The Fundamentals of Economic Dynamics and Policy Analyses: Learning through Numerical Examples

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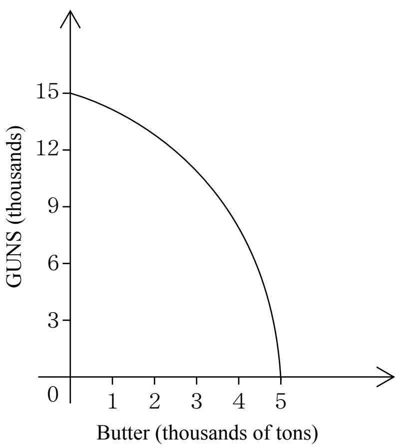
The objective of this paper is to present the fundamentals of economic dynamics and policy analyses. Since 1980s', it is widely recognized that there are two main objectives of macroeconomics; (i) to explicitly construct the micro-foundations of macroeconomic models, and (ii) to analyze the statistical properties of macroeconomic variables. Because macroeconomic variables are described as equilibrium time series sequences chosen by optimizing microeconomic agents, mathematical tools for stochastic dynamic optimization are therefore appropriate to accomplish these objectives. The economic policies in this framework must be described as time series sequences of policy variables. The policy analyses in macroeconomic models which lack explicit treatment of time will often yield misleading outcomes. This is pointed out by R. Lucas and other economists who propose the rational expectations hypothesis. In the following, the importance of dynamic analyses in macroeconomics will be explained in section 1 and section 2 through two examples; section 1 discusses economic growth, and section 2 analyzes tax policies in lifecycle models. Section 3 is an exercise for numerical dynamic policy analyses using "Mathematica", a computer software for numerical calculation.

1. Economic Growth

Figure 1-1 depicts a typical production possibility frontier (PPF) often used in rudimentary textbook of economics. (In fact, figure 1-1 is a replication of figure 2-2 of Samuelson (1980).) The horizontal axis measures the number of guns, and the vertical axis measures the tons of butter. The PPF is a locus of combination (portfolio) of goods and services that can be produced by using the most efficient combination of the factors of production at a given point in time. The adjective "efficient" means that a point on PPF is Pareto efficient. At the point, any one good or service cannot be increased without reducing the other goods and services.

Figure 1-1. The Production-Possibility Fronter

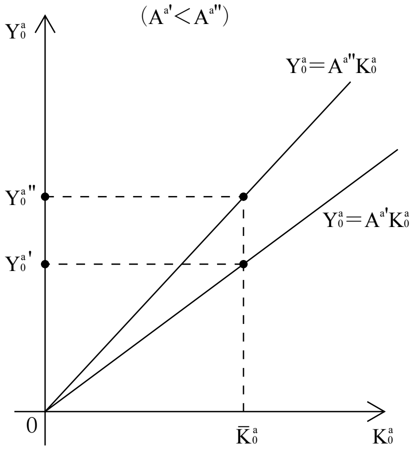


Now consider the following System of National Accounts (SNA) of two imaginary countries, *a* and *b*. The demand side of national income is defined by

$$(1.1)\quad Y_t = C_t + I_t + G_t + X_t - M_t \ ,$$

where Y_t is GDP, C_t is private consumption, I_t is private investment, G_t is public spending, X_t is export, and M_t is import. In the following example, we abstract public sector and foreign sector. In time period $t=0$, country *a* has capital stock K^a_0, and produces $Y^a_0 = A^a K^a_0$. A^a is a constant technology parameter, and K^a_0 is the sole factor of production. Figure 1-2 depicts the relationship between the capital K^a_0 and the output Y^a_0 for two values of the technology parameter, $A^{a}{'}$ and $A^{a}{''}$, such that $A^{a}{'} < A^{a}{''}$. For a given capital $\bar{K}^a_0 > 0$, larger value of the technology parameter implies larger output, i.e., $Y^a_0{'} = A^{a}{'}\bar{K}^a_0 < Y^a_0{''} = A^{a}{''}\bar{K}^a_0$.

Figure 1-2



The output Y_0^a is divided between consumption $C_0^a = (1 - s^a)\ Y_0^a$ and investment $K_1^a = s^a Y_0^a$. $s^a \in [0, 1]$ is a constant investment rate. Because country *a* is closed to international trade, s^a is also a saving rate. The capital K_0^a depreciates 100%. Therefore, the investment in $t = 0$, K_1^a, is also the capital stock in $t = 1$. In $t = 1$, country *a* produces $Y_1^a = A^a K_1^a$, consumes $C_1^a = (1 - s^a)\ Y_1^a$, and invests $K_2^a = s^a Y_1^a$. This process is repeated for $t = 0, 1, 2, \cdots$. Therefore, in any time period *t*, country *a* has capital stock K_t^a, produces $Y_t^a = A^a K_t^a$, consumes $C_t^a = (1 - s^a)\ Y_t^a$, and invests $K_{t+1}^a = s^a Y_t^a$. Similarly, the SNA of country *b* is described as follows. Given the capital stock K_0^b in the initial period $t = 0$, for $t = 0, 1, 2, \cdots$, country *b* has capital stock K_t^b, produces $Y_t^b = A^b K_t^b$, consumes $C_t^b = (1 - s^b)\ Y_t^b$, and invests $K_{t+1}^b = s^b Y_t^b$. A^b is the constant technology parameter, and s^b is the investment rate.

Suppose the two countries have same capital stock, $K_0^a = K_0^b = 50$, in the initial period $t = 0$. Furthermore, assume that they have same production technology, $A^a = A^b = 2$, but country *b* has higher investment rate than country *a*, $s^a = 0.4 < s^b = 0.6$. Then, the economic dynamics of country *a* and country *b* for $t = 0, 1, 2$ are calculated as follows.

t = 0;

Country *a*	Country *b*
$Y_0^a = A^a K_0^a$ (100 = 2 × 50)	$Y_0^b = A^b K_0^b$ (100 = 2 × 50)
$C_0^a = (1 - s^a)\, Y_0^a$ (60 = (1 − 0.4) × 100)	$C_0^b = (1 - s^b)\, Y_0^b$ (40 = (1 − 0.6) × 100)
$K_1^a = s^a Y_0^a$ (40 = 0.4 × 100)	$K_1^b = s^b Y_0^b$ (60 = 0.6 × 100)

t = 1;

Country *a*	Country *b*
$Y_1^a = A^a K_1^a$ (80 = 2 × 40)	$Y_1^b = A^b K_1^b$ (120 = 2 × 60)
$C_1^a = (1 - s^a)\, Y_1^a$ (48 = (1 − 0.4) × 80)	$C_1^b = (1 - s^b)\, Y_1^b$ (48 = (1 − 0.6) × 120)
$K_2^a = s^a Y_1^a$ (32 = 0.4 × 80)	$K_2^b = s^b Y_1^b$ (72 = 0.6 × 120)

t = 2;

Country *a*	Country *b*
$Y_2^a = A^a K_2^a$ (64 = 2 × 32)	$Y_2^b = A^b K_2^b$ (144 = 2 × 72)
$C_2^a = (1 - s^a)\, Y_2^a$ (38.4 = (1 − 0.4) × 64)	$C_2^b = (1 - s^b)\, Y_2^b$ (57.6 = (1 − 0.6) × 144)
$K_3^a = s^a Y_2^a$ (25.6 = 0.4 × 64)	$K_3^b = s^b Y_2^b$ (86.4 = 0.6 × 144)

Figure 1-3 is the graph of country *a*'s GDP Y_t^a and country *b*'s GDP Y_t^b for $t = 0, 1, 2$, figure 1-4 is the graph of country *a*'s consumption C_t^a and country *b*'s consumption C_t^b for $t = 0, 1, 2$, and figure 1-5 is the graph of country *a*'s capital K_t^a and country *b*'s capital K_t^b for $t = 0, 1, 2, 3$. Country *a* and country *b* have the same output $Y_0^a = Y_0^b = 100$ in period $t = 0$. Because of the higher investment rate, country *b* consumes less than country *a* in period $t = 0$, $C_0^a = 60 > C_0^b = 40$. However, for the same reason, country *b*'s capital K_t^b and hence output Y_t^b increase faster than country *a*'s. Country *b*'s consumption C_t^b catches up with country *a*'s consumption C_t^a in $t = 1$, and surpasses after $t = 2, 3, 4, \cdots$.

