+1}(x)|^2}{(1+|x|)^l}\,dx$$

and

$$\|\vec{f}\|_{H_m(D)} = \|\vec{f}\|_{H_{m,0}(D)}.$$

Note that for any $m \in \mathbb{N}$ we have $D(L^m) \subset H_m(\Omega)$ and the graph norm $\|\cdot\|_{D(L^m)}$ is estimated as $C_m^{-1}\|\vec{f}\|_{H_m(\Omega)} \leq \|\vec{f}\|_{D(L^m)} \leq C_m\|\vec{f}\|_{H_m(\Omega)}$ $(m = 0, 1, 2, \ldots)$.

From the weighted energy estimates obtained in Theorem 2.1 and Corollary 2.2, we have the following estimates for $\{U_0(t)\}$ immediately:

Proposition 5.2. *There exists a constant* $C > 0$ *such that for* $t \geq 0$, $l \geq 0$ *and* $\vec{g} \in H_0$ *the following hold:*

(i) $\displaystyle\int_0^t \|U_0(s)\vec{g}\|^2_{H_{0,l}(\mathbb{R}^n)}ds \leq Cq_l(t)\|\vec{g}\|^2_{H_0}$,

(ii) $\displaystyle\int_0^t \|(1+|\cdot|)^{-(l+2)}[U_0(s)\vec{g}]_1(\cdot)\|^2_{L^2(\mathbb{R}^n)}ds \leq Cq_l(t)\|\vec{g}\|^2_{H_0}$,

where the $q_l(t)$ *are introduced in Section 1.*

We put $H_m(\Omega) = (\dot{H}^1_{\mathcal{B}}(\Omega) \cap \dot{H}^{m+1}(\Omega)) \times H^m(\Omega)$ $(m = 0, 1, 2, \ldots)$. Using Proposition 5.2, we have the following estimates of higher order derivatives of the solutions:

Proposition 5.3. *For any integer* $m \geq 0$*, there exists a constant* $C(m) > 0$ *such that*

$$\int_0^t \|U_0(s)\vec{g}\|^2_{H_{m,l}(\mathbb{R}^n)} ds \leq C(m) q_l(t) \|\vec{g}\|^2_{H_m(\mathbb{R}^n)} \quad (t \geq 0, l \geq 0, \vec{g} \in H_m(\mathbb{R}^n)).$$

Indeed, noting $\bigcap_{j=0}^m D(L_0^j) = \dot{H}^{m+1}(\mathbb{R}^n) \times H^m(\mathbb{R}^n)$, the property $\partial_x^\alpha U_0(t) = U_0(t)\partial_x^\alpha$ for the free system and the fact that $\sum_{|\alpha| \leq m} \|\partial_x^\alpha \vec{g}\|^2_{H_0} \leq C_m \|\vec{g}\|^2_{H_m(R^n)}$, we conclude Proposition 5.3 from Proposition 5.2.

Now we show Theorem 1.1. We begin with estimating the time integral of local energy norms.

Proposition 5.4. *Assume that* $n \geq 3$ *and* (E1) *is satisfied. Then there exists a constant* $C > 0$ *such that for* $t \geq 0$ *and* $\vec{f} \in D(L^m))$ *the following hold:*

(i) $\displaystyle\int_0^t \|U(s)\vec{f}\|^2_{H(\Omega \cap B_{R_0+\tilde{R_0}}(0))} ds \leq C\|\vec{f}\|^2_{H_m(\Omega)}$,

(ii) $\displaystyle\int_0^t \|[U(s)\vec{f}]_1\|^2_{L^2(\Omega \cap B_{R_0+\tilde{R_0}}(0))} ds \leq C\|\vec{f}\|^2_{H_m(\Omega)}$.

Proof. We choose an arbitrary $\vec{g} \in H_m(\mathbb{R}^n)$. From (2) of Lemma 5.1, it follows that

$$\begin{aligned}
&\int_0^t \|U(s)(\psi\vec{g})\|^2_{H(\Omega \cap B_{R_0+\tilde{R_0}}(0))} ds \\
&\quad \leq 2\int_0^t \|\psi U_0(s)\vec{g}\|^2_{H(\Omega \cap B_{R_0+\tilde{R_0}}(0))} ds \\
&\qquad + 2\int_0^t \left(\int_0^s \|U(s-\tau)Q_\psi U_0(\tau)\vec{g}\|_{H(\Omega \cap B_{R_0+\tilde{R_0}}(0))} d\tau\right)^2 ds.
\end{aligned} \tag{5.2}$$

Note that for the first term on the right-hand side of (5.2), there exists a constant $C > 0$ such that

$$\begin{aligned}
&\|\psi U_0(s)\vec{g}\|_{H(\Omega \cap B_{R_0+\tilde{R_0}}(0))} \\
&\quad \leq C\left\{\|U_0(s)\vec{g}\|_{H_{0,2}(\mathbb{R}^n)} + \|[U_0(s)\vec{g}]_1\|_{L^2(\Omega \cap (B_{R_0+\tilde{R_0}}(0) \setminus B_{R_0}(0)))}\right\} \\
&\quad \leq C\left\{\|U_0(s)\vec{g}\|_{H_{0,2}(\mathbb{R}^n)} + \|(1+|\cdot|)^{-4}[U_0(s)\vec{g}]_1(\cdot)\|_{L^2(\Omega)}\right\}.
\end{aligned}$$

Hence the case of $l = 2$ in Proposition 5.2 implies

$$\int_0^t \|\psi U_0(s)\vec{g}\|^2_{H(\Omega\cap B_{R_0+\tilde{R}_0}(0))} ds \le C\|\vec{g}\|^2_{H_0} \quad (t \ge 0, \vec{g} \in H_0). \tag{5.3}$$

For the second term of the right-hand side in (5.2), we need assumption (E1) for the decay rate of the local energy. Note that for any $\vec{h} \in H_0$ we have $Q_\psi \vec{h}(x) = 0$ for $|x| \le R_0$, and for any $\vec{g} \in H_m(\mathbb{R}^n)$, it follows that $Q_\psi \vec{g} \in D(L^m)$ and $\operatorname{supp} Q_\psi \vec{g} \subset B_{R_0+\tilde{R}_0}$. Since $\vec{f} \in D(L^m)$ if and only if $\vec{f} = {}^t(f_1, f_2) \in H_m(\Omega)$ and the pair $(f_1, f_2, 0)$ satisfies the compatibility condition of order m, from assumption (E1), there exists a function $p \in C([0,\infty)) \cap L^1([0,\infty))$ such that

$$\begin{aligned} &\|U(t)Q_\psi \vec{h}\|_{H(\Omega\cap B_{R_0+\tilde{R}_0}(0))} + \|[U(t)Q_\psi \vec{h}]_1\|_{L^2(\Omega\cap B_{R_0+\tilde{R}_0}(0))} \\ &\qquad \le p(t)\|Q_\psi \vec{h}\|_{H_m(\Omega)} \quad (t \ge 0, \vec{h} \in H_m(\mathbb{R}^n)). \end{aligned} \tag{5.4}$$

