On a study of Hilbert functions and Betti numbers of Artinian Gorenstein graded algebras

by

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DOCTORAL DISSERTATION, 1998

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Abstract

The study of Hilbert functions and Betti numbers of Gorenstein algebras is an old topic in commutative ring theory. We consider the following two important open problems.

PROBLEM 1. What are the possible Hilbert functions of Gorenstein algebras ?

PROBLEM 2. What are the possible Betti numbers of Gorenstein algebras ?

At a time when there were not many known examples of Hilbert functions of Artinian Gorenstein algebras, it was conjectured that all Artinian Gorenstein algebras have unimodal Hilbert functions, that is, the Hilbert function $(h_0, h_1, \ldots, h_s)$ of any Artinian Gorenstein algebra has the property $h_0 \leq h_1 \leq \cdots \leq h_t \geq h_{t+1} \geq \cdots \geq h_s$. However the first counterexample to this conjecture was given by R. Stanley [36, Example 4.3]. This example has codimension thirteen. Later, D. Bernstein and A. Iarrobino [2, Theorem 1] gave some examples of such Artinian Gorenstein algebras with codimension five, and further M. Boij and D. Laksov constructed a large class of Artinian Gorenstein algebras, including the examples of R. Stanley and D. Bernstein-A. Iarrobino (cf. [5] and [6]). Most of these algebras have non-unimodal Hilbert functions with many extremal values. In the case of codimension three, the possible Hilbert functions of Gorenstein algebras are completely characterized by a numerical condition, and it is well-known that there are no non-unimodal Hilbert functions. This famous characterization theorem was obtained by R. Stanley [36, Theorem 4.2]. The method of his proof make use of the well-known structure theorem for Gorenstein ideals of height three due to D. A. Buchsbaum and D. Eisenbud [8, Theorem 2.1]. The study of unimodality is closely related to the problem of characterizing the possible f-vectors of pure simplicial complexes (cf. [4], [35] and [37], for example). The question of unimodality has not yet settled in the following cases: codimension four, integral domains and licci algebras (i.e., algebras which are in the linkage class of a complete intersection).

The Betti numbers give much more information of an algebra than the Hilbert function. Therefore it is natural that the study of Betti numbers gets more complicated than the study of Hilbert functions. But, thanks to D. A. Buchsbaum and D. Eisenbud's structure theorem, the possible Betti numbers of Gorenstein algebras for codimension three are completely characterized, and there are some well-known results on the problem of constructing Gorenstein algebras with given possible Betti numbers (cf. [14], [18] and [25], for example). J. Herzog, N. V. Trung and G. Valla [25, page 63] constructed an explicit example of alternating matrixes defining Gorenstein algebras with given possible Betti numbers. S. J. Diesel [14, Theorem 3.2] gave an algorithm of describing all possible Betti numbers of Gorenstein algebras with a fixed Hilbert function. Hence we can observe that, in the case of codimension three, the Hilbert function of a Gorenstein algebra determine all possible Betti numbers of the algebra. Moreover, she gave an interesting observation that there exist

both the maximum and the minimum among all possible Betti numbers which determine the fixed Hilbert function. A. V. Geramita and J. C. Migliore [18, Theorem 2.1] showed that any set of given possible Betti numbers actually occurs for a "reduced" Gorenstein algebra of codimension three.

In this way, the study of the case of codimension three is a very well developed area concerning the above problems, and the starting point for these results is the well-known structure theorem due to D. A. Buchsbaum and D. Eisenbud. But, in the case of any codimension which is greater or equal to four, there is no structure theorem ! So we recall a basic fact of linkage theory [33, Remarque 1.4] that the sum of the saturated ideals of two geometrically linked arithmetically Cohen-Macaulay closed subschemes of projective space is the saturated ideal of an arithmetically Gorenstein closed subscheme of codimension one greater than codimension of the previous closed subschemes. This provides a way to construct Gorenstein ideals of height one greater than height of these ideals from Cohen-Macaulay ideals of smaller height. Using this construction, for example, we can get a special class of Artinian Gorenstein algebras constructed by the sums of the ideals of two finite sets of points in projective space such that the intersection of these sets is empty and the union is a complete intersection.

In this paper, we study some systematic constructions of such Artinian Gorenstein algebras controlling two finite sets being geometrically linked, and show the following three main results on the above problems concerning Hilbert functions and Betti numbers.

1) At first, we study Hilbert functions of certain Gorenstein algebras constructed by the linkage theory. In the first main result of Section 3, for two given geometrically linked arithmetically Cohen-Macaulay closed subschemes of projective space, we can construct a number of Gorenstein algebras whose Hilbert functions can be recovered from the Hilbert functions of the given closed subschemes (Theorem 3.2). This theorem is an important key of making our study concerning Hilbert functions and Betti numbers of Artinian Gorenstein algebras. By virtue of this theorem, we can observe that the Hilbert function of any Artinian Gorenstein algebra in the certain class obtained above is described in terms of the Hilbert functions of the given finite sets of points in projective space. So we would like to find some algorithmic ways for calculating the Hilbert function of any finite set of points in projective space. In fact, it is difficult to calculate the Hilbert function of any given finite set of points. But, in this paper, we construct finite sets of points called k-configurations which are in a special geometric configuration, and using a similar idea of the proof of [16, Theorem 4.1], we can give an algorithmic way for calculating the Hilbert functions of these sets (Remarks 3.6 and 3.10). Consequently, as an application of Theorem 3.2, we can give a new method of an explicit construction of Artinian Gorenstein algebras with a given possible Hilbert function for codimension three (Theorem 3.7). Then, by virtue of R. Stanley's characterization theorem, we can show that the sequence associated with the Hilbert function of an Artinian Gorenstein algebra is always the Hilbert function of a finite set of points in projective plane (Remark 3.4). This observation is a key to prove Theorem 3.7. Hence using the algorithmic way obtained by Remark 3.6, we can construct a k-configuration whose Hilbert function is equal to the sequence associated with any given possible Hilbert function, and further using Theorem 3.2, we can produce a desired Artinian Gorenstein algebra as the sum of the two ideals of this k-configuration and another controlled set. Later, A. V. Geramita, M. Pucci and Y. S. Shin [19] made an interesting observation to our construction, that is, they observed that the Betti numbers of Artinian Gorenstein algebras constructed by Theorem 3.7 are maximum among all possible Betti numbers which determine the same Hilbert function. Furthermore, using a similar idea of Theorem 3.7 to the case of codimension four, we give some examples of unimodal Gorenstein sequences of codimension four (Proposition 3.11 and Corollary 3.12).

2) We say that a finite sequence of positive integers is an SI-sequence if this sequence is symmetric and the first difference of the "first half" is an O-sequence (Definition 3.1). Here we can check that there exist Gorenstein SI-sequences which can not be constructed by the construction of Section 3 (Example 3.13). In Section 4, we go further, and give a method of an explicit construction of Artinian Gorenstein algebras whose Hilbert functions are equal to any given SI-sequence (Lemmas 4.4 and 4.6). We can get this construction by making use of an algebraic technique of the proof of [41, Theorem 3.8] controlling two geometrically linked sets of points in projective space, as a similar idea of the construction of Section 3. Also, the notion of "weak Stanley property" for Artinian algebras, which is introduced by J. Watanabe [41], plays an important role in a process of constructing Artinian Gorenstein algebras with a given SI-sequence. Cosequently, we can get the main result (Theorem 4.2) of Section 4 which give a characterization of Hilbert functions of Artinian Gorenstein algebras with the weak Stanley property. That is, it is showed that a given finite sequence of positive integers is the Hilbert function of an Artinian Gorenstein algebra with the weak Stanley property if and only if this sequence is an SI-sequence. J. Watanabe discovered a large class of Artinian Gorenstein algebras with unimodal Hilbert functions [41, Theorem 3.8 and Example 3.9]. That is, he showed a very interesting result that most Artinian Gorenstein algebras have a property which is stronger than this weak Stanley property. Hence, combining these results, we can observe that Hilbert functions of most Artinian Gorenstein algebras are SI-sequences.

3) The famous characterization theorem due to R. Stanley [36, Theorem 4.2] says that, in the case of codimension three, a symmetric sequence is the Hilbert function of an Artinian Gorenstein algebra if and only if the first difference of the "first half" is an O-sequence. The main purpose of Section 5 is to give an algebraic explanation of the essence of this formulation in terms of the first difference. At first, we can give a new method of an explicit construction of Artinian Gorenstein algebras with given possible Betti numbers for codimension three (Theorem 5.4). This construction is obtained by studying the construction

of Theorem 3.7 in detail. That is, we consider special finite sets of points in projective plane (called pure configurations) which are in configuration of lattice points, and describe minimal generators of Artinian Gorenstein ideals which are obtained as the sums of the ideals of two geometrically linked pure configurations in projective plane (Lemma 5.5). In the proof of this lemma, we can give a geometric observation that a pure configuration is the finite set of points defined by a lifting of an Artinian ideal generated by monomials. Hence we can see that all Artinian Gorenstein ideals of Lemma 5.5 are the sums of liftings of two geometrically linked monomial ideals, and this observation is a key for proving Lemma 5.5. Next, using a similar idea of Lemmas 4.4 and 4.5, we add the following important observation to this construction of Theorem 5.4, that is, we show that for any Artinian Gorenstein algebra of codimension three, there exists an Artinian Gorenstein algebra with the weak Stanley property which has the same Hilbert function (Theorem 5.8 and Remark 5.12). Consequently, by virtue of this observation, we can give another proof of the famous theorem due to R. Stanley which gives a characterization of the Hilbert functions of Gorenstein algebras of codimension three (Theorem 5.11). In this proof, we see an algebraic explanation of the essence of R. Stanley's formulation.

Key words. Graded ring, Graded algebra, Artinian ring, Artinian algebra, Cohen-Macaulay ring, Cohen-Macaulay algebra, Gorenstein ring, Gorenstein algebra, Hilbert function, Gorenstein sequence, Unimodality, Free resolution, Betti numbers, Linkage, Weak Stanley property.

Contents

1 Introduction

1-1 Hilbert functions and Betti numbers

We begin by introducing some standard notation and terminology of graded algebras that will be used throughout this paper.

Let A be a *standard graded algebra* over a field k, namely, A is a graded ring $\bigoplus_{i\geq 0} A_i$ satisfying $A_0 = k$, $A = k[A_1]$ and $\dim_k A_1 < \infty$. This means that there exists a positive integer n such that $A = R/I$ where $R = k[x_0, x_1, \ldots, x_n]$ is the polynomial ring with the standard grading, i.e., each $\deg x_i = 1$, and I is a homogeneous ideal of R.

The *Hilbert function* of A is defined by the numerical function $H(A, -) : \mathbf{N} \longrightarrow \mathbf{N}$ with

$$H(A, i) = \dim_k A_i \quad \text{for all} \quad i \geq 0,$$

in particular $H(A, 0) = 1$. The Hilbert function $H(A, i)$ measures the dimension of the i-th homogeneous piece of a graded algebra A. The *Hilbert series* of A is defined by the formal power series

$$F(A, \lambda) = \sum_{i=0}^{\infty} H(A, i)\lambda^i \in \mathbf{Z}[[\lambda]].$$

Let $d = \dim A$. It is a classical result that $F(A, \lambda)$ is a rational function in λ of the form

$$F(A, \lambda) = \frac{h_0 + h_1\lambda + \cdots + h_s\lambda^s}{(1-\lambda)^d}$$

for certain integers $h_0, h_1, \cdots, h_s$ satisfying $\sum_{i=0}^{s} h_i \neq 0$ and $h_s \neq 0$. Furthermore $H(A, i)$ is a polynomial for i sufficiently large values and the degree of this polynomial is $d - 1$. We call the sequence

$$h(A) = (h_0, h_1, \ldots, h_s)$$

the *h-sequence* (or the *h-vector*) of A. A sequence $(h_0, h_1, \ldots, h_s)$ of non-negative integers is said to be a *Gorenstein sequence* if this sequence is the h-sequence of some Gorenstein algebra. A sequence is *unimodal* if there exists an integer t such that

$$h_0 \leq h_1 \leq \cdots \leq h_t \geq h_{t+1} \geq \cdots \geq h_s.$$

We note that if $a \in A_u$ is not a zero-divisor, then $F(A/aA, \lambda) = (1 - \lambda^u)F(A, \lambda)$. Hence we get

$$F(A/(a_1, \ldots, a_n), \lambda) = (1 - \lambda)^n F(A, \lambda)$$

for any linear regular sequence $\{a_1, \ldots, a_n\}$. Furthermore we note that if k' is a field containing k, then

$$F(A \otimes_k k', \lambda) = F(A, \lambda).$$

Therefore we can study all possible Hilbert functions for Cohen-Macaulay algebras by studying only the Hilbert functions for Artinian algebras, at least when there is a linear maximal regular sequence, e.g., for infinite fields. When $A = \bigoplus_{i \ge 0} A_i$ is an Artinian algebra, we put

$$c(A) = \mathrm{Max}\{i \mid A_i \neq (0)\}$$

and we call $c(A)$ the *socle degree* of A. In this case, we denote by the finite sequence

$$H(A) = (H(A,0), H(A,1), \ldots, H(A, c(A)))$$

the Hilbert function of A. Obviously this finite sequence is the h-sequence of A.

Let

$$0 \longrightarrow \bigoplus_{j=1}^{b_r} R(-j)^{\beta_{r,j}} \longrightarrow \cdots \longrightarrow \bigoplus_{j=1}^{b_1} R(-j)^{\beta_{1,j}} \longrightarrow R(0) \longrightarrow A \longrightarrow 0$$

be a graded minimal free resolution of A as a graded module over R, where $R(\ell)_i = R_{\ell+i}$ for all $i \ge 0$. The numbers $\{\beta_{i,j}\}$ are uniquely determined by A, namely,

$$\beta_{i,j} = \dim_k \mathrm{Tor}_i^R(A,k)_j$$

for all (i,j). So we call $\{\beta_{i,j}\}$ the (i,j)-th graded *Betti numbers* of A. We note that if $y \in A_1$ is not a zero-divisor, then

$$\mathrm{Tor}_i^{R/YR}(A/yA, k) \cong \mathrm{Tor}_i^R(A,k),$$

where $Y \in R_1$ is a pre-image of y. Furthermore if k' is a field containing k, then

$$\mathrm{Tor}_i^{R\otimes_k k'}(A \otimes_k k', k') \cong \mathrm{Tor}_i^R(A,k) \otimes_k k'.$$

Therefore we can also study all possible graded Betti numbers for Cohen-Macaulay algebras by studying only the Betti numbers for Artinian algebras.

