Cointegration Rank Tests In Vector ARMA Models

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Abstract

This paper proposes several parametric hypothesis tests to determine the cointegration rank in multivariate time series expressed as vector ARMA models. The tests are constructed based on the instrumental variables (Ⅳ) method. It is established that adopting critical values for the standard Johansen likelihood ratio (LR) test is valid in that the same limiting distribution reached. Some Monte Carlo experiments also show that the proposed tests exhibit sufficiently desirable finite sample performances.

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1 Introduction

The cointegration rank has been, in general, detected by the methodology based on likelihood ratio (LR) tests proposed by Johansen (1988, 1992) (Johansen methodology). However, its asymptotic validity has not been established for the data generating processes (DGPs) other than finite lag-order vector autoregressions (VARs).

Some semiparametric and nonparametric approaches to the rank determination may be considered to compensate for limitation of LR tests. For the case in which the VAR lag-order is infinite, Saikkonen (1992) and Qu and Perron (2007) have discussed the applicability of the Johansen methodology based on a finite lag-order approximation of the infinite lag-order or the determination of an optimal lag-order.

In contrast, Shintani (2001) proposed nonparametric tests for rank determination without formulating any vector autoregression scheme. However, these tests need a kernel estimator with its bandwidth parameter in construction, and the tests comparatively slow rate of convergence/divergence of some statistics forming the tests as some defect in using a kernel estimator deteriorates the test performances, as shown in Shintani (2001).

This paper proposes several testing methods following a parametric approach in the DGPs that are not expressed as finite order VARs. For this purpose, the DGPs considered are confined to vector ARMA (VARMA) models, and we tried to construct some tests such that the Johansen LR tests and their asymptotics are utilized as much as possible. We also discuss and impose conditions to exclude several VARMA structures, including pure VMA models, as one of the requirements to make the proposed tests valid. One reason why LR tests are not applicable to VARMA models with some validity in the asymptotic sense is that the lagged differenced series as explanatory variables in reduced rank regression are correlated with the error term formed as a VMA. In this study, we develop two methods constructed based on a technique known as the method of instrumental variables (Ⅳ). For this purpose, we first estimate time series that are not correlated with the error term (and the error correction term) but are correlated with the lagged differenced series. In so far as possible, it proceeds in a manner similar to that used for LR tests based on reduced rank regression, apart from estimation of the long-run covariance matrix of the residuals using a fixed bandwidth kernel such that the bandwidth is equal to the lag-order of the vector MA (VMA).

This paper establishes that one of the tests proposed is the same limiting distribution as for the LR test in the most standard case and that the critical values based on the limiting distribution are applicable to the tests with some asymptotic validity. It is also shown that the tests are consistent and are as asymptotically powerful as the LR test.

Monte Carlo experiments are executed in selected DGPs and sample sizes 200 and 500 in order to investigate finite sample performances of the tests proposed, comparing the LR test or Shintani tests. The experimental results show that the tests proposed perform comparatively well and are stable throughout the whole DGPs, particularly when the sample size is 500, and sufficiently reflects the asymptotics. Results also reveal that one of the tests proposed exhibits satisfactory performances even in DGPs that are 'nearly' non-invertible in the sense that the characteristic equation of the VMA structure possesses one root close to -1, whereas the performances of most remaining tests deteriorate under such DGPs.

The paper is organized as follows. Section 2 formulates the DGP as a VARMA model and discuss the conditions mentioned above as some preliminaries. The tests proposed are presented in Section 3, and asymptotics for the tests are established in Section 4. Section 5 deals with Monte Carlo experiments. The remaining issues, along with some concluding remarks, are discussed in Section 6. The proofs of a lemma and theorems in the text are provided in Appendix.

2 The DGP and some preliminaries

Consider a k-dimensional time series y_t expressed as

$$\Delta y_t = \alpha\beta' y_{t-1} + \sum_{i=1}^{p} H_i \Delta y_{t-i} + \epsilon_t + \sum_{i=1}^{q} D_i \epsilon_{t-i}, \quad (1)$$

where $t \geq 1$, α and β are $k\times r$ full column rank matrices with r such that $1 \leq r \leq k-1$ and $rank\, \beta'\alpha = r$, p and q are positive integers, H_i and D_j are $k\times k$ matrices such that $H_p \neq 0$ and $D_q \neq 0$, $\{\varepsilon_t\}$is i.i.d.(0, Λ) with a $k\times k$ positive definite matrix Λ and finite fourth order cumulants of elements of ε_t, and it is assumed that $y_{-i} = O_p(1)$ or $O(1)$ for any $i \geq 0$, that det $A(z) \neq 0$ for $\forall |z| \leq 1$ except $z=1$ and det $D(z)\neq 0$ for $\forall |z| \leq 1$, where

$$A(z) = -\alpha\beta' z + (1-z)(I - \sum_{i=1}^{p} H_i z^i),$$

$$D(z) = I + \sum_{i=1}^{q} D_i z^i,$$

with I denoting the identity matrix, and that $A(z)$ and $D(z)$ have no common left divisors as $U(z)$ satisfying

$A(z) = U(z)\, A_1(z)$, $\quad D(z) = U(z)\, D_1(z)$ *and either* $A_1(z) = I + A_{1;1} z$ *or* $D_1(z) = I$,

where $A_{1;1}$ is a $k\times k$ matrix. We also make the assumption that

$$rank\, \delta' \left(I - \sum_{i=1}^{p} H_i\right) \gamma = k - r,$$

for any $k\times(k-r)$ full column rank matrices δ and γ such that $\delta'\alpha = 0$ and $\beta'\gamma = 0$.

(1) is a VRAMA model for the vector time series system y_t that is cointegrated with the cointegration rank r and may correspond to the expression derived in the Granger representation theorem by Engle and Granger (1987) if $D(z) = d(z)I$ with a scalar polynomial $d(z) = 1 + \sum_{i=1}^{q} d_i z^i$, although not being any finite order VAR. Note that neither a pure VAR nor a pure VMA is dealt with as the DGP, and also note that the last one of the assumptions above is made to exclude the occurrence of multicointegration (Granger and Lee (1990) e.g.), which is equivalent to the one imposed in Johansen (1996, pp. 55-57) or Assumption A3 in Banerjee et.al (1993, p. 147). We shall include the case in which y is not cointegrated (i.e. the case of $r=0$) in our analysis. However, this case is trivial, and this paper does not provide its formulation.

It is assured by the above assumptions that

$$\det \left[A(z)\beta,\ (I - \sum_{i=1}^{p} H_i z^i)\gamma\right] \neq 0$$

for $\forall |z| \leq 1$. Putting

$$C(z) = [\beta,\ \gamma] \begin{bmatrix} (1-z)I & 0 \\ 0 & I \end{bmatrix} \cdot \left[A(z)\beta,\ (I - \sum_{i=1}^{p} H_i z^i)\gamma \right]^{-1},$$

we can derive an infinite order VMA expression for Δy_t (Johansen (1996, p. 55 Theorem 4.5) e.g.):

$$\begin{aligned} \Delta y_t &= C(B)D(B)\epsilon_t \\ &= C(1)D(1)\epsilon_t + \{C^{(1)}(B)D(B) \\ &\quad + C(1)D^{(1)}(B)\}(\epsilon_t - \epsilon_{t-1}), \quad (2) \end{aligned}$$

where B is the backward operator, $C^{(1)}(z) = \sum_{i=0}^{\infty} (-\sum_{j=i+1}^{\infty} C_j)\, z^i$ with C_i that decay exponentially as i increases and such that

$$C(z) = I + \sum_{i=1}^{\infty} C_i z^i, \quad D^{(1)}(z) = \sum_{i=0}^{q-1} (-\sum_{j=i+1}^{q} D_j) z^i.$$

It should be noted that $C(1) = \gamma\tau\delta$ with γ and δ defined above and a nonsingular matrix τ of $(k-r)\times(k-r)$. Without losing generality, suppose that $\delta' D(1) \Lambda D(1)'\delta = I$.

(1) may be written as

$$\Delta y_t = \alpha\beta' y_{t-q-1} + \sum_{i=1}^{m} \bar{H}_i \Delta y_{t-i} + \epsilon_t + \sum_{i=1}^{q} D_i \epsilon_{t-i}, \quad (3)$$

where $m = max\{p, q, 2\}$, $\bar{H}_i = H_i + \alpha\beta'$ if $i \leq min\{p, q\}$, $\bar{H}_i = \alpha\beta'$ if $p < i \leq q$, $\bar{H}_i = H_i$ if $q < i \leq p$ and $\bar{H}_2 = 0$ if $p = q = 1$. We shall develop our discussion under (3) rather (1).

