Universality of one-dimensional reversible and number-conserving cellular automata

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Outline of the Talk

- **Objective:** Finding universal 1-d *reversible* and *number-conserving* CAs (RNCCAs).
- We give a method of converting a 2-neighbor s-state reversible partitioned CA (RPCA) into a 4-neighbor $4s$-state RNCCA.
- Since there is a universal 2-neighbor 24-state RPCA [Morita, 2011], we can obtain a computationally universal 4-neighbor 96-state RNCCA.

Contents

1. Preliminaries
   - Number-Conserving CAs (NCCAs)
   - Reversible NCCAs (RNCCAs)
   - Partitioned CAs (PCAs)

Number-Conserving CA (NCCA)

- An abstract spatiotemporal model having a property similar to the conservation laws in physics.
  - Each cell has an integer value.
  - Their sum in a configuration is conserved.

												Sum
t	$\cdots$	0	0	5	2	11	3	7	0	0	$\cdots$	28
$t+1$	$\cdots$	0	0	3	4	9	4	5	3	0	$\cdots$	28

- So far, several definitions and characterization of NCCAs have been given [Boccara, Fuks, 2000], [Durand, et al., 2003].

Several Notions on NCCAs

- Periodic-number-conserving
- Finite-number-conserving
- Number-conserving (for infinite configurations)

Durand, Formenti, Róka (2003) proved the above are all equivalent.

- Note: In our proof, we use the definition of *finite-number-conserving* CAs.

Finite-Number-Conserving CAs

Definition [Durand, et al., 2003]

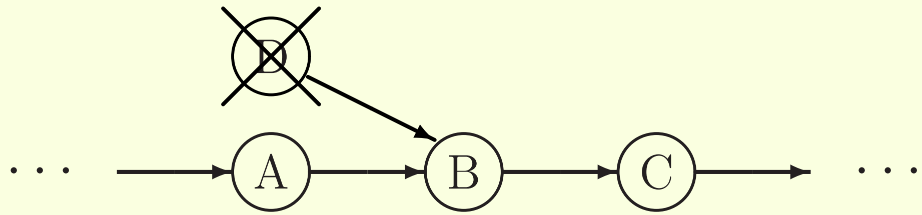
- Let A be a 1-d CA whose state set is $Q = \{0, \ldots, s-1\}$.
- Let F be the global function of A.
- Let $\mathrm{Conf}_{\mathrm{fin}}(Q)$ be the set of all finite configurations over Q.

A is called *finite-number-conserving*, if the following condition holds.

$$\forall \alpha \in \mathrm{Conf}_{\mathrm{fin}}(Q) : \sum_{x \in \mathbb{Z}} \alpha(x) = \sum_{x \in \mathbb{Z}} F(\alpha)(x)$$

Reversible Cellular Automaton (RCA)

- It is a CA whose global function is one-to-one.
- Hence, there is no pair of configurations that go to the same configuration.



- In the following, we investigate *reversible NCCAs* (RNCCAs).

Past Studies on 1-D RNCCAs

- 2-neighbor (radius 1/2) case:
  Every RNCCA is a *shift-identity product CA*.
  [García-Ramos, 2012]
- 3-neighbor (radius 1) case:
  Some RNCCAs show nontrivial behaviors.
  [Imai, Martin, Saito, 2012]

2-Neighbor (Radius 1/2) RNCCAs

- They are *all* shift-identity product CAs (SIPCAs).

[García-Ramos, 2012]

Shift CA (SCA)

t										
0	2				1		2			
1		2				1		2		
2			2				1		2	
3				2				1		2
4					2				1	
5						2				1

Identity CA (ICA)

		2			1		2		
		2			1		2		
		2			1		2		
		2			1		2		
		2			1		2		
		2			1		2		

Product of SCA and ICA

2		6		1	3	2	6		
	2	6			4		8		
		8			3	1	6	2	
		6	2		3		7		2
		6		2	3		6	1	
		6			5		6		1

- In a SIPCA, signals do not interact each other.
- Hence, it cannot be computationally universal.