Figure 1-6 is the PPF of country *a* and the PPF of country *b* in $t = 0$. Instead of the number of guns and the tons of butter, the horizontal axis measures

Figure 1-3

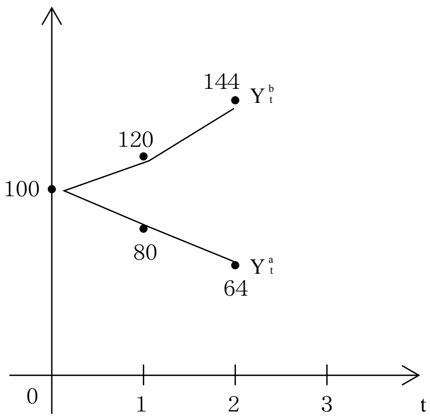


Figure 1-4

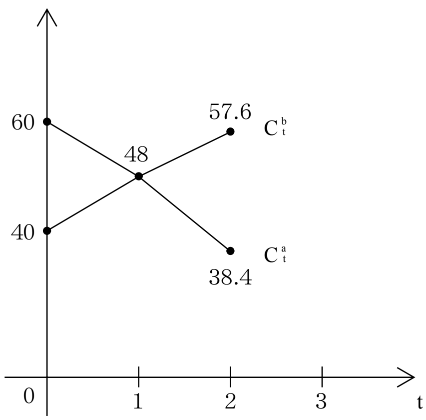
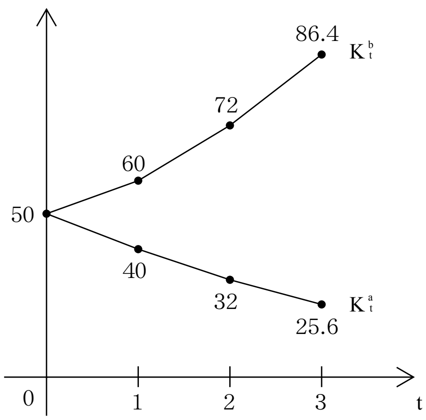
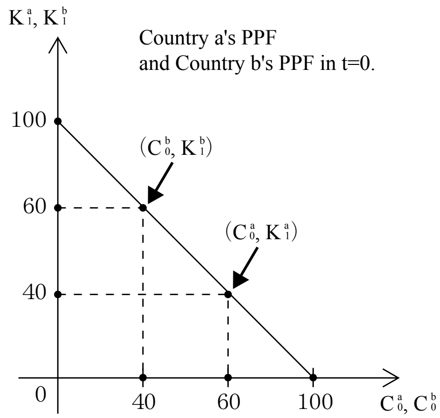


Figure 1-5



consumption C_0 and the vertical axis measures investment K_1 that becomes the capital in $t = 1$. The PPF is a locus of combination of consumption C_0 and investment K_1 that can be produced by the efficient use of GDP Y_0. Because country a and country b have the same GDP, $Y_0^a = Y_0^b = 100$, they have the same PPF; $100 = Y_0^a = C_0^a + K_1^a$ for country a, and $100 = Y_0^b = C_0^b + K_1^b$ for country b. In $t = 0$, country a chooses a point $(C_0^a, K_1^a) = (60, 40)$ on its PPF, and country b chooses a point $(C_0^b, K_1^b) = (40, 60)$ on its PPF.

Figure 1-6



The choice of each country causes the PPF to shift between $t = 0$ and $t = 1$. Figure1-7 is country a's PPF in $t = 1$, and figure 1-8 is country b's PPF in $t = 1$. Because country a's investment rate $s^a = 0.4$ is smaller than country b's $s^b = 0.6$, country a's PPF shifts inward, while country b's PPF shifts outward. As time period passes $t = 0, 1, 2, \cdots$, country a's PPF shrinks further toward the origin, while country b's PPF expands outward.

Figure 1-7

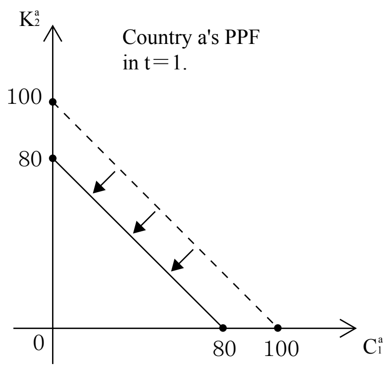
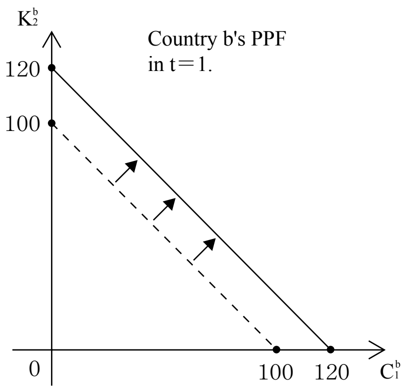


Figure 1-8



The simple economic growth model described above is expressed by the following equations. For $t = 0, 1, 2, \cdots$, given the initial capital K_0,

$$(1.2)\quad Y_t = AK_t$$

$$(1.3)\quad Y_t = C_t + K_{t+1}$$

$$(1.4)\quad C_t = (1-s)Y_t$$

$$(1.5)\quad K_{t+1} = sY_t$$

where $s \in [0, 1]$. This model can be analyzed as follows. By equations (1.2) and (1.5), we have

$$(1.6)\quad K_{t+1} = sAK_t \ , \quad t = 0, 1, 2, \cdots .$$

Given K_0, K_1 is determined by equation (1.6) as

$$(1.7)\quad K_1 = sAK_0 \ .$$

Then, K_2 is also determined by equation (1.6) as

$$(1.8)\quad K_2 = sAK_1 \ .$$

By repetition, K_3 is

$$(1.9)\quad K_3 = sAK_2 \ .$$

This process is repeated to calculate capital K_{t+1} in any $t = 0, 1, 2, \cdots$. Furthermore, by inserting K_1 of equation (1.7) into the right-hand side of equation (1.8), K_2 is expressed as

$$(1.10)\quad K_2 = sA(sAK_0) = (sA)^2 K_0 \ .$$

Similarly, by inserting K_2 of equation (1.10) into the right-hand side of equation (1.9), K_3 is expressed as

$$(1.11)\quad K_4 = sA((sA)^2 K_0) = (sA)^3 K_0 \ .$$

By equations (1.7), (1.10), and (1.11), we conjecture that K_t in any $t = 0, 1, 2, \cdots$ is expressed as

$$(1.12)\quad K_t = (sA)^t K_0 \ , \quad t = 0, 1, 2, \cdots .$$

We can verify that the conjecture is true by inserting equation (1.12) into the left-hand side and the right-hand side of equation (1.6). The left-hand side is

$$(1.13)\quad K_{t+1} = (sA)^{t+1} K_0 \ ,$$

and the right-hand side is

$$(1.14)\quad sAK_t = sA(sA)^t K_0 = (sA)^{t+1} K_0 .$$

Therefore, equation (1.12) satisfies equation (1.6). In fact, equation (1.6) is said to be a first-order homogenous linear difference equation with respect to K_t, and equation (1.12) is said to be a particular solution of equation (1.6).

Once the sequence of capital $\{K_t \ ; t = 0, 1, 2, \cdots\}$ is determined, the other variables $\{Y_t, C_t \ ; t = 0, 1, 2, \cdots\}$ are calculated by equations (1.2) and (1.4). Equation (1.6) implies

$$(1.15)\quad K_{t+1} \gtreqless K_t \quad \text{as} \quad sA \gtreqless 1 \ .$$

Because Y_t and C_t are proportional to K_t,

$$(1.16)\quad Y_{t+1} \gtreqless Y_t \quad \text{and} \quad C_{t+1} \gtreqless C_t \quad \text{as} \quad sA \gtreqless 1$$

also holds. In the above example, it is assumed $s^a = 0.4$ and $A^a = 2$. Then $s^aA^a = 0.8$ for country a. For country b, $s^b = 0.6$ and $A^b = 2$ so that $s^bA^b = 1.2$. Therefore, country a exhibits economic contraction, while country b exhibits economic expansion.

From equation (1.6), it is also clear that $(s^a, A^a) = (0.4, 2)$ for country a, and $(s^b, A^b) = (0.4, 3)$ for country b also result in the same outcome because $s^aA^a = 0.8$ and $s^bA^b = 1.2$. (In this case, however, unlike the previous one, country b's consumption C_t^b always exceeds country a's consumption C_t^a for all $t = 0, 1, 2, \cdots$.) Equation (1.6) exhibits the importance of investment (s) and technology (A) for economic growth.

In the above example, without explanation, we assumed country a's investment rate is $s^a = 0.4$ and country b's investment rate is $s^b = 0.6$. Some readers may wonder why the investment rates are different. Furthermore, some readers may ask if $s^a = 0.4$ is good for country a, because it could consume more in the future if it saves more like country b. $s^a = 0.4$ for country a and $s^b = 0.6$ for country b are optimal for each country in the following situation. In fact, what we are going to do is to provide micro-foundations to the above example.

Consider the following two-periods consumption-saving planning problem of an individual.

(1.17) $U = C_0^{1-\alpha} C_1^{\alpha}$, $0 < \alpha < 1$

(1.18) $C_0 + K_1 = Y_0$

(1.19) $C_1 = AK_1$

Equation (1.17) is the utility as a function of the first period consumption C_0 and the second-period consumption C_1. α is a parameter that measures the relative importance of C_1 to C_0. (Equation (1.17) implies that the utility function is of Cobb-Douglas form.) Equation (1.18) is the first period budget constraint. Y_0 is the first period income that will be divided between the first period consumption C_0 and the first period investment (saving) K_1. Equation (1.19) is the second period budget constraint. The second period income is AK_1 that is consumed as the second period consumption C_1. Notice that the first period and the second period budget constraints are the same as the SNA of the two countries in the previous example. The individual chooses a consumption-saving plan $\{C_0^*, C_1^*, K_1^*\}$ that maximizes utility U subject to the first period and the second period budget constraints.

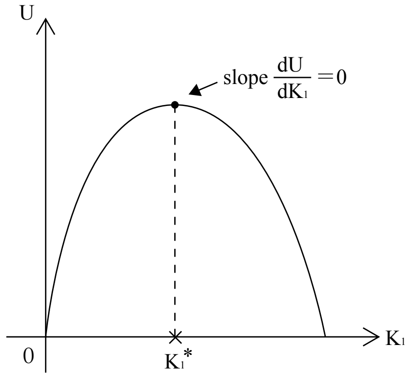
This problem can be solved as follows. By using the first period and the second period budget constraints, the utility is expressed as a function of saving K_1 as follows.