Now, consider conditions to exclude several VARMA structures, although the condition to fully identify the VARMA for weakly stationary series (Priestley (1981, p. 802) e.g.) is not imposed. Put

$$\Delta z_{t-i;j} = \left(\Delta y'_{t-i}, \cdots, \Delta y'_{t-i-j+1}\right)', \quad \bar{H}' = \left[\bar{H}_1, \cdots, \bar{H}_m\right],$$

let $P(\bar{x}_{t-i;0} \,|\, \{\bar{x}_{t-j;1}\ ; j = \bar{m}, \cdots, \bar{n}\})$ stand for the linear least-square predictor of a vector time series $\bar{x}_{t-j;0}$ onto $\{\bar{x}_{t-j;1}\ ; j = \bar{m}, \cdots, \bar{n}\}$ as the Hilbert space spanned by vector time series $\bar{x}_{t-j;1}\ ; j = \bar{m}, \cdots, \bar{n}$, and let $\mathcal{F}_t$, $\mathcal{G}_{t-1}$ and $\mathcal{H}_{t-q-1}$ denote the Hilbert spaces given as

$$\begin{aligned} \mathcal{F}_t &= \{\beta' y_{t-q-1},\ \epsilon_{t-i},\ i = 0, 1, \cdots, q\}, \\ \mathcal{G}_{t-1} &= \{\beta' y_{t-q-1},\ \epsilon_{t-i},\ i = 1, \cdots, q\}, \\ \mathcal{H}_{t-q-1} &= \{\epsilon_{t-q-i} - P(\epsilon_{t-q-i} \,|\{\beta' y_{t-q-1}\}),\ \forall i \geq 1\}. \end{aligned}$$

We first consider the following condition:

Condition (A): *There exist a k-dimensional nonzero vector $\bar{f}$ and a matrix $\bar{F}$ of $mk \times k$ such that:*

(i) *$\bar{f}'\Delta y_t$ is spanned not only by elements in $\mathcal{F}_t$ but also by those in $\mathcal{H}_{t-q-1}$, and*

(ii) *Any nonzero linear combination of $\bar{F}'\Delta z_{t-1;m}$ is spanned not only by elements in $\mathcal{G}_{t-1}$ but also by those in $\mathcal{H}_{t-q-1}$.*

Condition (A) (i) is put to exclude the case in which Δy_t is expressed as

$$\epsilon_t + \sum_{i=1}^{q} \Phi_i \epsilon_{t-i} + \bar{\alpha}\beta' y_{t-q-1},$$

where Φ_i are $k \times k$ matrices and $\bar{\alpha}$ is a $k \times r$ matrix, in view of (2) and (3). It is then easy to see from (3) that $\bar{H} \neq 0$, and (i) is equivalently expressed as the one based on $\bar{H}'\Delta z_{t-1;m}$, i.e., there exists at least one $\bar{f}$ such that $\bar{f}'\bar{H}'\Delta z_{t-1;m}$ is spanned not only by elements in $\mathcal{G}_{t-1}$ but also by those in $\mathcal{H}_{t-q-1}$. Therefore, the case in which Condition (A) (i) does not hold may substantially render meaningless the role of $\bar{H}'\Delta z_{t-1;m}$ in (3). Similarly, the absence of Condition (A) (ii) implies that there exist no k linearly independent linear combinations of $\Delta z_{t-1;m}$ that are spanned not only by elements in $\mathcal{G}_{t-1}$ but also by those in $\mathcal{H}_{t-q-1}$, suggesting that there exist k-dimensional VARM models with lower VAR and VMA orders.

Second, we focus our attention on:

Condition (B): *There exist no k-dimensional vectors such that $\bar{f}'\bar{H}' \neq 0$ and $\bar{f}'\bar{H}'\Delta z_{t-1;m}$ is expressed as a linear combination of elements in $\mathcal{G}_{t-1}$.*

This condition states that for a k-dimensional nonzero vector $\bar{f}$, $\bar{f}'\Delta y_t$ is expressed as a linear combination of elements in $\mathcal{F}_t$ if and only if $\bar{f}'\bar{H}' = 0$. Now, consider the case in which there exists a $k \times \bar{h}_1$ matrix $\bar{F}_1$ of $k \times \bar{h}_1$ such that $\bar{h}_1 \leq k-1$, $rank\ \bar{F}_1'\bar{H}' = \bar{h}_1$ and

$$\bar{F}_1'\bar{H}'\Delta z_{t-1;m} = \sum_{i=1}^{q} \bar{\Phi}_i' \epsilon_{t-i} + \bar{\Phi}\beta' y_{t-q-1},$$

where $\bar{\Phi}_i$ are $k \times \bar{h}_1$ matrices and $\bar{\Psi}$ is a $\bar{h}_1 \times r$ matrix, provided that Condition (A) holds. Such case does not satisfy the identifiability of the VARMA parameters of (1), since we have a VARMA model given as

$$\Delta y_t = \alpha_* \beta' y_{t-q-1} + \sum_{i=1}^{m} \bar{H}_{i;*} \Delta y_{t-i} + \epsilon_t + \sum_{i=1}^{q} D_{i;*} \epsilon_{t-i}, \quad (4)$$

where

$$\alpha_* = \alpha - \bar{F}_1 \bar{\Phi}, \quad \bar{H}_{i;*} = \bar{H}_i + \bar{F}_1 \bar{F}_1' \bar{H}_i \quad and \quad D_{i;*} = D_i - \bar{F}_1 \bar{\Phi}_i'.$$

Next, let $k_1(n)$ and $\bar{k}(n)$ be the integers given as

$$k_1(n) = rank\, E\{\Delta z_{t-q-1;n} - P(\Delta z_{t-q-1;n} \mid \{\beta' y_{t-q-1}\})\} \Delta z_{t-1;m}' \bar{H},$$
$$\bar{k}(n) = rank\, E\{\Delta z_{t-q-1;n} - P(\Delta z_{t-q-1;n} \mid \{\beta' y_{t-q-1}\})\} \Delta z_{t-1;m}',$$

where n is a positive integer either equal to or strictly greater than m We then have the following lemma.

Lemma: *Suppose that y_t is generated by* (1) *and that Conditions* (*A*) *and* (*B*) *hold. Then, for any positive integer n either equal to or strictly greater than m,* $k_1(n) = rank\, \bar{H}$ *and* $\bar{k}(n) \geq k$.

This lemma implies that any nonzero linear combination of $\bar{H}' \Delta z_{t-1;m}$ is spanned not by only elements in $\{\beta' y_{t-q-1},\ \epsilon_{t-i}\ ;\ i=1, \cdots, q\}$ but also by those in $\{\Delta z_{t-q-1;n} - P(\Delta z_{t-q-1;n} | \{\beta' y_{t-q-1}\})\}$ and that there exist k linearly independent linear combinations of $\Delta z_{t-1;m}$ that are correlated with $\Delta z_{t-q-1;n} - P(\Delta z_{t-q-1;n} | \{\beta' y_{t-q-1}\})$. Owing to this, an appropriate Ⅳ will be constructed based on $\Delta z_{t-q-1;n}$, as clarified in the proof of Theorem 1. As seen later, a concrete value of n is used for the test construction. This lemma states that m is the minimum of n satisfying $k_1(n) = rank\, \bar{H}$ and $\bar{k}(n) \geq k$.

3 Test statistics

Given T observations $y_1, \cdots, y_T$ in (1) or (3), this section presents testing methods to determine the cointegration rank r. We first introduce a series of data matrix notations:

$$\hat{Y}_{-h} = [\Delta y_{2q+n-h+2}, \cdots, \Delta y_{T-h}]', \quad Y_{-h} = [y_{2q+n+2-h}, \cdots, y_{T-h}]',$$

which are of $\bar{T} \times k$, where $\bar{T} = T - (2q+n) - 1$ and $h = 0, 1, \cdots, 2q+n$,

$$\hat{Z}_{-1-h;m} = \left[\hat{Y}_{-1-h}, \cdots, \hat{Y}_{-m-h}\right],$$
$$\hat{Z}_{-q-1-h;n} = \left[\hat{Y}_{-q-1-h}, \cdots, \hat{Y}_{-q-n-h}\right],$$

of $\bar{T} \times mk$ and $\bar{T} \times nk$ respectively, with the supposition that hereafter h takes values in $\{0, 1, \cdots, q\}$. Second, put.

$$\tilde{S} = I - Y_{-q-1} \left(Y_{-q-1}' Y_{-q-1}\right)^{-1} Y_{-q-1}',$$
$$\hat{N}_{-2} = I - \hat{Z}_{-2;m} \left(\hat{Z}_{-2;m}' \hat{Z}_{-2;m}\right)^{-1} \hat{Z}_{-2;m}',$$

and let each of $\tilde{B}$, $\hat{G}$ and $\hat{B}$ be the matrix whose columns are the eigenvectors, of the respective matrix below, which correspond to the k largest eigenvalues arranged in descending order: For $\tilde{B}$ of $nk \times k$,

$$(T^{-1} \hat{Z}_{-q-1;n}' \tilde{S} \hat{Y}_0)\, (T^{-1} \hat{Y}_0' \tilde{S} \hat{Z}_{-q-1;n}),$$

$\hat{G}$ of $mk \times k$,

$$(T^{-1} \hat{Z}_{-1;m}' \tilde{S} \hat{Z}_{-q-1;n}) \tilde{B}\, \tilde{B}' (T^{-1} \hat{Z}_{-q-1;n}' \tilde{S} \hat{Z}_{-1;m}),$$

and for $\hat{B}$ of $nk \times k$,

$$(T^{-1} \hat{Z}_{-q-1;n}' \tilde{S} \hat{Z}_{-1;m}) \hat{G}\, \hat{G}' (T^{-1} \hat{Z}_{-1;m}' \tilde{S} \hat{Z}_{-q-1;n}).$$

Third, put

$$\hat{M}_{-h} = I - \hat{Z}_{-1-h;m} \hat{G} \left(\hat{B}' \hat{Z}_{-q-1-h;n}' \hat{Z}_{-1-h;m} \hat{G}\right)^{-1} \cdot \hat{B}'\, \hat{Z}_{-q-1-h;n}',$$

and define the notations S_{ij} and $S_{00;0}$ as

$$S_{11} = T^{-1} Y_{-q-1}' \hat{N}_{-2} Y_{-q-1},$$
$$S_{10} = T^{-1} Y_{-q-1}' \hat{M}_0 \hat{Y}_0, \qquad S_{10} = S_{01}',$$
$$S_{00} = T^{-1} \hat{Y}_0' \hat{M}_0' \hat{M}_0 \hat{Y}_0 + \sum_{h=1}^{q} \left(T^{-1} \hat{Y}_0' \hat{M}_0' \hat{M}_{-h} \hat{Y}_{-h} + T^{-1} \hat{Y}_{-h}' \hat{M}_{-h}' \hat{M}_0 \hat{Y}_0\right),$$
$$S_{00;0} = T^{-1} \hat{Y}_0' \hat{M}_0' \hat{M}_0 \hat{Y}_0.$$

Finally, let $\hat{\lambda}_1, \cdots, \hat{\lambda}_k$ be the eigenvalues which are obtained by solving the eigenvalue problem of the equation

$$\det\{\lambda\, S_{11} - S_{10}\, S_{00}^{-1}\, S_{01}\} = 0,$$

such that $|\hat{\lambda}_1| \geq \cdots \geq |\hat{\lambda}_k|$.