Past Studies on 1-D RNCCAs

- 2-neighbor (radius 1/2) case:
  Every RNCCA is a *shift-identity product CA*.
  [García-Ramos, 2012]
- 3-neighbor (radius 1) case:
  Some RNCCAs show nontrivial behaviors.
  [Imai, Martin, Saito, 2012]

3-Neighbor (Radius 1) RNCCAs

- Some RNCCAs show nontrivial behaviors.

[Imai, Martin, Saito, 2012]

4-state RNCCA No.2

t									
0			2		1	2			1
1			2	1	2			1	
2			3	2			1		
3		1	2	2		1			
4	1	2		2	1				
5	2			3					
6			1		2				
7		1			2				

4-state RNCCA No.3

t									
0			2		1	2			1
1			2	1		2		1	
2			1	2		2	1		
3		1		2		1	2		
4	1			2	1		2		
5				1	2		2		
6			1		2		2		
7		1			2		2		

- In these CAs, signals can interact each other.
- It is not known whether there is a universal one.

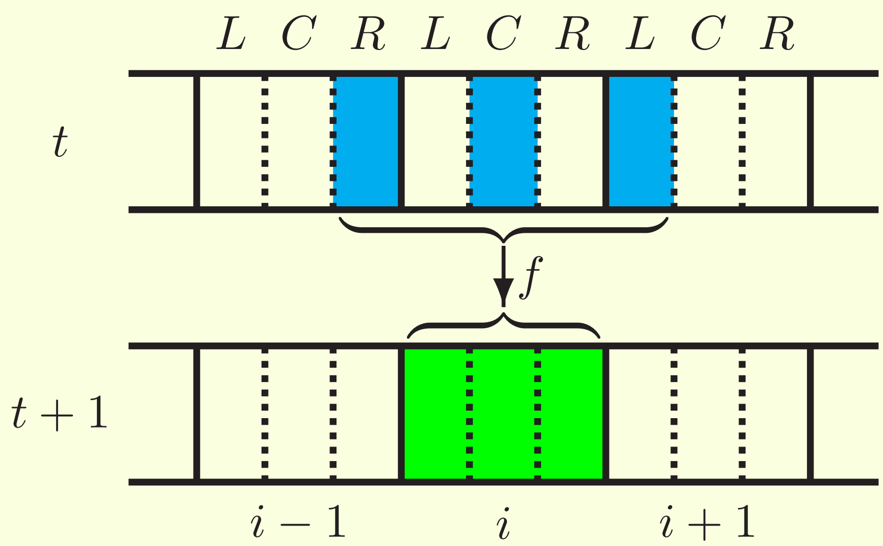
Studies on 1-D RNCCAs

- 2-neighbor (radius 1/2) case:
  Every RNCCA is a *shift-identity product CA*.
  [García-Ramos, 2012]
- 3-neighbor (radius 1) case:
  Some RNCCAs show nontrivial behaviors.
  [Imai, Martin, Saito, 2012]
- 4-neighbor (radius 3/2) case:
  In this talk, we will show there is a *universal* RNCCA.

Partitioned Cellular Automaton (PCA)

- 1-d 3-neighbor PCA

$$P = (\mathbb{Z}, (L, C, R), (1, 0, -1), f)$$



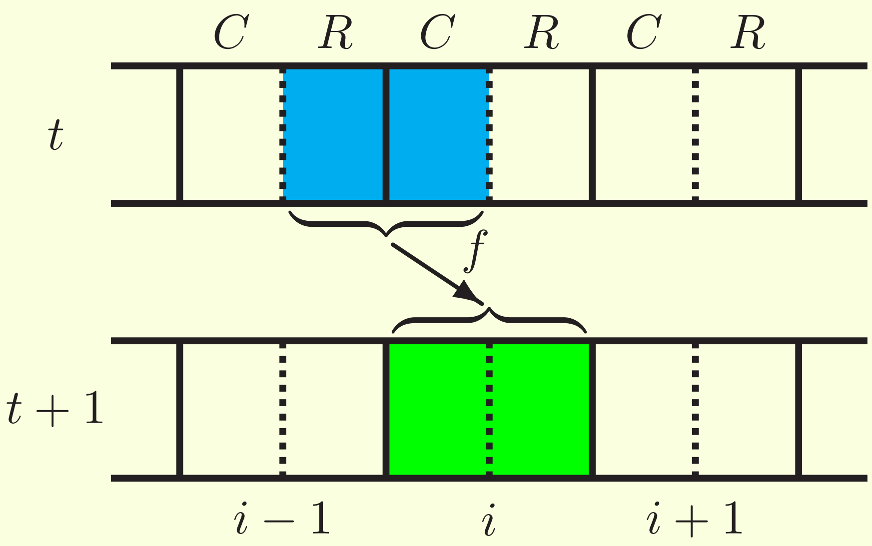
f: local function

- To construct an RCA, it is sufficient to give a PCA whose local function f is one-to-one.