(1.20) $U = (Y_0 - K_1)^{1-\alpha}(AK_1)^{\alpha}$.

The optimal saving K_1^* is a solution to the following first-order condition. (See figure 1-9 for the graphical implication of the optimal saving K_1^*.)

(1.21) $$0 = \frac{dU}{dK_1} = -(1-\alpha)(Y_0 - K_1)^{-\alpha}(AK_1)^{\alpha} + (Y_0 - K_1)^{1-\alpha}\alpha(AK_1)^{\alpha-1}A$$

Figure 1-9. Optimal Saving K_1^*



Equation (1.21) implies that the optimal saving is

(1.22) $K_1^* = \alpha Y_0$.

By the first period budget constraint, the optimal first period consumption is

(1.23) $C_0^* = Y_0 - K_1^* = Y_0 - \alpha Y_0 = (1-\alpha)Y_0$,

and by the second period budget constraint, the optimal second period consumption is

(1.24) $C_1^* = AK_1^* = A\alpha Y_0$.

Equation (1.22) implies that α is the optimal saving rate for the individual. This example can be used as a micro-foundation for the previous economic growth model. If the utility function of consumers in country a is $U^a = C_0^{0.6}C_1^{0.4}$, they save 40% of the first period income $K_1^a = 0.4 \times Y_0^a$. Similarly, if the utility function of consumers in country b is $U^b = C_0^{0.4}C_1^{0.6}$, they save 60% of the first period income $K_1^b = 0.6 \times Y_0^b$. Therefore, the answer to the question "why the saving rates are different" might be "because they place different weights on inter-temporal consumptions." The consumers in country a put heavier weights on consumptions in earlier periods, while the consumers in country b puts heavier weights on consumptions in later periods. Unfortunately, the analysis stops at this point. The readers may want to ask why these countries put different weights on inter-temporal consumption. Macroeconomists treat parameters of utility function (like α in equation (1.17)) and production function (like A in equation (1.19)) as the ultimate givens (deep parameters), and leave the explanations of their differences to differences in history, culture, ethnicity, religion, and so on. Recently, however, macroeconomists began to seek the determinants of the deep parameters. For example, the endogenous economic growth theories try to explain cross-country differences in production technologies. Economists also collaborate with psychologists, sociologists, and even neurologists to seek the determinants of consumers behavior which can be applied to the analyses of utility functions.

2. Dynamic Economic Policy Analyses and Lucas' Critic

The 2011 Nobel Prize of Economics was bestowed to T. J. Sargent of New York University and C. Sims of Princeton University for their contribution to the development of rational expectations theories that changed the way economists and government analyze the effects of economic policies. The most important element of economic policy analyses in the rational expectations theories is the explicit treatment of time. Households make economic decisions (consumption demands and production factors supplies) across time to maximize their utilities. Likewise, firms make economic decisions (goods and services supplies and production factors demands) across time to maximize their values. Obviously, they have to take economic policies, not only the present but also the future, into account when they make dynamic decisions. The same is also true to public sectors. When public sectors design economic policies, they have to take the dynamic reactions of households and firms into account. In other words, economic policies that fail to take the dynamic reactions of households and firms into account will result in suboptimal performance. Prior to the rational expectations theories, economists and governments used Keynesian macroeconomic models to analyze the effects of economic policies. A typical Keynesian macroeconomic model is a system of simultaneous equations with respect to macroeconomic variables and policy variables. The parameters of the equations system are assumed to be invariant to policy changes. The economists who advocate the rational expectations theories criticize Keynesian models for their lack of explicit treatment of time. Because of the lack of explicit treatment of time, Keynesian models cannot be solved for dynamic variables such as inflation rate which is intrinsically a dynamic variable. In other words, Keynesian models are not closed with respect to dynamic variables, i.e., the number of structural equations is smaller than the number of variables. (Chapter 1 of Sargent (1987) extensively discusses this issue.)

Furthermore, the rational expectations theorists criticized that the parameters of the structural equations system can be affected by changes in economic policies once Keynesian models explicitly incorporate time. This point is known as "Lucas' critic". In the following, we demonstrate Lucas' critic through simple examples. (See Lucas (1987) for an extensive treatment of the issue.) One of the key structural equations of Keynesian macroeconomic model is a consumption function.

$$(2.1)\quad C = \alpha + \beta(Y - T)$$

C is aggregate private consumption, Y is aggregate income, and T is aggregate tax on private sector. $Y - T$, hence, is aggregate disposable income. α and β are parameters that are assumed to be invariant to policy changes. A typical Keynesian macroeconomic model starts with proposing structural equations like equation (2.1) which describes the relationship between macroeconomic variables, C, Y, and T. For this reason, Keynesian models are said to lack micro-foundations. In equation (2.1), β is the marginal propensity to consumption. For example, consider a decrease in tax from T' to T'' ($T' > T''$). This causes the disposable income to increase as follows.

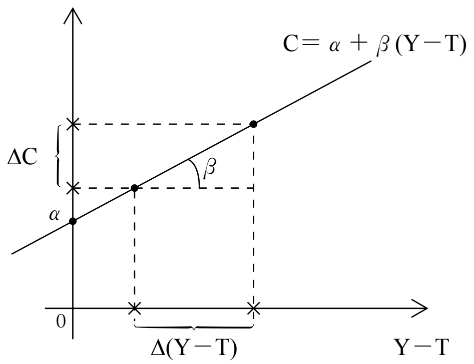
$$(2.2)\quad \Delta(Y-T) = (Y-T') - (Y-T'') = T'' - T' > 0$$

Then, by equation (2.1), the increase in consumption caused by the increase in disposable income is

(2.3) $\Delta C = [\alpha + \beta(Y - T')] - [\alpha + \beta(Y - T'')]$
$= \beta\Delta(Y - T)$.

Figure 2-1 depicts graphically the implication of the marginal propensity to consumption β.

Figure 2-1. Consumption Function



The marginal propensity to consumption β plays important roles in economic policy analyses. For example, in the demand side of national income (equation (1.1)), a marginal increase in public spending ΔG causes a proportional increase in national income ΔY with coefficient $1/(1-\beta)$.

(2.4) $\Delta Y = \dfrac{1}{1-\beta}\Delta G$

$1/(1-\beta)$ is called the public spending multiplier. The larger is the marginal propensity to consumption β, the larger is the multiplier $1/(1-\beta)$. For example, if $\beta = 0.8$, then the multiplier is $1/(1-\beta) = 5$, i.e., an increase in public spending by $\Delta G =$ ¥1 million causes an increase in national income by $\Delta Y =$ ¥5 million. A marginal increase in disposable income by tax reduction also causes a proportional increase in national income with coefficient $\beta/(1-\beta)$.

(2.5) $\Delta Y = \dfrac{\beta}{1-\beta}(-\Delta T)$

$\beta/(1-\beta)$ is called the tax reduction multiplier. For example, if $\beta = 0.8$, then $\beta/(1-\beta) = 4$, i.e., a marginal decrease in tax by $\Delta T = -$¥1 million causes an increase in national income by $\Delta Y =$ ¥4 million.

In the following we demonstrate by simple examples that the marginal propensity to consumption is affected by changes in public policies. The examples we use to demonstrate above point are "lifecycle models of consumption". The simplest form of the lifecycle model of consumption is the following two-periods consumption-saving planning of a representative consumer. (Readers may notice the similarity between lifecycle models and the consumption-investment planning models presented at the previous section 1.)

(2.6) $U = C_1 \times C_2$

(2.7) $C_1 + S_2 = Y_1$

(2.8) $C_2 = Y_2 + S_2$

Equation (2.6) is utility of the consumer as a function of the first period income C_1 and the second period income C_2. Equation (2.7) is the first period budget constraint showing the division of the first period income Y_1 between the first period consumption C_1 and the first period saving S_2. Equation (2.8) is the second period budget constraint showing that the second period income Y_2 and the first period saving S_2 are used for the second period consumption C_2. For simplicity, the interest rate on the saving is assumed to be zero. The consumer makes an optimal consumption-saving plan $\{C_1^*, C_2^*, S_2^*\}$ that maximizes the utility (2.6) subject to the budget constraints {(2.7), (2.8)}.

This utility maximization problem is solved as follows. By eliminating the first period savings S_2 from the first and the second period budget constraints, we have a single budget constraint with respect to C_1 and C_2.

(2.9) $C_1 + C_2 = Y_1 + Y_2$

In lifecycle models, equation (2.9) is often called the lifetime budget constraint for the two periods are interpreted as the first half and the second half of the consumer's life. It may be assumed that the consumer works in the first half of her life, and spends the rest as a retiree in the second half. Therefore, in the two-periods model, one period may consist of 20 ~ 30 years. Equation (2.9) shows that the combination of the first period consumption and the second period consumption $\{C_1, C_2\}$ cannot exceed the lifetime income $Y_1 + Y_2$.

Suppose $\{Y_1 = 10, Y_2 = 0\}$ so that $Y_1 + Y_2 = 10$. There are many combinations of C_1 and C_2 that satisfy the lifetime budget constraint (2.9). The following table shows the combinations of C_1 and C_2 that satisfy the lifetime budget constraint (2.9), and corresponding values of utility.