A test statistic for the null $r = j$ and the alternative $r \geq j+1$ is now given as

$$\hat{Q}_j = T \sum_{i=j+1}^{k} |\hat{\lambda}_i|, \quad for\ j = 0, 1, \cdots, k-2.$$

The value of r is determined by the test executed in ascending order with respect to the value of j, implying that this is done in the same manner as the Johansen methodology (Johansen (1992) or Johansen (1996, pp. 98-100) e.g.). Note that $\hat{B}'\hat{Z}'_{-q-1;n}$ plays the role of an IV in some reduced rank regression used for the construction of $\hat{Q}_j$, in the sense that this is asymtotically correlated not with $\hat{Z}_{-1;m}$ but with $\hat{Z}_{-1;m}\bar{H}$, as seen clearly in the proof of Theorem 1 later.

Note that S_{00} used for the construction of $\hat{Q}_j$ is an estimator of the long-run covariance matrix of some residual series using the truncated kernel of a finite-lag equal to q. In this connection, a consistent estimator for $(\delta' D(1)\Lambda D(1)'\delta)^{-1}$ is formed based on S_{00}^{-1} under the null, as seen in the proof of Theorem 1. Therefore, the accuracy of the consistent estimator may be severely impaired by the presence of one root close to -1 in det $D(z)=0$, and consequently such a root may considerably lower the performance of $\hat{Q}_j$. As mentioned in the introduction and shown by Monte Carlo experiments later, the performance of $\hat{Q}_j$ becomes far from satisfactory in finite samples when such a DGP is adopted.

We provide a test to handle such 'near' non-invertibility issues. Put

$$S_{00;0} = T^{-1}\hat{Y}_0'\hat{M}_0'\hat{M}_0\hat{Y}_0,$$

let $\tilde{\lambda}_1, \cdots \tilde{\lambda}_k$ be the eigenvalues corresponding to the equation

$$\det\{\lambda S_{11} - S_{10}S_{00;0}^{-1}S_{01}\} = 0,$$

such that $|\tilde{\lambda}_{*;1}| \geq \cdots \geq |\tilde{\lambda}_{*;k}|$ and again put

$$\tilde{Q}_{*;j} = T\sum_{i=j+1}^{k}|\hat{\lambda}_{*;i}|, \quad for\ j = 0, \cdots, k-2.$$

Another statistic for the null $r=j$ and the alternative $r \geq j+1$ is now given as

$$\hat{Q}_{*;j} = min\{\hat{Q}_j,\ \tilde{Q}_{*;j}\}, \quad for\ j = 0, \cdots, k-2.$$

For the case in which Condition (A) does not hold, we may provide similar tests: Replace S_{ij} with the counterpart constructed by substituting I for all the $\hat{M}_{-h}$. Then, construct $\hat{Q}_j$ based on such S_{ij}.

It should be also noted that these statistics can be easily extended to more general models that allow a drift or deterministic trends, by replacing $\tilde{S}$, $\hat{N}_{-2}$ and $\hat{M}_{-h}$ with the counterparts dealing with such extension. For example, let τ be the $\bar{T}$-dimensional vector whose all elements are equal to 1 and put

$$\tilde{M} = I - \tau(\tau'\tau)^{-1}\tau'.$$

Then, based on these, redefine $\tilde{S}$, $\hat{N}_{-2}$ and $\hat{M}_{-h}$ as

$$\tilde{S} = \tilde{M} - \tilde{M}Y_{-q-1}\left(Y_{-q-1}'\tilde{M}Y_{-q-1}\right)^{-1}Y_{-q-1}'\tilde{M},$$
$$\hat{N}_{-2} = \tilde{M} - \tilde{M}\hat{Z}_{-2;m}\left(\hat{Z}_{-2;m}'\tilde{M}\hat{Z}_{-2;m}\right)^{-1}\hat{Z}_{-2;m}'\tilde{M},$$
$$\hat{M}_{-h} = \tilde{M} - \tilde{M}\hat{Z}_{-1-h;m}\hat{G} \cdot \left(\hat{B}'\hat{Z}_{-q-1-h;n}'\tilde{M}\hat{Z}_{-1-h;m}\hat{G}\right)^{-1}\hat{B}'\hat{Z}_{-q-1-h;n}'\tilde{M},$$

which are used in forming the test statistics corresponding to (3) that allows a nonzero constant vector, implying that y_t is allowed to possess a linear deterministic trend.

4 Asymptotics

In order to derive asymptotics for the test statistics, let the symbol $W_{k-r}(u)$ stand for a $(k-r)$-dimensional standard Brownian motion on [0, 1]. In addition, put

$$u_t = \epsilon_t + \sum_{i=1}^{q}D_i\epsilon_i, \quad \bar{U}_{-h} = [u_{2q+n+2-h}, \cdots, u_{T-h}]',$$
$$w_t = \{C^{(1)}(B)D(B) + C(1)D^{(1)}(B)\}\epsilon_t,$$
$$\Sigma_{11;0} = Ew_tw_t', \quad \Sigma_{12;0} = Ew_{t-q}\Delta z_{t;m}',$$
$$\Sigma_{31} = E\Delta z_{t;n}w_t' \quad \Sigma_{32} = E\Delta z_{t-q-1;n}\Delta z_{t-1;m}',$$
$$\bar{\Sigma}_{11} = E\{w_{t-q-1} - P(w_{t-q-1}\,|\,\{\Delta z_{t-2;m}\})\} \cdot \{w_{t-q-1}' - P(w_{t-q-1}'\,|\,\{\Delta z_{t-2;m}\})\},$$

Moreover, put

$$\tilde{G} = \hat{G}\left(\hat{B}'\Sigma_{32}\hat{G}\right)^{-1}\hat{B}'\Sigma_{31}\beta,$$
$$\tilde{R}_* = \beta'\Sigma_{11;0}\beta - \beta'\Sigma_{12;0}\tilde{G},$$
$$\hat{R}_{1;h} = T^{-1}\left(\beta'Y_{-q-1}' - \tilde{G}'\hat{Z}_{-1;m}'\right) \cdot \left(Y_{-q-1-h}\beta - \hat{Z}_{-1-h;m}\tilde{G}\right),$$
$$\hat{R}_{2;h} = T^{-1}\left(\beta'Y_{-q-1}' - \tilde{G}'\hat{Z}_{-1;m}'\right)\bar{U}_{-h},$$
$$\hat{R}_{3;h} = -(T^{-1}\bar{U}_0'\hat{Z}_{-1-h;m})\tilde{G}.$$

Based on these, define $\hat{\Sigma}$ and $\hat{\Sigma}_*$ as

$$\hat{\Sigma} = D(1)\Lambda D(1)' + \alpha\{\hat{R}_{1;0} + \sum_{h=1}^{q}(\hat{R}_{1;h} + \hat{R}_{1;h}')\}\alpha' + \alpha\{\hat{R}_{2;0} + \sum_{h=1}^{q}(\hat{R}_{2;h} + \hat{R}_{3;h}')\} + \{\hat{R}_{3;0} + \sum_{h=1}^{q}(\hat{R}_{3;h} + \hat{R}_{2;h}')\}\alpha',$$

$$\hat{\Sigma}_* = \Lambda + \sum_{i=1}^{q} D_i \Lambda D_i' + \alpha \hat{R}_{1;0}\alpha' + \alpha \hat{R}_{2;0} + \hat{R}_{3;0}\alpha',$$

and $\hat{\Omega}$ and $\hat{\Omega}_*$ as

$$\hat{\Omega} = \left(\beta' \bar{\Sigma}_{11} \beta'\right)^{-1/2} \tilde{R}_* \alpha' \hat{\Sigma}^{-1} \alpha \tilde{R}_*' \left(\beta' \bar{\Sigma}_{11} \beta'\right)^{-1/2},$$

$$\hat{\Omega}_* = \left(\beta' \bar{\Sigma}_{11} \beta'\right)^{-1/2} \tilde{R}_* \alpha' \hat{\Sigma}_*^{-1} \alpha \tilde{R}_*' \left(\beta' \bar{\Sigma}_{11} \beta'\right)^{-1/2}$$

We now set up the following theorem:

Theorem 1: *Suppose that y_t is generated by* (1) *and that Conditions* (*A*) *and* (*B*) *hold. Then, for $\forall n \geq m$ and $\hat{Q}_j$ corresponding to it, we have:*

(i) *The limiting distribution of $\hat{Q}_r$ is given as*

$$\breve{M}_r = tr\left\{\left(\int_0^1 dW_{k-r}(u) W_{k-r}'(u)\right) \times \left(\int_0^1 W_{k-r}(u) W_{k-r}'(u) du\right)^{-1} \times \left(\int_0^1 W_{k-r}(u) dW_{k-r}'(u)\right)\right\}.$$

(ii) *For the case $r \geq 1$ and $j=0, \cdots, r-1$, $T^{-1}\hat{Q}_j = \sum_{i=1+j}^{r} |\hat{\nu}_i| + O_p(T^{-1/2})$, where $\hat{\nu}_1, \cdots, \hat{\nu}_r$ are the eigenvalues of $\hat{\Omega}$ such that either $\hat{\nu}_i = O_p(1)$ and*

$$\hat{\nu}_i^{-1} = O_p(1) \ or \ \hat{\nu}_i = O_p(T^{1/2}) \ and \ (T^{-1/2}\hat{\nu}_i)^{-1} = O_p(1).$$

Theorem 1 ensures that the limiting distribution of $\hat{Q}_r$ is equal to that of the LR (trace) test statistic in the most standard case. We also see that the rate of divergence of the test statistic under the alternative is either equal to or greater than the divergence for the LR test. This implies that the test by $\hat{Q}_j$ is consistent and is at least as powerful as the LR test in an asymptotic sense. Thus, it is established that the critical values for the LR test, given as upper percentage points of $\breve{M}_r$, are valid for the tests by $\hat{Q}_j$.