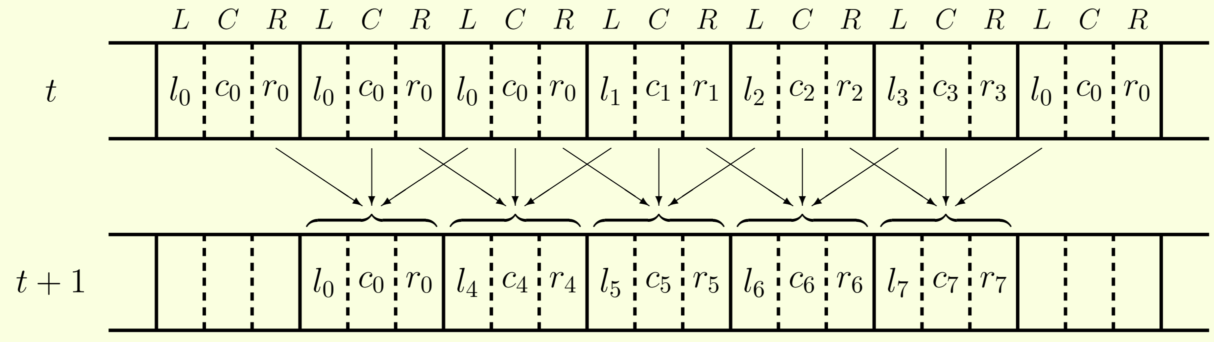
1-D 2-Neighbor PCA

- It is a special case of a 3-neighbor PCA where the left state set is a singleton.

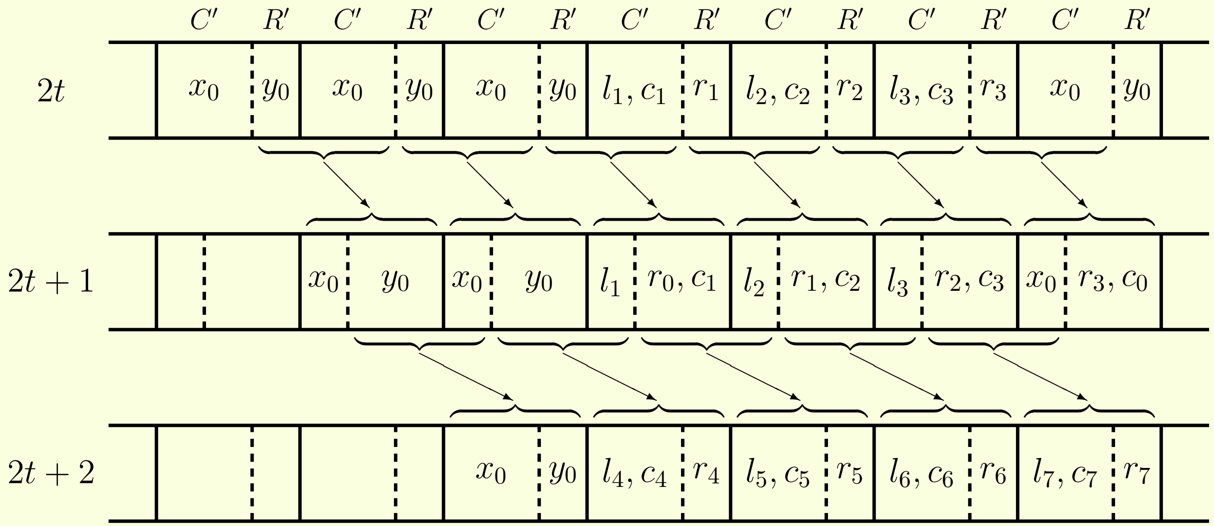
$$P = (\mathbb{Z}, (C, R), (0, -1), f)$$