Putting $\vec{h} = U_0(\tau)\vec{g}$ in (5.4), and noting that $U_0(\tau)\vec{g} \in H_m(\mathbb{R}^n)$ for $\vec{g} \in H_m(\mathbb{R}^n)$, we have

$$\begin{aligned} &\|U(t)Q_\psi U_0(\tau)\vec{g}\|_{H(\Omega\cap B_{R_0+\tilde{R}_0}(0))} + \|[U(t)Q_\psi U_0(\tau)\vec{g}]_1\|_{L^2(\Omega\cap B_{R_0+\tilde{R}_0}(0))} \\ &\qquad \le p(t)\|Q_\psi U_0(\tau)\vec{g}\|_{H_m(\Omega)} \quad (t \ge 0, \tau \in \mathbb{R}, \vec{g} \in H_m(\mathbb{R}^n)). \end{aligned} \tag{5.5}$$

From (5.5), Schwarz's inequality implies that

$$\begin{aligned} &\int_0^t \left(\int_0^s \|U(s-\tau)Q_\psi U_0(\tau)\vec{g}\|_{H(\Omega\cap B_{R_0+\tilde{R}_0}(0))} d\tau \right)^2 ds \\ &\qquad \le \int_0^t \int_0^s p(s-\tau)d\tau \int_0^s p(s-\tau)\|Q_\psi U_0(\tau)\vec{g}\|^2_{H_m(\mathbb{R}^n)} d\tau ds \\ &\qquad \le \|p\|_{L^1([0,\infty))} \int_0^t \int_\tau^t p(s-\tau)\|Q_\psi U_0(\tau)\vec{g}\|^2_{H_m(\mathbb{R}^n)} ds d\tau \\ &\qquad \le \|p\|^2_{L^1([0,\infty))} \int_0^t \|Q_\psi U_0(\tau)\vec{g}\|^2_{H_m(\mathbb{R}^n)} d\tau. \end{aligned} \tag{5.6}$$

Noting that

$$\begin{aligned} &\|Q_\psi U_0(\tau)\vec{g}\|^2_{H_m(\Omega)} \\ &\qquad \le C\left\{ \|U_0(\tau)\vec{g}\|^2_{H_{m,2}(\mathbb{R}^n)} + \|(1+|\cdot|)^{-4}|[U_0(\tau)\vec{g}]_1(\cdot)|\|^2_{L^2(\mathbb{R}^n)} \right\}, \end{aligned}$$

from the case $l = 2$ in Propositions 5.2 and 5.3, we have

$$\int_0^t \|Q_\psi U_0(\tau)\vec{g}\|^2_{H_m(\Omega)} d\tau \le C\|\vec{g}\|^2_{H_m(\mathbb{R}^n)} \quad (t \ge 0, \vec{g} \in H_m(\mathbb{R}^n)). \tag{5.7}$$

The estimates (5.7) and (5.6) imply

$$\int_0^t \left(\int_0^s \|U(s-\tau)Q_\psi U_0(\tau)\vec{g}\|_{H(\Omega\cap B_{R_0+\tilde{R}_0}(0))} d\tau\right)^2 ds \le C\|\vec{g}\|^2_{H_m(\mathbb{R}^n)} \quad (t \ge 0, \vec{g} \in H_m(\mathbb{R}^n)). \tag{5.8}$$

Combining (5.8), (5.3) and (5.2), we obtain

$$\int_0^t \|U(s)(\psi\vec{g})\|^2_{H(\Omega\cap B_{R_0+\tilde{R}_0}(0))} ds \le C\|\vec{g}\|^2_{H_m(\mathbb{R}^n)} \quad (t \ge 0, \vec{g} \in H_m(\mathbb{R}^n)). \tag{5.9}$$

For the cutoff function $\psi \in C^\infty(\mathbb{R}^n)$, we choose $\tilde{\psi} \in C^\infty(\mathbb{R}^n)$ with supp $\tilde{\psi} \subset \mathbb{R}^n \setminus B_{R_0+\tilde{R}_0/12}(0)$ and $\tilde{\psi}(x) = 1$ for $|x| \ge R_0 + \tilde{R}_0/6$. For any $\vec{f} \in D(L^m)$, we have $\tilde{\psi}\vec{f} \in H_m(\mathbb{R}^n)$. We put $\vec{g} = \tilde{\psi}\vec{f}$ in (5.9). Since $\psi\vec{g} = \psi\tilde{\psi}\vec{f} = \psi\vec{f}$, it follows that

$$\int_0^t \|U(s)(\psi\vec{f})\|^2_{H(\Omega\cap B_{R_0+\tilde{R}_0}(0))} ds \le C\|\tilde{\psi}\vec{f}\|^2_{H_m(\mathbb{R}^n)} \le C\|\vec{f}\|^2_{H_m(\Omega)} \quad (t \ge 0, \vec{f} \in D(L^m)). \tag{5.10}$$

Since $(1-\psi)\vec{f} \in D(L^m)$, supp$(1-\psi)\vec{f} \subset B_{R_0+\tilde{R}_0}$, from assumption (E1) and Hardy's inequality (3.9), there exist a function $p \in C^0([0,\infty)) \cap L^1([0,\infty))$ and a constant $C > 0$ such that

$$\begin{aligned} &\|U(t)(1-\psi)\vec{f}\|_{H(\Omega\cap B_{R_0+\tilde{R}_0}(0))} + \|[U(t)(1-\psi)\vec{f}]_1\|_{L^2(\Omega\cap B_{R_0+\tilde{R}_0}(0))} \\ &\quad \le p(t)\|(1-\psi)\vec{f}\|_{H_m(\Omega)} \le Cp(t)\|\vec{f}\|_{H_m(\Omega)} \quad (t \ge 0, \vec{f} \in D(L^m)). \end{aligned} \tag{5.11}$$

We put $\vec{h} = (1-\psi)\vec{f}$. The above estimate implies

$$\begin{aligned} &\int_0^t \|U(s)\vec{h}\|^2_{H(\Omega\cap B_{R_0+\tilde{R}_0}(0))} ds \\ &\quad \le C\int_0^t \|U(s)\vec{h}\|_{H(\Omega\cap B_{R_0+\tilde{R}_0}(0))} \|U(s)\vec{h}\|_{H(\Omega)} ds \\ &\quad \le C\int_0^t p(s)\|\vec{f}\|_{H_m(\Omega)}\|\vec{f}\|_{H(\Omega)} ds, \end{aligned}$$

which yields

$$\int_0^t \|U(s)(1-\psi)\vec{f}\|^2_{H(\Omega\cap B_{R_0+\tilde{R_0}}(0))}ds \le C\|\vec{f}\|^2_{H_m(\Omega)} \quad (t \ge 0, \vec{f} \in D(L^m)). \tag{5.12}$$

Combining the estimates (5.12) and (5.10), we obtain (i) of Proposition 5.4.