The asymptotics for $\hat{Q}_{*;j}$ are established in the following theorem:

Theorem 2: *Suppose that y_t is generated by (1) and that the same conditions as in Theorem 1 hold. Then, for $\forall n \geq m$ and $\hat{Q}_{*;j}$ corresponding to it, we have:*

(i) *$Pr(\hat{Q}_{*;r} \geq \bar{c}) \leq Pr(\hat{Q}_r \geq \bar{c})$, where the notation $Pr(\cdot)$ denotes the probability and $\bar{c}$ is a positive number given arbitrarily.*

(ii) *For the case $r \geq 1$ and $j=0, \cdots, r-1$, $T^{-1}\hat{Q}_{*;j} = \tilde{\psi}_j + Op(T^{-1/2})$, where*

$$\tilde{\psi}_j = min\left\{\sum_{i=1+j}^{r} |\hat{\nu}_i|, \sum_{i=1+j}^{r} |\tilde{\nu}_i|\right\},$$

with $\tilde{\nu}_1, \cdots, \tilde{\nu}_r$ as the eigenvalues of $\hat{\Omega}_*$, such that $\tilde{\nu}_i = O_p(1)$, $\tilde{\nu}_i^{-1} = O_p(1)$ and $|\tilde{\nu}_1| \geq \cdots \geq |\tilde{\nu}_r| > 0$.

Theorem 2 establishes that any upper percentage point of $\hat{Q}_{*;j}$ are below it for $\hat{Q}_j$. It also establishes that the test by $\hat{Q}_{*;j}$ is consistent and is as powerful as the LR test. Similarly, the critical values for the LR test are applicable to the test by $\hat{Q}_{*;j}$, as are those by $\hat{Q}_j$, although the limiting distribution $\hat{Q}_{*;r}$ is not $\breve{M}_r$.

Theorems 1 and 2 may hold for the test statistics provided to handle the case in which Condition (A) is not satisfied in the previous section. We also note that the asymptotics in the above theorems can be established for the statistics constructed by substituting $\tilde{B}$ for $\hat{B}$ and that $\hat{N}_{-2}$ used in S_{11} does not play any essential role in the asymptotics, as seen in the proof of Theorem 1. Furthermore, in the previous section, we mentioned the statistics reconstructed for more general models containing a drift or deterministic trends. It will be straightforward to establish that they possess asymptotics as an extension of those in Theorems 1 and 2.

5 Monte Carlo Experiments

In this section, we execute Monte Carlo experiments for the cointegration rank determination based on $\hat{Q}_j$ or $\hat{Q}_{*;j}$ proposed in several DGPs as special cases of (1). The purpose of the experiments is to observe how the tests are performed in finite samples in connection with the asymptotics established theoretically in the previous section. We focus our attention on finite sample properties not for an individual test but for the entire procedure to determine the rank value.

The DGPs presented below are of 4-variates systems ($k=4$) with $p=1$ or 2 and $q=1$ or 2, implying that $m=2$. They are constructed on the basis of ε_t as Gaussian with mean zero and covariance matrix I ($\Lambda = I$) and $y_{-j}=0$ for any $j \geq 0$, so that the condition to exclude the occurrence of multicointegration as well as the requirements for $A(z)$ and $D(z)$ in (1) are satisfied. It is found that Conditions (A) and (B) hold. We note that it can be seen by evaluating the coefficient matrices of ε_{t-q-1} and ε_{t-q-2} in the VMA expressions of Δy_t and $\beta' y_{t-q-1}$.

All DGPs are classified into three groups such that a part of scalar parameter values is common within each group. In the first and second groups for which $p=1$, there are three variations for the VAR structure. They are diversified according to the value of r (i.e. $r=0, 1, 2$), whereas the third group is specified by an identical VAR structure as $p=2$ and diversifies the DGPs via VMA only.

Six VMA structures corresponding to one VAR structure are specified with the coefficient matrices D_i written as

$$D_1 = \begin{bmatrix} d_1 & 0 & 0 & 0 \\ 0 & d_2 & 0 & d_5 \\ 0 & 0 & d_3 & 0 \\ 0 & 0 & 0 & d_4 \end{bmatrix}, \qquad D_2 = \begin{bmatrix} d_6 & 0 & 0 & 0 \\ 0 & d_7 & 0 & 0 \\ 0 & 0 & d_8 & 0 \\ 0 & 0 & 0 & d_9 \end{bmatrix},$$

where

(i): $q=1$, $d_1=d_2=d_3=d_4=0.6$ *and* $d_5=d_6=d_7=d_8=d_9=0$,

(ii): $q=1$, $d_1=d_2=d_3=d_4=-0.6$ *and* $d_5=d_6=d_7=d_8=d_9=0$,

(iii): $q=2$, $d_1=d_2=d_3=d_4=1.5$, $d_5=0$ *and* $d_5=d_6=d_7=d_8=d_9=0.56$,

(iv): $q=2$, $d_1=d_2=d_3=d_4=0.2$, $d_5=0$ *and* $d_5=d_6=d_7=d_8=d_9=-0.48$,

(v): $q=2$, $d_1=1.5$, $d_2=d_4=0.9$, $d_3=0.4$, $d_5=0.5$, $d_6=0.56$, $d_7=d_9=0.2$ *and* $d_8=0.04$,

(vi): $q=2$, $d_1=1.5$, $d_2=0.9$, $d_3=0.4$, $d_4=-0.4$, $d_5=0.5$, $d_6=0.56$, $d_7=0.2$ *and* $d_8=d_9=0$.

The first group, in addition to the VMA specification above, formulates the VAR structure as follows:

$r=0$: $H_1=\bar{K}-I$,

$r=1$: $H_1=\bar{K}-I-\alpha\beta'$,

$\alpha=-[0.2, 0.2, 0.5, 0.2]'$, $\beta=[1, 1, 1, 1]'$,

$r=2$: $H_1=\bar{K}-I-\alpha\beta'$,

$$\alpha = -\begin{bmatrix} 0.2 & 0.2 & 0.5 & 0.2 \\ 0 & -1.5 & 0 & 0.5 \end{bmatrix}',$$

$$\beta = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix}',$$

where

$$\bar{K} = \begin{bmatrix} 0.8 & 0 & 0 & 0 \\ -0.2 & 0.8 & 0 & 0 \\ -0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0.8 \end{bmatrix}.$$

Thus, the group consists of eighteen DGPs.

In the second group, the VAR formulation is given as

$r=0$: $H_1=\bar{K}_0-I$,

$r=1$: $H_1=\bar{K}_1-I-\alpha\beta'$,

$\alpha=-[0.5, 0.5, 0, 0]'$, $\beta=[1, 1, 0, 0]'$,

$r=2$: $H_1=\bar{K}_2-I-\alpha\beta'$,

$$\alpha = -\begin{bmatrix} 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0.1 & 0 \end{bmatrix}',$$

$$\beta = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 2.5 & 2.5 & 6 & -6 \end{bmatrix}',$$

where

$$\bar{K}_i = \begin{bmatrix} 0.5 & -0.5 & 0.7 & \bar{k}_{i;1} \\ \bar{k}_{i;2} & 0.5 & -0.6 & -0.2 \\ 0 & 0 & 0.4 & 0.6 \\ 0 & \bar{k}_{i;3} & -0.2 & \bar{k}_{i;4} \end{bmatrix},$$

$\bar{k}_{0;1}=0$, $\bar{k}_{0;2}=0.5$, $\bar{k}_{0;3}=0$ *and* $\bar{k}_{0;4}=0.3$,

$\bar{k}_{1;1}=-0.7$, $\bar{k}_{1;2}=-0.5$, $\bar{k}_{1;3}=0$ *and* $\bar{k}_{1;4}=0.7$,

$\bar{k}_{2;1}=-0.7$, $\bar{k}_{2;2}=-0.5$, $\bar{k}_{2;3}=-2$ *and* $\bar{k}_{2;4}=0.7$.

This group also consists of eighteen DGPs.

The VAR structure for the third group is as follows:

$$H_1 = \begin{bmatrix} 0 & 0.2 & 0.2 & -0.4 \\ 0 & 0 & 0.2 & 0.2 \\ 0 & 0.5 & 0 & 0.5 \\ 0.2 & 0.2 & 0.2 & 0 \end{bmatrix},$$

$$H_2 = \begin{bmatrix} 0 & 0 & 0.2 & -0.4 \\ 0 & 0 & 0 & 0.2 \\ 0 & 0 & 0 & 0 \\ 0.2 & 0.2 & 0 & 0 \end{bmatrix},$$

$\alpha=-[0.2, 0.2, 0.5, 0.2]'$, $\beta=[1, 1, 1, 1]$.

This group consists of six DGPs.