C_1	0	1	2	3	4	5	6	…	10
C_2	10	9	8	7	6	5	4	…	0
U	0	9	16	21	24	25	24	…	0

For example, $\{C_1 = 0, C_2 = 10\}$ satisfies $C_1 + C_2 = 10$. The utility, however, is only $U = C_1 \times C_2 = 0 \times 10 = 0$. The other extreme is $\{C_1 = 10, C_2 = 0\}$ which also gives $U = C_1 \times C_2 = 10 \times 0 = 0$. Two-periods consumption plans that spend too much in one period and too little in another do not give high utility. The above table indicates that the smooth path for consumptions $\{C_1 = 5, C_2 = 5\}$ maximizes utility at $U = C_1 \times C_2 = 5 \times 5 = 25$. This property, known as "consumption smoothing", holds under more general settings. Denote the "optimal consumptions" by

$$(2.10)\quad C_1^* = C_2^* = \frac{Y_1 + Y_2}{2} = 5.$$

From the first period budget constraint (2.7), the optimal saving is

$$(2.11)\quad S_2^* = Y_1 - C_1^* = 10 - 5 = 5.$$

Instead of $\{Y_1 = 10, Y_2 = 0\}$, suppose $\{Y_1 = 0, Y_2 = 10\}$. In this case, the lifetime budget constraint is same as before, i.e., $C_1 + C_2 = Y_1 + Y_2 = 10$. Therefore, the optimal consumption plan is same, i.e., $\{C_1^* = 5, C_2^* = 5\}$. On the other hand, the optimal saving is

$$(2.12)\quad S_2^* = Y_1 - C_1^* = 0 - 5 = -5.$$

Because the first period income is small, the consumer has to borrow $S_2^* = -5$ to consume $C_1^* = 5$ in the first period. In the second period, the consumer redeems $S_2^* = -5$ out of the second period income $Y_2 = 10$. Hence, the consumer's second period consumption is

$$(2.13)\quad C_2^* = Y_2 + S_2^* = 10 + (-5) = 5.$$

These analyses are graphically presented by using the graph of indifference curves and the graph of lifetime budget constraint. Figure 2-2 is the graph of indifference curves of utility function (2.6). The horizontal axis measures the first period consumption C_1 and the vertical axis measures the second period consumption C_2. Because the utility function is assumed to be a product of C_1 and C_2, the graph of indifference curves are orthogonal hyperbolic curves. For example, the indifference curve for utility $U = 4$ is $C_2 = 4/C_1$. It is the locus of (C_1, C_2) that gives same utility $U = 4$. $(C_1, C_2) = \{(0.5, 8), (1, 4), (2, 2), (4, 1), (8, 0.5)\}$ are some of the points on the indifference curve. The further away from the origin, the higher the utility of indifference curves. The downward sloping line in figure 2-3 is the graph of the lifetime budget constraint (2.9). The graph has intercepts at $Y_1 + Y_2$ on the horizontal axis, and $Y_1 + Y_2$ on the vertical axis.

Figure 2-2. Indifference Curves of $U = C_1 \times C_2$

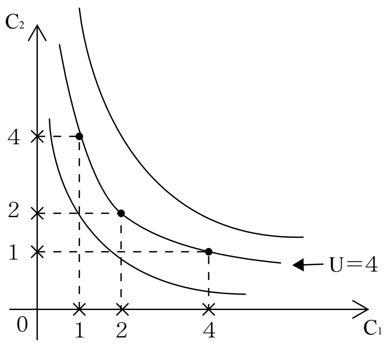
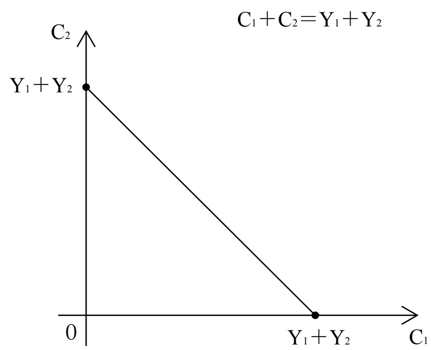
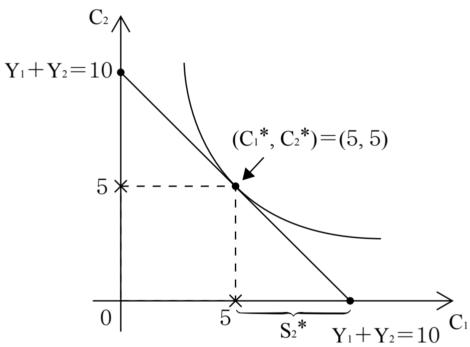


Figure 2-3. Lifetime Budget Constraint



The optimal consumption plan (C_1*, C_2*) maximizes utility subject to lifetime budget constraint. Figure 2-4 shows that the optimal consumption plan is {C_1* = 5, C_2* = 5} when the lifetime income is $Y_1 + Y_2 = 10$ which is a numerical example we used before. The figure shows that the optimal consumption plan is a tangent point of an indifference curve and the lifetime budget constraint $C_1 + C_2 = Y_1 + Y_2 = 10$. Because C_1* = 5 and C_2* = 5, the indifference curve that touches the graph of lifetime budget constraint must be the one representing $U = C_1 \times C_2 = 25$.

Figure 2-4. The Optimal Consumption $\{C_1^*, C_2^*\}$ and the Optimal Saving S_2^*.



In figure 2-4, we can also measure the first period income $Y_1 = 10$ on the horizontal axis and the second period income $Y_2 = 0$ on the vertical axis. By the first period budget constraint, the optimal saving is S_2* = $Y_1 - C_1$* = 10 − 5 = 5. Hence, figure 2-4 can also show on the horizontal axis the optimal saving S_2* as the difference between the period income $Y_1 = 10$ and the first period optimal consumption C_1* = 5. Readers may understand that the optimal consumption plan (C_1*, C_2*) for {$Y_1 = 10$, $Y_2 = 0$} is same as that for {$Y_1 = 0$, $Y_2 = 10$} because these cases have the same lifetime budget constraint. Therefore, we can also use figure 2-4 to express the (C_1*, C_2*). The optimal saving S_2* for {$Y_1 = 0$, $Y_2 = 10$}, however, is different from that for {$Y_1 = 10$, $Y_2 = 0$}. It is S_2* = $Y_1 - C_1$* = 0 − 5 = − 5. Therefore, in figure 2-4, it is the difference between the first period optimal consumption C_1* = 5 and the first period income $Y_1 = 0$ on the horizontal axis. This example shows that, despite the variation in income stream across time {Y_1, Y_2}, if the lifetime income is same, the consumption opportunity is same, so the optimal consumption plan is same. For this claim to hold, however, there must be no restriction on lending and borrowing in credit market. In other words, credit market must be "perfect". Otherwise, consumers with same lifetime income may face different consumption opportunities if their income streams across time are different. This implies that credit market imperfections will prevent consumers from making the optimal consumption-saving plans.

The above two-periods model can be extended to three-periods model as follows.

$$(2.14)\quad U = C_1 \times C_2 \times C_3$$

$$(2.15)\quad C_1 + S_2 = Y_1$$

$$(2.16)\quad C_2 + S_3 = Y_2 + S_2$$

$$(2.17)\quad C_3 = Y_3 + S_3$$

As before, the initial saving is assumed to be $S_1 = 0$, and the third period saving must be $S_4 = 0$ for utility maximization. By eliminating savings {S_2, S_3} from the first, the second, and the third period budget constraints, equations {(2.15), (2.16), (2.17)}, we have the lifetime budget constraint.

$$(2.18)\quad C_1 + C_2 + C_3 = Y_1 + Y_2 + Y_3$$

It can be shown, by using the same reasoning of the two-periods model, that the optimal consumption plan is

$$(2.19)\quad C_1^* = C_2^* = C_3^* = \frac{1}{3}(Y_1 + Y_2 + Y_3).$$

As before, if the lifetime income $Y_1 + Y_2 + Y_3$ is same, then the optimal consumption plan {C_1*, C_2*, C_3*} is same. The optimal saving plan {S_2*, S_3*}, however, as before depends on the pattern of income stream {Y_1, Y_2, Y_3} across time periods. Individuals with smaller income in earlier periods and larger income in later periods tend to borrow in earlier periods, and redeem the debts in later periods. On the other hand, individuals with larger income in earlier periods and smaller income in later periods tend to lend in earlier periods and use the capital income for consumption in later periods. The optimal savings are calculated by using the budget constraints as follows.

(2.20) $S_2{}^* = Y_1 - C_1{}^*$

(2.21) $S_3{}^* = Y_2 + S_2{}^* - C_2{}^*$

Instead of equation (2.21), $S_3{}^*$ is also calculated by using equation (2.17) as follows.

(2.22) $S_3{}^* = Y_3 - C_3{}^*$

Now consider the following ten-periods consumption-saving planning problem with taxation. It is a straightforward generalization of the above two-periods and three-periods models.