Next we show (ii). We divide the term $\int_0^t \|[U(s)(\psi\vec{g})]_1\|^2_{L^2(\Omega\cap B_{R_0+\tilde{R_0}}(0))}ds$ similarly to (5.2). From (5.5), the argument for (5.8) implies

$$\int_0^t \left(\int_0^s \|[U(s-\tau)Q_\psi U_0(\tau)\vec{g}]_1\|_{L^2(\Omega\cap B_{R_0+\tilde{R_0}}(0))}d\tau\right)^2 ds \le C\|\vec{g}\|^2_{H_m(\mathbb{R}^n)}$$
$$(t \ge 0, \vec{g} \in H_m(\mathbb{R}^n)).$$

Combining these estimates with the case $l = 2$ in (2) of Proposition 5.2, we obtain

$$\int_0^t \|[U(s)(\psi\vec{g})]_1\|^2_{L^2(\Omega\cap B_{R_0+\tilde{R_0}}(0))}ds \le C\|\vec{g}\|^2_{H_m(\mathbb{R}^n)} \quad (t \ge 0, \vec{g} \in H_m(\mathbb{R}^n)).$$

Hence noting (5.11), we get (ii) similarly to (i). This completes the proof of Proposition 5.4. □

For the estimate in x far from the boundary, we have the following one:

Proposition 5.5. *Assume that* $n \ge 3$ *and* (E1) *is satisfied. Then there exists a constant* $C > 0$ *such that*

$$\int_0^t \|\psi U(s)\vec{f}\|^2_{H_{0,l}(\mathbb{R}^n)}ds + \int_0^t \|(1+|\cdot|)^{-(l+2)}[\psi U(s)\vec{f}]_1(\cdot)\|^2_{L^2(\mathbb{R}^n)}ds$$
$$\le Cq_l(t)\|\vec{f}\|^2_{H_m(\Omega)} \quad (t \ge 0, l \ge 0, \vec{f} \in D(L^m)).$$

Proof. We choose an arbitrary $\vec{f} \in D(L^m)$ and put

$$W(t)\vec{f} = \int_0^t U_0(t-s)Q_\psi U(s)\vec{f}ds.$$

From (1) of Lemma 5.1, we have $\psi U(t)\vec{f} = U_0(t)(\psi\vec{f}) + W(t)\vec{f}$ in H_0, which yields

$$\int_0^t \|\psi U(s)\vec{f}\|^2_{H_{0,l}(\mathbb{R}^n)}ds$$
$$\le 2\int_0^t \|U_0(s)(\psi\vec{f})\|^2_{H_{0,l}(\mathbb{R}^n)}ds + 2\int_0^t \|W(s)\vec{f}\|^2_{H_{0,l}(\mathbb{R}^n)}ds.$$

From Proposition 5.2 and Hardy's inequality (3.9), the above estimate implies

$$\int_0^t \|\psi U(s)\vec{f}\|^2_{H_{0,l}(\mathbb{R}^n)} ds \le C q_l(t)\|\vec{f}\|^2_H + 2\int_0^t \|W(s)\vec{f}\|^2_{H_{0,l}(\mathbb{R}^n)} ds. \quad (5.13)$$

Note that the same argument gives the following estimate:

$$\begin{aligned}&\int_0^t \|(1+|\cdot|)^{-(l+2)}[\psi U(s)\vec{f}]_1(\cdot)\|^2_{L^2(\mathbb{R}^n)} ds \\ &\quad \le C q_l(t)\|\vec{f}\|^2_H + 2\int_0^t \|(1+|\cdot|)^{-(l+2)}[W(s)\vec{f}]_1(\cdot)\|^2_{L^2(\mathbb{R}^n)} ds.\end{aligned} \quad (5.14)$$

Now we estimate $W(t)\vec{f}$. Note that for any $\vec{f} \in D(L)$, $W(t)\vec{f} \in D(L_0)$ and $W(t)\vec{f}$ satisfies

$$\frac{d}{dt}W(t)\vec{f} = L_0 W(t)\vec{f} + Q_\psi U(t)\vec{f} \text{ in } H_0, \quad W(0)\vec{f} = 0.$$

We express $W(t)\vec{f}$ as $W(t)\vec{f}(x) = {}^t(w_1(t,x), w_2(t,x))$. Then we have $w_1 \in C^0(\mathbb{R}; \dot{H}^2(\mathbb{R}^n)) \cap C^1(\mathbb{R}; \dot{H}^1(\mathbb{R}^n))$, $w_2 \in C^0(\mathbb{R}; H^1(\mathbb{R}^n)) \cap C^1(\mathbb{R}; L^2(\mathbb{R}^n))$ and

$$\begin{aligned}\partial_t w_1(t,x) &= w_2(t,x),\\ \partial_t w_2(t,x) &= \triangle w_1(t,x) - (2\nabla_x\psi\cdot\nabla_x u(t,x) + (\triangle\psi)u(t,x)),\end{aligned}$$

where $u(t,x)$ is the solution of (1.1) with the initial data $\vec{f} = {}^t(f_1, f_2)$ and inhomogeneous data $f(t,x) = 0$.