A 2-Neighbor RPCA Can Simulate a 3-Neighbor RPCA [Morita, 1992]



3-Neighbor PCA P_3



2-Neighbor PCA P_2

Computationally Universal 1-D Reversible CAs

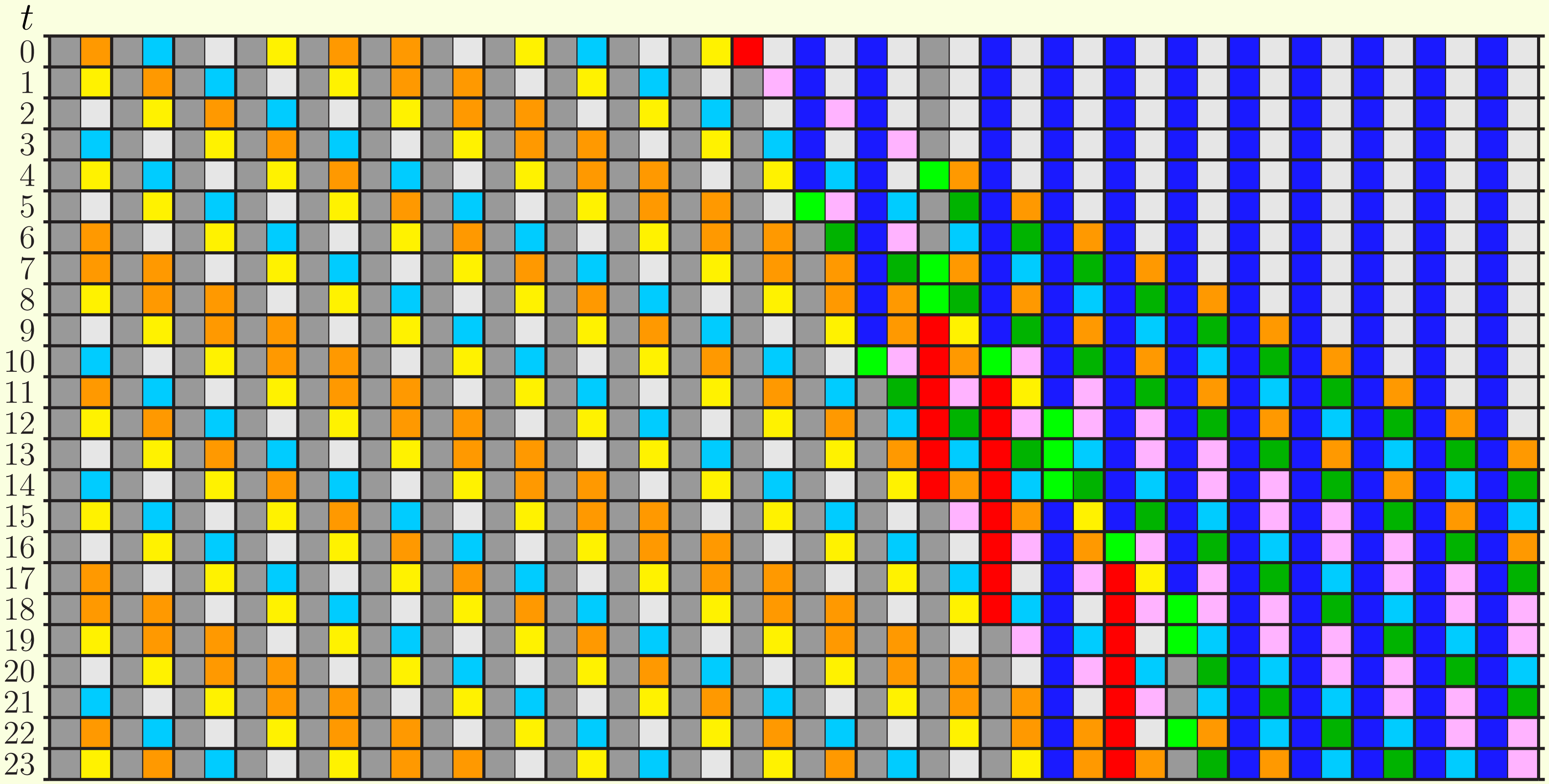
- On infinite configurations:
  24-state 2-neighbor RPCA [Morita, 2011]
- On finite configurations:
  98-state 3-neighbor RPCA [Morita, 2007]