(2.23) $U = C_1 \times C_2 \times \cdots \times C_{10}$

(2.24) $C_1 + S_2 = Y_1 - T_1$

(2.25) $C_2 + S_3 = Y_2 - T_2 + S_2$
$\cdots$

(2.26) $C_9 + S_{10} = Y_9 - T_9 + S_9$

(2.27) $C_{10} = Y_{10} - T_{10} + S_{10}$

As before, equation (2.23) is utility function of the consumer that is assumed to be a product of consumptions across time periods $\{C_1, C_2, \cdots, C_{10}\}$. Equations (2.24) ~ (2.27) are budget constraints for each period $t = 1, 2, \cdots, 10$. Because of taxes, the disposable income in each period is $Y_t - T_t$, $t = 1, 2, \cdots, 10$. As before, the initial saving is assumed to be $S_1 = 0$, and $S_{11} = 0$ by optimization. By using the same logic of the analyses of two-periods and three-periods utility maximization problems, the optimal consumption plan of this ten-periods utility maximization problem is shown to be the following.

(2.28)
$$C_1{}^* = C_2{}^* = \cdots = C_{10}{}^* = \frac{1}{10}[(Y_1 - T_1) + (Y_2 - T_2) + \cdots + (Y_{10} - T_{10})]$$

As before, by eliminating savings $\{S_2, S_3, \cdots, S_{10}\}$ from the budget constraints (2.24) ~ (2.27), we obtain the lifetime budget constraint of the ten-periods model as follows.

(2.29)
$$C_1 + C_2 + \cdots + C_{10} = (Y_1 - T_1) + (Y_2 - T_2) + \cdots + (Y_{10} - T_{10})$$

Then, we can calculate the optimal savings by using the period-wise budget constraints (2.24) ~ (2.27) as follows.

(2.30) $S_2{}^* = Y_1 - T_1 - C_1{}^*$

(2.31) $S_3{}^* = Y_2 - T_2 + S_2{}^* - C_2{}^*$
$\cdots$

(2.32) $S_9{}^* = Y_8 - T_8 + S_8{}^* - C_8{}^*$

(2.33) $S_{10}{}^* = Y_9 - T_9 + S_9{}^* - C_9{}^*$

The optimal saving of the last period $S_{10}{}^*$ is also calculated from the last period budget constraint (2.27) as follows.

(2.34) $S_{10}{}^* = Y_{10} - T_{10} - C_{10}{}^*$

In the following, by using specific numerical examples, we will demonstrate that the marginal propensity to consumption is not a constant parameter, that it is affected by public policy changes.

Assume that $Y_1 = Y_2 = \cdots = Y_{10} = 10$. Let us specify the benchmark case as follows.

Case 1 (Benchmark Case):

(2.35) $T_1(1) = T_2(1) = \cdots = T_{10}(1) = 0$

By equation (2.28), the optimal consumptions in case 1, denoted as $\{C_1{}^*(1), C_2{}^*(1), \cdots, C_{10}{}^*(1)\}$, are

(2.36)
$$C_1{}^*(1) = C_2{}^*(1) = \cdots = C_{10}{}^*(1) = \frac{1}{10}[(Y_1 - T_1(1)) + (Y_2 - T_2(1)) + \cdots + (Y_{10} - T_{10}(1))]$$
$$= \frac{1}{10}[(10 - 0) + (10 - 0) + \cdots + (10 - 0)]$$
$$= \frac{1}{10}[10 \times 10] = 10.$$

By the period-wise budget constraints, equations (2.24) ~ (2.27), the optimal saving in case 1, denoted as $\{S_2{}^*(1), S_3{}^*(1), \cdots, S_{10}{}^*(1)\}$, are

(2.37)
$$S_2{}^*(1) = Y_1 - T_1(1) - C_1{}^*(1) = 10 - 0 - 10 = 0$$

$$(2.38)\quad S_3^*(1) = Y_2 - T_2(1) + S_2^*(1) - C_2^*(1) = 10 - 0 + 0 - 10 = 0$$

$$\cdots$$

$$(2.39)\quad S_{10}^*(1) = Y_9 - T_9(1) + S_9^*(1) - C_9^*(1) = 10 - 0 + 0 - 10 = 0$$

Therefore, in case 1, the individual consumes all the disposable income, does not save nor borrow, in every period $t = 1, 2, \cdots, 10$.

In the next case, there is a permanent tax increase from zero to one in every period.

Case 2 (Permanent Tax Increase):

$$(2.40)\quad T_1(2) = T_2(2) = \cdots = T_{10}(2) = 1$$

By equation (2.28), the optimal consumptions in case 2, denoted as $\{C_1^*(2), C_2^*(2), \cdots, C_{10}^*(2)\}$, are

$$(2.41)\quad C_1^*(2) = C_2^*(2) = \cdots = C_{10}^*(2) = \frac{1}{10}[(Y_1 - T_1(2)) + (Y_2 - T_2(2)) + \cdots + (Y_{10} - T_{10}(2))] = \frac{1}{10}[(10-1) + (10-1) + \cdots + (10-1)] = \frac{1}{10}[10 \times 9] = 9.$$

By the period-wise budget constraints, equations (2.24) ~ (2.27), the optimal savings in case 2, denoted as $\{S_2^*(2), S_3^*(2), \cdots, S_{10}^*(2)\}$, are shown to be zero in every period. Therefore, in case 2, just like in case 1, the individual consumes all the disposable income, does not save nor borrow, in every period.

Relative to the benchmark case (case 1), the marginal propensity to consumption in the first period in case 2, denoted as $\beta(2)$, is calculated as follows.

$$(2.42)\quad \beta(2) = \frac{C_1^*(2) - C_1^*(1)}{(Y_1 - T_1(2)) - (Y_1 - T_1(1))} = \frac{-1}{-1} = 1.$$

Next, consider a quasi-permanent tax increase, described as case 3.

Case 3 (Quasi-Permanent Tax Increase):

$$(2.43)\quad T_1(3) = T_2(3) = \cdots = T_5(3) = 2, \quad T_6(3) = T_7(3) = \cdots = T_{10}(3) = 0.$$

By equation (2.28), the optimal consumptions in case 3, denoted as $\{C_1^*(2), C_2^*(2), \cdots, C_{10}^*(2)\}$ are

$$(2.44)\quad C_1^*(3) = C_2^*(3) = \cdots = C_{10}^*(3) = \frac{1}{10}[(Y_1 - T_1(3)) + (Y_2 - T_2(3)) + \cdots + (Y_5 - T_5(3)) + (Y_6 - T_6(3)) + (Y_7 - T_7(3)) + \cdots + (Y_{10} - T_{10}(3))] = \frac{1}{10}[(10-2) + (10-2) + \cdots + (10-2) + (10-0) + (10-0) + \cdots + (10-0)] = \frac{1}{10}[5 \times 8 + 5 \times 10] = 9.$$

Notice that the optimal consumption plan in case 3 is same as that in case 2. This is because the total taxes across time periods in case 2 and case 3 are same, i.e.,

$$(2.45)\quad T_1(2) + T_2(2) + \cdots + T_{10}(2) = T_1(3) + T_2(3) + \cdots + T_{10}(3) = 10,$$

and hence the lifetime disposable incomes in case 2 and case 3 are same, i.e.,

$$(2.46)\quad (Y_1 - T_1(2)) + (Y_2 - T_2(2)) + \cdots + (Y_{10} - T_{10}(2)) = (Y_1 - T_1(3)) + (Y_2 - T_2(3)) + \cdots + (Y_{10} - T_{10}(3)) = 90.$$

Because the optimal consumptions require the equal division of lifetime disposable income across time periods $t = 1, 2, \cdots, 10$, the optimal consumption plan in case 3 must be same as that in case 2.

Remark: Equations (2.45) and (2.46) must be interpreted as the present value of total taxes and the present value of lifetime disposable incomes. In these examples, we assume that the interest rates are zero. When the interest rates are not zero, the taxes and the disposable incomes in future periods must be discounted to convert them into the same accounting units.

On the other hand, in case 3, because the disposable incomes in $t = 1, 2, \cdots, 5$ are smaller than those in $t = 6, 7, \cdots, 10$, the individual must borrow in the earlier periods to achieve consumption smoothing. In fact, by the period-wise budget constraints, the optimal savings in case 3, denoted as $\{S_2^*(3), S_3^*(3), \cdots, S_{10}^*(3)\}$, are calculated as follows.

$$(2.47)\quad S_2^*(3) = Y_1 - T_1(3) - C_1^*(3) = 10 - 2 - 9 = -1$$

$$(2.48)\quad S_3^*(3) = Y_2 - T_2(3) + S_2^*(3) - C_2^*(3) = 10 - 2 + (-1) - 9 = -2$$

$$(2.49)\quad S_4^*(3) = Y_3 - T_3(3) + S_3^*(3) - C_3^*(3) = 10 - 2 + (-2) - 9 = -3$$

$$(2.50)\quad S_5^*(3) = Y_4 - T_4(3) + S_4^*(3) - C_4^*(3) = 10 - 2 + (-3) - 9 = -4$$

$$(2.51)\quad S_6^*(3) = Y_5 - T_5(3) + S_5^*(3) - C_5^*(3) = 10 - 2 + (-4) - 9 = -5$$

$$(2.52)\quad S_7^*(3) = Y_6 - T_6(3) + S_6^*(3) - C_6^*(3) = 10 - 0 + (-5) - 9 = -4$$

$$(2.53)\quad S_8^*(3) = Y_7 - T_7(3) + S_7^*(3) - C_7^*(3) = 10 - 0 + (-4) - 9 = -3$$

$$(2.54)\quad S_9^*(3) = Y_8 - T_8(3) + S_8^*(3) - C_8^*(3) = 10 - 0 + (-3) - 9 = -2$$

$$(2.55)\quad S_{10}^*(3) = Y_9 - T_9(3) + S_9^*(3) - C_9^*(3) = 10 - 0 + (-2) - 9 = -1$$

Figure 2-5. The Optimal Saving in Case 3.