We put $G(t,x) = -(2\nabla_x\psi\cdot\nabla_x u(t,x) + (\triangle\psi)u(t,x))$ $(= [Q_\psi U(t)\vec{f}]_2(x))$. Then we have

$$\begin{cases}(\partial_t^2 - \triangle)w_1(t,x) = G(t,x) & \text{in } \mathbb{R}\times\mathbb{R}^n,\\ w_1(0,x) = 0, \quad \partial_t w_1(0,x) = 0 & \text{on } \mathbb{R}^n.\end{cases}$$

This fact and Theorem 4.1 imply that there exists a constant $C > 0$ such that

$$\begin{aligned}&\int_0^t \|W(s)\vec{f}\|^2_{H_{0,l}(\mathbb{R}^n)} ds + \int_0^t \|(1+|\cdot|)^{-(l+2)}[W(s)\vec{f}]_1(\cdot)\|^2_{L^2(\mathbb{R}^n)} ds\\ &\quad \le C q_l(t)\int_0^t\int_{\mathbb{R}^n}(1+|x|)^2|G(s,x)|^2 dx ds \quad (t\ge 0, l\ge 0).\end{aligned} \quad (5.15)$$

Since $Q_\psi\vec{g} = {}^t(0, -[\triangle,\psi])\vec{g} = {}^t(0, -(2\nabla_x\psi\cdot\nabla_x g_1 + (\triangle\psi)g_1))$, noting that

$$\|Q_\psi\vec{g}\|_{H_0} \le C\left\{\|\vec{g}\|_{H(\Omega\cap B_{R_0+\tilde{R_0}}(0))} + \|g_1\|_{L^2(\Omega\cap B_{R_0+\tilde{R_0}}(0))}\right\} \quad (\vec{g}\in H),$$

from Proposition 5.4, we obtain

$$\int_0^t \|Q_\psi U(s)\vec{f}\|_{H_0}^2 ds \le C\|\vec{f}\|_{H_m(\Omega)}^2 \quad (t \ge 0, \vec{f} \in D(L^m)). \tag{5.16}$$

The fact $\operatorname{supp} G(t,\cdot) \subset \{x \in \mathbb{R}^n \mid R_0 \le |x| \le R_0 + \tilde{R}_0\}$ implies

$$\begin{aligned}\int_{\mathbb{R}^n}(1+|x|)^2|G(s,x)|^2 dx &\le (1+R_0+\tilde{R}_0)^2\int_{\mathbb{R}^n}|G(s,x)|^2 dx \\ &\le C\|Q_\psi U(s)\vec{f}\|_{H_0}^2.\end{aligned} \tag{5.17}$$

Combining (5.15)–(5.17), we obtain

$$\begin{aligned}&\int_0^t \|W(s)\vec{f}\|_{H_{0,l}(\mathbb{R}^n)}^2 ds + \int_0^t \|(1+|\cdot|)^{-(l+2)}[W(s)\vec{f}]_1(\cdot)\|_{L^2(\mathbb{R}^n)}^2 ds \\ &\quad \le Cq_l(t)\|\vec{f}\|_{H_m(\Omega)}^2 \quad (t \ge 0, l \ge 0, \vec{f} \in D(L^m)).\end{aligned}$$

The above estimate, (5.13) and (5.14) imply the estimates in Proposition 5.5, which completes the proof of Proposition 5.5. □

Proof of Theorem 1.1. We divide the proof into three steps.

Step 1 (the case that $f=0$). We choose an arbitrary $\vec{f} \in D(L^m)$. Noting that $\psi = 1$ for $|x| \ge R_0 + 2\tilde{R}_0/3$, we have

$$\begin{aligned}&\int_0^t \|U(s)\vec{f}\|_{H_{0,l}(\Omega)}^2 ds \\ &\quad \le \int_0^t \|U(s)\vec{f}\|_{H(\Omega\cap B_{R_0+\tilde{R}_0}(0))}^2 ds + \int_0^t \|\psi U(s)\vec{f}\|_{H_{0,l}(\mathbb{R}^n)}^2 ds\end{aligned}$$

for any $l \ge 0$. From the above estimate, Propositions 5.4 and 5.5, we obtain

$$\int_0^t \|U(s)\vec{f}\|_{H_{0,l}(\Omega)}^2 ds \le Cq_l(t)\|\vec{f}\|_{H_m(\Omega)}^2 \quad (t \ge 0, l \ge 0, \vec{f} \in D(L^m)). \tag{5.18}$$

Noting that

$$\begin{aligned}&\|(1+|\cdot|)^{-(l+2)}[U(s)\vec{f}]_1\|_{L^2(\Omega)}^2 \\ &\quad \le C\|(1+|\cdot|)^{-(l+2)}[U(s)\vec{f}]_1\|_{L^2(\Omega\cap B_{R_0+\tilde{R}_0}(0))}^2 \\ &\qquad + \|(1+|\cdot|)^{-(l+2)}[\psi U(s)\vec{f}]_1\|_{L^2(\mathbb{R}^n)}^2 \quad (l \ge 0),\end{aligned}$$

we can similarly show the following estimate:

$$\int_0^t \|(1+|\cdot|)^{-(l+2)}[U(s)\vec{f}]_1(\cdot)\|_{L^2(\Omega)}^2 ds \le Cq_l(t)\|\vec{f}\|_{H_m(\Omega)}^2 \qquad (t \ge 0, l \ge 0, \vec{f} \in D(L^m)). \tag{5.19}$$

Hence we have the estimate in Theorem 1.1 for the case that $f(t,x) = 0$.

Step 2 (the case that $m = 0$ with non-zero inhomogeneous data). In this case, we can assume that the initial data f_1 and f_2 satisfy $f_1 = 0$ and $f_2 = 0$. From (5.1), we have

$$\begin{aligned}\|V(t,\vec{0},f)\|_{H_{0,l}(\Omega)} &\le \int_0^t \|U(t-s)F(s)\|_{H_{0,l}(\Omega)} ds \\ &= \int_0^t \tilde{\chi}(t-s)\|U(t-s)F(s)\|_{H_{0,l}(\Omega)} ds,\end{aligned}$$

where $\tilde{\chi}$ is the function defined by $\tilde{\chi}(\tau) = 1$ for $\tau \ge 0$, $\tilde{\chi}(\tau) = 0$ for $\tau < 0$. Thus Minkowski's inequality implies

$$\begin{aligned}\int_0^t \|V(s,\vec{0},f)\|_{H_{0,l}(\Omega)}^2 ds &\le \int_0^t \left(\int_0^s \tilde{\chi}(s-\tau)\|U(s-\tau)F(\tau)\|_{H_{0,l}(\Omega)} d\tau\right)^2 ds \\ &\le \left\{\int_0^t \left(\int_0^{t-\tau} \|U(s)F(\tau)\|_{H_{0,l}(\Omega)}^2 ds\right)^{1/2} d\tau\right\}^2,\end{aligned} \tag{5.20}$$

Using (5.18) with $m = 0$, which is a part of Theorem 1.1 and has already been shown, we obtain

$$\begin{aligned}\int_0^{t-\tau} \|U(s)F(\tau)\|_{H_{0,l}(\Omega)}^2 ds &\le Cq_l(t-\tau)\|F(\tau)\|_{H_0(\Omega)}^2 \\ &\le Cq_l(t-\tau)\|f(\tau,\cdot)\|_{L^2(\Omega)}^2 \quad (t, \tau \ge 0, l \ge 0).\end{aligned}$$

Since q_l is a non-decreasing function, it follows that

$$\int_0^t \|V(s,\vec{0},f)\|_{H_{0,l}(\Omega)}^2 ds \le Cq_l(t)\left\{\int_0^t \|f(\tau,\cdot)\|_{L^2(\Omega)} d\tau\right\}^2,$$

which shows the estimate in Theorem 1.1 for weighted energy norms of the solutions with inhomogeneous data. From (5.19), we can similarly show the estimate of weighted L^2-norm. This completes the task in Step 2.