For each of the DGPs, we ran 10, 000 simulations using $T=200$ and 500 as well as pseudo normal random variables for ε_t. We obtained the relative frequency for each of $\hat{Q}_j$ and $\hat{Q}_{*;j}$ calculated as $n=m=2$ to determine correctly r over 10, 000 simulations. Relative frequency results of the tests executed consecutively at 5% level are provided in the tables below. For the sake of comparison, the results of the most standard LR test (LR) and Shintani's $P(n, s)$ and $P_*(n, s)$ tests ($P(k, j)$ and $P_*(k, j)$ in terms of the present paper respectively) are also tabulated. For Shintani tests, the Parzen kernel is used with the bandwidth parameters $K_T=4$ and 8, noting that the automatic bandwidths are not adopted since less favorable results are reported in Shintani (2001). All calculations were made in Gauss, and the 5% critical values for the tests $\hat{Q}_j$, $\hat{Q}_{*;j}$ and LR are from Table II in MacKinnon *et al*. (1999). Those for Shintani tests are from Table IVa in Phillips and Ouliaris (1990).

Now, let us survey finite sample performances of these testing methods through tabular comparison. As observed, performances by $\hat{Q}_j$ and $\hat{Q}_{*;j}$ are stable in comparison with others: $\hat{Q}_j$ exhibits performance not greatly different from the asymptotics in all DGPs except the cases in which (ii) or (iv) is adopted as the VMA structure. However, $\hat{Q}_{*;j}$ is free from the severe performances even in such cases. This indicates that the tests proposed generally show satisfactory finite sample performances, although their performances become comparatively worse as the VMA structure becomes more complex, as in (v) or (iv).

Performance of the LR test is inferior to those of $\hat{Q}_j$ or $\hat{Q}_{*;j}$, particularly for $T=500$, although not so worse in spite of the absence of the asymptotic validity.

On the other hand, Shintani tests seem to be unstable, particularly for the case in which $K_T=8$ is used. They or some of them exhibit remarkably admirable performance within some DGPs and remarkably poor performance with others. It is also noted that the results generally tend to be worse as r increases, particularly for $T=200$. This indicates that such deterioration mainly occurs in their power performances and suggests that it may originate from a defect in using a kernel estimator, as stated already.

We should pay attention to the case in which one root in det $D(z)=0$ is close to -1, mentioned as the 'near' non-invertible case in the introduction. The tables reveal that such cases tend to result in severely poor performance for each of the tests except $\hat{Q}_{*;j}$. This is most conspicuous for DGPs associated with (ii) but is less so for DGPs associated with (iv). We note that such issue has been discussed similarly in unit root testing for time series with serially correlated errors (Perron and Ng (1996) e.g.).

Finally, note that $\hat{Q}_{*;j}$ displays exceptionally robustness in DGPs associated with (ii), In addition, it shows the best performance among all methods on the whole.

6 Concluding remarks

We have proposed two tests to determine the cointegration rank in the framework of VARMA models. In addition, we have discussed the validity through the asmptotics established theoretically and finite sample performances by Monte Carlo experiments. It is established that the Johansen methodology based on the standard LR test and its critical values are asymptotically applicable to the tests proposed. Further, we have shown that the tests are consistent and are as asymptotically powerful as the LR. The Monte Carlo experiments also support the conclusion that the performance of the tests are not on the whole unsatisfactory and are generally superior to the LR and nonparametric tests by Shintani (2001), particularly in the stability of performances. Findings also indicate that our tests seem to exhibit the desirability in the DGPs with comparatively large rank ($r=2$) and that $\hat{Q}_{*;j}$ showed the best performance with robustness among DGPs that are 'nearly' non-invertible.

The experiments were not executed for the tests constructed by n other than m or substitution of $\tilde{B}$ for $\hat{B}$, and the issue on how finite sample performances of the tests are improved by these alterations remains for future research.

The tests proposed are parametric, need the values of p and q for construction, and do not cover a wide variety of DGPs with serially correlated errors, unlike Shintani tests. However, it may be asserted that these succeed by utilizing the structure and characteristics of the model.

This paper has not discussed how values of p and q are determined prior to the rank tests nor how parameters of (3) are estimated. It may be relatively simple to estimate u_t and its long-run covariance matrix using or advancing the arguments and techniques shown in the proof of Theorem 1. Those estimates obtained for different values of p and q may provide us a clue to find the true values of p and q. The latter issue may be discussed with the identification of the parameters of (1)/(3), which is partly mentioned in Section 2. The results in the proof of Theorem 1 (particularly $\hat{G}$) may be applicable to finding consistent estimators of the VAR and VMA coefficient matrices. Formal discussion of these issues remains for future research.

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Appendix

Proof of Lemma First, let $\mathcal{J}_{t-j-1}$ and $\mathcal{K}_{t-q-1}$ denote the Hilbert spaces given as

$$\mathcal{J}_{t-j-1}=\{\beta' y_{t-j-1},\ \epsilon_{t-j-i},\ i=1,\cdots,q\}\quad j=0,q,$$
$$\mathcal{K}_{t-q-1}=\{\epsilon_{t-q-i}-P(\epsilon_{t-q-i}\,|\,\{\beta' y_{t-1}\}),\ \forall i\geq 1\}.$$

For $\bar{F}$ satisfying Condition (A) (ii), we have

$$\bar{F}'\Delta z_{t-1;m}=\sum_{i=1}^{\infty}\bar{\Theta}_i'\epsilon_{t-i}\,+\,\bar{\Psi}\beta' y_{t-q-1},$$

with suitable matrices $\bar{\Theta}_i$ and $\bar{\Psi}$. Now, put $\check{F}'=\bar{F}'+\bar{\Psi}'$, where $\bar{\Psi}$ is a $mk\times m$ matrix such as

$$\begin{aligned}\check{\Psi}\ &=\ \left[\bar{\Psi}\beta',\cdots,\bar{\Psi}'\beta'\right]\quad if\ m=q,\\ &=\ \left[\bar{\Psi}\beta',\cdots,\bar{\Psi}'\beta',\ 0\right]\quad otherwise.\end{aligned}$$

Now, suppose that there exists a k-dimensional nonzero vector $\check{f}$. such that $\check{f}'\check{F}'\Delta z_{t-1;m}$ is expressed as a linear combination of elements in $\mathcal{J}_{t-1}$, and note that $\check{f}'\check{F}'\neq 0$, in view of the supposition of $\bar{F}$. It is then seen clearly that $\check{f}'\check{F}'\Delta z_{t-1;m}$ must be expressed as a linear combination of elements in $\mathcal{G}_{t-1}$, which contradicts the supposition of $\bar{F}$. Hence any nonzero linear combination of $\check{F}'\Delta z_{t-1;m}$ must be spanned not only by elements in $\mathcal{J}_{t-1}$ but also by those in $\mathcal{K}_{t-q-1}$. Consequently, we have

$$rank\,\check{F}'E\{\Delta z_{t-q-1;m}-P(\Delta z_{t-q-1;m}\,|\,\mathcal{J}_{t-q-1})\}\times\{\Delta z'_{t-q-1;m}-P(\Delta z'_{t-q-1;m}\,|\,\mathcal{J}_{t-q-1})\}\check{F}=k. \quad (A.1)$$

Next, suppose that for a k-dimensional nonzero vector $\bar{f}$,

$$\check{F}'E\{\Delta z_{t-q-1;m}\,-\,P(\Delta z_{t-q-1;m}\,|\,\mathcal{J}_{t-q-1})\}\cdot\Delta z'_{t-1;m}\bar{K}\bar{f}\neq 0, \quad (A.2)$$

where $\bar{K}$ denotes either $\bar{H}$ or $\bar{F}$. Since

$$\begin{aligned}&\check{F}'E\{\Delta z_{t-q-1;m}-P(\Delta z_{t-q-1;m}\,|\,\mathcal{J}_{t-q-1})\}\Delta z'_{t-1;m}\bar{K}\bar{f}\\&=\check{F}'E\{\Delta z_{t-q-1;m}-P(\Delta z_{t-q-1;m}\,|\,\mathcal{J}_{t-q-1})\}\\&\quad\times\{\Delta z'_{t-1;m}-P(\Delta z'_{t-1;m}\,|\,\{\beta' y_{t-q-1}\})\}\bar{K}\bar{f},\end{aligned}$$

$\bar{f}'\bar{K}'\{\Delta z_{t-1;m}-P(\Delta z_{t-1;m}|\{\beta' y_{t-q-1}\})\}$ must be spanned not only by elements in $\mathcal{J}_{t-q-1}\cup\{\varepsilon_{t-i}\,;\,i=1,\cdots,q\}$ but also by those in $\{\check{F}'\Delta z_{t-q-1;m}\}$. Obviously, this is equivalent to

$$\check{F}'E\Delta z_{t-q-1;m}\{\Delta z'_{t-1;m}-P(\Delta z'_{t-1;m}|\{\beta' y_{t-q-1}\})\}\cdot\bar{K}\bar{f}\neq 0.$$

Thus we have

$$\check{F}'E\{\Delta z_{t-q-1;m}-P(\Delta z_{t-q-1;m}|\{\beta' y_{t-q-1}\})\}\cdot\Delta z'_{t-1;m}\bar{K}\bar{f}\neq 0, \quad (A.3)$$

for any $\bar{f}$ satisfying (A.2).