A 24-State 1-D 2-Neighbor RPCA That Simulates Any Cyclic Tag System

t																																								
0	—	n	—	y	—	—	—	*	—	n	—	n	—	—	—	*	—	y	—	—	—	*	N	—	Y	—	Y	—	—	—	Y	—	Y	—	Y	—	Y	—	Y	—
1	—	*	—	n	—	y	—	—	—	*	—	n	—	n	—	—	—	*	—	y	—	—	—	/	Y	—	Y	—	—	—	Y	—	Y	—	Y	—	Y	—	Y	—
2	—	—	—	*	—	n	—	y	—	—	—	*	—	n	—	n	—	—	—	*	—	y	—	—	Y	/	Y	—	—	—	Y	—	Y	—	Y	—	Y	—	Y	—
3	—	y	—	—	—	*	—	n	—	y	—	—	—	*	—	n	—	n	—	—	—	*	—	y	Y	—	Y	/	—	—	Y	—	Y	—	Y	—	Y	—	Y	—
4	—	*	—	y	—	—	—	*	—	n	—	y	—	—	—	*	—	n	—	n	—	—	—	*	Y	y	Y	—	+	n	Y	—	Y	—	Y	—	Y	—	Y	—
5	—	—	—	*	—	y	—	—	—	*	—	n	—	y	—	—	—	*	—	n	—	n	—	—	+	/	Y	y	—	+	Y	n	Y	—	Y	—	Y	—	Y	—
6	—	n	—	—	—	*	—	y	—	—	—	*	—	n	—	y	—	—	—	*	—	n	—	n	—	+	Y	/	—	y	Y	+	Y	n	Y	—	Y	—	Y	—
7	—	n	—	n	—	—	—	*	—	y	—	—	—	*	—	n	—	y	—	—	—	*	—	n	—	n	Y	+	+	n	Y	y	Y	+	Y	n	Y	—	Y	—
8	—	*	—	n	—	n	—	—	—	*	—	y	—	—	—	*	—	n	—	y	—	—	—	*	—	n	Y	n	+	+	Y	n	Y	y	Y	+	Y	n	Y	—
9	—	—	—	*	—	n	—	n	—	—	—	*	—	y	—	—	—	*	—	n	—	y	—	—	—	*	Y	n	N	*	Y	+	Y	n	Y	y	Y	+	Y	n
10	—	y	—	—	—	*	—	n	—	n	—	—	—	*	—	y	—	—	—	*	—	n	—	y	—	—	+	/	N	n	+	/	Y	+	Y	n	Y	y	Y	+
11	—	n	—	y	—	—	—	*	—	n	—	n	—	—	—	*	—	y	—	—	—	*	—	n	—	y	—	+	N	/	N	*	Y	/	Y	+	Y	n	Y	y
12	—	*	—	n	—	y	—	—	—	*	—	n	—	n	—	—	—	*	—	y	—	—	—	*	—	n	—	y	N	+	N	/	+	/	Y	/	Y	+	Y	n
13	—	—	—	*	—	n	—	y	—	—	—	*	—	n	—	n	—	—	—	*	—	y	—	—	—	*	—	n	N	y	N	+	+	y	Y	/	Y	/	Y	+
14	—	y	—	—	—	*	—	n	—	y	—	—	—	*	—	n	—	n	—	—	—	*	—	y	—	—	—	*	N	n	N	y	+	+	Y	y	Y	/	Y	/
15	—	*	—	y	—	—	—	*	—	n	—	y	—	—	—	*	—	n	—	n	—	—	—	*	—	y	—	—	—	/	N	n	Y	*	Y	+	Y	y	Y	/
16	—	—	—	*	—	y	—	—	—	*	—	n	—	y	—	—	—	*	—	n	—	n	—	—	—	*	—	y	—	—	N	/	Y	n	+	/	Y	+	Y	y
17	—	n	—	—	—	*	—	y	—	—	—	*	—	n	—	y	—	—	—	*	—	n	—	n	—	—	—	*	—	y	N	—	Y	/	N	*	Y	/	Y	+
18	—	n	—	n	—	—	—	*	—	y	—	—	—	*	—	n	—	y	—	—	—	*	—	n	—	n	—	—	—	*	N	y	Y	—	N	/	+	/	Y	/
19	—	*	—	n	—	n	—	—	—	*	—	y	—	—	—	*	—	n	—	y	—	—	—	*	—	n	—	n	—	—	—	/	Y	y	N	—	+	y	Y	/
20	—	—	—	*	—	n	—	n	—	—	—	*	—	y	—	—	—	*	—	n	—	y	—	—	—	*	—	n	—	n	—	—	Y	/	N	y	—	+	Y	y
21	—	y	—	—	—	*	—	n	—	n	—	—	—	*	—	y	—	—	—	*	—	n	—	y	—	—	—	*	—	n	—	n	Y	—	N	/	—	y	Y	+
22	—	n	—	y	—	—	—	*	—	n	—	n	—	—	—	*	—	y	—	—	—	*	—	n	—	y	—	—	—	*	—	n	Y	n	N	—	+	n	Y	y
23	—	*	—	n	—	y	—	—	—	*	—	n	—	n	—	—	—	*	—	y	—	—	—	*	—	n	—	y	—	—	—	*	Y	n	N	n	—	+	Y	n