Step 3 (the case that $m \geq 1$). In this case, we cannot use the argument in Step 2, since we have to handle the compatibility conditions. For $\vec{f} \in H_m(\Omega)$ and $f \in \bigcap_{j=0}^{m-1} W_{\mathrm{loc}}^{j+1,1}([0,\infty); H^{m-1-j}(\Omega))$, we put $F(s) = {}^t(0, f(s,\cdot))$, $G_0 = \vec{f}$ and $G_j = LG_{j-1} + \partial_t^{j-1} F(0)$ $(j = 1, 2, \ldots, m)$. In what follows, we assume that (f_1, f_2, f) satisfies the compatibility condition of order m. Note that this is equivalent to the fact that $G_j \in D(L)$ $(j = 0, 1, \ldots, m-1)$ holds.

Since L is skew self-adjoint, $(L+I)^{-1} \in B(H)$ exists. From the fact that $f \in W_{\mathrm{loc}}^{m,1}([0,\infty); L^2(\Omega))$, we have $F \in W_{\mathrm{loc}}^{m,1}([0,\infty); H)$. For any $k = 1, 2, \ldots, m$, we put $F_k(t) = (L+I)^{-k}(\partial_t + I)^{k-1} F(t) \in W_{\mathrm{loc}}^{m+1-k,1}([0,\infty); D(L^k)) \subset C^{m-k}([0,\infty); D(L^k))$ and $\vec{g}_k = \vec{f} + \sum_{j=1}^k F_j(0)$.

Lemma 5.6. *For $\vec{f}$ and f as above, the following hold:*

(1) *We have the identity*

$$V(t, \vec{f}, f) = U(t)\Big(\vec{f} + \sum_{j=1}^m F_j(0)\Big) - \sum_{j=1}^m F_j(t) + \int_0^t U(t-s)(\partial_s + I) F_m(s)\,ds.$$

(2) *For $p = 1, 2, \ldots, m$, we have $\vec{g}_p \in D(L^p)$ and*

$$(L+I)^p \vec{g}_p = \sum_{j=0}^p \binom{p}{j} G_{p-j}.$$

(3) *There exists a constant $C > 0$ such that $\|\vec{g}_m\|_{D(L^m)}^2 \leq C I_m(t)$ $(t \geq 1)$, where*

$$I_m(t) = \|\vec{f}\|_{H_m(\Omega)}^2 + \sum_{k=0}^1 \sum_{p=0}^{m-1} \Big(\int_0^t \|\partial_s^{p+k} f(s,\cdot)\|_{H^{m-1-p}(\Omega)} ds\Big)^2.$$

Proof. We consider the case of $m = 1$. From $F_1 \in C^0(\mathbb{R}; D(L))$, it follows that $U(t-s)F(s) = (L+I)U(t-s)(L+I)^{-1}F(s) = \big((-\partial_s + I)(U(t-s))\big)F_1(s)$. Therefore applying integration by parts to (5.1), we have

$$V(t, \vec{f}, f) = U(t)(\vec{f} + F_1(0)) - F_1(t) + \int_0^t U(t-s)(\partial_s + I)F_1(s)\,ds.$$

Hence we obtain the case of $m = 1$. Repeating integration by parts for the above identity, we have (1) of Lemma 5.6.

To show (2), we use induction. Since $F_1(0) = (L+I)^{-1}F(0) \in D(L)$, noting that $G_0 = \vec{f} \in D(L)$, it follows that $\vec{g}_1 \in D(L)$ and $(L+I)\vec{g}_1 = (L+I)\vec{f} +$

$F(0) = G_1 + G_0$. Thus we obtain (2) for $p = 1$. Assume that (2) holds for some p with $1 \le p \le m-1$. Then it follows that $\vec{g}_{p+1} = \vec{g}_p + F_{p+1}(0) \in D(L^p)$ and

$$(L+I)^p \vec{g}_{p+1} = \sum_{j=0}^{p} \binom{p}{j} G_{p-j} + (L+I)^{-1}(\partial_t + I)^p F(0).$$

Noting that $G_j \in D(L)$ $(j = 0, 1, \ldots, m-1)$, we have $\vec{g}_{p+1} \in D(L^{p+1})$ and

$$\begin{aligned}(L+I)^{p+1}\vec{g}_{p+1} &= \sum_{j=0}^{p} \binom{p}{j} \left\{(L+I)G_{p-j} + \partial_t^{p-j} F(0)\right\} \\ &= \sum_{j=0}^{p} \binom{p}{j} \left\{G_{p+1-j} + G_{p-j}\right\} \\ &= \sum_{j=0}^{p+1} \binom{p+1}{j} G_{p+1-j}.\end{aligned}$$

Thus we have the case of $p+1$, which implies (2) of Lemma 5.6.