The Betti numbers of A determine the Hilbert series (i.e., the Hilbert function) of A by

$$F(A,\lambda) = \frac{1 + \sum_{i=1}^{r} \sum_{j=1}^{b_i} (-1)^i \lambda^{\beta_{i,j}}}{(1-\lambda)^{n+1}}.$$

The converse is not necessarily true. But it is also well-known that the Hilbert function of an algebra give some restrictions on the Betti numbers of the algebra. For example, the Hilbert function of a Gorenstein algebra of codimension three determine all possible Betti numbers of the algebra (cf. Diesel [14] for the details). The Betti numbers provide much more information about a graded algebra than just the Hilbert function. We are not only given the number of generators in various degree, but also their relations due to the module structure over a polynomial ring. Therefore it is natural that the study of Betti numbers gets more complicated than the study of Hilbert functions.

In the following section, we recall some well-known results on the problem of characterizing possible Hilbert functions and Betti numbers for a subclass of graded algebras. No

doubt, we see that it is very hard to find a characterization of a subclass of graded algebras, like Gorenstein algebras, Gorenstein domains, Cohen-Macaulay domains.

1-2 Characterization of possible Hilbert functions and Betti numbers

A central object of the study about Hilbert functions and Betti numbers of graded algebras and the numerical invariants derived from them is to see what kind of conditions the structure of an algebra A imposes on the numerical invariants of A, and conversely what we can deduce about A from knowledge of the numerical invariants of A.

Many results is known about the problem of characterizing possible Hilbert functions and Betti numbers for a subclass of graded algebras.

1) HILBERT FUNCTIONS OF GRADED ALGEBRAS: A characterization of Hilbert functions of graded algebras was first made by F.S. Macaulay. If a and i are positive integers then a can be written uniquely in the form, called the *i-binomial expansion of a*,

$$a = \binom{n_i}{i} + \binom{n_{i-1}}{i-1} + \cdots + \binom{n_j}{j},$$

where $n_i > n_{i-1} > \cdots > n_j \ge j \ge 1$. We put $0^{<i>} = 0$ and

$$a^{<i>} = \binom{n_i+1}{i+1} + \binom{n_{i-1}+1}{i} + \cdots + \binom{n_j+1}{j+1}.$$

A finite or infinite sequence $(a_0, a_1, \ldots)$ of positive integers is called an *O-sequence* if $a_0 = 1$ and $a_{i+1} \le a_i^{<i>}$ for all $i \ge 1$.

F.C. Macaulay described all possible Hilbert functions of graded algebras by a numerical condition. That is, he showed that a given sequence of positive integers is the Hilbert function of a graded algebra if and only if this sequence is an O-sequence (cf. [30], [36, Theorem 2.2] and [7, Theorem 4.2.10]). In the proof of this theorem, he showed that the Hilbert function of a graded algebra arises as the Hilbert function of a polynomial ring modulo an ideal which is defined by monomials.

2) HILBERT FUNCTIONS OF GORENSTEIN ALGEBRAS OF CODIMENSION THREE: At some stage, it was conjectured that all Artinian Gorenstein algebras have unimodal Hilbert functions. R. Stanley and A. Iarrobino independently conjectured that a sequence $(h_0, h_1, \ldots, h_s)$ is a Gorenstein sequence if and only if

(i) $h_{s-i} = h_i$ for all $i = 0, 1, \ldots, [s/2]$, i.e., this sequence is symmetric, and

(ii) $(h_0, h_1 - h_0, h_2 - h_1, \ldots, h_t - h_{t-1})$ is an O-sequence, where $t = [s/2]$.

For details, see [35, Conjecture 2] and [36, page 68]. Here we note that any sequence satisfying the above conditions (i) and (ii) is always unimodal. We call a sequence satisfying the conditions (i) and (ii) an *SI-sequence*.

R. Stanley [36, Theorems 4.1 and 4.2] showed that all Gorenstein sequences satisfy the condition (i), and further by using D. A. Buchsbaum and D. Eisenbud's structure theorem [8, Theorem 2.1] for Gorenstein ideals of height three, that this conjecture is true if $h_1 \leq 3$. Hence we can observe that there exist no Artinian Gorenstein algebras with non-unimodal Hilbert functions for codimension three.

3) Unimodality of Gorenstein sequences: The above conjecture is not necessarily true in general. The first example $(1, 13, 12, 13, 1)$ of a non-unimodal Gorenstein sequence was found by R. Stanley [36, Example 4.3]. Later, D. Bernstein and A. Iarrobino [2, Theorem 1] gave some examples of non-unimodal Gorenstein sequences with codimension five. The case of codimension four is still open now. On the other hand, it is well-known that there exist some classes of Artinian Gorenstein algebras with unimodal Hilbert functions (cf. [1], [4], [34], [35], [36], [37]). For example, R. Stanley [37] showed, by using the Hard Lefschetz Theorem, that the h-vector of a Gorenstein algebra, which is the Stanley-Reisner ring of a simplicial polytope, satisfies the conditions (i) and (ii) above. The converse was proved by L. J. Billera and C. W. Lee [4].

4) Gorenstein property for Cohen-Macaulay domains: R. Stanley showed the beautiful theorem [36, Theorem 4.4] characterizing graded Gorenstein domains by their Hilbert functions. Precisely speaking, if A is a graded Cohen-Macaulay domain of Krull-dimension d, then the following conditions are equivalent.

(a) A is Gorenstein.

(b) There exists an integer ρ such that $F(A, 1/\lambda) = (-1)^d \lambda^\rho F(A, \lambda)$, that is, the h-sequence of A is symmetric.

5) Hilbert functions of reduced algebras: The O-sequence $\{b_i\}$ is called *differentiable* if its first difference $\{c_i\}$, where $c_0 = 1$ and $c_i = b_i - b_{i-1}$ for all $i \geq 1$, is again an O-sequence.

A. V. Geramita, P. Maroscia and L. Roberts [16, Theorem 4.1] characterized Hilbert functions of reduced algebras as follows : A given O-sequence $\{b_i\}$ is the Hilbert function of a reduced algebra if and only if $\{b_i\}$ is differentiable.

6) Hilbert functions of Gorenstein domains of codimension three: Let $(h_0, h_1, \ldots, h_s)$ be a Gorenstein sequence with $h_1 = 3$. We put

$$a = \mathrm{Min}\{i \mid h_i \neq \binom{2+i}{2}\} \quad \text{and} \quad \sum_{i=0}^{s+2} q_i \lambda^i = (1-\lambda)^2 \sum_{i=0}^{s} h_i \lambda^i.$$

Then E. de Negri and G. Valla showed in [32, Theorem 5] that there exists a Gorenstein domain of codimension three which has this sequence as the h-sequence if and only if the following conditions are satisfied:

(i) $q_i \leq 0$ for every i such that $a \leq i \leq [s/2] + 1$, and

(ii) it does not happen that $q_t < 0$, $q_v = 0$ and $q_r < 0$ with $a \leq t < v < r < [s/2] + 1$.

7) Betti numbers of Gorenstein algebras of codimension three: There are some well-known results on the problem of constructing Artinian Gorenstein algebras having an assigned set of graded Betti numbers which are possible for some Artinian Gorenstein algebra of codimension three (cf. [14], [18] and [25], for example). An explicit construction can be found in the paper by J. Herzog, N. V. Trung and G. Valla [25, page 63] and the paper by S. J. Diesel [14, Proposition 3.1]. Using this construction, we can give an explicit example of alternating matrixes defining Gorenstein algebras with given possible Betti numbers. Furthermore, S. J. Diesel [14, Theorem 3.2] gave an algorithm of describing all possible Betti numbers of Gorenstein algebras with a fixed Hilbert function. As a result of this theorem, we can observe that in the case of codimension three, the Hilbert function of a Gorenstein algebra determine all possible Betti numbers of the algebra. A. V. Geramita and J. C. Migliore [18, Theorem 2.1] showed that any set of given possible Betti numbers in fact occurs for a reduced set of points in $\mathbf{P}^3$.

1-3 Main results

We consider the following two important open problems concerning Hilbert functions and Betti numbers of Gorenstein algebras.

Problem 1. What are the possible Hilbert functions of Gorenstein algebras ?

Problem 2. What are the possible Betti numbers of Gorenstein algebras ?

As can be seen from the above section, thanks to the well-known D. A. Buchsbaum and D. Eisenbud's structure theorem [8, Theorem 2.1], the study of the case of codimension three is a very well developed area concerning these problems. But, in the case of any codimension which is greater or equal to four, our knowledge is much more limited because there is no structure theorem. So, we recall a standard fact of linkage theory [33, Remarque 1.4] that the sum of the saturated ideals of two geometrically linked arithmetically Cohen-Macaulay closed subschemes of projective space is the saturated ideal of an arithmetically Gorenstein closed subscheme of codimension one greater than codimension of the previous closed subschemes. This fact is a starting point for our study concerning Hilbert functions and Betti numbers of Artinian Gorenstein algebras.

In this paper, we study Hilbert functions and Betti numbers of Artinian Gorenstein algebras constructed by the linkage theory, and show the following three main results concerning the above problems.

The main results of this paper to the first problem are:

1) We study Hilbert functions of Gorenstein algebras with the ideals constructed from Cohen-Macaulay ideals of smaller height by the above fact. A main purpose of Section 3 is, for two given geometrically linked arithmetically Cohen-Macaulay closed subschemes, to give a way to construct a number of Gorenstein algebras whose Hilbert functions can be recovered from the Hilbert functions of the given closed subschemes (Theorem 3.2). This theorem provides a way to construct many Gorenstein algebras with unimodal Hilbert functions. As an application of this construction, we give a new method of an explicit construction of Artinian Gorenstein algebras achieving all possible Hilbert functions for codimension three (Theorem 3.7). A key idea for finding this construction is to construct a finite set of points, called a k-configuration which is a special geometric configuration, with a given differentiable O-sequence. Consequently, we can construct a desired Artinian Gorenstein algebra as the sum of the two ideals of this k-configuration and another controlled set. This idea of Theorem 3.7 was further exploited by A. V. Geramita, M. Pucci and Y. S. Shin [19] to find good points in the parameterizing space for Gorenstein algebras of codimension three. Furthermore using a similar idea of Theorem 3.7, we give some examples of unimodal Gorenstein sequences of codimension four. But we can check that there exist Gorenstein SI-sequences which can not be constructed by this construction (Example 3.13). In Section 4, we go further and give a method of an explicit construction of Artinian Gorenstein algebras with a given SI-sequence.

2) At some stage, it was conjectured that all Artinian Gorenstein algebras have unimodal Hilbert functions. However, in the cases of any codimension which is greater or equal to five, it is showed that there exist some examples of Artinian Gorenstein algebras with non-unimodal Hilbert functions (cf. [2], [5], [6] and [36]). For example, M. Boij [5] gave some examples of Artinian Gorenstein algebras having non-unimodal Hilbert functions with many extremal values. On the other hand, J. Watanabe discovered a large class of Artinian Gorenstein algebras with unimodal Hilbert functions [41, Theorem 3.8 and Example 3.9]. That is, he showed a very interesting result that most Artinian Gorenstein algebras have the strong Stanley property. In Section 4, we study Hilbert functions of Artinian Gorenstein algebras with a property which is weaker than this Stanley property. The main purpose of this section is to give a characterization of Hilbert functions of Artinian Gorenstein algebras with the weak Stanley property. That is, we show that a given sequence of integers is the Hilbert function of an Artinian Gorenstein algebra with the weak Stanley property if and only if this sequence is symmetric and the first difference of the "first half" is an O-sequence (Theorem 4.2). An idea for proving this characterization is to use the technique of the proof of [41, Theorem 3.8], controlling two geometrically linked sets of points in projective space. Then using this idea, we can give a method of an explicit construction of Artinian Gorenstein algebras whose Hilbert functions are equal to any given SI-sequence (Lemmas 4.4 and 4.6), and we can be led to this theorem. Hence, combining these results,

we can observe that most unimodal Gorenstein sequences satisfy these conditions. But the author does not know whether there exists an Artinian Gorenstein algebra with a unimodal Hilbert function not satisfying these conditions. And further, it remains open an important question of characterizing unimodal Gorenstein sequences. The author believe that all unimodal Gorenstein sequences are always SI-sequences.

As for the second problem, the main result of this paper is:

3) In the final section, we give a new method of an explicit construction of Artinian Gorenstein algebras achieving all possible graded Betti numbers for codimension three (Theorem 5.4). This construction is obtained by studying the construction of Theorem 3.7 in detail controlling two geometrically linked finite sets of points in projective plane, called pure configurations which are in configuration of lattice points. Using this construction, we can produce our Artinian Gorenstein algebras as the sums of ideals of two geometrically linked pure configurations formulated completely in terms of the diagonal degrees defined by given graded Betti numbers. Hence we can observe that all possible resolutions for Gorenstein ideals of height three can be obtained as those of the sums of two geometrically linked Cohen-Macaulay ideals of height two. Moreover, we add an important observation to this construction, that is, we show that for any Artinian Gorenstein algebra of codimension three, there exists an Artinian Gorenstein algebra with the weak Stanley property which has the same Hilbert function (Theorem 5.8 and Remark 5.12). Hence, as a result of this observation, we can give another proof of the famous theorem due to R. Stanley [36, Theorem 4.2] which gives a characterization of the Hilbert functions of Artinian Gorenstein algebras of codimension three (Theorem 5.11). In this proof, we see an algebraic explanation of the essence of R. Stanley's formulation (in terms of the first difference) for Hilbert functions.