On the other hand, suppose that there exists a k-dimensional nonzero vector $\bar{f}$ such that $\bar{f}'\bar{K}'\neq 0$ (note that this always holds for $\bar{K}=\bar{F}$) but neither (A.2) nor (A.3) is satisfied. In view of (A.2), $\bar{f}'\bar{K}'\Delta z_{t-1;m}$ must be expressed as a linear combination of elements in $\mathcal{J}_{t-q-1}\cup\{\varepsilon_{t-i}\,;\,i=1,\cdots,q\}$. In addition, it must be spanned not only by elements in $\mathcal{G}_{t-1}$ but also by those in

$$\mathcal{L}_{t-q-1}=\{\epsilon_{t-q-i}\,-\,P(\epsilon_{t-q-i}\,|\,\mathcal{M}_{t-q-1})\,;\,\forall\, i\geq 1\}$$

where

$$\mathcal{M}_{t-q-1}=\{\check{F}'\Delta z_{t-q-1;m}-P(\check{F}'\Delta z_{t-q-1;m}\,|\,\{\beta' y_{t-q-1}\})\},$$

in view of (A.3) and Condition (A) (ii) or (B). It is not difficult to see from (A.1) that any element of $\mathcal{L}_{t-q-1}$ is not spanned by only elements in $\mathcal{J}_{t-q-1}$ and vice versa. Thus we have a contradiction. Conclusively, $\bar{f}'\bar{H}'\Delta z_{t-1;m}$ satisfies (A.3) if and only if $\bar{f}'\bar{H}\neq 0$. Similarly, $\bar{f}'\bar{F}'\Delta z_{t-1;m}$ satisfies (A.3) for any nonzero $\bar{f}$. Thus, it is established that $k_1(m)=rank\ \bar{H}$ and $\bar{k}(m)\geq k$. It is trivial that $k_1(n)=rank\ \bar{H}$ and $\bar{k}(n)\geq k$ for any positive integer n greater than m.

Proof of Theorem 1 For the proof of Theorem 1, it suffices to show only the case of $r\geq 1$ since the case of $r=0$ is trivial. One of the essential parts of the proof is to establish that $\hat{B}'\Delta z_{t-q-1;n}$ is correlated with $\bar{H}'\Delta z_{t-1;m}$ as the 'explanatory' variables in an asymptotic sense, and for other parts, arguments similar to those used for the LR test (Johansen (1988) or Johansen (1996, ch. 11)) and the well-known asymptotics on $I(0)$ and $I(1)$ series (Hamilton (1994, p. 548) e.g.) are used.

(i) It follows from the asymptotics on $I(0)$ and $I(1)$ mentioned above that

$$\hat{Z}'_{-q-1-h;n}\hat{Z}_{-q-1-h;m}=\Sigma_{32}\,+\,O_p(T^{-1/2}), \quad (A.4)$$

$$T^{-1}\hat{Z}'_{-q-1;n}\tilde{S}\hat{Z}_{-1;m}=\bar{\Sigma}_{32}\,+\,O_p(T^{-1/2}), \quad (A.5)$$

where

$$\bar{\Sigma}_{32}=E\{\Delta z_{t-q-1;n}\,-\,P(\Delta z'_{t-q-1;n}\,|\,\{\beta' y_{t-q-1}\})\}\cdot\Delta z'_{t-1;m},$$

and several results similar to (A.4) or (A.5). From (3), (A.5) and several asymptotics derived similarly we also obtain

$$T^{-1}\hat{Z}'_{-q-1;n}\tilde{S}\hat{Y}_0=\bar{\Sigma}_{32}\bar{H}\,+\,O_p(T^{-1/2}). \quad (A.6)$$

Note that the identifiablity of the VARMA parameters in (3) is not assured. There may exist a power series with matrix coefficients as $U(z)$ such that $U(z)\neq I$,

$$A(z)=U(z)A_*(z),\qquad D(z)=U(z)D_*(z),$$

$$A_*(z) = \alpha_* \beta' z^{q+1} + (1-z)(I - \sum_{i=1}^{m} \bar{H}_{*;i} z^i), \quad and$$

$$D_*(z) = I + \sum_{i=1}^{q} D_{*;i} z^i,$$

where α_* is a $k \times r$ matrix satisfying $\alpha = U(1)\, \alpha_*$, and $\bar{H}_{*;i}$ and $D_{*;i}$ are $k \times k$ matrices. Defining $\bar{H}_*$ in a manner similar to that used for $\bar{H}$, it can be established that $\bar{H}_*$ and its related parameters and equations corresponding to $A_*(z)$ and $D_*(z)$ either reasonably substitute for the counterparts corresponding to $A(z)$ and $D(z)$ in the following arguments or are reduced to the counterparts corresponding to another VARMA structure that are applicable to the following arguments.

Next, put $k_1 = rank\, \bar{\Sigma}_{32} \bar{H}$ and we shall establish that

$$\left(T^{-1} \hat{B}' \hat{Z}'_{-q-1-h;n} \hat{Z}_{-1-h;m} \hat{G}\right)^{-1} = \left(\hat{B}' \Sigma_{32} \hat{G}\right)^{-1} + O_p(T^{-1/2}), \qquad (A.7)$$

$$\left(\hat{B}' \Sigma_{32} \hat{G}\right)^{-1} = O_p(1), \qquad (A.8)$$

$$\hat{G}'_* \bar{H} = O_p(T^{-1/2}), \qquad (A.9)$$

where $\hat{G}_*$ is the matrix whose columns are the eigenvectors, of

$$(T^{-1} \hat{Z}'_{-1;m} \tilde{S} \hat{Z}_{-q-1;n} \tilde{B})\, (T^{-1} \tilde{B}'\, \hat{Z}'_{-q-1;n} \tilde{S} \hat{Z}_{-1;m}),$$

which correspond to the eigenvalues other than those corresponding to $\hat{G}$.

First, consider the case $k_1 = k$, and let $\breve{B}$ be the matrix whose columns are the eigenvectors, of $\bar{\Sigma}_{32} \bar{H} \bar{H}' \bar{\Sigma}'_{32}$, which correspond to the k nonzero eigenvalues arranged in descending order. In addition, let $\breve{B}_*$ be the matrix whose columns are the eigenvectors corresponding to the $(n-1)k$ zero eigenvalues. It is not difficult to see from (A.6) combined with the definition of $\tilde{B}$ that

$$\tilde{B} = \breve{B} + O_p(T^{-1/2}).$$

Since

$$\breve{B}'_* \bar{\Sigma}_{32} \bar{H} = 0, \qquad \breve{B}'_* \breve{B} = 0,$$
$$rank\, \bar{\Sigma}_{32} \bar{H} = rank\, \breve{B} = k \quad and \quad rank\, \breve{B}_* = (n-1)k,$$

there must exist a nonsingular matrix $\bar{Q}_1$ of $k \times k$ such that $\breve{B} = \bar{\Sigma}_{32} \bar{H} \bar{Q}_1$. Similarly, we have

$$T^{-1} \hat{Z}'_{-1;m} \tilde{S} \hat{Z}_{-q-1;n} \tilde{B} = \bar{\Sigma}'_{32} \breve{B} + O_p(T^{-1/2}).$$

Therefore, using arguments similar to those for $\tilde{B}$ leads to

$$\hat{G} = \bar{G} + O_p(T^{-1/2}), \qquad \hat{G}_* = \bar{G}_* + O_p(T^{-1/2}),$$

where $\bar{G}$ and $\bar{G}_*$ be the matrices whose columns are the eigenvectors, of $\bar{\Sigma}'_{32} \breve{B} \breve{B}' \bar{\Sigma}_{32}$, which correspond to the k nonzero eigenvalues arranged in descending order, as in the above definition of $\breve{B}$, and are occupied by the remainder of the eigenvectors, respectively. Since

$$\bar{G}'_* \bar{\Sigma}'_{32} \breve{B} = 0, \qquad \bar{G}'_* \bar{G} = 0,$$
$$rank\, \bar{\Sigma}'_{32} \breve{B} = rank\, \bar{G} = k \quad and \quad rank\, \bar{G}_* = (m-1)k,$$

there must exist a nonsingular matrix $\bar{Q}_2$ of $k \times k$ such that

$$\bar{G} = \bar{\Sigma}'_{32} \breve{B} \bar{Q}_2 = \bar{M} \bar{H} \bar{Q}_1 \bar{Q}_2,$$

where $\bar{M} = \bar{\Sigma}'_{32} \bar{\Sigma}_{32}$. Now, let $\bar{P}$ be the $k \times k$ matrix whose columns are the eigenvectors of $\bar{K}' \bar{M} \bar{K}$, where $\bar{K} = \bar{H}\left(\bar{H}'\bar{H}\right)^{-1/2}$. This implies that $\bar{K}\bar{P}$ is the $mk \times k$ matrix whose columns are the eigenvectors, of $\bar{M}$, which correspond to the k nonzero eigenvalues. Then, there must exist the $mk \times (m-1)k$ matrix $\bar{H}_*$ whose columns are the eigenvectors, of $\bar{M}$, which correspond to the eigenvalues other than those corresponding to $\bar{K}\bar{P}$, such that

$$\bar{H}'_* \bar{M} \bar{K} \bar{P} = 0, \quad \bar{H}'_* \bar{K} \bar{P} = 0,$$
$$therefore, \quad \bar{H}'_* \bar{G} = \bar{H}'_* \bar{M} \bar{H} \bar{Q}_1 \bar{Q}_2 = 0.$$

Since

$$rank\, \bar{G} = rank\, \bar{M} \bar{H} = rank\, \bar{K} \bar{P} = k \quad and \quad rank\, \bar{H}_* = (m-1)k,$$

it must be asserted that

$$\bar{G} = \bar{H} \bar{Q},$$

with a $k \times k$ nonsingular matrix $\bar{Q}$. This leads to (A.9). Similarly, we have

$$\hat{B} = \bar{B} + O_p(T^{-1/2}),$$

where $\bar{B}$ is the matrix whose columns are the eigenvectors, of $\bar{\Sigma}_{32}\bar{G}\bar{G}'\bar{\Sigma}'_{32}$, which correspond to the k nonzero eigenvalues arranged in descending order. Noting that

$$rank\,\bar{\Sigma}_{32}\bar{G} = rank\,\bar{B}'\bar{\Sigma}_{32}\bar{G} = k,$$

it is clearly seen that any nonzero linear combination of $\bar{G}'\Delta z_{t-1;m}$ is spanned not only by elements in $\mathcal{G}_{t-1}$ but also by those in $\{\bar{B}'\Delta z_{t-q-1;n}\}$, implying that

$$rank\,\Sigma_{32}\bar{G} = rank\,\bar{B}'\Sigma_{32}\bar{G} = k.$$

These results ensure that (A.7) and (A.8) hold.