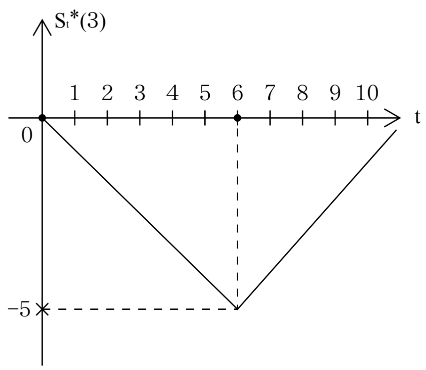


Figure 2-5 shows the graph of the optimal savings in case 3. The individual accumulates debt through borrowings in the earlier periods, $t = 1, 2, 3, 4, 5$, because the consumptions (9) are larger than the disposable incomes ($10 - 2 = 8$) in these periods. In the later periods, $t = 6, 7, 8, 9, 10$, the individual reduces debt through repayment because the consumptions (9) are smaller than the disposable incomes ($10 - 0 = 10$) in these periods. In the last period $t = 10$, the individual pays off all the debt to balance the last period budget constraint.

$$(2.56)\quad \underset{(9)}{C_{10}^*(3)} = \underset{(10)}{Y_{10}} - \underset{(0)}{T_{10}(3)} + \underset{(-1)}{S_{10}^*(3)}$$

On the other hand, the disposable incomes in case 2 are constant across all the time periods $t = 1, 2, \cdots, 10$. Therefore, there is no need for consumption smoothing through either lending or borrowing.

Relative to the benchmark case (case 1), the marginal propensity to consumption in the first period in case 3, denoted as $\beta(3)$, is calculated as follows.

$$(2.57)\quad \beta(3) = \frac{C_1^*(3) - C_1^*(1)}{(Y_1 - T_1(3)) - (Y_1 - T_1(1))} = \frac{9 - 10}{(10-2)-(10-0)} = \frac{-1}{-2} = 0.5$$

Remember that the marginal propensity to consumption in case 2 was $\beta(2) = 1$. In these two cases, the utility functions and pre-tax incomes are same. The difference between $\beta(2)$ and $\beta(3)$ is caused by the difference between tax policies. In case 3, in the first period, the increase in tax by 2 causes the disposable income to decrease by the same amount. This causes the first period consumption to decrease. However, it decreases half the size of the decrease in disposable income ($\beta(3) = 0.5$). On the other hand, in case 2, in the first period, the increase in tax by 1 causes the consumption to decrease by the full amount of the decrease in disposable income ($\beta(2) = 1$). Because the tax increase in case 3 is not permanent, the first period consumption needs not decrease by the full amount of the decrease in the first period disposable income. Although the consumer has to borrow to finance the first period consumption, she can redeem the debt when the disposable incomes increase by the removal of taxes in later periods. On the other hand, in case 2, the consumer has to decrease the first period consumption by the full amount of decrease in the first period disposable income to achieve consumption smoothing because the tax increase is permanent. Consider a hypothetical world in which consumers behave as those in lifecycle models. In this world, tax policies designed by using Keynesian models may fail to achieve expected outcomes, as the following case 4 suggests.

Case 4 (Temporary Tax Increase)

(2.58) $T_1(4) = 1,\ T_2(4) = T_3(4) = \cdots,\ T_{10}(4) = 0.$

By equation (2.28), the optimal consumption plan, denoted as $\{C_1*(4), C_2*(4),\cdots, C_{10}*(4)\}$, is calculated as follows.

(2.59) $$C_1*(4) = C_2*(4) = \cdots = C_{10}*(4)$$
$$= \frac{1}{10}[(Y_1 - T_1(4)) + (Y_2 - T_2(4)) + \cdots + (Y_{10} - T_{10}(4))]$$
$$= \frac{1}{10}[(10-1) + (10-0) + \cdots + (10-0)]$$
$$= \frac{1}{10}[9 + 9 \times 10] = 9.9$$

Relative to the benchmark case (case 1), the marginal propensity to consumption in the first period in case 4, denoted as $\beta(4)$, is calculated as follows.

(2.60) $$\beta(4) = \frac{C_1*(4) - C_1*(1)}{(Y_1 - T_1(4)) - (Y_1 - T_1(1))}$$
$$= \frac{9.9 - 10}{(10-1) - (10-0)} = \frac{-0.1}{-1} = 0.1$$

Suppose there are two countries, A and B. The government of country A plans a permanent tax increase which is described by case 2. However, it worries negative effects of the tax increase on consumption. In order to estimate the negative impact, country A's government looks at country B whose government also implemented a tax increase. By applying statistical analysis to Keynesian consumption function (2.1) on country B's data, country A's government concludes that the marginal propensity to consumption is $\beta = 0.1$, i.e., a tax increase by ¥10 million causes consumption to decrease by ¥1 million. We already know such a conclusion is misleading. The marginal propensity to consumption is not a constant parameter. It may be affected not only by current policies but also by future policies. $\beta = 0.1$ in country B may suggest that the tax increase is temporary. If country A's government implements permanent tax increase, consumption may decrease, not by 10%, but by full amount of the tax increase as case 2 suggests.

3. Exercise: The Effects of Dynamic Tax Policies on the Optimal Consumption-Saving Plan in Lifecycle Models.

In this exercise, we analyze the effects of tax policies on consumers' consumption-saving behavior in lifecycle models under more general assumptions than those in the models presented in the main texts. In the exercise, we use "Mathematica", a computer software for numerical calculation, to simulate alternative tax policies. A brief rudimentary guide for using Mathematica to solve simultaneous equations is provided in the appendix at the end of the paper.

Question 1. Two-Periods Lifecycle Model

In question1, we analyze consumption-saving behavior of an individual in a two-periods lifecycle model described by the following equations (3.1) ~ (3.4).

(3.1) $U = u(C_1) + Zu(C_2)\ ,\quad Z > 0$

(3.2) $u(C) = AC - \frac{B}{2}C^2\ ,\quad A > 0,\quad B > 0$

(3.3) $C_1 + S_2 = Y_1 - T_1$

(3.4) $C_2 = Y_2 - T_2 + R_2 S_2$

Equation (3.1) is utility of the individual as a weighted sum of the utility $u(C_1)$ of the first period consumption C_1 and the utility $u(C_2)$ of the second period consumption C_2. Z is a weighing parameter for measuring the importance of the second period utility relative to the first period utility. Equation (3.2) implies that the period-wise utility $u(C)$ is a quadratic function of consumption C, with two parameters $\{A, B\}$. By equation (3.2), the marginal utility of consumption is

(3.5) $u'(C) = A - BC$.

In addition, $u''(C) = -B < 0$. Therefore, for $0 < C < A/B$, the marginal utility is positive and decreasing in C. Equation (3.3) is the first period budget constraint. Y_1 is the first period income, T_1 is the first period tax, and $Y_1 - T_1$ the first period disposable income. The first period disposable income is divided between the first period consumption C_1 and the first period saving S_2. Equation (3.4) is the second period budget constraint. $Y_2 - T_2$ is the second period disposable income. R_2 is the second period gross interest rate. If the second period interest rate is r_2, R_2 is expressed as $R_2 = 1 + r_2$. Therefore, $R_2S_2 = S_2 + r_2S_2$ implies the sum of the principal S_2 and the interest income r_2S_2. The second period budget constraint (3.4) implies that the second period after-tax total income $Y_2 - T_2 + r_2S_2$ and the principal S_2 are used for the second period consumption C_2.

The individual makes an optimal consumption-saving plan $\{C_1^*, C_2^*, S_2^*\}$ that maximizes the utility (3.1) subject to the budget constraints {(3.3), (3.4)}. In the following, we will explain how to solve this constrained optimization problem by using Mathematica through three steps.

Step 1. Derive the first-order condition for the utility maximization that the optimal saving S_2^* must satisfy.

Use the first period and the second period budget constraints, equations (3.3) and (3.4) , to express the utility, equation (3.1), as a function of saving S_2, as following equation (3.6).

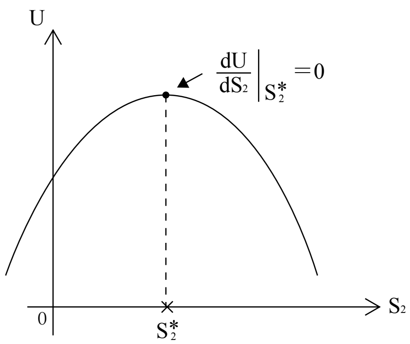
$$(3.6)\quad U = u(Y_1 - T_1 - S_2) + Z u(Y_2 - T_2 + R_2 S_2)$$

The optimal saving S_2^* that maximizes the utility U subject to the budget constraints is a solution to the following equation (3.7).

$$(3.7)\quad \frac{dU}{dS_2} = -[A - B(Y_1 - T_1 - S_2)] + Z R_2[A - B(Y_2 - T_2 + R_2 S_2)] = 0$$

Figure 3-1 depicts the optimal saving as the one that makes the slope of the graph of utility function to be zero.

Figure 3-1. The Optimal Saving in S_2^* in Two-Periods Lifecycle Model.