Acknowledgements. I thank members of our seminar in Hiroshima University for many valuable comments and discussions, and furthermore I would like to express my sincere gratitude to Professor H. Sumihiro, Professor S. Itoh, Professor A. Ooishi, Professor H. Okuyama and my friend S. Isogawa for many useful advices on my study. Finally I am deeply grateful to my wife for her great help and encouragement.

2 Preliminaries

This section is devoted to recall some standard facts of linkage theory and basic properties of Hilbert functions of arithmetically Cohen-Macaulay (aCM for short) closed subschemes of projective space $\mathbf{P}^n = \mathbf{P}^n_k$ that will be used later.

Throughout this paper, we assume that k is always an infinite field. Let X be a closed subscheme of $\mathbf{P}^n$ and let $I(X)$ be the saturated homogeneous ideal defining X in the homogeneous coordinate ring $R = k[x_0, x_1, \ldots, x_n]$ of $\mathbf{P}^n$.

2-1 Some standard facts of linkage theory

Linkage was first formally introduced by Peskine and Szpiro in [33], however essentially equivalent ideas have been studied since the nineteenth century. Linkage provides a technique to construct large and interesting classes of C-M ideals and of Gorenstein ideals (see e.g. [27], [38] and [39]).

Definition 2.1 (cf. Peskine and Szpiro [33]). (1) Let I and J be two homogeneous ideals of $R = k[x_0, x_1, \ldots, x_n]$. Then I and J are said to be *(algebraically) linked* (with respect to (α)) if there exists a homogeneous R-regular sequence $\alpha = \{\alpha_1, \ldots, \alpha_g\}$ in $I \cap J$ such that $J = (\alpha) : I$ and $I = (\alpha) : J$. Furthermore I and J are said to be *geometrically linked* if I and J are linked and if in addition I and J have no common associated primes.

(2) Let X and Y be two closed subschemes of $\mathbf{P}^n$. Then X and Y are said to be *(algebraically) linked* (with respect to (α)) if the two ideals $I(X)$ and $I(Y)$ are linked. Furthermore X and Y are said to be *geometrically linked* if $I(X)$ and $I(Y)$ are geometrically linked.

The following three lemmas are well-known, so we omit the proofs (cf. [33]).

Lemma 2.2 *Let X and Y be closed subschemes of $\mathbf{P}^n$ which are linked with respect to (α). Then we have the following.*

(1) *$I(X)$ and $I(Y)$ are unmixed ideals of the same grade. Furthermore we have*

$$Ass(R/(\alpha)) = Ass(R/I(X)) \cup Ass(R/I(Y)).$$

(2) *X is aCM if and only if Y is aCM.*

Lemma 2.3 *Let X and Y be d-dimensional aCM closed subschemes of $\mathbf{P}^n$ which have no common irreducible components. Then the following conditions are equivalent, and in this case X and Y are linked with respect to $I(X \cup Y)$.*

(a) *X and Y are geometrically linked.*

(b) *$X \cup Y$ is a complete intersection, i.e., $I(X \cup Y)$ is generated by a homogeneous R-regular sequence.*

Lemma 2.4 *Let X and Y be d-dimensional aCM closed subschemes of $\mathbf{P}^n$ which are geometrically linked. Then $A = R/I(X) + I(Y)$ is a d-dimensional Gorenstein graded algebra.*

This standard fact of linkage theory, providing a way to construct Gorenstein ideals from Cohen-Macaulay ideals of smaller height, is a starting point for the main results of this paper.

2-2 Some basic properties of Hilbert functions

Let X be a closed subscheme of $\mathbf{P}^n$. The Hilbert function of X is defined by

$$H(X,i) = \begin{cases} H(R/I(X),i) & \text{for all } i \geq 0, \\ 0 & \text{for all } i < 0. \end{cases}$$

Furthermore we put

$$\Delta H(X,i) = \begin{cases} 1 & \text{for } i = 0, \\ H(X,i) - H(X,i-1) & \text{for all } i \geq 1, \\ 0 & \text{for all } i < 0. \end{cases}$$

Inductively, for all $t \geq 2$, we put

$$\Delta^t H(X,i) = \Delta(\Delta^{t-1} H(X,i)) \quad \text{for all } i.$$

The Hilbert series of X is defined by $F(X,\lambda) = F(R/I(X),\lambda)$. We denoted by $e(X)$ the multiplicity of X. If X is a d-dimensional aCM closed subscheme of $\mathbf{P}^n$, then we put

$$c(X) = \mathrm{Max}\{i \mid \Delta^{d+1} H(X,i) \neq 0\}.$$

Furthermore, if $I(X) = (F_1, \ldots, F_{n-d})$ for some $F_i \in R_{e_i}$ $(1 \leq i \leq n-d)$, then X is said to be d-dimensional *complete intersection of type* $(e_1, \ldots, e_{n-d})$ (written as $X = C.I.(e_1, \ldots, e_{n-d})$).

We state some basic properties of Hilbert functions of aCM closed subschemes. Refer to [12] for the proof, for example.

Lemma 2.5 *Let X be a d-dimensional aCM closed subscheme of $\mathbf{P}^n$.*

(1) *If $a = \{a_1, \ldots, a_t\}$ is a linear sequence in R_1 which is regular on $R/I(X)$, then $H(R/(I(X),a),i) = \Delta^t H(X,i)$ for all i.*

(2) $\Delta^d H(X, i) < \Delta^d H(X, i+1)$ *for all* $0 \le i < c(X)$.

(3) $\Delta^d H(X, i) = e(X)$ *for all* $i \ge c(X)$.

(4) *If* X *is a 0-dimensional reduced closed subscheme, then*

$$e(X) = \mid X \mid \quad and \quad c(X) = \mathrm{Min}\{i \mid H(X, i) = \mid X \mid\},$$

where $\mid X \mid$ *denote the number of points in* X.

(5) *If* Y *be a d-dimensional aCM closed subscheme of* X, *then* $c(Y) \le c(X)$.

(6) *If* $X = C.I.(e_1, \ldots, e_{n-d})$, *then* $F(X, \lambda) = \dfrac{\prod_{i=1}^{n-d}(1 - \lambda^{e_i})}{(1-\lambda)^{n+1}}$.

(7) *If* $X = C.I.(e_1, \ldots, e_{n-d})$, *then* $e(X) = \prod_{i=1}^{n-d} e_i$ *and* $c(X) = e_1 + \cdots + e_{n-d} - (n-d)$.

The following lemma is an important result concerning Hilbert functions under linkage. By the duality of Gorenstein algebras we can prove this lemma, see [13] for details.

Lemma 2.6 (cf. Davis, Geramita and Orecchia [13, Theorem 3]). *Let* X *and* Y *be d-dimensional aCM closed subschemes of* $\mathbf{P}^n$ *which are linked with respect to* (α). *Put* $c = \mathrm{Max}\{i \mid \Delta^{d+1} H(R/(\alpha), i) \ne 0\}$. *Then*

$$\Delta^{d+1} H(R/(\alpha), i) = \Delta^{d+1} H(X, i) + \Delta^{d+1} H(Y, c-i)$$

for all i.

The following is clear from Lemmas 2.3 and 2.6.

Corollary 2.7 *Let* X *and* Y *be d-dimensional aCM closed subschemes of* $\mathbf{P}^n$ *which are geometrically linked. Then*

$$\Delta^{d+1} H(X \cup Y, i) = \Delta^{d+1} H(X, i) + \Delta^{d+1} H(Y, c(X \cup Y) - i)$$

for all i.

Let X and Y be d-dimensional aCM closed subschemes of $\mathbf{P}^n$ which are geometrically linked. Then there is the following well-known exact sequence preserving degree:

$$0 \to R/I(X \cup Y) \to R/I(X) \oplus R/I(Y) \to R/I(X) + I(Y) \to 0.$$

Combining this exact sequence and the preceding corollary, we get in the following section that if $2c(X) \le c(X \cup Y) - 1$, then the Hilbert function of $R/I(X) + I(Y)$ can be described completely by the Hilbert functions of X and Y (Theorem 3.2). This theorem, which is an important key of making our study of Hilbert functions and Betti numbers, is also an interesting result concerning Hilbert functions under linkage.

3 Some examples of unimodal Gorenstein sequences

We consider the following finite sequences stated in Introduction 1-2.

Definition 3.1 An *SI-sequence* is a finite sequence $h = (h_0, h_1, \ldots, h_s)$ of positive integers which satisfies the following two conditions:

(i) $h_i = h_{s-i}$ for all $0 \le i \le s$, i.e., h is symmetric;

(ii) $(h_0, h_1 - h_0, h_2 - h_1, \ldots, h_t - h_{t-1})$ is an O-sequence, where $t = [s/2]$.

Here we note that every SI-sequence is always unimodal.

In this section, which is the major part of the paper [21], we study h-sequences of certain Gorenstein algebras obtained by the construction of Lemma 2.4, and construct a large class of unimodal Gorenstein sequences. Consequently, for a given SI-sequence with $h_1 = 3$, we construct explicit examples of Artinian Gorenstein algebras whose Hilbert functions are this SI-sequence. Furthermore using a similar idea, we give some examples of Gorenstein SI-sequences with $h_1 = 4$.

3-1 Hilbert functions of certain Gorenstein algebras

In the following theorem, which is one of the main results of this section, we construct a number of Gorenstein algebras with Hilbert functions described in terms of the Hilbert functions of two given aCM closed subschemes.

Theorem 3.2 *Let* X *and* Y *be d-dimensional aCM closed subschemes of* $\mathbf{P}^n$ *which are geometrically linked. Furthermore, let* $(h_0, \ldots, h_s)$ *be the h-sequence of* $R/I(X) + I(Y)$. *Then we have the following.*

(1) $R/I(X) + I(Y)$ *is a d-dimensional Gorenstein graded algebra.*

(2) $h_i = \Delta^d H(X, i) + \Delta^d H(X, c(X \cup Y) - 1 - i) - e(X)$ *for all* $0 \le i \le s$.

(3) $c(R/I(X) + I(Y)) = c(X \cup Y) - 1$, *i.e.,* $s = c(X \cup Y) - 1$.

(4) *Assume that* $2c(X) \le c(X \cup Y) - 1$. *Then* $h(R/I(X) + I(Y))$ *is a Gorenstein SI-sequence as follows:*

$$h_i = \begin{cases} \Delta^d H(X, i) & \text{for all } 0 \le i \le c(X) - 1, \\ e(X) & \text{for all } c(X) \le i \le c(X \cup Y) - 1 - c(X), \\ \Delta^d H(X, c(X \cup Y) - 1 - i) & \text{for all } c(X \cup Y) - c(X) \le i \le c(X \cup Y) - 1. \end{cases}$$

PROOF. (1) This follows from Lemma 2.4.

(2) From Corollary 2.7, we obtain

$$\Delta^{d+1}H(X\cup Y,i)=\Delta^{d+1}H(X,c(X\cup Y)-i)+\Delta^{d+1}H(Y,i)$$

for all i. Therefore

$$\sum_{i=0}^{t}\Delta^{d+1}H(X\cup Y,i)=\sum_{i=0}^{t}\Delta^{d+1}H(X,c(X\cup Y)-i)+\sum_{i=0}^{t}\Delta^{d+1}H(Y,i)$$

for every $t\geq 0$. Hence

$$\sum_{i=0}^{t}\Delta^{d+1}H(X,c(X\cup Y)-i)=\Delta^{d}H(X\cup Y,t)-\Delta^{d}H(Y,t).$$

Furthermore, by using Lemma 2.5 (3),

$$\sum_{i=0}^{c(X\cup Y)}\Delta^{d+1}H(X,i)=\Delta^{d}H(X,c(X\cup Y))=e(X),$$

because $c(X)\leq c(X\cup Y)$. Therefore

$$\begin{aligned}\sum_{i=0}^{t}\Delta^{d+1}H(X,c(X\cup Y)-i) &= \sum_{i=0}^{c(X\cup Y)}\Delta^{d+1}H(X,i)-\sum_{i=0}^{c(X\cup Y)-1-t}\Delta^{d+1}H(X,i)\\ &= e(X)-\Delta^{d}H(X,c(X\cup Y)-1-t).\end{aligned}$$

Hence it turns out that

$$\Delta^{d}H(X\cup Y,t)-\Delta^{d}H(Y,t)=e(X)-\Delta^{d}H(X,c(X\cup Y)-1-t).$$

On the other hand, since the ring $R/I(X\cup Y)$ is a $d+1$-dimensional CM standard graded algebra, there exists a homogeneous sequence $a=\{a_1,\dots,a_d\}\subset R_1$ which is regular on $R/I(X\cup Y)$. Put $\overline{R}=R/aR$, and let $\overline{I},\overline{J}$ and $\overline{K}$ be the images of $I(X),I(Y)$ and $I(X\cup Y)$ respectively. Then, by virtue of [27, Lemma 1.10], the two ideals $\overline{I}$ and $\overline{J}$ are geometrically linked with respect to $\overline{K}$, and a is a regular sequence on $R/I(X)+I(Y)$. We note that $\overline{R}/\overline{I}+\overline{J}$ is Artinian, and

$$h_i=H(\overline{R}/\overline{I}+\overline{J},i)\quad\text{for all } 0\leq i\leq s.$$

Furthermore from the following exact sequence

$$0\to\overline{R}/\overline{K}\to\overline{R}/\overline{I}\oplus\overline{R}/\overline{J}\to\overline{R}/\overline{I}+\overline{J}\to 0,$$

it follows that

$$H(\overline{R}/\overline{I}+\overline{J},i)=H(\overline{R}/\overline{I},i)+H(\overline{R}/\overline{J},i)-H(\overline{R}/\overline{K},i).$$

Hence we conclude

$$\begin{aligned}h_i &= \Delta^{d}H(X,i)+\Delta^{d}H(Y,i)-\Delta^{d}H(R/I(X\cup Y),i)\\ &= \Delta^{d}H(X,i)+\Delta^{d}H(X,c(X\cup Y)-1-i)-e(X).\end{aligned}$$