From (2) of Lemma 5.6, it follows that $\vec{g}_m = \vec{f} + \sum_{j=1}^{m} F_j(0) \in D(L^m) \subset H_m(\Omega)$ and

$$\begin{aligned}\|\vec{g}_m\|_{D(L^m)} &\le C\left\{\|\vec{f}\|_{H_m(\Omega)} + \sum_{j=1}^{m} \|(\partial_s + I)^{j-1} F(0)\|_{H_{m-j}(\Omega)}\right\} \\ &\le C\left\{\|\vec{f}\|_{H_m(\Omega)} + \sum_{p=0}^{m-1} \|\partial_t^p f(0,\cdot)\|_{H^{m-p-1}(\Omega)}\right\}\end{aligned}$$

since $\|(L+I)^{-j}(\partial_s + I)^{j-1} F(0)\|_{H_m(\Omega)} \le C\|(\partial_s + I)^{j-1} F(0)\|_{H_{m-j}(\Omega)}$. Choose $\chi \in C^\infty(\mathbb{R})$ with $\chi(t) = 1$ $(t < 1/3)$ and $\chi(t) = 0$ $(t > 2/3)$. For $t \ge 1$, we have

$$\begin{aligned}&\left\|\partial_t^p f(0,\cdot)\right\|_{H^{m-1-p}(\Omega)} \\ &\quad = \left\| -\int_0^t \partial_s\big(\chi(s)\partial_s^p f(s,\cdot)\big) ds \right\|_{H^{m-1-p}(\Omega)} \\ &\quad \le \max_{t\in\mathbb{R}}\{|\chi(t)| + |\partial_t \chi(t)|\} \sum_{k=0}^{1} \int_0^t \|\partial_s^{p+k} f(s,\cdot)\|_{H^{m-1-p}(\Omega)} ds.\end{aligned} \tag{5.21}$$

Combining these estimates, we obtain (3) of Lemma 5.6. □

Now we give the estimate for $\int_0^t \|V(s,\vec{f},f)\|^2_{H_{0,l}(\Omega)}ds$. It suffices to consider the estimate for $t \geq 1$ since for $0 \leq t \leq 1$, we can obtain the estimate from the usual energy inequality for problem (1.1). From (1) of Lemma 5.6, we have

$$\begin{aligned}\int_0^t \|V(s,\vec{f},f)\|^2_{H_{0,l}(\Omega)}ds \\ \leq C\Bigg\{\int_0^t \|U(s)\vec{g}_m\|^2_{H_{0,l}(\Omega)}ds + \int_0^t \Bigg\|\sum_{j=1}^m F_j(s)\Bigg\|^2_{H_{0,l}(\Omega)} ds \\ + \int_0^t \Bigg\|\int_0^s U(s-\tau)(\partial_\tau + I)F_m(\tau)d\tau\Bigg\|^2_{H_{0,l}(\Omega)} ds\Bigg\}.\end{aligned} \tag{5.22}$$

From (5.18) and (3) of Lemma 5.6, the first term on the right-hand side of (5.22) is estimated by

$$\int_0^t \|U(s)\vec{g}_m\|^2_{H_{0,l}(\Omega)}ds \leq Cq_l(t)I_m(t) \quad (t \geq 0, l \geq 0, (f_1,f_2,f) \in D_m(\Omega)), \tag{5.23}$$

where $(f_1,f_2,f) \in D_m(\Omega)$ means that (f_1,f_2,f) satisfies the compatibility condition of order m.

For the second term on the right-hand side of (5.22), since $\|F_j(s)\|_{H_{0,l}(\Omega)} \leq \|F_j(s)\|_{H_m(\Omega)} \leq \|(\partial_s+I)^{j-1}F(s)\|_{H_{m-j}(\Omega)} \leq C\sum_{p=0}^{j-1}\|\partial_s^p f(s,\cdot)\|_{H^{m-1-p}(\Omega)}$ we have

$$\begin{aligned}\int_0^t \Bigg\|\sum_{j=1}^m F_j(s)\Bigg\|^2_{H_{0,l}(\Omega)} ds \\ \leq C\sum_{p=0}^{m-1}\int_0^t \|\partial_s^p f(s,\cdot)\|^2_{H^{m-1-p}(\Omega)}ds \\ \leq C\sum_{p=0}^{m-1}\sup_{0\leq s\leq t}\|\partial_s^p f(s,\cdot)\|_{H^{m-1-p}(\Omega)}\int_0^t \|\partial_s^p f(s,\cdot)\|_{H^{m-1-p}(\Omega)}ds.\end{aligned}$$

Replacing χ in the proof of (5.21) with the function $s \mapsto \chi(s-(t-1))$, from the same argument as for (5.21), we have

$$\Big\|\partial_t^p f(s,\cdot)\Big\|_{H^{m-1-p}(\Omega)} \leq C\sum_{k=0}^{1}\int_0^t \|\partial_s^{p+k} f(s,\cdot)\|_{H^{m-1-p}(\Omega)}ds$$
$$(t \geq 1, 0 \leq s \leq t-1/2, p = 0,1,\ldots,m-1).$$

Using $1-\chi(s)$ in the above, we can see that the above estimate also hold for $t-1/2 \le s \le t$. Combining the above estimates, we obtain

$$\int_0^t \left\| \sum_{j=1}^m F_j(s) \right\|_{H_{0,l}(\Omega)}^2 ds \le C \sum_{p=0}^{m-1} \sum_{k=0}^{1} \left(\int_0^t \|\partial_s^{p+k} f(s,\cdot)\|_{H^{m-1-p}(\Omega)} ds \right)^2$$
$$\left(t \ge 1,\ f \in \bigcap_{j=0}^{m-1} W_{\mathrm{loc}}^{j+1,1}([0,\infty); H^{m-1-j}(\Omega))\right). \tag{5.24}$$

We consider the third term on the right side of (5.22). Since $(\partial_\tau + I)F_m(\tau) = (L+I)^{-m}(\partial_\tau + I)^m F(\tau) \in L^1_{\mathrm{loc}}([0,\infty); D(L^m))$, from (5.18) and the monotonicity q_l, it follows that

$$\begin{aligned}
&\int_0^{t-\tau} \|U(s)(\partial_\tau + I)F_m(\tau)\|_{H_{0,l}(\Omega)}^2 ds \\
&\quad \le C q_l(t-\tau) \|(\partial_\tau + I)F_m(\tau)\|_{H_m(\Omega)}^2 \\
&\quad \le C q_l(t) \|(\partial_\tau + I)^m F(\tau)\|_{H_0(\Omega)}^2 \\
&\quad \le C q_l(t) \sum_{p=0}^{m-1} \sum_{k=0}^{1} \|\partial_s^{p+k} f(\tau,\cdot)\|_{H^{m-1-p}(\Omega)}^2.
\end{aligned}$$

Hence the argument for obtaining (5.20) implies that

$$\begin{aligned}
&\int_0^t \left\| \int_0^s U(s-\tau)(\partial_\tau + I)F_m(\tau) d\tau \right\|_{H_{0,l}(\Omega)}^2 ds \\
&\quad \le \left\{ \int_0^t \left(\int_0^{t-\tau} \|U(s)(\partial_\tau + I)F_m(\tau)\|_{H_{0,l}(\Omega)}^2 ds \right)^{1/2} d\tau \right\}^2 \\
&\quad \le C q_l(t) I_m(t).
\end{aligned} \tag{5.25}$$