If we specify the values of parameters $\{A, B, Z\}$, the interest rate R_2, incomes $\{Y_1, Y_2\}$, and taxes $\{T_1, T_2\}$, then we can calculate the value of the optimal saving S_2^* from equation (3.7).

Step 2. Use Mathematica to calculate the value of the optimal saving S_2^*.

Here we present an example program in which $\{A = 100, B = 1, Z = 0.9, R_2 = 1.1, Y_1 = 30, Y_2 = 10, T_1 = 0, T_2 = 0\}$.

```
A = 100;
B = 1;
Z = 0.9;
R2 = 1.1;
Y1 = 30;
Y2 = 10;
T1 = 0;
T2 = 0;
F = -(A - B*(Y1 - T1 - S2))+ Z*R2*(A - B*(Y2 - T2 + R2*S2));
Solve [F == 0, S2]
```

We run this program to get $S_2^* = 9.14$.

Comment 1. Equation (3.7) is explicitly solved to the optimal saving as follows.

$$(3.8)\quad S_2^* = \frac{Z R_2[A - B(Y_2 - T_2)] - [A - B(Y_1 - T_1)]}{B[1 + Z(R_2)^2]}$$

Therefore, by putting $\{A = 100, B = 1, Z = 0.9, R_2 = 1.1, Y_1 = 30, Y_2 = 10, T_1 = 0, T_2 = 0\}$ into equation (3.8), we can get $S_2^* = 9.14$. In this case, we can solve equation (3.7) for the optimal saving by using a handy calculator. However, we showed above the solution by using Mathematica so that we can apply the method to cases in which the number of variables and the number of equations are larger in simultaneous equations system which describes the first-order conditions for utility maximization.

Step 3. Use S_2^* of step 2 above to calculate the optimal consumption plan $\{C_1^*, C_2^*\}$ from the budget constraints.

By equation (3.3),

(3.9) $C_1{}^* = Y_1 - T_1 - S_2{}^* = 30 - 0 - 9.14 = 20.86$,

and by equation (3.4),

(3.10) $C_2{}^* = Y_2 - T_2 + R_2 S_2{}^*$
$= 10 - 0 + 1.1 \times 9.14 = 20.1$.

Question 1-1. In the Mathematica program of step 2, change the first period tax from $T_1 = 0$ to $T_1 = 10$. Denote the corresponding optimal consumption-saving plan as $\{C_1{}^*(1), C_2{}^*(1), S_2{}^*(1)\}$. Calculate $\{C_1{}^*(1), C_2{}^*(1), S_2{}^*(1)\}$.

Question 1-2. In the Mathematica program of step 2, change the first-period and the second period taxes from $T_1 = T_2 = 0$ to $T_1 = T_2 = 11/2.1$. Denote the corresponding optimal consumption-saving plan as $\{C_1{}^*(2), C_2{}^*(2), S_2{}^*(2)\}$. Calculate $\{C_1{}^*(2), C_2{}^*(2), S_2{}^*(2)\}$.

Question 1-3. For the optimal consumption plan $\{C_1{}^*(1), C_2{}^*(1)\}$ under $\{T_1 = 10, T_2 = 0\}$ and the optimal consumption plan $\{C_1{}^*(2), C_2{}^*(2)\}$ under $\{T_1 = 11/2.1, T_2 = 11/2.1\}$, choose the correct relationship among the following (A) ~ (E).

(A) $C_1{}^*(1) > C_1{}^*(2)$ and $C_2{}^*(1) > C_2{}^*(2)$
(B) $C_1{}^*(1) > C_1{}^*(2)$ and $C_2{}^*(1) < C_2{}^*(2)$
(C) $C_1{}^*(1) < C_1{}^*(2)$ and $C_2{}^*(1) > C_2{}^*(2)$
(D) $C_1{}^*(1) < C_1{}^*(2)$ and $C_2{}^*(1) < C_2{}^*(2)$
(E) $C_1{}^*(1) = C_1{}^*(2)$ and $C_2{}^*(1) = C_2{}^*(2)$

Question 1-4. About your answer to question 1-3 above, explain briefly why that relationship holds.

Question 1-5. Suppose the taxes change from $\{T_1 = 0, T_2 = 0\}$ to $\{T_1 = 10, T_2 = 0\}$. Use your answer to question 1-1to calculate the first period marginal propensity to consumption.

Question 1-6. Suppose the taxes change from $\{T_1 = 0, T_2 = 0\}$ to $\{T_1 = 11/2.1, T_2 = 11/2.1\}$. Use your answer to question 1-2 to calculate the first period marginal propensity to consumption.

Question 2. Three-Periods Lifecycle Model

The two-periods lifecycle model of question 1 above is extended to a three-periods lifecycle model as follows.

(3.11) $U = u(C_1) + Z u(C_2) + Z^2 u(C_3)\ , \quad Z > 0$

(3.12) $u(C) = AC - \frac{B}{2}C^2\ , \quad A > 0\ , \quad B > 0$

(3.13) $C_1 + S_2 = Y_1 - T_1$

(3.14) $C_2 + S_3 = Y_2 - T_2 + R_2 S_2$

(3.15) $C_3 = Y_3 - T_3 + R_3 S_3$

The individual makes an optimal three periods consumption-saving plan $\{C_1{}^*, C_2{}^*, C_3{}^*, S_2{}^*, S_3{}^*\}$ that maximizes the utility (3.11) subject to the budget constraints {(3.13), (3.14), (3.15)}. Just like the above two-periods utility maximization problem, the method to solve this utility maximization problem by using Mathematica consists of the following three steps.

Step 1. Derive the first-order conditions for the utility maximization that the optimal savings $\{S_2{}^*, S_3{}^*\}$ must satisfy.

Use the budget constraints {(3.13), (3.14), (3.15)} to express the utility, equation (3.11), as a function of savings $\{S_2, S_3\}$, as following equation (3.16).

(3.16) $U = u(Y_1 - T_1 - S_2) + Z u(Y_2 - T_2 + R_2 S_2 - S_3)$
$+ \ Z^2 u(Y_3 - T_3 + R_3 S_3)$

The optimal savings $\{S_2{}^*, S_3{}^*\}$ that maximize utility U subject to the budget constraints are solution to the following simultaneous equations {(3.17), (3.18)}.

(3.17) $\frac{dU}{dS_2} = -[A - B(Y_1 - T_1 - S_2)]$
$+ Z R_2 [A - B(Y_2 - T_2 + R_2 S_2 - S_3)] = 0$

(3.18) $\frac{dU}{dS_3} = -Z[A - B(Y_2 - T_2 + R_2 S_2 - S_3)]$
$+ \ Z^2 R_3 [A - B(Y_3 - T_3 + R_3 S_3)] = 0$

Equation (3.17) and equation (3.18) form a system of simultaneous equations with respect to $\{S_2, S_3\}$. If we specify the values of parameters $\{A, B, Z\}$, the interest rates $\{R_2, R_3\}$, incomes $\{Y_1, Y_2, Y_3\}$ and taxes $\{T_1, T_2, T_3\}$, then we can calculate the value of the optimal savings $\{S_2{}^*, S_3{}^*\}$ from equation (3.13) and equation

(3.14) (or (3.15)).

Step 2. Use Mathematica to calculate the value of the optimal savings $\{S_2^*, S_3^*\}$.

Here we present an example program in which $\{A = 100, B = 1, Z = 0.9, R_2 = R_3 = 1.1, Y_1 = 30, Y_2 = 20, Y_3 = 10, T_1 = 0, T_2 = 0, T_3 = 0\}$.

```
A = 100;
B = 1;
Z = 0.9;
R2 = 1.1;
R3 = 1.1;
Y1 = 30;
Y2 = 20;
Y3 = 10;
T1 = 0;
T2 = 0;
T3 = 0;
F1 =-(A - B*(Y1 - T1 - S2))+ Z*R2*(A -
    B*(Y2 - T2 + R2*S2 - S3));
F2 =-(A - B*(Y2 - T2 + R2*S2 - S3))+
    Z*R3*(A - B*(Y3 - T3 + R3*S3));
Solve [{F1 == 0, F2 == 0}, {S2, S3}]
```

Comment 2. The thirteenth row of the above program corresponds to equation (3.18), where both sides are divided by Z.

We run this program to get $\{S_2^* = 8.62, S_3^* = 8.89\}$.

Step 3. Use $\{S_2^*, S_3^*\}$ of step 2 above to calculate the optimal consumption plan $\{C_1^*, C_2^*, C_3^*\}$ from the budget constraints.

By equations {(3.13), (3.14), (3.15)},

$$(3.19)\quad C_1^* = Y_1 - T_1 - S_2^* = 21.38 \text{ 、}$$

$$(3.20)\quad C_2^* = Y_2 - T_2 + R_2 S_2^* - S_3^* = 20.59$$

$$(3.21)\quad C_3^* = Y_3 - T_3 + R_3 S_3^* = 19.78.$$

Question 2-1. In the Mathematica program of step 2, change the first period tax from $T_1 = 0$ to $T_1 = 10$. Denote the corresponding optimal consumption-saving plan as $\{C_1^*(1), C_2^*(1), C_3^*(1), S_2^*(1), S_3^*(1)\}$. Calculate $\{C_1^*(1), C_2^*(1), C_3^*(1), S_2^*(1), S_3^*(1)\}$.