(3) We note that

$$c(R/I(X)+I(Y))=c(\overline{R}/\overline{I}+\overline{J}).$$

Therefore it is enough to show that

$$H(\overline{R}/\overline{I}+\overline{J},c(X\cup Y)-1)\neq 0\quad\text{and}\quad H(\overline{R}/\overline{I}+\overline{J},c(X\cup Y))=0.$$

When $i=0$, we have from (2)

$$h_0=\Delta^{d}H(X,0)+\Delta^{d}H(X,c(X\cup Y)-1)-e(X).$$

Since $h_0=1$ and $\Delta^{d}H(X,0)=1$, it follows that $\Delta^{d}H(X,c(X\cup Y)-1)=e(X)$. Hence when $i=c(X\cup Y)-1$, we obtain

$$\begin{aligned}H(\overline{R}/\overline{I}+\overline{J},c(X\cup Y)-1) &= \Delta^{d}H(X,c(X\cup Y)-1)+\Delta^{d}H(X,0)-e(X)\\ &= e(X)+1-e(X)\\ &= 1.\end{aligned}$$

Furthermore when $i=c(X\cup Y)$, it turns out that

$$\begin{aligned}H(\overline{R}/\overline{I}+\overline{J},c(X\cup Y)) &= \Delta^{d}H(X,c(X\cup Y))+\Delta^{d}H(X,-1)-e(X)\\ &= e(X)+0-e(X)\\ &= 0.\end{aligned}$$

(4) From $2c(X)\leq c(X\cup Y)-1$, we see $c(X)\leq c(X\cup Y)-1-i$ for all $0\leq i\leq c(X)-1$. Therefore

$$\Delta^{d}H(X,c(X\cup Y)-1-i)=e(X)\quad\text{for all}\quad 0\leq i\leq c(X)-1.$$

Hence, by using (2),

$$h_i=\Delta^{d}H(X,i)\quad\text{for all}\quad 0\leq i\leq c(X)-1.$$

Furthermore we note that

$$\Delta^{d}H(X,i)=e(X)\quad\text{and}\quad\Delta^{d}H(X,c(X\cup Y)-1-i)=e(X)$$

for all $c(X)\leq i\leq c(X\cup Y)-1-c(X)$. Therefore we have from (2)

$$h_i=e(X)\quad\text{for all}\quad c(X)\leq i\leq c(X\cup Y)-1-c(X).$$

Also since $\Delta^d H(X,i) = e(X)$ for all $i \geq c(X \cup Y) - c(X)$, it follows that

$$h_i = \Delta^d H(X, c(X \cup Y) - 1 - i) \quad \text{for all} \quad c(X \cup Y) - c(X) \leq i \leq c(X \cup Y) - 1.$$

Next we note that $[(c(X \cup Y) - 1)/2] \leq c(X \cup Y) - 1 - c(X)$, because $2c(X) \leq c(X \cup Y) - 1$. Hence, the above equation implies that $\Delta h_i = \Delta^{d+1} H(X,i)$ for all $0 \leq i \leq [(c(X \cup Y)/2]$. That is, $(\Delta h_0, \Delta h_1, \cdots, \Delta h_{[(c(X \cup Y)/2]})$ is an O-sequence. Also it is clear from the above equation that $h(R/I(X) + I(Y))$ is symmetric. Thus $h(R/I(X) + I(Y))$ is a Gorenstein SI-sequence. Q.E.D.

CONJECTURE. Let X and Y be aCM closed subschemes of $\mathbf{P}^n$ which are geometrically linked. Then the h-sequence of $R/I(X) + I(Y)$ is an SI-sequence.

3-2 Gorenstein sequences of codimension three

The aim of this subsection is to give a new method of an explicit construction of Artinian Gorenstein algebras whose Hilbert functions are a given SI-sequence with $h_1 = 3$.

Definition 3.3 Let $h = (h_0, h_1, \ldots, h_s)$ be an SI-sequence, and put $b_i = h_i$ for each $0 \leq i \leq [s/2]$ and $b_i = h_{[s/2]}$ for all $i > [s/2]$. We call this sequence $\{b_i\}$ *the sequence associated with h.*

Remark 3.4 It is clear that the sequence $\{b_i\}$ above is a 0-dimensional differentiable O-sequence (in the terminology of [16, Definitions 2.8 and 2.9]), i.e.,$b_i = b_{i+1}$ for all $i \gg 0$ and the sequence $(b_0, b_1 - b_0, b_2 - b_1, \ldots)$ is an O-sequence.

Next we consider the configurations of points in $\mathbf{P}^2$ as follows.

Definition 3.5 (cf. Roberts and Roitman [34]). A finite set X of points in $\mathbf{P}^2$ which satisfies the following conditions is called a *k-configuration.*

There exist integers $1 \leq d_1 < d_2 < \ldots < d_m$, subsets $X_1, \ldots, X_m$ of X, and distinct lines $L_1, \ldots, L_m$ such that

(i) X is the union of the X_i's,

(ii) $\mid X_i \mid = d_i$ for each $1 \leq i \leq m$, where $\mid X_i \mid$ denote the number of points in X_i,

(iii) Any point of X_i lies on L_i for each $1 \leq i \leq m$, and

(iv) $L_i (1 < i \leq m)$ does not contain any point of X_j for all $j < i$.

In this case, the *type* of X is defined by $type(X) = (d_1, \ldots, d_m)$.

In the following remark, using a similar idea of [16, Theorem 4.1], we give an algorithmic way for calculating the Hilbert functions of k-configurations in $\mathbf{P}^2$.

Remark 3.6 (1) All k-configurations in $\mathbf{P}^2$ of type $(d_1, \ldots, d_m)$ have the same Hilbert function, which will be denoted by $H^{(d_1, \ldots, d_m)}$. $H^{(d_1, \ldots, d_m)}$ can be obtained as follows: For any $d \geq 1$, let $\tau(d)$ be the infinite sequence $1, 2, \ldots, d, d, \rightarrow$ (continuing with this constant value d). Write down the sequences $\tau(d_1), \ldots, \tau(d_m)$, successively shifted to the left and add:

$$\begin{array}{lcccc}
\tau(d_1): & & & 1, 2, \ldots, d_1, \rightarrow & \\
\tau(d_2): & & 1, 2, 3, \ldots, & d_2, \rightarrow & \\
\cdots & & \cdots & \cdots & \\
\tau(d_m): & 1, 2, 3, & \cdots & , & d_m, \rightarrow \\
\hline
H^{(d_1, \ldots, d_m)}: & 1, 3, \ldots & & &
\end{array}$$

Therefore we obtain

$$H^{(d_1, \ldots, d_m)}(i) = \sum_{j=1}^{m} \tau(d_j)(j + i - m).$$

Furthermore it follows that

$$H^{(d_1, \ldots, d_m)}(i) = d_1 + \cdots + d_m = \mid X \mid \quad \text{for all} \quad i \gg 0$$

and

$$\min\{i \mid H^{(d_1, \cdots, d_m)}(i) = \mid X \mid\} = d_m - 1, \quad \text{i.e.,} \quad c(X) = d_m - 1.$$

(2) Let $b = (b_0, b_1, b_2, \ldots)$ be a 0-dimensional differentiable O-sequence with $b_1 = 3$. Then there exist integers $1 \leq d_1 < \ldots < d_m$ such that $H^{(d_1, \ldots, d_m)}(i) = b_i$ for all $i \geq 0$. Since the integers $d_1, \ldots, d_m$ for a given 0-dimensional O-sequence b are uniquely determined, we call $(d_1, \ldots, d_m)$ the *type* of b, which will be denoted by $type(b)$. The proof of [16, Theorem 4.1] gives a process of calculating the type of b. Thus if X is a k-configuration in $\mathbf{P}^2$ of type $(d_1, \ldots, d_m)$, then we obtain $H(X,i) = b_i$ for all $i \geq 0$.

Now as an application of Theorem 3.2, we prove the following theorem.

Theorem 3.7 *Let $h = (h_0, h_1, \ldots, h_s)$ be an SI-sequence with $h_1 = 3$, and let $b = \{b_i\}$ be the sequence associated with h. Furthermore let $type(b) = (d_1, \ldots, d_m)$ be the type of b. Then we have the following.*

(1) There exist a k-configuration X of type $(d_1, \ldots, d_m)$ in $\mathbf{P}^2$ and a finite set Y of points in $\mathbf{P}^2$ such that $X \cap Y = \phi$ and $X \cup Y = C.I.(e_1, e_2)$, where $e_1 = d_m$ and $e_2 = s + 3 - d_m$.

(2) Moreover the h-sequence of $R/I(X) + I(Y)$ is equal to the given SI-sequence h.

PROOF. (1) It is enough to show $m \leq e_1$ and $d_m \leq e_2$. Since $1 \leq d_1 < \cdots < d_m$, we have $m \leq e_1$. Furthermore we note that $b_{[s/2]} = d_1 + \cdots + d_m$. Therefore by Remark 3.6, $d_m - 1 \leq [s/2]$, i.e., $2(d_m - 1) \leq s$. Hence $d_m < e_2$.

(2) We note that $c(X) = d_m - 1$ and $c(X \cup Y) = e_1 + e_2 - 2$. Since $d_m \leq e_1 < e_2$, we have $2(d_m - 1) \leq e_1 + e_2 - 3$. Therefore $2c(X) \leq c(X \cup Y) - 1$. Hence it follows by Theorem

3.2 that $H(R/I(X)+I(Y),i) = H(X,i) = b_i$ for all $0 \le i \le e_1+e_2-2-d_m$. Also it is easy to show that $[s/2] \le e_1+e_2-2-d_m$. Therefore $H(R/I(X)+I(Y),i) = b_i = h_i$ for all $0 \le i \le [s/2]$. Hence since $e_1+e_2-3=s$, we conclude $H(R/I(X)+I(Y),i) = h_i$ for all $0 \le i \le s$. Q.E.D.

We illustrate this theorem with the following example.

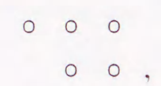
Example 3.8 When $n = 5, 6$ or 7, $h = (1,3,5,n,5,3,1)$ is an SI-sequence, and the sequence b associated with h is $1,3,5,n,n,\rightarrow$. By the construction in [16, Theorem 4.1], we obtain

$$type(b) = \begin{cases} (2,3) & \text{if } n=5 \\ (n-4,4) & \text{if } n=6 \text{ or } 7. \end{cases}$$

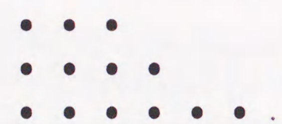
Then we put, as in Theorem 3.7

$$e_1 = \begin{cases} 3 & \text{if } n=5 \\ 4 & \text{if } n=6 \text{ or } 7 \end{cases} \quad \text{and} \quad e_2 = \begin{cases} 6 & \text{if } n=5 \\ 5 & \text{if } n=6 \text{ or } 7 \end{cases}.$$

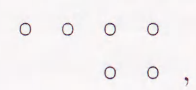
For $n=5$, let X be the following set of points in $\mathbf{P}^2$



and let Y be

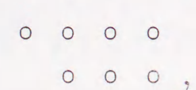


For $n=6$, let X be the following set of points in $\mathbf{P}^2$

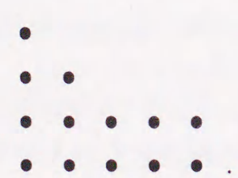


and let Y be

Furthermore for $n=7$, let X be the following set of points in $\mathbf{P}^2$



and let Y be



Then we have, by the construction in Theorem 3.7 (2),

$$F(A,\lambda) = 1 + 3\lambda + 5\lambda^2 + n\lambda^3 + 5\lambda^4 + 3\lambda^5 + \lambda^6$$

where $A = k[x,y,z]/I(X)+I(Y)$.

Recently, this idea of Theorem 3.7 was further exploited by A. V. Geramita, M. Pucci and Y. S. Shin [19] to find good points in the parameterizing space for Gorenstein codimension three ideals.

3-3 Unimodal Gorenstein sequences of codimension four

In this subsection, using a similar idea of Theorem 3.7, we give some examples of unimodal Gorenstein sequences of codimension four.

First we introduce the notion of k-configurations of points in $\mathbf{P}^3$ as follows.

Definition 3.9 A finite set X of points in $\mathbf{P}^3$ which satisfies the following conditions is called a *k-configuration.*

There exist subsets $X_1, \ldots, X_u$ of X and distinct hyperplanes $H_1, \ldots, H_u$ such that

(i) X is the union of the X_i's,

(ii) For each $i = 1, \ldots, u$, any point of X_i lies on H_i,

(iii) $H_i (1 < i \le u)$ does not contain any point of X_j for all $j < i$ and

(iv) $X_i (1 < i \le u)$ is a k-configuration in H_i of type $(d_{i1}, \ldots, d_{im_i})$ with $d_{im_i} < m_{i+1}$ for $1 \le i < u$.

In this case, the *type* of X is defined by $type(X) = (d_{11}, \ldots, d_{1m_1}; \ldots; d_{u1}, \ldots, d_{um_u})$. For the simplicity of notation, (d_{ij}) denote the integers $(d_{11}, \ldots, d_{1m_1}; \ldots; d_{u1}, \ldots, d_{um_u})$ with $d_{im_i} < m_{i+1}$ for $1 \le i < u$.

In the following remark, we give an algorithmic way for calculating the Hilbert functions of k-configurations in $\mathbf{P}^3$.