Next, consider the case $k_1 < k$. Put $\bar{H}_1 = H_1(H_1'H_1)^{-1/2}$ with H_1 defined as a $mk \times k_1$ matrix that consists of k_1 linearly independent columns of $\bar{H}$. In view of Condition (B) and Lemma, there must exist a $k \times k$ nonsingular matrix $\breve{A}$ such taht

$$\bar{H}\breve{A} = \left[\bar{H}_1,\ 0\right].$$

Putting

$$\begin{aligned}\bar{F}_1 &= \bar{\Sigma}_{32}\bar{H}_1,\\ \tilde{F}_2 &= \bar{N}_0(T^{-1/2}\hat{Z}'_{-q-1;n}\hat{S}\bar{U}\breve{A}\,[0,\ I]'),\end{aligned}$$

where $\bar{N}_0 = I - \bar{F}_1(\bar{F}_1'\bar{F}_1)^{-1}\bar{F}_1'$, from (3), (A.5) and several asymptotics derived similarly we have

$$\begin{aligned}&(T^{-1}\hat{Z}'_{-q-1;n}\tilde{S}\hat{Y}_0)\tilde{A}\\ &= \left[\bar{F}_1 + O_p(T^{-1/2}),\ T^{-1/2}\tilde{F}_2\right] + O_p(T^{-1}),\end{aligned} \qquad (A.10)$$

where $\tilde{A}$ is a $k \times k$ matrix such that $\tilde{A} = O_p(1)$ and $\tilde{A}^{-1} = O_p(1)$. The standard of matrix algebra then leads to

$$\tilde{B} = \left[\bar{F}_1,\ \tilde{F}_2\right]\tilde{\Theta}_*\tilde{L} + O_p(T^{-1/2}), \qquad (A.11)$$

where

$$\tilde{\Theta}_* = \begin{bmatrix} \left(\bar{F}_1'\bar{F}_1\right)^{-1/2} & 0 \\ 0 & \left(\tilde{F}_2'\tilde{F}_2\right)^{-1/2} \end{bmatrix},$$

and $\tilde{L}$ is a $k \times k$ nonsingular matrix such that $\tilde{L} = O_p(1)$ and $\tilde{L}^{-1} = O_p(1)$. Noting that

$$k_1 = rank\,\bar{H}_1'\bar{\Sigma}'_{32}\bar{\Sigma}_{32}\bar{H}_1 = rank\,\bar{\Sigma}'_{32}\bar{F}_1$$

and recalling that $\tilde{F}_2$ is a random matrix of $nk \times k_2$, it is not difficult to see that

$$\left(\begin{bmatrix}\bar{F}_1'\\ \tilde{F}_2'\end{bmatrix}\bar{\Sigma}_{32}\bar{\Sigma}'_{32}\left[\bar{F}_1,\ \tilde{F}_2\right]\right)^{-1} = O_p(1). \qquad (A.12)$$

Similarly, from (A.5) and (A.11) we have

$$\begin{aligned}&(T^{-1}\hat{Z}'_{-1;m}\tilde{S}\hat{Z}_{-q-1;n})\tilde{B}\\ &= \left[\bar{\Sigma}'_{32}\bar{F}_1,\ \tilde{N}_1\bar{\Sigma}'_{32}\tilde{F}_2\right]\tilde{\Theta} + O_p(T^{-1/2}),\end{aligned} \qquad (A.13)$$

where

$$\bar{N}_1 = I - \bar{\Sigma}'_{32}\bar{F}_1\left(\bar{F}_1'\bar{\Sigma}_{32}\bar{\Sigma}'_{32}\bar{F}_1\right)^{-1}\bar{F}_1'\bar{\Sigma}_{32}$$

and $\tilde{\Theta}$ is a $k \times k$ nonsingular matrix such that $\tilde{\Theta} = O_p(1)$ and $\tilde{\Theta}^{-1} = O_p(1)$. (A.13), along with (A.12), implies that $\hat{G}$ is the matrix whose columns are the eigenvectors, of

$$\left[\bar{\Sigma}'_{32}\bar{F}_1,\ \bar{N}_1\bar{\Sigma}'_{32}\tilde{F}_2\right]\tilde{\Theta}\tilde{\Theta}'\begin{bmatrix}\bar{F}_1'\bar{\Sigma}_{32}\\ \tilde{F}_2'\bar{\Sigma}_{32}\bar{N}_1\end{bmatrix} + O_p(T^{-1/2}),$$

which correspond to the k largest (nonzero) eigenvalues arranged in descending order. In addition, put

$$\bar{G}_1 = \bar{\Sigma}'_{32}\bar{F}_1, \qquad \tilde{G}_2 = \bar{N}_1\bar{\Sigma}'_{32}\tilde{F}_2,$$

Similarly to the derivation of $\tilde{B}$, we obrain

$$\hat{G} = \left[\bar{G}_1,\ \tilde{G}_2\right]\hat{\Theta} + O_p(T^{-1/2}), \qquad (A.14)$$

where $\hat{\Theta}$ is a $k \times k$ matrix such that $\hat{\Theta} = O_p(1)$ and $\hat{\Theta}^{-1} = O_p(1)$.

Next, put

$$\bar{M} = \bar{\Sigma}'_{32}\bar{\Sigma}_{32},$$

as done in the proof of the case $k_1 = k$. It is then easily checked that $\bar{G}_1 = \bar{M}\bar{H}_1$. It also follows from (A.14) that

$$\hat{G}_*'\left[\bar{G}_1,\ \tilde{G}_2\right] = O_p(T^{-1/2}),$$

implying that $\hat{G}_*'\bar{M}\bar{H}_1 = O_p(T^{-1/2})$. Based on the eigenvalues and eigenvectors of $\bar{H}_1'\bar{M}\bar{H}_1$ and by arguments similar to those used for the case $k_1 = k$, it can be shown that there exists a $k_1 \times k_1$ nonsingular matrix $\bar{Q}$ such that

$$\bar{M}\bar{H}_1 = \bar{H}_1\bar{Q}.$$

Thus we obtain

$$\hat{G}_*'\bar{H}_1 = O_p(T^{-1/2}),$$

which is equivalent to (A.9) in this case.

Now, put $\bar{k} = rank\ \bar{\Sigma}_{32}$. For the case $mk > \bar{k}$, there must exist a nonrandom full column rank matrix $\check{G}_*$ such that $\bar{\Sigma}_{32}\bar{N}_1\check{G}_* = 0$. For a $mk \times k_2$ full column rank matrix $\bar{G}_{*;3}$ expressed as

$$\bar{G}_{*;3}' = \bar{G}_{21}'\bar{G}_{*;2}'\bar{N}_1 + \bar{G}_{22}'\check{G}_*'\bar{N}_1 \quad or \quad = \bar{G}_{21}'\bar{G}_{*;2}'\bar{N}_1,$$

where $\bar{G}_{21}$ is a full column rank matrix of $(\bar{k} - k_1) \times k_2$, and $\bar{G}_{22}$ is a matrix of $(mk - \bar{k}) \times k_2$, it is clearly seen that

$$rank \begin{bmatrix} \bar{G}_1'\bar{\Sigma}_{32}' \\ \bar{G}_{*;3}'\bar{\Sigma}_{32}'\bar{N}_2 \end{bmatrix} \bar{\Sigma}_{32}\left[\bar{G}_1,\ \bar{G}_{*;3}\right] = k. \qquad (A.15)$$

where

$$\bar{N}_2 = I - \bar{\Sigma}_{32}\bar{G}_1(\bar{G}_1'\bar{\Sigma}_{32}'\bar{\Sigma}_{32}\bar{G}_1)^{-1}\bar{G}_1'\bar{\Sigma}_{32}'.$$

Using arguments similar to those used to derive (A.8) in the case $k_1 = k$ and from (A.15), it follows that

$$rank \begin{bmatrix} \bar{G}_1'\bar{\Sigma}_{32}' \\ \bar{G}_{*;3}'\bar{\Sigma}_{32}'\bar{N}_2 \end{bmatrix} \Sigma_{32}\left[\bar{G}_1,\ \bar{G}_{*;3}\right] = k. \qquad (A.16)$$

Now, recall that $\tilde{F}_2'\bar{\Sigma}_{32}$ is a random matrix of $k_2 \times mk$ and $\tilde{G}_2' = \tilde{F}_2'\bar{\Sigma}_{32}\bar{N}_1$, which, along with (A.12), ensures that

$$\left(\bar{G}_{21}'\bar{G}_{*;2}\bar{N}_1\tilde{G}_2\right)^{-1} = O_p(1).$$

It follows from this and (A.15) that

$$\left(\begin{bmatrix} \bar{G}_1'\bar{\Sigma}_{32}' \\ \tilde{G}_2'\bar{\Sigma}_{32}'\bar{N}_2 \end{bmatrix} \bar{\Sigma}_{32}\left[\bar{G}_1,\ \tilde{G}_2\right]\right)^{-1} = O_p(1). \qquad (A.17)$$

Using the same argument as that used for the derivation of (A.16) from (A.15), (A.17) leads to

$$\left(\begin{bmatrix} \bar{G}_1'\bar{\Sigma}_{32}' \\ \tilde{G}_2'\bar{\Sigma}_{32}'\bar{N}_2 \end{bmatrix} \Sigma_{32}\left[\bar{G}_1,\ \tilde{G}_2\right]\right)^{-1} = O_p(1). \qquad (A.18)$$

In view of (A.17) and through arguments similar to those used for the derivation of $\hat{G}$, we can attain to

$$\hat{B} = \left[\bar{\Sigma}_{32}\bar{G}_1,\ \bar{N}_2\bar{\Sigma}_{32}\tilde{G}_2\right]\hat{\Upsilon} + O_p(T^{-1/2}), \qquad (A.19)$$

where $\hat{\Upsilon}$ is a $k \times k$ matrix such that $\hat{\Upsilon} = O_p(1)$ and $\hat{\Upsilon}^{-1} = O_p(1)$. (A.18) and (A.19), along with the definitions of $\bar{G}_i$, lead to (A.7) and (A.8).