Question 2-2. In the Mathematica program of step 2, change the first-period and the second period taxes from $T_1 = T_2 = T_3 = 0$ to $T_1 = T_2 = T_3 = 12.1/3.31$. Denote the corresponding optimal consumption-saving plan as $\{C_1^*(2), C_2^*(2), C_3^*(2), S_2^*(2), S_3^*(2)\}$. Calculate $\{C_1^*(2), C_2^*(2), C_3^*(2), S_2^*(2), S_3^*(2)\}$.

Question 3. Ten-Periods Lifecycle Model.

The utility maximization problems of question 1 and question 2 are extended further to the following ten-periods problem.

$$(3.22)\quad U = u(C_1) + Zu(C_2) + Z^2 u(C_3) + \cdots + Z^8 u(C_9) + Z^9 u(C_{10}),\quad Z > 0$$

$$(3.23)\quad u(C) = AC - \frac{B}{2}C^2,\quad A > 0,\quad B > 0$$

$$(3.24)\quad C_1 + S_2 = Y_1 - T_1$$

$$(3.25)\quad C_2 + S_3 = Y_2 - T_2 + R_2 S_2$$

$$(3.26)\quad C_3 + S_4 = Y_3 - T_3 + R_3 S_3$$

$$\cdots$$

$$(3.27)\quad C_9 + S_{10} = Y_9 - T_9 + R_9 S_9$$

$$(3.28)\quad C_{10} = Y_{10} - T_{10} + R_{10} S_{10}$$

Question 3-1. Describe a Mathematica program for calculating the optimal savings $\{S_2^*, S_3^*, \cdots, S_{10}^*\}$. In the program, assume $\{A = 100, B = 1, Z = 1, R_2 = R_3 = \cdots = R_{10} = 1, Y_1 = Y_2 = \cdots = Y_{10} = 10, T_1 = T_2 = \cdots = T_{10} = 0\}$.

Question 3-2. Run the program of question 3-1 to get the value of the optimal savings $\{S_2^*, S_3^*, \cdots, S_{10}^*\}$. Then use the budget constraints, equations (3.24) ~ (3.28), to calculate the optimal consumptions $\{C_1^*, C_2^*, \cdots, C_{10}^*\}$.

Question 3-3. In the Mathematica program of question 3-1, change the values of $\{Z, R_2, R_3, \cdots, R_{10}\}$ from $\{Z = 1, R_2 = R_3 = \cdots = R_{10} = 1\}$ to $\{Z = 0.9, R_2 = R_3 = \cdots = R_{10} = 1.1\}$. Denote the corresponding optimal consumption-saving plan as $\{C_1^*(1), C_2^*(1), \cdots, C_{10}^*(1), S_2^*(1), S_3^*(1), \cdots, S_{10}^*(1)\}$. Calculate $\{C_1^*(1), C_2^*(1), \cdots, C_{10}^*(1), S_2^*(1), S_3^*(1), \cdots, S_{10}^*(1)\}$.

Question 3-4. Use MS-Excel to draw the graphs of the optimal consumption-saving plans of question 3-2 and question 3-3 as follows.

(i) In graph 1, measure the time periods $\{t = 1, 2, \cdots, 10\}$ by the horizontal axis, and plot the optimal savings $\{\{S_2^*, S_3^*, \cdots, S_{10}^*\}, \{S_2^*(1), S_3^*(1), \cdots, S_{10}^*(1)\}\}$ by the vertical axis.

(ii) In graph 2, measure the time periods $\{t = 1, 2, \cdots, 10\}$ by the horizontal axis, and plot the optimal consumptions $\{\{C_1^*, C_2^*, \cdots, C_{10}^*\}, \{C_1^*(1), C_2^*(1), \cdots, C_{10}^*(1)\}\}$ by the vertical axis.

Question 3-5. In the assumption of question 3-3, $\{A = 100, B = 1, Z = 0.9, R_2 = R_3 = \cdots = R_{10} = 1.1, Y_1 = Y_2 = \cdots = Y_{10} = 10, T_1 = T_2 = \cdots = T_{10} = 0\}$, consider the changes in taxes described as the following two cases.

Case 1. (Permanent Tax Increase) $T_1 = T_2 = \cdots = T_{10} = 1$

Case 2. (Temporary Tax Increase) $T_1 = 1, T_2 = \cdots = T_{10} = 0$

Calculate the optimal consumption plans for case 1 and case 2. Then, as you did in question 1-5, ca Calculate the optimal consumption plans for case 1 and case 2. Then, as you did in question 1-5, calculate the marginal propensities to consumption of the first period for case 1 and case 2.

Question 3-6. The present value of taxes, denoted as VT, in the ten-periods lifecycle model is defined as follows.

$$(3.29)\quad VT = T_1 + \frac{T_2}{R_2} + \frac{T_3}{R_2 \times R_3} + \frac{T_4}{R_2 \times R_3 \times R_4} + \cdots + \frac{T_{10}}{R_2 \times R_3 \times \ \ldots \ \times R_{10}}$$

Specify the quasi-permanent tax increase as the following case 3.

Case 3. (Quasi-Permanent Tax Increase)

$$T_1 = T_2 = \cdots = T_5 = \tilde{T},\ T_6 = T_7 = \cdots = T_{10} = 0$$

Calculate the value of $\tilde{T}$ that equates the VT of case 1 and the VT of case 2.

Question 3-7. Calculate the optimal consumption-saving plan for case 3. Then, as you did in question 3-4, draw the graphs of the optimal consumption-saving plans of case 1 and case 3.

Question 3-8. Calculate the first period marginal propensity to consumption for case 3.

Question 3-9. In the assumption of question 3-3, $\{A = 100, B = 1, Z = 0.9, R_2 = R_3 = \cdots = R_{10} = 1.1, Y_1 = Y_2 = \cdots = Y_{10} = 10, T_1 = T_2 = \cdots = T_{10} = 0\}$, change the value of Z from $Z = 0.9$ to $Z = 0.92$. Denote the corresponding optimal consumption-saving plan as $\{C_1^*(2), C_2^*(2), \cdots, C_{10}^*(2), S_2^*(2), S_3^*(2), \cdots, S_{10}^*(2)\}$. Calculate $\{C_1^*(2), C_2^*(2), \cdots, C_{10}^*(2), S_2^*(2), S_3^*(2), \cdots, S_{10}^*(2)\}$. Then, as you did in question 3-4, draw the graphs of the optimal consumption-saving plans of question 3-3, $\{C_1^*(1), C_2^*(1), \cdots, C_{10}^*(1), S_2^*(1), S_3^*(1), \cdots, S_{10}^*(1)\}$ and $\{C_1^*(2), C_2^*(2), \cdots, C_{10}^*(2), S_2^*(2), S_3^*(2), \cdots, S_{10}^*(2)\}$.

Question 3-10. Explain the properties of the graphs of question 3-9, and explain why the graphs look like the way you draw.

Appendix: Solving Equations by Mathematica

A1. Solving Single Variable Equation.

Example 1. The solution to a single variable equation $f(x) = -3x + 6 = 0$ is $x = 2$. (See figure A-1.)

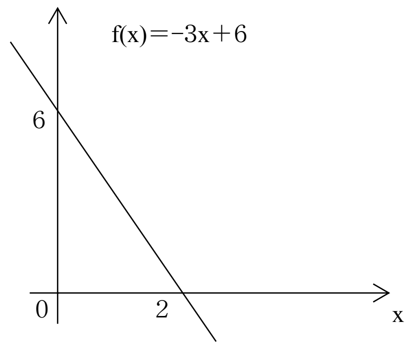
Step 1. Start Mathematica, and type the following lines.

```
f =- 3*x + 6;
Solve[f == 0, x]
```

Remark 1. When you run the above program, Mathematica will display all the lines that do not have " ; " at the end.

Remark 2. In Mathematica program, distinction

Figure A-1



between upper-case letters and lower-case letters is important. For example, “Solve” is a Mathematica command, but “solve” is not.

Step 2. Press [Shift] key and [Enter] key to run the above program.

A2. Solving Simultaneous Equations.

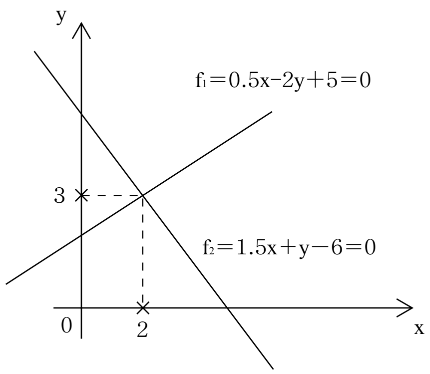
Example 2. Two Equations with two variables $\{x, y\}$

$$f_1(x, y) = 0.5x - 2y + 5 = 0$$

$$f_2(x, y) = 1.5x + y - 6 = 0$$

The solution to this simultaneous equations system is $\{x = 2, y = 3\}$. (See figure A-2.)

Figure A-2



Step 1. Start Mathematica, and type the following lines.

```
f1 = 0.5*x - 2*y + 5;
f2 = 1.5*x + y - 6;
Solve[{f1 == 0, f2 == 0}, {x, y}]
```

Step 2. Press [Shift] key and [Enter] key to run the above program.

References.

[1] Lucas, R. E., Models of Business Cycles, Blackwell, 1987..

[2] Samuelson, P. A., Economics, 11th edition, McGraw-Hill, 1980.

[3] Sargent, T. J., Macroeconomic Theory, 2nd edition, Academic Press, 1987.