Remark 3.10 (1) All k-configurations in $\mathbf{P}^3$ of type (d_{ij}) have the same Hilbert function, which will be denoted by $H^{(d_{ij})}$. $H^{(d_{ij})}$ can be obtained as follows. Let $\varrho(i)$ be the Hilbert function of a k-configuration of type $(d_{i1}, \ldots, d_{im_i})$. Then, following the same way as in the Remark 3.6 (1), we obtain

$$H^{(d_{ij})}(i) = \sum_{j=1}^{u} \varrho(d_j)(j+i-u).$$

Hence it turns out that

$$H^{(d_{ij})}(i) = \mid X \mid \quad \text{for all } \ i \gg 0,$$
$$\min\{i \mid H^{(d_{ij})}(i) = \mid X \mid\} = d_{um_u} - 1$$
$$\text{and} \quad c(X) = d_{um_u} - 1.$$

(2) Let $b = (b_0, b_1, b_2, \ldots)$ be a 0-dimensional differentiable O-sequence with $b_1 = 4$, and we put

$$\alpha(b) = \min\{i \mid b_i < \binom{3+i}{i}\},$$
$$\beta(b) = \min\{i \mid b_i - \binom{2+i}{i} > b_{i+1} - \binom{2+i+1}{i+1}\}$$
$$\text{and} \quad \gamma(b) = \min\{i \mid b_i = b_{i+1}\}\ .$$

Then by the construction in [16, Theorem 4.1], there exist integers $(d_{11}, \ldots, d_{1m_1}; \ldots; d_{u1}, \ldots, d_{um_u})$ with $u = \alpha(b)$, $m_u = \beta(b) + 1$ and $d_{um_u} = \gamma(b) + 1$ such that $H^{(d_{ij})}(i) = b_i$ for all $i \geq 0$. The proof of [16, Theorem 4.1] gives a process of calculating the integers (d_{ij}) associated with the sequence b. Hence if X is a k-configuration in $\mathbf{P}^3$ of type (d_{ij}), then we have $H(X, i) = b_i$ for all $i \geq 0$.

Proposition 3.11 *Let $h = (h_0, \ldots, h_s)$ be an SI-sequence with $h_1 = 4$, and put*

$$\alpha(h) = \min\{i \mid h_i < \binom{3+i}{i}\},$$
$$\beta(h) = \min\{i \mid h_i - \binom{2+i}{i} > h_{i+1} - \binom{2+i+1}{i+1}\}$$
$$\text{and} \quad \gamma(h) = \min\{i \mid h_i \geq h_{i+1}\}.$$

If $\alpha(h) + \beta(h) + \gamma(h) \leq s + 2$, then h is a Gorenstein sequence.

Proof. Let $b = \{b_i\}$ be the sequence associated with h. Then by Remark 3.10 (2), there exist integers $(d_{11}, \ldots, d_{1m_1}; \ldots; d_{u1}, \ldots, d_{um_u})$ with $u = \alpha(b), m_u = \beta(b) + 1$ and $d_{um_u} = \gamma(b) + 1$ such that $H^{(d_{ij})}(i) = b_i$ for all $i \geq 0$. On the other hand, put $e_1 = u$, $e_2 = \beta(b) + 1$ and $e_3 = s + 3 - \alpha(b) - \beta(b)$. Noting that $\alpha(h) = \alpha(b), \beta(h) = \beta(b)$ and $\gamma(h) = \gamma(b)$, it follows from the assumption that $\alpha(h) + \beta(h) + \gamma(h) \leq s + 2$, $m_u \leq e_2$ and $d_{um_u} \leq e_3$. Hence it is clear that there exist a k-configuration X of type (d_{ij}) in $\mathbf{P}^3$ and a finite set Y of points in $\mathbf{P}^3$ such that $X \cap Y = \phi$ and $X \cup Y = C.I.(e_1, e_2, e_3)$. Next we note that $c(X) = d_{um_u} - 1 = \gamma(h)$ and $c(X \cup Y) = e_1 + e_2 + e_3 - 3 = s + 1$. Therefore from $\gamma(h) \leq [s/2]$, i.e., $2\gamma(h) \leq s$, we obtain $2c(X) \leq c(X \cup Y) - 1$. Hence it follows from Theorem 3.2 (4) that $H(R/I(X) + I(Y), i) = b_i$ for all $0 \leq i \leq s - \gamma(h)$. That is, $H(R/I(X) + I(Y), i) = b_i = h_i$ for all $0 \leq i \leq [s/2]$, because $[s/2] \leq s - \gamma(h)$. Thus since $c(X \cup Y) - 3 = s - 1$, we conclude $H(R/I(X) + I(Y), i) = h_i$ for all $0 \leq i \leq s$. Q.E.D.

Corollary 3.12 *Let $h = (h_0, h_1, \ldots, h_\gamma, \ldots, h_{\gamma+\theta-1}, h_{\gamma+\theta}, \ldots, h_s)$ be an SI-sequence with $h_1 = 4$ and $h_0 < \cdots < h_\gamma = \cdots = h_{\gamma+\theta-1} > \cdots > h_s$. If $\alpha(h) + \beta(h) \leq \gamma(h) + \theta + 1$, $\alpha(h) \leq \theta + 1$ or $\beta(h) \leq \theta$, then h is a Gorenstein sequence.*

Proof. In this case, it is clear from Definition 3.1 of the SI-sequences that $s = 2\gamma + \theta - 1$. Since $\gamma = \gamma(h)$, it is also obvious that the two conditions "$\alpha(h) + \beta(h) + \gamma(h) \leq s + 2$" and "$\alpha(h) + \beta(h) \leq \gamma(h) + \theta + 1$" are equivalent. Hence by noting $\alpha(h) \leq \gamma(h) + 1$ and $\beta(h) \leq \gamma(h)$, it is easy to show our claim. Q.E.D.

The following example shows that there exists a Gorenstein SI-sequence with $h_1 = 4$ which can not be constructed by Proposition 3.11.

Example 3.13 When $n = 4, 5, \ldots, 10$, it is easy to show that $h = (1, 4, n, 4, 1)$ is an SI-sequence. Then

$$\alpha(h) = \begin{cases} 2 & (n \neq 10) \\ 3 & (n = 10) \end{cases}, \quad \beta(h) = \begin{cases} 1 & (n = 4, 5, 6) \\ 2 & (n = 7, 8, 9, 10) \end{cases} \quad \text{and} \quad \gamma(h) = \begin{cases} 1 & (n = 4) \\ 2 & (n \neq 4)\ . \end{cases}$$

Therefore

$$\alpha(h) + \beta(h) + \gamma(h) = \begin{cases} \text{less than } 7 & (n \neq 10) \\ 7 & (n = 10)\ . \end{cases}$$

Hence when $n = 4, \ldots, 9$, it follows by Proposition 3.11 that $(1, 4, n, 4, 1)$ is a Gorenstein sequence. But $(1, 4, 10, 4, 1)$ is also a Gorenstein sequence. In fact, we put

$$A = \frac{\mathbf{C}[x, y, z, w]}{\begin{pmatrix} x^3, & x^2z, & x^2w, & y^3 \\ y^2z, & y^2w, & xz^2, & yz^2 \\ z^3, & xw^2, & yw^2, & w^3 \\ x^2y - yzw, & xy^2 - xzw, & z^2w - xyw, & zw^2 - xyz \end{pmatrix}}.$$

Then we have $F(A, \lambda) = 1 + 4\lambda + 10\lambda^2 + 4\lambda^3 + \lambda^4$ and

$$A = k \oplus \overline{R_1} \oplus \overline{R_2} \oplus (k\overline{xyz} \oplus k\overline{xyw} \oplus k\overline{xzw} \oplus k\overline{yzw}) \oplus k\overline{xyzw}\ .$$

Next we show that the standard graded algebra A is Gorenstein. It is enough to show that

$$Soc(A) = k\overline{xyzw}, \text{ where } Soc(A) = \{a \in A \mid \overline{x}a = 0, \overline{y}a = 0, \overline{z}a = 0 \text{ and } \overline{w}a = 0\ \}.$$

It is easy to check that $Soc(A) = k\overline{xyzw}$, so we omit the proof.

In the following section, we go further and show how to construct Artinian Gorenstein algebras whose Hilbert functions are a given SI-sequence with any codimension. The notion of "weak Stanley property" for Artinian graded algebras, which is introduced in [41] by J. Watanabe, plays an important role in a process of constructing these Artinian Gorenstein algebras. One of the key ideas for finding our construction is to make use of the technique of the proof of [41, Theorem 3.8].

4 Artinian Gorenstein algebras with the weak Stanley property

In this section we study Hilbert functions of Artinian Gorenstein algebras with the following property. For details of the background and motivation concerning this property, see [29], [41] and [42].

Definition 4.1 (cf. J. Watanabe [41]). Let $A = \bigoplus_{i=0}^{c} A_i$ be an Artinian algebra, where $A_c \neq (0)$. We say that A has the *weak Stanley property* (WSP for short) if A satisfies the following two conditions:

(i) The Hilbert function of A is unimodal,

(ii) There exists $g \in A_1$ such that the k-vector space homomorphism $g : A_i \longrightarrow A_{i+1}$ defined by $f \longmapsto gf$ is either injective or surjective for every $0 \leq i \leq c-1$.

In this case, we say that the pair (A, g) has the WSP.

The main purpose of this section is to prove the following theorem which is the main result of the paper [22].

Theorem 4.2 *Let $h = (h_0, h_1, \ldots, h_s)$ be a sequence of positive integers. Then h is the Hilbert function of an Artinian Gorenstein algebra with the WSP if and only if h is an SI-sequence.*

First we recall some basic properties of Hilbert functions of points in $\mathbf{P}^n$. The following lemma is clear from Lemma 2.5 and [16], so we omit the proof.

Lemma 4.3 *Let X be a finite set of points in $\mathbf{P}^n$.*

(1) $H(X,i) < H(X,i+1)$ *for all* $0 \leq i < c(X)$.

(2) $H(X,i) = \mid X \mid$ *for all* $i \geq c(X)$, *where* $\mid X \mid$ *denote the number of points in* X.

(3) *If* $Y \subset X$, *then* $c(Y) \leq c(X)$.

(4) $\mid X \mid \geq 2 \Longleftrightarrow c(X) \geq 1$.

(5) $\mathrm{Min}\{i \mid \Delta H(X,i) = 0\} = c(X) + 1$.

(6) $(\Delta H(X,0), \ldots, \Delta H(X, c(X))$ *is an O-sequence.*

4-1 A construction of a number of Artinian Gorenstein algebras with the weak Stanley property

We prepare the following four lemmas which are the key to the proof of Theorem 4.2. That is, for two given geometrically linked sets of points, we give a construction of a number of

Artinian Gorenstein algebras with the WSP whose Hilbert functions can be recovered from the Hilbert functions of the given sets.

Lemma 4.4 *Let X and Y be two finite sets of points in $\mathbf{P}^n$ such that $X \cap Y = \phi$ and $X \cup Y$ is complete intersection, and put $A = R/I(X) + I(Y)$. Furthermore put $a = c(X)$, $b = c(X \cup Y) - c(X) - 1$ and $c = c(X \cup Y) - 1$. Assume that $2c(X) \leq c(X \cup Y) - 1$ and $\mid X \mid \geq 2$. Then $H(A)$ is a Gorenstein SI-sequence as follows.*

$$(4.4.1) \qquad H(A,i) = \begin{cases} H(X,i) & \text{for all } 0 \leq i \leq a-1, \\ \mid X \mid & \text{for all } a \leq i \leq b, \\ H(X, c-i) & \text{for all } b+1 \leq i \leq c, \end{cases}$$

i.e., $H(A) = (1, h_1, \ldots, h_{a-1}, \mid X \mid, \ldots, \mid X \mid, h_{a-1}, \ldots, h_1, 1)$, where $h_i = H(X,i)$, and we have $c(A) = c(X \cup Y) - 1$.

Proof. It follows from Lemma 2.3 that the given X and Y are 0-dimensional closed subschemes of $\mathbf{P}^n$ which are geometrically linked. Hence Theorem 3.2 (4) implies the equality (4.4.1). Furthermore $H(A)$ is an SI-sequence. Q.E.D.

Lemma 4.5 *With the same notation as in Lemma 4.4, let $L \subset \mathbf{P}^n$ be a hyperplane defined by a polynomial $G \in R_1$, and let $g \in A_1$ be the image of G. Assume that $2c(X) \leq c(X \cup Y) - 1$ and $X \cap L = \phi$. Then (A, g) has the WSP.*

Proof. It is enough to show that $g : A_i \longrightarrow A_{i+1}$ is either injective or surjective for every i. Put $B = R/I(X) = \oplus_{i \geq 0} B_i$, and let $\overline{G} \in B_1$ be the image of G. Consider the following commutative diagram:

$$\begin{array}{ccccccccccc} B_0 & \xrightarrow{\overline{G}} & B_1 & \xrightarrow{\overline{G}} & \cdots & \xrightarrow{\overline{G}} & B_c & \xrightarrow{\overline{G}} & B_{c+1} & \cdots \\ \varphi \downarrow & & \varphi \downarrow & & & & \varphi \downarrow & & \varphi \downarrow & \\ A_0 & \xrightarrow{g} & A_1 & \xrightarrow{g} & \cdots & \xrightarrow{g} & A_c & \xrightarrow{g} & 0 & \cdots \end{array}$$

where φ is the canonical homomorphism $B \longrightarrow A$. It follows immediately from $X \cap L = \phi$ that $\overline{G}$ is not a zero-divisor in B . Therefore since $\overline{G} : B_i \longrightarrow B_{i+1}$ is injective for all i, we have from Lemma 4.3 (2) that $H(B,i) = \mid X \mid$ for all $i \geq a$. Hence $\overline{G} : B_i \longrightarrow B_{i+1}$ is bijective for all $i \geq a$. Furthermore we get from (4.4.1) that $H(A,i) = H(B,i)$ for all $0 \leq i \leq b$. Hence since the homogeneous part of the canonical homomorphism $\varphi : B \longrightarrow A$ is surjective, $\varphi : B_i \longrightarrow A_i$ is bijective for all $0 \leq i \leq b$. Thus it follows immediately that

$$(4.5.1) \qquad g : A_i \longrightarrow A_{i+1} \text{ is } \begin{cases} \text{injective} & \text{for all } 0 \leq i \leq a-1, \\ \text{bijective} & \text{for all } a \leq i \leq b-1, \\ \text{surjective} & \text{for all } b \leq i \leq c. \end{cases}$$

Q.E.D.