Combining (5.22)–(5.25), we obtain

$$\int_0^t \|V(s,\vec{f},f)\|_{H_{0,l}(\Omega)}^2 ds \le C q_l(t) I_m(t)$$
$$(t \ge 0, l \ge 0, (f_1, f_2, f) \in D_m(\Omega)).$$

From (5.19), we can similarly show the estimate for

$$\int_0^t \|(1+|\cdot|)^{-(l+2)}[V(s,\vec{f},f)]_1(\cdot)\|_{L^2(\Omega)}^2 ds.$$

Thus we have finished Step 3. This completes the proof of Theorem 1.1. □

Remark. We consider the following uniform decay rate $\tilde{p}_{m,R}(t)$ of the local energy of solutions $u(t,x)$ of (1.1) with zero inhomogeneous data $f(t,x) = 0$:

$$\tilde{p}_{m,R}(t) = \sup\left\{ \frac{E(u,\Omega\cap B_R,t) + \|u(t,\cdot)\|^2_{L^2(\Omega\cap B_R)}}{\|\nabla_x f_1\|^2_{H^m(\Omega)} + \|f_2\|^2_{H^m(\Omega)}} \;\middle|\; 0 \neq f_1, f_2 \in C_0^\infty(\overline{\Omega}\cap B_R),\, f_1 \in \dot{H}^1_{\mathcal{B}}(\Omega) \right\}.$$

In the definition of the rate $\tilde{p}_{m,R}(t)$, we do not care about the compatibility condition, but for the rate $p_{m,R}(t)$ in (E1) it is considered. This is the difference between them. Note that $p_{m,R}(t) \leq \tilde{p}_{m,R}(t)$ is obvious, however for large m, $\tilde{p}_{m,R}(t)$ does not seem to be coincide with $p_{m,R}(t)$. The lack of the compatibility condition may produce singularities which contain energy remaining near the boundary. This may cause slowness of $\tilde{p}_{m,R}(t)$.

In the case of $m > 0$, the argument of Walker [31] for showing $p_{m,R}(t) \to 0$ as $t \to \infty$ also implies that $\tilde{p}_{m,R}(t) \to 0$ as $t \to \infty$. Hence this rate also decay. Note that the decay rates $p_{m,R}(t)$ and $\tilde{p}_{m,R}(t)$ are also defined for non-integers $m \geq 0$. Note also that for $m \leq 1/2$, we have $\tilde{p}_{m,R}(t) = p_{m,R}(t)$ since we do not have boundary values of functions belonging to $H^m(\Omega)$ in the trace sense. From this fact and the interpolation theorems, we can also have estimates for $\tilde{p}_{m,R}(t)$.

In the results of Ikawa [4, 5], which is introduced in Section 1, the solutions with compatibility conditions for the case of the Dirichlet condition are treated. This corresponds to considering the decay rate $p_{m,R}(t)$. Recall the estimate of Ikawa [5], that is, $p_{m,R}(t) = O(e^{-\alpha t})$ with $m \geq 2$ for some $\alpha > 0$. From this, it follows that

$$\begin{aligned} &\|U(t)\vec{f}\|_{H_0(\Omega\cap B_R)} + \|[U(t)\vec{f}]_1\|^2_{L^2(\Omega\cap B_R)} \\ &\quad \leq Ce^{-\alpha t}\big(\|f_1\|_{H_0^{1+m}(\Omega)} + \|f_2\|_{H_0^m(\Omega)}\big) \\ &\qquad (f_1 \in H_0^{1+m}(\Omega),\, f_2 \in H_0^m(\Omega),\, \operatorname{supp} f_1 \cup \operatorname{supp} f_2 \subset \overline{\Omega}\cap\overline{B_R}) \end{aligned} \tag{5.26}$$

for $m \geq 2$ since the rate $p_{m,R}(t)$ contains the compatibility condition and we have $H_0^{1+m}(\Omega)\times H_0^m(\Omega) \subset D(L^m)$ $(m \in \mathbb{N}\cup\{0\})$. From the energy conservation law, we also have estimate (5.26) for $m = 0$. Hence the interpolation theorem implies that the estimates replaced $e^{-\alpha t}$ in (5.26) with $e^{-m\alpha t/2}$ also hold for $0 \leq m \leq 2$, $m \neq 1/2, 3/2$. Noting that the set $\{(f_1, f_2) \in H \mid f_1, f_2 \in C_0^\infty(\overline{\Omega}\cap B_R)$ and $f_1 \in \dot{H}^1_{\mathcal{B}}(\Omega)\}$ is dense in $H^{1+m}(\Omega)\times H^m(\Omega)$ if $0 \leq m \leq 1/2$, we obtain $\tilde{p}_{m,R}(t) = p_{m,R}(t) = O(e^{-(m\alpha/2)t})$ for $0 \leq m < 1/2$. For $m \geq 1/2$, we also have $\tilde{p}_{m,R}(t) \leq \tilde{p}_{(1-\delta)/2,R}(t) = O(e^{-((1-\delta)\alpha/4)t})$ for $0 < \delta < 1$. Thus, in this case, we still have exponential decay estimates for the uniform decay rate $\tilde{p}_{m,R}(t)$.

Now we consider the condition obtained by replacing $p_{m,R}(t)$ to $\tilde{p}_{m,R}(t)$ in (E1). If we assume that this condition holds, then the argument for (5.18) and (5.19) imply that

$$\int_0^t \|U(s)\vec{f}\|^2_{H_{0,l}(\Omega)} ds + \int_0^t \|(1+|\cdot|)^{-(l+2)}[U(s)\vec{f}]_1(\cdot)\|^2_{L^2(\Omega)} ds$$
$$\leq C q_l(t) \|\vec{f}\|^2_{H_m(\Omega)} \quad (t \geq 0, l \geq 0, \vec{f} \in H_m(\Omega)).$$

Note that in this argument, we do not use the compatibility conditions. Hence from the argument for Step 2 of the proof of Theorem 1.1, it follows that there exists a constant $C > 0$ such that

$$\int_0^t \int_\Omega (1+|x|)^{-l} e(s,x;u) dx ds + \int_0^t \int_\Omega (1+|x|)^{-(l+2)} |u(s,x)|^2 dx ds$$
$$\leq C q_l(t) \left\{ \|\nabla_x f_1\|^2_{H^m(\Omega)} + \|f_2\|^2_{H^m(\Omega)} + \left(\int_0^t \|f(s,\cdot)\|_{H^m(\Omega)} ds \right)^2 \right\}$$
$$(t \geq 0, l \geq 0, f_1 \in \dot{H}^{m+1}(\Omega) \cap \dot{H}^1_{\mathcal{B}}(\Omega), f_2 \in H^m(\Omega),$$
$$f \in L^1_{\mathrm{loc}}([0,\infty), H^m(\Omega))).$$

Thus, we can avoid the derivatives for t. Instead, we need one more spatial derivative than those in Theorem 1.1.

