Supplementary Material for "Linear Discriminative Image Processing Operator Analysis"

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4.2. α step

Suppose that a sample $\boldsymbol{x}$ in class ω_i is transformed to $\boldsymbol{x}^{(k)}$ by $G^{(k)}$ at the kth step as $\boldsymbol{x}^{(k)} = G^{(k)}\boldsymbol{x}$. Let

$$F^{(k)} = \sum_{l=0}^{k-1} G^{(l)}, \tag{1}$$

and

$$\bar{G}^{(k)} = \frac{1}{k+1}\sum_{l=0}^{k} G^{(l)}. \tag{2}$$

Now $D = PA$ are known, and the scatter matrices $\widetilde{S}_W'^{(k)}$, $\widetilde{S}_B'^{(k)}$ are given as follows:

$$\begin{aligned}\widetilde{S}_W'^{(k)} &= D^T\bar{G}^{(k)}\left(S_W - R_W\right)\bar{G}^{(k)^T}D \\ &\quad + \frac{1}{k+1}\sum_{l=0}^{k} D^T G^{(l)} R_W G^{(l)^T} D\end{aligned} \tag{3}$$

$$\widetilde{S}_B'^{(k)} = D^T\bar{G}^{(k)} S_B \bar{G}^{(k)^T} D \tag{4}$$

Now the criterion $E(A, P, \boldsymbol{\alpha}^{(k)})$ can be rewritten as the ratio of

$$\mathrm{tr}\left(\widetilde{S}_W'^{(k)}\right) = \boldsymbol{\alpha}^{(k)^T} H_W^{(k)} \boldsymbol{\alpha}^{(k)} + 2\boldsymbol{q}_W^{(k)^T}\boldsymbol{\alpha}^{(k)} + \pi_W^{(k)}, \tag{5}$$

$$\mathrm{tr}\left(\widetilde{S}_B'^{(k)}\right) = \boldsymbol{\alpha}^{(k)^T} H_B^{(k)} \boldsymbol{\alpha}^{(k)} + 2\boldsymbol{q}_B^{(k)^T}\boldsymbol{\alpha}^{(k)} + \pi_B^{(k)}, \tag{6}$$

where

$$H_W^{(k)} = \left\{h_{Wjl}^{(k)}\right\}, \tag{7}$$

$$h_{Wjl}^{(k)} = \frac{\mathrm{tr}\left\{D^T G_j \left(S_W + kR_W\right) {G_l}^T D\right\}}{(k+1)^2}, \tag{8}$$

$$\boldsymbol{q}_W^{(k)} = \frac{(q_{W\,1}^{(k)}, q_{W\,2}^{(k)}, \ldots, q_{W\,J}^{(k)})^T}{(k+1)^2}, \tag{9}$$

$$q_{W\,j}^{(k)} = \mathrm{tr}\left\{D^T G_j \left(S_W - R_W\right) F^{(k)^T} D\right\}, \tag{10}$$

$$\begin{aligned}\pi_W^{(k)} &= \frac{\mathrm{tr}\left\{D^T F^{(k)}\left(S_W - R_W\right) F^{(k)^T} D\right\}}{(k+1)^2} \\ &\quad + \frac{\sum_{l=0}^{k-1}\mathrm{tr}\left\{D^T G^{(l)} R_W G^{(l)^T} D\right\}}{k+1},\end{aligned} \tag{11}$$

$$H_B^{(k)} = \left\{h_{Bjl}^{(k)}\right\}, \tag{12}$$

$$h_{Bjl}^{(k)} = \frac{\mathrm{tr}\left\{D^T G_j S_B {G_l}^T D\right\}}{(k+1)^2}, \tag{13}$$

$$\boldsymbol{q}_B^{(k)} = \frac{(q_{B\,1}^{(k)}, q_{B\,2}^{(k)}, \ldots, q_{B\,J}^{(k)})^T}{(k+1)^2}, \tag{14}$$

$$q_{B\,j}^{(k)} = \mathrm{tr}\left\{D^T G_j S_B F^{(k)^T} D\right\}, \tag{15}$$

$$\pi_B^{(k)} = \frac{1}{(k+1)^2}\mathrm{tr}\left\{D^T F^{(k)} S_B F^{(k)^T} D\right\}. \tag{16}$$

Since this is not a usual form of the Rayleigh quotient, we prove the following proposition.

Proposition 3 *Let $\boldsymbol{\beta}^{(k)}$ be a $J+1$ dimensional vector $\boldsymbol{\beta}^{(k)} = (\boldsymbol{\alpha}^{(k)^T}, 1)^T$. Then, the solution that maximizes the ratio of the equations above is given by the solution to the eigenvalue problem for $Q_W^{(k)^{-1}} Q_B^{(k)}$, which maximize*

$$\frac{\boldsymbol{\beta}^{(k)^T} Q_B^{(k)} \boldsymbol{\beta}^{(k)}}{\boldsymbol{\beta}^{(k)^T} Q_W^{(k)} \boldsymbol{\beta}^{(k)}}, \tag{17}$$

where

$$Q_B^{(k)} = \begin{bmatrix} H_B^{(k)} & \boldsymbol{q}_B^{(k)} \\ \boldsymbol{q}_B^{(k)^T} & \pi_B^{(k)} \end{bmatrix}, \quad Q_W^{(k)} = \begin{bmatrix} H_W^{(k)} & \boldsymbol{q}_W^{(k)} \\ \boldsymbol{q}_W^{(k)^T} & \pi_W^{(k)} \end{bmatrix}. \tag{18}$$

Proof Let $\boldsymbol{e}$ be the eigenvector of the largest eigenvalue of $Q_W^{(k)^{-1}} Q_B^{(k)}$, and e_{last} its last element. Then, $\frac{1}{e_{\mathrm{last}}}\boldsymbol{e} = \boldsymbol{\beta}^{(k)} = (\boldsymbol{\alpha}^{(k)^T}, 1)^T$ and $\boldsymbol{\alpha}^{(k)}$ is obtained.