Lemma 4.6 *With the same notation as in Lemma 4.5, let d be an integer such that $1 \le d \le c(X \cup Y) - 1 - 2c(X)$ and let $(0 : g^d)$ denote the homogeneous ideal generated by homogeneous elements $f \in A$ such that $g^d f = 0$. Then $H(A/(0 : g^d))$ is a Gorenstein SI-sequence as follows.*

$$(4.6.1) \quad H(A/(0:g^d), i) = \begin{cases} H(X,i) & \text{for all } 0 \le i \le a-1, \\ |X| & \text{for all } a \le i \le b-d, \\ H(X, c-i-d) & \text{for all } b+1-d \le i \le c-d, \end{cases}$$

and $c(A/(0:g^d)) = c(X \cup Y) - 1 - d$.

PROOF. Put $\overline{A} = A/(0 : g^d)$. Note that the i-th graded piece of $\overline{A}$ is $A_i/\ker[g^d : A_i \longrightarrow A_{i+d}]$. Since $1 \le d \le c(X \cup Y) - 1 - 2c(X)$, i.e., $a \le b - d$, we obtain from (4.5.1) that

$$g^d : A_i \longrightarrow A_{i+d} \text{ is } \begin{cases} \text{injective} & \text{for all } 0 \le i \le b-d, \\ \text{surjective} & \text{for all } i \ge b+1-d. \end{cases}$$

Therefore we get the following identification

$$(4.6.2) \quad \overline{A} \cong A_0 \oplus A_1 \oplus \cdots \oplus A_{b-d} \oplus A_{b+1} \oplus A_{b+2} \oplus \cdots \oplus A_c \oplus 0 \oplus \cdots.$$

Obviously $c(\overline{A}) = c - d = c(X \cup Y) - d$ and

$$H(\overline{A}, i) = \begin{cases} H(A,i) & \text{for all } 0 \le i \le b-d, \\ H(A, i+d) & \text{for all } b+1-d \le i \le c-d. \end{cases}$$

Hence, from (4.4.1), we are led to the equality (4.6.1). Thus $H(\overline{A})$ is an SI-sequence.

Next we check that $\overline{A}$ is Gorenstein. Put $Soc(\overline{A}) = \{\overline{y} \in \overline{A} \mid \overline{A}_1 \overline{y} = (0)\}$, where $\overline{y}$ is the image of $y \in A$. We note that $c(\overline{A}) = c - d$ and $\dim_k(\overline{A})_{c-d} = 1$. It is enough to show that $Soc(\overline{A}) = (\overline{A})_{c-d}$. Let $\overline{y} \in \overline{A}_i$ $(y \in A_i, i < c-d)$ be an element such that $\overline{y} \in Soc(\overline{A})$. Then $A_1 y \subset (0 : g^d)$. Since $Soc(A) = A_c$, we have $yg^d \in A_c$. On the other hand, $yg^d \in A_{i+d}$. Since $i + d < c$, we get $yg^d = 0$, i.e., $\overline{y} = 0$. Thus $Soc(\overline{A}) = (\overline{A})_{c-d}$. Q.E.D.

Lemma 4.7 *With the same notation as in Lemma 4.6, let $\overline{g}$ be the image of g in $A/(0 : g^d)$. Then $(A/(0 : g^d), \overline{g})$ is an Artinian Gorenstein algebra with the WSP.*

PROOF. It is enough to show that $\overline{g} : \overline{A}_i \longrightarrow \overline{A}_{i+d}$ is either injective or surjective for every i. Noting the identification (4.6.2), it is easy to show that the multiplication $\overline{g} : \overline{A} \longrightarrow \overline{A}$ is described as follows

$$A_0 \xrightarrow{g} A_1 \xrightarrow{g} \cdots \xrightarrow{g} A_{b-d} \xrightarrow{g^{d+1}} A_{b+1} \xrightarrow{g} A_{b+2} \xrightarrow{g} \cdots \xrightarrow{g} A_c.$$

Therefore the only part which is not clear is $A_{b-d} \xrightarrow{g^{d+1}} A_{b+1}$. But, by using (4.5.1), we have $A_{b-d} \xrightarrow{g^{d+1}} A_{b+1}$ is surjective, because $b - d \ge a$. Q.E.D.

Remark 4.8 Let X be a finite set of points in $\mathbf{P}^n$ and let j be an integer. Then it is easy to construct a finite set Y of points in $\mathbf{P}^n$ such that $X \cap Y = \phi$, $X \cup Y$ is complete intersection and $c(X \cup Y) \ge j$. For example we construct Y as follows. We may assume that $X \cap L = \phi$, where L is the hyperplane defined by the equation $x_0 = 0$. Obviously there exist distinct elements $a_{i,j} \in k$ $(1 \le i \le n,\ 1 \le j \le m)$ such that $X \subset Z$, where $Z = \{[1; a_{1,j}; \ldots; a_{n,j}] \mid 1 \le j \le m\}$. Then it turns out that Z is complete intersection and $c(Z) = nm - n$. Furthermore for a sufficiently large m, we get $c(Z) \ge j$. We put $Y = \{P \in Z \mid P \notin X\}$. Then Y satisfy the conditions above.

4-2 A characterization of Hilbert functions of Artinian Gorenstein algebras with the weak Stanley property

We now start to prove Theorem 4.2.

PROOF OF THEOREM 4.2. Assume that h is the Hilbert function of an Artinian Gorenstein algebra (A, g) with the WSP. Then the Hilbert function of A/gA is the sequence $(h_0, h_1 - h_0, h_2 - h_1, \ldots, h_t - h_{t-1})$, where $h_i = H(A, i)$ and $t = \min\{i \mid H(A,i) \ge H(A, i+1)\}$. Furthermore from [36, Theorem 4.1], h is symmetric. Hence $h = H(A)$ is an SI-sequence.

Conversely assume that h is an SI-sequence.

If $h_1 = 1$, then it is easy to show that $h_i = 1$ for all $0 \le i \le s$. Hence h is the Hilbert function of $A = k[x_0]/(x_0^{s+1})$ which is an Artinian Gorenstein algebra with the WSP.

Assume $h_1 \ge 2$. We put $t = \min\{i \mid h_i \ge h_{i+1}\}$. Hence $h = (1, h_1, \ldots, h_{t-1}, h_t, \ldots, h_t, h_{t-1}, \ldots, h_1, 1)$. Furthermore, by virtue of the proof of [16, Theorem 4.1], there exists a finite set X of points in $\mathbf{P}^{h_1 - 1}$ whose Hilbert function is the 0-dimensional differentiable O-sequence associated with h. We may assume that $X \cap L = \phi$, where L is the hyperplane defined by the equation $x_0 = 0$. Furthermore by Remark 4.8, there exists a finite set Y of points in $\mathbf{P}^n$ such that $X \cap Y = \phi$, $X \cup Y$ is complete intersection and $c(X \cup Y) \ge s + 1$. Since $2t \le s$ and $c(X) = t$, it follows that $2c(X) \le c(X \cup Y) - 1$. Put $A = R/I(X) + I(Y)$ and $d = c(X \cup Y) - s - 1$, and let g be the image of x_0 in A. Note that $d \ge 0$. Furthermore, we put $a = c(X) - 1$, $b = c(X \cup Y) - c(X) - 1$ and $c = c(X \cup Y) - 1$. Then, by adding Lemma 4.4 and Lemma 4.6, we get

$$H(A/(0:g^d), i) = \begin{cases} H(X,i) & \text{for all } 0 \le i \le a-1, \\ |X| & \text{for all } a \le i \le b-d, \\ H(X, c-i-d) & \text{for all } b+1-d \le i \le c-d. \end{cases}$$

where $g^0 = 1$. It is easy to show that $a \le [s/2] \le b - d$. Therefore $H(A/(0 : g^d), i) = H(X, i) = b_i = h_i$ for all $0 \le i \le [s/2]$. Hence since $H(A/(0 : g^d))$ is symmetric and $c - d = s$, it follows immediately that $H(A/(0 : g^d)) = h$. Furthermore by Lemma 4.5 and Lemma 4.7, $A/(0 : g^d)$ is an Artinian Gorenstein algebra with the WSP. Q.E.D.

Remark 4.9 In the final of this section, we pose a few conjectures concerning "unimodality" for Gorenstein sequences and "weak Stanley property" for Artinian Gorenstein algebras.

We conjecture the following which remains open as an important question on characterization of unimodal Gorenstein sequences.

CONJECTURE. Unimodal Gorenstein sequences are always SI-sequences.

It is not known whether there exists a Gorenstein sequence which is not an SI-sequence. But we believe that unimodal Gorenstein sequences are always SI-sequences.

Now it is clear that the Hilbert functions of Artinian Gorenstein algebras with the weak Stanley property are always unimodal. Therefore it is natural to ask a question of whether Artinian Gorenstein algebras with unimodal Hilbert functions have the weak Stanley property. H. Ikeda [29, Example 4.4] constructed an example of a unimodal Artinian Gorenstein algebra of codimension four not having the weak Stanley property. Here we note that the Hilbert function of this example is an SI-sequence. But the case of codimension three is still open now. That is, it is not known whether there exists an Artinian Gorenstein algebra of codimension three not having the weak Stanley property. Furthermore we are much interested in the following question for the case of any codimension.

QUESTION. What sort of Artinian Gorenstein algebras have the weak Stanley property ?

There are a few answers to this question (see [28], [41], [42] and [43], for details).

CONJECTURE. The Artinian Gorenstein algebras arising from the construction of Lemma 2.4 have the weak Stanley property.

If this conjecture is true, then the following conjecture is also true. Because it follows easily that every complete intersection ideal can be obtained as a sum of two geometrically linked Cohen-Macaulay ideals.

CONJECTURE. Complete intersections have the weak Stanley property.

Another tempting conjecture (cf. [18, Remark 2.3]) is that for any Artinian Gorenstein ideal $I \subset R$ (i.e., R/I is Artinian Gorenstein) with an SI-sequence, there exists a Cohen-Macaulay ideal I' such that

$$I' \subset I, \ \dim R/I' = 1 \quad \text{and} \quad H(R/I', i) = H(R/I, i)$$

for all $0 \le i \le [c(R/I)/2]$. But, while this conjecture seems to be true, it is not true in general. One checks easily that most complete intersections offer counterexamples. On the other hand, there are some affirmative answers to this conjecture for the case of codimension three (see [28]).

5 Artinian Gorenstein algebras of codimension three

In this section, we give a new method of an explicit construction of Artinian Gorenstein algebras achieving all possible graded Betti numbers for codimension three, which is the major part of the paper [23]. Furthermore we add an important observation to this construction, that is, we show that for any Artinian Gorenstein algebra of codimension three, there exists an Artinian Gorenstein algebra with the weak Stanley property which has the same Hilbert function. Consequently we give another proof of the famous theorem of Stanley [36, Theorem 4.2] which gives a characterization of Gorenstein sequences for codimension three.

5-1 The possible diagonal degrees

When I is a Gorenstein homogeneous ideal of height three in $R = k[x_0, x_1, \ldots, x_n]$, it is well-known that a graded minimal free resolution of $A = R/I$ has the form

$$0 \longrightarrow R(-s) \longrightarrow \bigoplus_{i=1}^{2m+1} R(-p_i) \longrightarrow \bigoplus_{i=1}^{2m+1} R(-q_i) \longrightarrow R(0) \longrightarrow A \longrightarrow 0,$$

where $q_1 \le \cdots \le q_{2m+1}$ and $p_1 \ge \cdots \ge p_{2m+1}$ (cf. [8]). Here we call this sequence

$$\{q_1, \ldots, q_{2m+1}; p_1, \ldots, p_{2m+1}; s\}$$

the *numerical characters* of A, and in particular, we put $s(A) = s$.

Definition 5.1 (cf. Buchsbaum and Eisenbud [8]). With the notation as above, we define a new sequence $\{r_i\}$, where

$$r_i = p_i - q_i \quad \text{for all} \quad 1 \le i \le 2m+1.$$

We call this sequence the *diagonal degrees* of A.

We recall a relation between the possible numerical characters and the possible diagonal degrees.

It follows from [8, page 466] that the diagonal degrees of A completely determine the numerical characters of A, that is,

(BE1) $s = \sum_{i=1}^{2m+1} r_i$,

(BE2) $q_i = \frac{1}{2}(s - r_i) = \frac{1}{2}\sum_{j \ne i} r_j$,

(BE3) $p_i = s - q_i = \frac{1}{2}(s + r_i)$.

Therefore there is a one-to-one correspondence between the possible numerical characters and the possible diagonal degrees. Hence it turns out that there is a one-to-one correspondence between the possible graded Betti numbers and the possible diagonal degrees.

Furthermore it follows from [14, Proposition 3.1] that the diagonal degrees of A must satisfy the following three conditions:

(D1) $r_1 \geq r_2 \geq \cdots \geq r_{2m+1}$,

(D2) the integers r_i are all even or all odd,

(D3) $r_1 > 0$, $r_2 + r_{2m+1} > 0$, $r_3 + r_{2m} > 0, \ldots, r_{m+1} + r_{m+2} > 0$.

Conversely, every sequence of integers satisfying the conditions (D1), (D2) and (D3) is realized as the diagonal degrees of a Gorenstein ideal of height three. This fact follows, for example, from [14, Proposition 3.1], [18, Theorem 2.1] and [25, Section 5].

5-2 A construction of Artinian Gorenstein algebras achieving all possible Betti numbers for codimension three

We prepare the following notation and definitions to state Theorem 5.4 which is one of the main results of this section.

Let $R = k[x, y, z]$ be the homogeneous coordinate ring of $\mathbf{P}^2$. We consider the following finite sets of points which are in position of lattice points in $\mathbf{P}^2$.

Definition 5.2 (1) A finite set X of points in $\mathbf{P}^2$ is called a *basic configuration of type* (d, e) if there exist distinct elements b_j, c_j in k such that

$$I(X) = (\prod_{j=1}^{d}(x - b_j z), \prod_{j=1}^{e}(y - c_j z)).$$

We write $X = B(d, e)$. Obviously $B(d, e)$ is complete intersection and $| B(d, e) |= de$.