6 Appendix

Proposition 6.1. *We have the following identities:*

(i) $$\begin{aligned}&\mathrm{Re}\left[F(x\cdot\nabla_x\overline{v})(\partial_t^2-\triangle)v\right]\\ &\quad= \mathrm{Re}\left[\partial_t\left(\partial_t v F(x\cdot\nabla_x\overline{v})\right) + (\nabla_x F\cdot\nabla_x v)x\cdot\nabla_x\overline{v}\right]\\ &\qquad + F|\nabla_x v|^2 + 2^{-1}\mathrm{div}(Fx)\cdot\left(|\partial_t v|^2 - |\nabla_x v|^2\right)\\ &\qquad + \mathrm{div}\left\{2^{-1}\left(|\nabla_x v|^2 - |\partial_t v|^2\right)Fx\right.\\ &\qquad \left. - \mathrm{Re}\left[F(x\cdot\nabla_x\overline{v})\nabla_x v\right]\right\},\end{aligned}$$

(ii) $$\begin{aligned}&\mathrm{Re}\left[F\overline{v}(\partial_t^2-\triangle)v\right]\\ &\quad= \mathrm{Re}\left[\partial_t\left(\partial_t v F\overline{v}\right)\right] - \mathrm{div}\left\{\mathrm{Re}\left[Fv\nabla_x\overline{v}\right] - 2^{-1}|v|^2\nabla_x F\right\}\\ &\qquad - F\left(|\partial_t v|^2 - |\nabla_x v|^2\right) - 2^{-1}(\triangle F)|v|^2.\end{aligned}$$

Proof. First we compute the term $\partial_t^2 v F(x\cdot\nabla_x\overline{v})$. Since

$$\partial_t^2 v F(x\cdot\nabla_x\overline{v}) = \partial_t\left(\partial_t v F(x\cdot\nabla_x\overline{v})\right) - \partial_t v F(x\cdot\nabla_x\overline{\partial_t v}),$$

we have

$$\begin{aligned}\operatorname{Re}\left[\partial_t^2 vF(x\cdot\nabla_x\overline{v})\right] &= \operatorname{Re}\left[\partial_t\left(\partial_t vF(x\cdot\nabla_x\overline{v})\right)\right] - 2^{-1}Fx\cdot\nabla_x(|\partial_t v|^2)\\ &= \operatorname{Re}\left[\partial_t\left(\partial_t vF(x\cdot\nabla_x\overline{v})\right)\right] - \operatorname{div}\left(2^{-1}F|\partial_t v|^2 x\right)\\ &\quad + 2^{-1}\operatorname{div}(Fx)|\partial_t v|^2.\end{aligned} \tag{6.1}$$

For the second term, since

$$\begin{aligned}\triangle v\cdot F(x\cdot\nabla_x\overline{v}) &= \sum_{j=1}^n \partial_{x_j}\left(\partial_{x_j}v\cdot F(x\cdot\nabla_x\overline{v})\right) - \sum_{j=1}^n \partial_{x_j}v\cdot\partial_{x_j}\left(F(x\cdot\nabla_x\overline{v})\right)\\ &= \operatorname{div}\left(F(x\cdot\nabla_x\overline{v})\nabla_x v\right) - \nabla_x v\cdot\nabla_x F(x\cdot\nabla_x\overline{v})\\ &\quad - \sum_{j=1}^n F|\partial_{x_j}v|^2 - \sum_{j=1}^n \partial_{x_j}vFx\cdot\nabla_x(\overline{\partial_{x_j}v}),\end{aligned}$$

it follows that

$$\begin{aligned}\operatorname{Re}\left[\triangle v\cdot F(x\cdot\nabla_x\overline{v})\right] &= \operatorname{div}\left(\operatorname{Re}\left[F(x\cdot\nabla_x\overline{v})\nabla_x v\right]\right) - \operatorname{Re}\left[\nabla_x v\cdot\nabla_x F(x\cdot\nabla_x\overline{v})\right]\\ &\quad - F|\nabla_x v|^2 - 2^{-1}Fx\cdot\nabla_x(|\nabla_x v|^2)\\ &= \operatorname{div}\left(\operatorname{Re}\left[F(x\cdot\nabla_x\overline{v})\nabla_x v\right]\right) - \operatorname{Re}\left[\nabla_x v\cdot\nabla_x F(x\cdot\nabla_x\overline{v})\right]\\ &\quad - F|\nabla_x v|^2 - 2^{-1}\operatorname{div}\left(|\nabla_x v|^2 Fx\right)\\ &\quad + 2^{-1}\operatorname{div}(Fx)|\nabla_x v|^2.\end{aligned} \tag{6.2}$$

Subtracting (6.2) from (6.1), we have proved (1) of Proposition 6.1.

We show (2). Note that

$$\partial_t^2 vF\overline{v} = \partial_t\left(\partial_t vF\overline{v}\right) - F\partial_t v\overline{\partial_t v} = \partial_t\left(\partial_t vF\overline{v}\right) - F|\partial_t v|^2. \tag{6.3}$$

Since $\triangle v\cdot F\overline{v} = \sum_{j=1}^n \partial_{x_j}(\partial_{x_j}v\cdot F\overline{v}) - \sum_{j=1}^n \partial_{x_j}v(\partial_{x_j}F)\overline{v} - \sum_{j=1}^n \partial_{x_j}vF\partial_{x_j}\overline{v}$ we have

$$\begin{aligned}\operatorname{Re}\left[\triangle v\cdot F\overline{v}\right] &= \operatorname{div}\left(\operatorname{Re}[Fv\nabla_x\overline{v}]\right) - 2^{-1}\sum_{j=1}^n \partial_{x_j}F\partial_{x_j}|v|^2 - F|\nabla_x v|^2\\ &= \operatorname{div}\left(\operatorname{Re}[Fv\nabla_x\overline{v}]\right) - 2^{-1}\operatorname{div}\left(|v|^2\nabla_x F\right)\\ &\quad + 2^{-1}|v|^2\triangle F - F|\nabla_x v|^2.\end{aligned} \tag{6.4}$$

Subtracting (6.4) from (6.3), we obtain (2) of Proposition 6.1. □

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