A 24-State 1-D 2-Neighbor RPCA That Simulates Any Cyclic Tag System

(Colored version)



How can we convert it into an RNCCA?

2. Converting a reversible PCA (RPCA) into an RNCCA

Main Result

Lemma

- Let $P = (\mathbb{Z}, (C, R), (0, -1), f)$ be a 1-d RPCA.
- We can construct a 1-d RNCCA $A = (\mathbb{Z}, \tilde{Q}, (-2, -1, 0, 1), \tilde{f}, 0)$ that simulates P, such that $\tilde{Q} = \{0, 1, \ldots, 4|C| \cdot |R| - 1\}$.

Proposition [Morita, 2011] There is a computationally universal 1-d 2-neighbor 24-state RPCA.

Theorem There is a computationally universal 1-d 4-neighbor 96-state RNCCA.

How to Construct an RNCCA from an RPCA
— A Proof Outline of the Lemma —

Given: A 1-d 2-neighbor s-state RPCA
$P = (\mathbb{Z}, (C, R), (0, -1), f)$

Construct: A 1-d 4-neighbor $4s$-state RNCCA
$A = (\mathbb{Z}, \tilde{Q}, (-2, -1, 0, 1), \tilde{f}, 0)$

Example:

If $C = \{Y, N, P, M\}$ and $R = \{y, n, +, -, *, /\}$, then $\tilde{Q} = \{0, \ldots, 95\}$, and $\tilde{f}$ is determined as below.

Heavy and Light Particles for the States in C and R

$$P = (\mathbb{Z}, (C, R), (0, -1), f)$$

- If $C = \{Y, N, P, M\}$ and $R = \{y, n, +, -, *, /\}$, then prepare the following 4 sets of integers:

  $\widehat{C} = \{\ 0, 12, 24, 36\}, \quad \widehat{R} = \{\ 0,\ 1,\ 2,\ 3,\ 4,\ 5\}$

  $\check{C} = \{84, 72, 60, 48\}, \quad \check{R} = \{11, 10, 9,\ 8,\ 7,\ 6\}$

  Elements of $\widehat{C} \cup \check{C}$ are heavy particles.

  Elements of $\widehat{R} \cup \check{R}$ are light particles.

  Each number represents the *mass* of the particle.

Correspondence among $C, R, \widehat{C}, \check{C}, \widehat{R},$ and $\check{R}$

$$P = (\mathbb{Z}, (C, R), (0, -1), f)$$

$$C = \{Y, N, P, M\}, \quad R = \{y, n, +, -, *, /\}$$

$\widehat{\varphi}_C : C \to \widehat{C}, \ \check{\varphi}_C : C \to \check{C}$

c	Y	N	P	M
$\widehat{\varphi}_C(c)$	0	12	24	36
$\check{\varphi}_C(c)$	84	72	60	48
$\widehat{\varphi}_C(c)$	$\widehat{Y}$	$\widehat{N}$	$\widehat{P}$	$\widehat{M}$
$\check{\varphi}_C(c)$	$\check{Y}$	$\check{N}$	$\check{P}$	$\check{M}$