(2) A finite set X of points in $\mathbf{P}^2$ is called a *pure configuration* if there exist finite basic configurations $B(d_1, e_1), \ldots, B(d_m, e_m)$, where $e_1 > \cdots > e_m$, which satisfy the following three conditions:

(i) $B(d_i, e_i) \cap B(d_j, e_j) = \phi$ if $i \neq j$,

(ii) $X = B(d_1, e_1) \cup \cdots \cup B(d_m, e_m)$,

(iii) $\varphi(B(d_i, e_i)) \supset \varphi(B(d_{i+1}, e_{i+1}))$ for all $1 \leq i \leq m-1$, where $\varphi : \mathbf{P}^2 \backslash \{(1, 0, 0)\} \to \mathbf{P}^1$ is the map defined by sending the point (x, y, z) to the point (y, z).

In this case, we write $X = \bigcup_{i=1}^{m} B(d_i, e_i)$.

Notation and Definition 5.3 For a sequence $\{r_1, \ldots, r_{2m+1}\}$ of integers satisfying the conditions (D1), (D2) and (D3) above, we define the following integers:

$$\begin{aligned}
d_i &= \frac{1}{2}(r_{m+2-i} + r_{m+1+i}) \quad \text{for all } 1 \leq i \leq m, \\
d_{m+1} &= \frac{1}{2}(r_1 + r_{m+1}), \\
e_m &= \frac{1}{2}(r_1 + r_{2m+1}), \\
e_i - e_{i+1} &= \frac{1}{2}(r_{m+1-i} + r_{m+1+i}) \quad \text{for all } 1 \leq i \leq m-1, \\
e &= e_1, \\
d &= \sum_{i=1}^{m+1} d_i.
\end{aligned}$$

It follows from the condition (D2) that all of d_i and e_i are integers. Furthermore we can check from the conditions (D1) and (D3) that

$$d_i > 0 \text{ for all } i, \text{ and } e_1 > e_2 > \cdots > e_m > 0.$$

Now there are a number of pairs

$$(X = \bigcup_{i=1}^{m} B(d_i, e_i), \ B = B(d, e))$$

of pure and basic configurations such that $X \subset B$. For such pairs (X, B), we put

$$Y = \{P \in B \mid P \notin X\},$$

and we consider the pairs (X, Y). We call such pairs (X, Y) the *G-pairs* of $\{r_i\}$.

We state the main theorem of this subsection.

Theorem 5.4 *Let (X, Y) be a G-pair of a sequence $\{r_1, \ldots, r_{2m+1}\}$ of integers satisfying the conditions* (D1), (D2) *and* (D3), *and put $A = R/I(X) + I(Y)$. Then the diagonal degrees of the Artinian Gorenstein algebra A is equal to the given sequence $\{r_i\}$.*

Thus we can produce our Gorenstein algebras as the sums of the ideals of two geometrically linked pure configurations formulated completely in terms of the given diagonal degrees.

In order to prove this theorem, we prepare a lemma.

Notation. Let $X = \bigcup_{i=1}^{m} B(d_i, e_i)$ be a pure configuration. Then there exist elements b_j, c_j in k such that

$$I(B(d_i, e_i)) = (\prod_{j=v_{i-1}+1}^{v_i} (x - b_j z), \prod_{j=1}^{e_i}(y - c_j z)),$$

where $v_0 = 0$ and $v_i = d_1 + \cdots + d_i$ for all $1 \le i \le m$. We put

$$g_i = \prod_{j=v_{i-1}+1}^{v_i} (x - b_j z) \quad \text{and} \quad h_i = \prod_{j=e_{i+1}+1}^{e_i} (y - c_j z)$$

for all $1 \le i \le m$, where $e_{m+1} = 0$. Note that $\deg g_i = d_i$ and $\deg h_i = e_i - e_{i+1}$ for all i. Furthermore let $B = B(d, e)$ be a basic configuration such that $d > \sum_{i=1}^{m} d_i$, $e = e_1$ and $X \subset B$. Obviously there exist elements b_j $(v_m + 1 \le j \le d)$ in k such that

$$I(B(d,e)) = (\ \prod_{j=1}^{d}(x - b_j z),\ \prod_{j=1}^{e}(y - c_j z)\ ).$$

We put

$$g_{m+1} = \prod_{j=v_m+1}^{d} (x - b_j z) \quad \text{and} \quad d_{m+1} = d - \sum_{i=1}^{m} d_i.$$

In the following lemma, we describe a set of minimal generators of an Artinian Gorenstein ideal of height three which is constructed as the sum of the ideals of two geometrically linked pure configurations in $\mathbf{P}^2$.

Lemma 5.5 *With the notation as above, we put* $Y = \{P \in B \mid P \notin X\}$.

(1) $I(X)$ *is minimally generated by the* $(m+1)$ *maximal minors of the* $m \times (m+1)$ *matrix* $U = (u_{ij})$ *as follows:*

$$U = \begin{pmatrix} g_1 & h_1 & & & \text{O} \\ & g_2 & h_2 & & \\ & & \ddots & \ddots & \\ \text{O} & & & g_m & h_m \end{pmatrix}.$$

(2) $I(X) + I(Y)$ *is an Artinian Gorenstein ideal of height three, minimally generated by the* $(2m+1)$ *pfaffians of the* $(2m+1) \times (2m+1)$ *alternating matrix* $M = (f_{ij})$ *as follows: For* $i \le j$,

$$f_{ij} = \begin{cases} u_{it} & \text{if } 1 \le i \le m \text{ and } j = m+t \text{ for } 1 \le t \le m+1, \\ g_{m+1} & \text{if } i = m+1 \text{ and } j = 2m+1, \\ 0 & \text{otherwise.} \end{cases}$$

PROOF. (1) The set of all maximal minors of U is

$$B = \{h_1 h_2 \cdots h_m,\ g_1 h_2 \cdots h_m,\ g_1 g_2 h_3 \cdots h_m,\ \ldots\ ,\ g_1 \cdots g_{m-1} h_m,\ g_1 g_2 \cdots g_m\}.$$

Hence we show that $I(X)$ is minimally generated by B. Let I be the ideal generated by B. Furthermore we consider the monomial ideal J in $S = k[x, y]$ generated by the $(m+1)$ maximal minors of the following matrix:

$$\begin{pmatrix} x^{d_1} & y^{e_1 - e_2} & & & \text{O} \\ & x^{d_2} & y^{e_2 - e_3} & & \\ & & \ddots & \ddots & \\ & & & x^{d_{m-1}} & y^{e_{m-1} - e_m} \\ \text{O} & & & x^{d_m} & y^{e_m} \end{pmatrix}.$$

That is , J is generated by the $(m+1)$ monomials

$$\{y^{e_1},\ x^{v_1} y^{e_2},\ x^{v_2} y^{e_3},\ \ldots\ ,\ x^{v_{m-1}} y^{e_m},\ x^{v_m}\}.$$

Since $e_1 > e_2 > \cdots > e_m > 0$ and $0 < v_1 < v_2 < \cdots < v_m$, it is easy to see that J is minimally generated by the above $(m+1)$ monomials. Moreover, it follows by virtue of the proof of [15, Theorem 2.2] that I is a lifting of J (see [15, Definition 1.7] for the definition of "lifting"). That is, I is the radical ideal, minimally generated by the $(m+1)$ maximal minors of U. And further we can easily check that

$$X = \{P \in \mathbf{P}^2 \mid F(P) = 0 \text{ for all } F \in I\}.$$

Thus we get $I = I(X)$.

(2) First of all, it follows from Lemma 2.4 that $I(X) + I(Y)$ is an Artinian Gorenstein ideal of height three.

Next we note that Y is a pure configuration. Hence similarly, it follows from (1) that $I(Y)$ is minimally generated by the $(m+1)$ maximal minors of the matrix

$$\begin{pmatrix} g_{m+1} & h_m & & & \text{O} \\ & g_m & h_{m-1} & & \\ & & \ddots & \ddots & \\ \text{O} & & & g_2 & h_1 \end{pmatrix},$$

i.e.,

$$B' = \{h_1 h_2 \cdots h_m,\ h_1 \cdots h_{m-1} g_{m+1},\ h_1 \cdots h_{m-2} g_m g_{m+1},\ \ldots\ ,\ h_1 g_3 \cdots g_{m+1},\ g_2 g_3 \cdots g_{m+1}\}.$$

Now let F_i be the pfaffian of the alternating matrix obtained by eliminating the i-th row and the i-th column from M for all $1 \le i \le 2m+1$. Then we can check that

$$F_1 = g_2 g_3 \cdots g_{m+1},\ F_2 = h_1 g_3 \cdots g_{m+1},\ \ldots\ ,\ F_m = h_1 \cdots h_{m-1} g_{m+1},$$
$$F_{m+1} = h_1 h_2 \cdots h_m,$$
$$F_{m+2} = g_1 h_2 \cdots h_m,\ \ldots\ ,\ F_{2m} = g_1 \cdots g_{m-1} h_m,\ F_{2m+1} = g_1 g_2 \cdots g_m.$$

Hence $B \cup B'$ is the set of all pfaffians of M. Thus we show that $I(X) + I(Y)$ is minimally generated by $B \cup B'$. We divide the proof of this claim into three cases.

Case 1. If $F_i \in (B \cup B' \backslash \{F_i\})R$ for some $1 \le i \le m$, then we have $F_i \in g_i R + h_i R$. Hence taking a point Q such that $g_i(Q) = h_i(Q) = 0$, we get $F_i(Q) = 0$. But obviously, $F_i(P) \ne 0$ for all $P \in \mathbf{P}^2$ such that $g_i(P) = h_i(P) = 0$. This is a contradiction.

Case 2. If $F_{m+1} \in (B \cup B' \backslash \{F_{m+1}\})R$, then we have $F_{m+1} \in g_1 R + g_{m+1} R$. Hence taking a point Q such that $g_1(Q) = g_{m+1}(Q) = 0$, we get $F_{m+1}(Q) = 0$. But obviously, $F_{m+1}(P) \ne 0$ for all $P \in \mathbf{P}^2$ such that $g_1(P) = g_{m+1}(P) = 0$. This is a contradiction.

Case 3. If $F_{m+1+i} \in (B \cup B' \backslash \{F_{m+1+i}\})R$ for some $1 \le i \le m$, then we have $F_{m+1+i} \in g_{i+1} R + h_i R$. Hence taking a point Q such that $g_{i+1}(Q) = h_i(Q) = 0$, we get $F_{m+1+i}(Q) = 0$. But obviously, $F_{m+1+i}(P) \ne 0$ for all $P \in \mathbf{P}^2$ such that $g_{i+1}(P) = h_i(P) = 0$. This is a contradiction. Q.E.D.

We now start to prove Theorem 5.4.

PROOF OF THEOREM 5.4. We prove this theorem with the notation introduced as above. From Lemma 5.5, the degrees of $\{F_i\}$ are as follows:

$$\deg F_1 = d_2 + d_3 + \cdots + d_{m+1}, \ \deg F_2 = (e_1 - e_2) + d_3 + \cdots + d_{m+1},$$
$$\deg F_3 = (e_1 - e_3) + d_4 + \cdots + d_{m+1}, \ \ldots, \ \deg F_m = (e_1 - e_m) + d_{m+1},$$
$$\deg F_{m+1} = e_1, \ \deg F_{m+2} = d_1 + e_2, \ \deg F_{m+3} = d_1 + d_2 + e_3, \ \ldots,$$
$$\deg F_{2m} = d_1 + \cdots + d_{m-1} + e_m, \ \deg F_{2m+1} = d_1 + d_2 + \cdots + d_m.$$

For the sake of convenience we put

$$G_i = \begin{cases} F_{2m+2-i} & \text{for all } 1 \le i \le m+1, \\ F_{i-(m+1)} & \text{for all } m+2 \le i \le 2m+1. \end{cases}$$

Hence from the definitions of d_i and e_i, a quick calculation gives that

$$\deg G_i = \frac{1}{2} \sum_{j \ne i} r_j = \frac{1}{2}(r_1 + \cdots + r_{i-1} + r_{i+1} + \cdots + r_{2m+1})$$

for all $1 \le i \le 2m+1$. Thus from the condition (D1) of the sequence $\{r_i\}$,

$$\deg G_1 \le \deg G_2 \le \cdots \le \deg G_{2m+1}.$$

Next we show that

$$s(A) = \sum_{i=1}^{2m+1} r_i.$$

Let

$$0 \longrightarrow R(-s(A)) \longrightarrow \bigoplus_{i=1}^{2m+1} R(-p_i) \longrightarrow \bigoplus_{i=1}^{2m+1} R(-q_i) \longrightarrow R(0) \longrightarrow A \longrightarrow 0$$

be a graded minimal free resolution of A, where

$$1 \le q_1 \le \cdots \le q_{2m+1} \quad \text{and} \quad p_1 \ge \cdots \ge p_{2m+1} \ge 1.$$

Then we have

$$(1-\lambda)^3 F(A, \lambda) = 1 - \sum_{i=0}^{2m+1} \lambda^{q_i} + \sum_{i=0}^{2m+1} \lambda^{p_i} - \lambda^{s(A)}.$$

On the other hand, it follows from the definition of Hilbert series that

$$F(A, \lambda) = \sum_{j=0}^{c(A)} H(A, j) \lambda^j.$$

Therefore

$$(1-\lambda)^3 \sum_{j=0}^{c(A)} H(A, j) \lambda^j = 1 - \sum_{i=0}^{2m+1} \lambda^{q_i} + \sum_{i=0}^{2m+1} \lambda^{p_i} - \lambda^{s(A)}.$$

Also we note that

$$s(A) = p_i + q_i \quad \text{for all } 1 \le i \le 2m+1$$

(cf. [8, page 466]), and hence we get

$$s(A) > q_i \quad \text{and} \quad s(A) > p_i \ (1 \le i \le 2m+1).$$

Thus it turns out that

$$s(A) = c(A) + 3.$$

Furthermore Theorem 3.2 (3) and Lemma 2.5 (7) imply that

$$c(A) = c(X \cup Y) - 1 = c(B(d, e)) - 1 = d + e - 3.$$

Hence we get

$$\begin{aligned} s(A) &= d + e = \{\sum_{i=1}^{m+1} d_i\} + e_1 \\ &= \{\sum_{i=1}^{m+1} d_i\} + \{\sum_{i=1}^{m-1} (e_i - e_{i+1})\} + e_m = \sum_{i=1}^{2m+1} r_i. \end{aligned}$$

Now let $\{r'_1, \ldots, r'_{2m+1}\}$ be the diagonal degrees of A. Then by noting that $\deg G_1 \le \cdots \le \deg G_{2m+1}$, it follows from the conditions (BE1) and (BE2) that

$$r'_i = s(A) - 2 \deg G_i$$

for all $1 \le i \le 2m+1$. Thus we conclude

$$r'_i = \sum_{j=1}^{2m+1} r_j - \sum_{j \ne i} r_j = r_i.$$

Q.E.D.