$\widehat{\varphi}_R : R \to \widehat{R}, \ \check{\varphi}_R : R \to \check{R}$

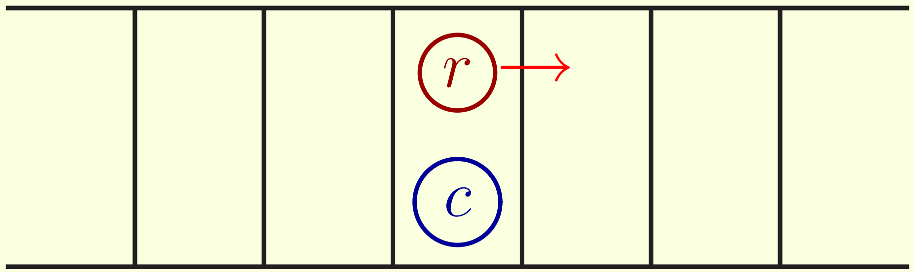
r	y	n	$+$	$-$	$*$	$/$
$\widehat{\varphi}_R(r)$	0	1	2	3	4	5
$\check{\varphi}_R(r)$	11	10	9	8	7	6
$\widehat{\varphi}_R(r)$	$\widehat{y}$	$\widehat{n}$	$\widehat{+}$	$\widehat{-}$	$\widehat{*}$	$\widehat{/}$
$\check{\varphi}_R(r)$	$\check{y}$	$\check{n}$	$\check{+}$	$\check{-}$	$\check{*}$	$\check{/}$

- Hereafter, we use notations $\widehat{c}$, $\check{c}$, $\widehat{r}$, $\check{r}$.
- ($\widehat{c}$, $\check{c}$) and ($\widehat{r}$, $\check{r}$) are *complementary pairs*, since $\widehat{c} + \check{c} = 84$ and $\widehat{r} + \check{r} = 11$.

Simulating RPCA P by RNCCA A (1)

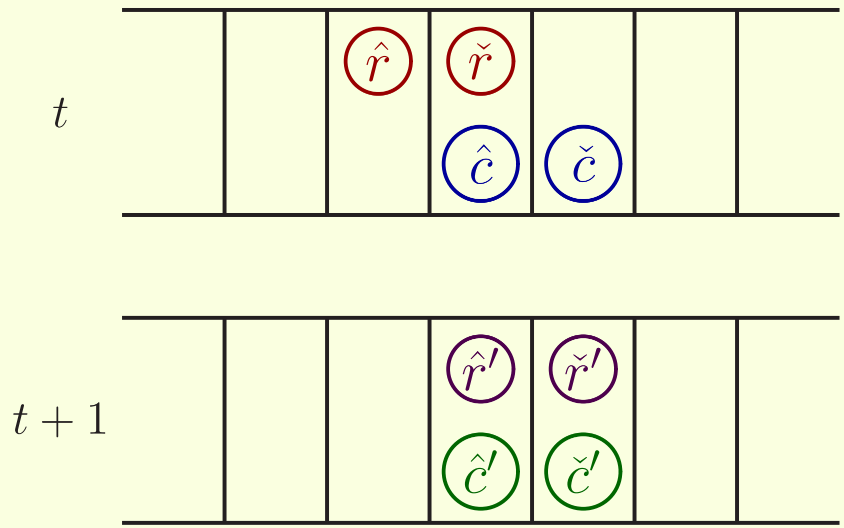
$$A = (\mathbb{Z}, \tilde{Q}, (-2, -1, 0, 1), \tilde{f}, 0)$$

- Each cell of A keeps one $\textcircled{c}$ and one $\textcircled{r}$.
- A light particle $\textcircled{r}$ always moves rightward at the unit speed.
- A heavy particle $\textcircled{c}$ is stationary.