We illustrate this theorem with the following example.

Example 5.6 S. J. Diesel described in [14, Example 3.7] all possible numerical characters among all Artinian Gorenstein ideals with the Hilbert function

$$T = (1, 3, 6, 10, 12, 12, 10, 6, 3, 1),$$

i.e., all sequences $\{r_i\}$ of integers satisfying the conditions (D1), (D2) and (D3) which determine T:

$$\{4,4,4\};\quad \{4,4,4,2,-2\};\quad \{4,4,4,0,0\};$$
$$\{4,4,4,2,2,-2,-2\};\quad \{4,4,4,2,0,0,-2\};$$
$$\{4,4,4,2,2,0,0,-2,-2\}.$$

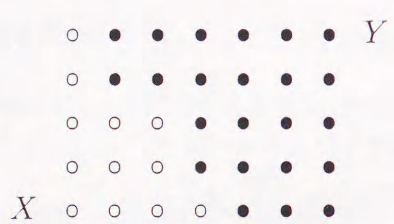
Here using our construction of Theorem 5.4, for example, we construct an example of an Artinian Gorenstein algebra with the diagonal degrees $\{4,4,4,2,0,0,-2\}$, i.e., with the numerical characters.

$$\{4,4,4,5,6,6,7;8,8,8,7,6,6,5;12\}.$$

So we put, as in Definition 5.3,

$$d_1 = 1,\quad d_2 = 2,\quad d_3 = 1,\quad d_4 = 3,\quad e_1 = 5,$$
$$e_2 = 3,\quad e_3 = 1,\quad d = 7 \quad \text{and} \quad e = 5.$$

Now, as a G-pair (X, Y) of $\{4,4,4,2,0,0,-2\}$, we take the following two pure configurations $X = B(1,5) \cup B(2,3) \cup B(1,1)$ and $Y = B(3,5) \cup B(1,4) \cup B(2,2)$ such that $X \cup Y = B(7,5)$:



Then it follows from Theorem 5.4 that $R/I(X) + I(Y)$ is an Artinian Gorenstein algebra with the diagonal degrees $\{4,4,4,2,0,0,-2\}$.

We saw that all possible resolutions for Gorenstein algebras of codimension three can be obtained by using the standard fact of linkage theory stated in Lemma 2.4. Therefore it is natural to ask whether all Gorenstein ideals (of height three) can be so obtained. We do not know any answers to this question (cf. [18, Remark 2.2]).

5-3 Artinian Gorenstein algebras defined by the saturated diagonal degrees

In this subsection, first we add an important observation to the construction of Theorem 5.4, that is, we show in Theorem 5.8 that an Artinian Gorenstein algebra defined by a set

of saturated diagonal degrees has the weak Stanley property and that the Hilbert function of the algebra is described in terms of the Hilbert functions of the corresponding G-pair.

We recall the definition of saturated diagonal degrees in [14].

Definition 5.7 (cf. Diesel [14, page 376]). For a sequence $\{r_1, \ldots, r_{2m+1}\}$ satisfying the conditions (D1), (D2) and (D3), we say that $\{r_i\}$ is *saturated* if

$$r_i + r_{2m+3-i} = 2 \quad \text{for all} \quad 2 \le i \le 2m+1.$$

We see from [14, section 3] that if the Hilbert function T is fixed, we can exhibit all sets of possible Betti numbers which determine T, as Example 5.6. Then we can easily check that the saturated diagonal degrees correspond the maximum Betti numbers among all possible Betti numbers which determine the same Hilbert function.

Theorem 5.8 *Let (X, Y) be a G-pair of a saturated sequence $\{r_1, \ldots, r_{2m+1}\}$, and put $A = R/I(X) + I(Y)$. Furthermore put $a = c(X)$, $b = c(X \cup Y) - c(X) - 1$ and $c = c(X \cup Y) - 1$. Then A has the weak Stanley property and the Hilbert function of A is recovered from the Hilbert function of X as follows:*

$$H(A,i) = \begin{cases} H(X,i) & \textit{for all} \quad 0 \le i \le a-1, \\ \mid X \mid & \textit{for all} \quad a \le i \le b, \\ H(X, c-i) & \textit{for all} \quad b+1 \le i \le c, \end{cases}$$

i.e., $H(A) = (1, h_1, \ldots, h_{a-1}, \mid X \mid, \ldots, \mid X \mid, h_{a-1}, \ldots, h_1, 1, 0, \ldots)$, where $h_i = H(X,i)$.

We need the following lemma to prove Theorem 5.8.

Lemma 5.9 *Let $X = \bigcup_{i=1}^{m} B(d_i, e_i)$ be a pure configuration.*

(1) $F(X, \lambda) = \sum_{i=1}^{m} \lambda^{v_{i-1}} \dfrac{(1-\lambda^{d_i})(1-\lambda^{e_i})}{(1-\lambda)^3}$, *where $v_0 = 0$ and $v_i = d_1 + \cdots + d_i$.*

(2) $c(X) = \operatorname{Max}\{e_i + v_i - 2 \mid 1 \le i \le m\}$.

PROOF. (1) We prove the equality by the induction on m. For the case $m = 1$, our assertion follows from Lemma 2.5 (6). Let $m > 1$. It follows from Lemma 5.5 (1) that

$$I(\bigcup_{i=1}^{m-1} B(d_i, e_i)) = (h_1 \cdots h_m, g_1 h_2 \cdots h_m, g_1 g_2 h_3 \cdots h_m, \ldots, g_1 \cdots g_{m-2} h_{m-1} h_m, g_1 \cdots g_{m-1}).$$

Therefore we have

$$I(\bigcup_{i=1}^{m-1} B(d_i, e_i)) + I(B(d_m, e_m)) = (g_1 \cdots g_{m-1}, g_m, h_m),$$

and see the following exact sequence

$$0 \longrightarrow R/I(X) \longrightarrow R/I(\bigcup_{i=1}^{m-1} B(d_i, e_i)) \oplus R/I(B(d_m, e_m)) \longrightarrow R/(g_1 \cdots g_{m-1}, g_m, h_m) \longrightarrow 0.$$

Hence

$$F(X, \lambda) = F(R/I(\bigcup_{i=1}^{m-1} B(d_i, e_i)), \lambda) + F(R/I(B(d_m, e_m)), \lambda) - F(R/(g_1 \cdots g_{m-1}, g_m, h_m), \lambda).$$

On the other hand, by the induction hypothesis,

$$F(R/I(\bigcup_{i=1}^{m-1} B(d_i, e_i)), \lambda) = \sum_{i=1}^{m-1} \lambda^{v_{i-1}} \frac{(1-\lambda^{d_i})(1-\lambda^{e_i})}{(1-\lambda)^3}.$$

Also since

$$\{P \in \mathbf{P}^2 \mid g_1 \cdots g_{m-1}(P) = g_m(P) = h_m(P) = 0\} = \phi,$$

it follows that $\{g_1 \cdots g_{m-1}, g_m, h_m\}$ is a homogeneous regular sequence. Hence by using Lemma 2.5 (6), it yields that

$$\begin{aligned} & F(R/I(B(d_m, e_m)), \lambda) - F(R/(g_1 \cdots g_{m-1}, g_m, h_m), \lambda) \\ = & \frac{(1-\lambda^{d_m})(1-\lambda^{e_m})}{(1-\lambda)^3} - \frac{(1-\lambda^{v_{m-1}})(1-\lambda^{d_m})(1-\lambda^{e_m})}{(1-\lambda)^3} \\ = & \frac{\lambda^{v_{m-1}}(1-\lambda^{d_m})(1-\lambda^{e_m})}{(1-\lambda)^3}. \end{aligned}$$

Thus we get the equality of (1).

(2) It follows from (1) that

$$F(X, \lambda) = \sum_{i=1}^{m} \lambda^{v_{i-1}} F(B(d_i, e_i), \lambda), \text{ i.e., } H(X, j) = \sum_{i=1}^{m} H(B(d_i, e_i), j - v_{i-1}).$$

Here we put

$$\tau(i) = \mathrm{Min}\{j \mid H(B(d_i, e_i), j - v_{i-1}) = \mid B(d_i, e_i) \mid\} \quad \text{for all } 1 \le i \le m.$$

Then by adding Lemma 2.5 (4) and (7), we see that

$$\tau(i) - v_{i-1} = c(B(d_i, e_i)) = d_i + e_i - 2, \text{ i.e., } \tau(i) = e_i + v_i - 2.$$

Hence we can check that

$$c(X) = \mathrm{Min}\{j \mid H(X, j) = \sum_{i=1}^{m} \mid B(d_i, e_i) \mid\} = \mathrm{Max}\{e_i + v_i - 2 \mid 1 \le i \le m\}.$$

Q.E.D.

PROOF OF THEOREM 5.8. If $2c(X) \le c(X \cup Y) - 1$, then our assertion follows from Lemmas 4.4 and 4.5. So we show that

$$2c(X) \le c(X \cup Y) - 1.$$

Since $\{r_i\}$ is saturated, we obtain $d_i = 1$ for all $1 \le i \le m$, i.e., $v_i = i$. Therefore from $e_1 > e_2 > \cdots > e_m$, it follows that $e_i + v_i - 1 \ge e_{i+1} + v_{i+1} - 1$. Hence from Lemma 5.9 (2) and the definition of e_1, we have

$$c(X) = e_1 + v_1 - 2 = e_1 - 1 = (\frac{1}{2} \sum_{i \ne m+1} r_i) - 1.$$

Furthermore from Lemma 2.5 (7) and Definition 5.3,

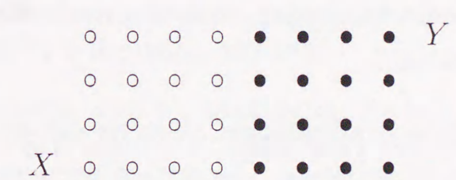
$$c(X \cup Y) - 1 = d + e - 3 = (\sum_{i=1}^{m+1} d_i) + e_1 - 3 = (\sum_{i=1}^{2m+1} r_i) - 3.$$

Also we see that $r_{m+1} > 0$, because $r_{m+1} \ge r_{m+2}$ and $r_{m+1} + r_{m+2} = 2$. Thus it yields that

$$c(X \cup Y) - 1 - 2c(X) = r_{m+1} - 1 \ge 0.$$

Q.E.D.

Remark 5.10 For any G-pair (X, Y) of any sequence $\{r_i\}$ satisfying the conditions (D1), (D2) and (D3), the author conjectures that $A = R/I(X) + I(Y)$ has the weak Stanley property. But, in general, A does not satisfy the equality concerning Hilbert functions stated in Theorem 5.8. For example, as a G-pair (X, Y) of $\{4, 4, 4\}$, we take the following:



Then it follows from Example 5.6 that

$$H(A) = (1, 3, 6, 10, 12, 12, 10, 6, 3, 1).$$

On the other hand, from Lemma 5.9 (1),

$$H(X) = (1, 3, 6, 10, 13, 15, 16, 16, \rightarrow).$$

Hence we see that $H(A)$ and $H(X)$ do not satisfy the equality of Theorem 5.8.

Now, combining Theorem 5.8 and the preceding Theorem 3.7, we can give another proof of the following famous theorem due to R. Stanley.

Theorem 5.11 (cf. R. Stanley [36, Theorem 4.2]). *Let $h = (h_0, h_1, \ldots, h_s)$ be a sequence of non-negative integers satisfying $h_1 \leq 3$. Then the following conditions are equivalent.*

(a) *There exists an Artinian Gorenstein graded algebra with the Hilbert function h.*

(b) *$h_i = h_{s-i}$ for all $0 \leq i \leq [s/2]$ and the sequence $(h_0, h_1 - h_0, h_2 - h_1, \ldots, h_{[s/2]} - h_{[s/2]-1})$ is the Hilbert function of an Artinian graded algebra.*

PROOF. (a) $\Rightarrow$ (b): We can easily check that there is a unique saturated sequence $\{r_i\}$ which determines h (cf. [14, Theorem 3.2] for example). So we take a G-pair (X, Y) of $\{r_i\}$, and put $A = k[x, y, z]/I(X) + I(Y)$. We note that $H(A) = h$. From Theorem 5.8, it is easy to see that $H(A, i) = H(A, s - i)$ for all $0 \leq i \leq [s/2]$. Furthermore from Theorem 5.8, we have that (A, g) has the weak Stanley property, that is, $(h_0, h_1 - h_0, h_2 - h_1, \ldots, h_{[s/2]} - h_{[s/2]-1})$ is the Hilbert function of A/gA.

(b) $\Rightarrow$ (a) follows from Theorem 3.7. Q.E.D.

Remark 5.12 In the proof of Theorem 5.11, we get an algebraic explanation of the essence of Stanley's formulation (in terms of the first difference) for Hilbert functions. That is, for any Artinian Gorenstein algebra of codimension three, there exists an Artinian Gorenstein algebra with the weak Stanley property which has the same Hilbert function. Therefore it is natural to ask, in view of Stanley's formulation, a question whether every Artinian Gorenstein algebra of codimension three has the weak Stanley property (cf. [28], [41] and [43]). And further we pose the following.

QUESTION. For an Artinian Gorenstein algebra with a unimodal Hilbert function, does there exist an Artinian Gorenstein algebra with the weak Stanley property which has the same unimodal Hilbert function ?

If the answer to this question is affirmative, then we can give an answer to an important question on characterization of unimodal Gorenstein sequences, that is, we conclude that all unimodal Gorenstein sequences are always SI-sequences.

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