Simulating RPCA P by RNCCA A (2)

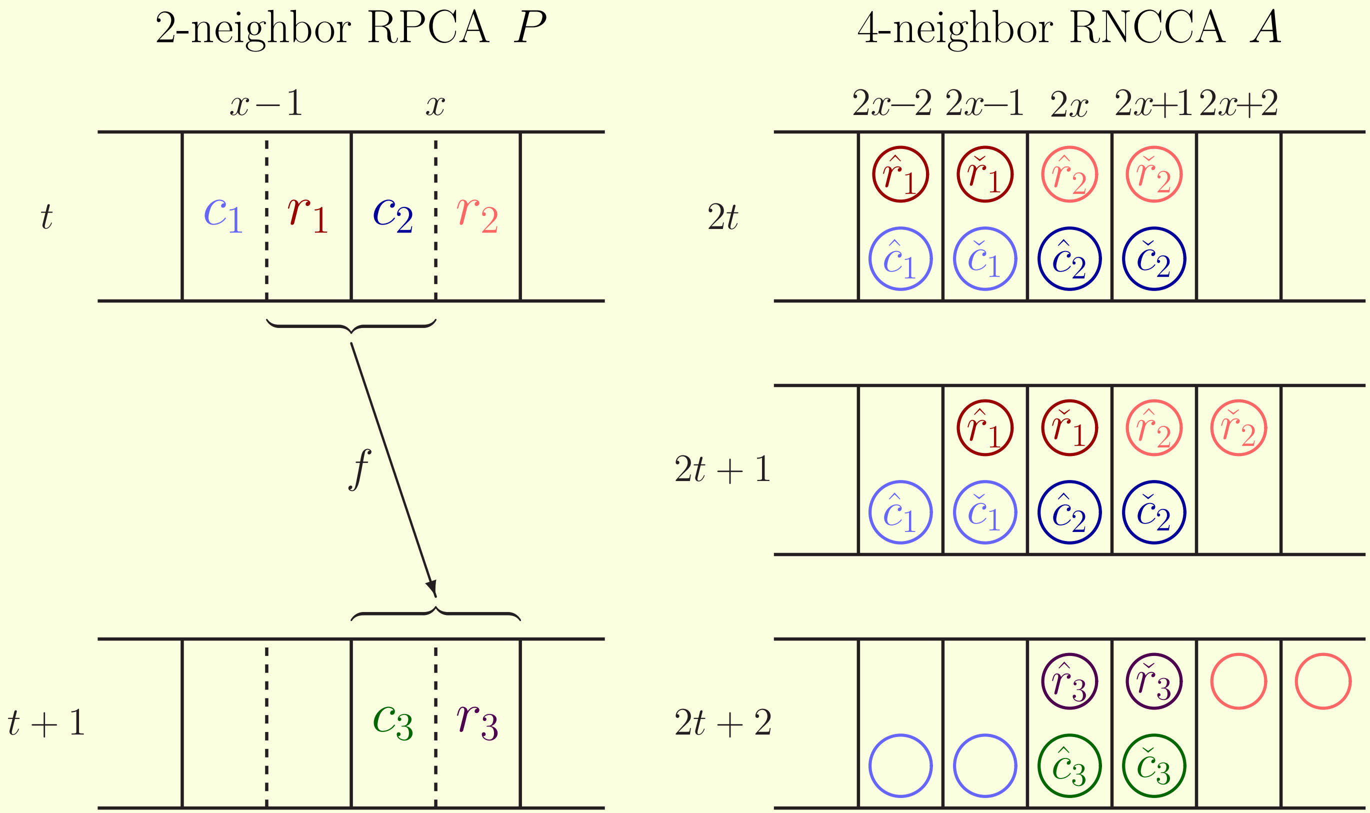
- If a complementary pair of light particles ($\widehat{r}$,$\check{r}$) meets that of heavy particles ($\widehat{c}$,$\check{c}$), state transition of P is simulated. Then, new pairs of complementary particles are created.



- In all other cases, a light particle r simply moves rightward without interacting heavy particles.

Simulation Process of P by A

$$A = (\mathbb{Z}, \tilde{Q}, (-2, -1, 0, 1), \tilde{f}, 0)$$



- A precise proof on number-conserving property and reversibility of A is found in the proceeding.

Concluding Remarks

- **Result:**
  A universal 4-neighbor 96-state RNCCA exists, in spite of the strong constraints of reversibility and number-conservation.
- **Open problem:**
  – Is there a universal 3-neighbor RNCCA?