Let us again consider the asmptotics on $I(0)$ and $I(1)$. It can be also established in the literature mentioned above or by a combination of the results therein that

$$T^{-1}\gamma' S_{11}\gamma \Rightarrow \bar{\Omega}^{1/2}\left(\int_0^1 W_{k-r}(u)W_{k-r}'(u)\right)\bar{\Omega}^{1/2}, \quad (A.20)$$

where the symbol $\Rightarrow$ stand for weak convergence of probability measures on the unit interval [0, 1] and $\bar{\Omega} = \gamma'\gamma\tau\delta' D(1)\Lambda D(1)'\delta\tau'\gamma'\gamma$,

$$\beta' S_{11}\beta = \beta'\bar{\Sigma}_{11}\beta + O_p(T^{-1/2}), \qquad (A.21)$$

$$\gamma' S_{11}\beta = O_p(1). \qquad (A.22)$$

Putting $\bar{v}_t = \sum_{i=0}^{q-1}\left(-\sum_{j=i+1}^{q} D_j\right)\varepsilon_{t-i}$, it is clearly seen that

$$T^{-1}\sum_{t=2q+n+2}^{T}\left(\sum_{s=1}^{t-q-1}\epsilon_s\right)(\bar{v}_t' - \bar{v}_{t-1}') = O_p(T^{-1/2}),$$

which, along with the supposition that $\delta' D(1)\Lambda D(1)'\delta = I$, leads to

$$T^{-1}\gamma' Y_{-q-1}'\bar{U}_0\delta \Rightarrow \bar{\Omega}^{1/2}\left(\int_0^1 W_{k-r}(u)dW_{k-r}'(u)\right). \quad (A.23)$$

On the other hand, in view of (A.7) to (A.9), it can be derived that

$$\begin{aligned}
&\left(T^{-1}X'\hat{Z}_{-1-h;m}\hat{G}\right)\left(T^{-1}\hat{B}'\hat{Z}_{-q-1-h;n}'\hat{Z}_{-1-h;m}\hat{G}\right)^{-1} \\
&\times\left(T^{-1}\hat{B}'\hat{Z}_{-q-1-h;n}'\hat{Y}_{-h}\right) \\
&= \left(T^{-1}X'\hat{Z}_{-1-h;m}\hat{G}\right)\left(T^{-1}\hat{B}'\hat{Z}_{-q-1-h;n}'\hat{Z}_{-1-h;m}\hat{G}\right)^{-1} \\
&\times\left(T^{-1}\hat{B}'\hat{Z}_{-q-1-h;n}'Y_{-q-1-h}\beta\right)\alpha' \\
&+ \left(T^{-1}X'\hat{Z}_{-1-h;m}\hat{G}\right)\hat{G}'\bar{H} + O_p(T^{-1/2}),
\end{aligned}$$

where X denotes any of $Y_{-q-1}\gamma$, $Y_{-q-1}\beta$, $\hat{Y}_0$, $\hat{M}_{-h}\hat{Y}_{-h}$, $\bar{U}_0$ or $(Y_{-q-1}\beta - \hat{Z}_{-1;m}\tilde{G})\alpha'$. It also follows similarly that

$$\begin{aligned}
&T^{-1}X'\hat{Y}_{-h} \\
&= T^{-1}X'\bar{U}_{-h} + T^{-1}X'Y_{-q-1-h}\beta\alpha' \\
&+ \left(T^{-1}X'\hat{Z}_{-1-h;m}\hat{G}\right)\hat{G}'\bar{H} + O_p(T^{-1/2}).
\end{aligned}$$

Combining (A.4), results similar to (A.4) or (A.6) and the above two results, it follows that

$$T^{-1}X'\hat{M}_{-h}\hat{Y}_{-h}$$
$$= T^{-1}X'\bar{U}_{-h} + T^{-1}X'\left(Y_{-q-1-h}\beta - \hat{Z}_{-1-h;m}\tilde{G}\right)\alpha' + O_p(T^{-1/2}), \quad (A.24)$$

Using (A.23) and $\alpha'\delta=0$ in (A.24) as $h=0$ and $X=Y_{-q-1}\gamma$, we have

$$\gamma' S_{10}\delta \Rightarrow \bar{\Omega}^{1/2}\left(\int_0^1 W_{k-r}(u)dW'_{k-r}(u)\right). \quad (A.25)$$

It can be also shown based on (A.24) as $h=0$ and $X=Y_{-q-1}\gamma$ that

$$\gamma' S_{10} = O_p(1). \quad (A.26)$$

Similarly, using $\beta'\gamma=0$ in (A.24) as $h=0$ and $X=Y_{-q-1}\beta$ and several asymptotics on $I(0)$ and $I(1)$,

$$\beta' S_{10} = \tilde{R}_*\alpha' + O_p(T^{-1/2}), \quad (A.27)$$

It is relatively easy to show that

$$\tilde{R}_* = O_p(1), \qquad \tilde{R}_*^{-1} = O_p(1),$$

since both

$$\tilde{R} = \begin{bmatrix} I & 0 \\ 0 & \hat{B}' \end{bmatrix}\{E\begin{bmatrix} \beta' w_{t-q-1} \\ \Delta z_{t-q-1;n} \end{bmatrix} \cdot \left[w'_{t-q-1}\beta,\ \Delta z'_{t-1;m}\right]\}\begin{bmatrix} I & 0 \\ 0 & \hat{G} \end{bmatrix}$$

and $\tilde{R}^{-1}$ are of $O_p(1)$.

Now, consider the asymptotics on S_{00}. (A.24) as $h=0$ leads to

$$T^{-1}\hat{Y}_0'\hat{M}_0'X = T^{-1}\bar{U}_0'X + T^{-1}\alpha\left(\beta'\bar{W}'_{-q-1} - \bar{F}'\hat{Z}'_{-1;m}\right)X + O_p(T^{-1/2})$$

as well. Substituting $\hat{M}_{-h}\hat{Y}_{-h}$ for X in the above equation, it is derived that

$$T^{-1}\hat{Y}_0'\hat{M}_0'\hat{M}_{-h}\hat{Y}_{-h}$$
$$= T^{-1}\bar{U}_0'\hat{M}_{-h}\hat{Y}_{-h} + T^{-1}\alpha\left(\beta' Y'_{-q-1} - \tilde{G}'\hat{Z}'_{-1;m}\right) \cdot \hat{M}_{-h}\hat{Y}_{-h} + O_p(T^{-1/2}).$$

Letting $X=\bar{U}_0$ in (A.24), we also have

$$T^{-1}\bar{U}_0'\hat{M}_{-h}\hat{Y}_{-h} = Eu_t u'_{t-h} + \hat{R}_{3;h}\alpha' + O_p(T^{-1/2}),$$

and letting $X=(Y_{-q-1}\beta - \hat{Z}_{-1;m}\tilde{G})\alpha'$ in (A.24), it follows that

$$T^{-1}\alpha\left(\beta' Y'_{-q-1} - \tilde{G}'\hat{Z}'_{-1;m}\right)\hat{M}_{-h}\hat{Y}_{-h}$$
$$= \alpha\hat{R}_{2;h} + \alpha\hat{R}_{1;h}\alpha' + O_p(T^{-1/2}).$$

Thus we obtain

$$T^{-1}\hat{Y}_0'\hat{M}_0'\hat{M}_{-h}\hat{Y}_{-h}$$
$$= Eu_t u'_{t-h} + \alpha\hat{R}_{1;h}\alpha' + \alpha\hat{R}_{2;h} + \hat{R}_{3;h}\alpha' + O_p(T^{-1/2}). \quad (A.28)$$

which, along with the definition of S_{00}, leads to

$$S_{00} = \hat{\Sigma} + O_p(T^{-1/2}). \quad (A.29)$$

In view of the definition of $\hat{\Sigma}$, it can be easily established by the standard theory that $\hat{\Sigma}=O_p(1)$. Next, note that $\delta'\hat{\Sigma}\delta=\delta' D(1)\Lambda D(1)'\delta$, and put

$$\hat{\Psi} = \alpha\{\tilde{R}_{1;0} + \sum_{h=1}^{q}(\hat{R}_{1;h} + \hat{R}'_{1;h})\}\alpha' + \alpha\{\hat{R}_{2;0} + \sum_{h=1}^{q}(\hat{R}_{2;h} + \hat{R}'_{3;h})\} + \{\hat{R}_{3;0} + \sum_{h=1}^{q}(\hat{R}_{3;h} + \hat{R}'_{2;h})\}\alpha',$$

and $\hat{\Psi}_* = \hat{\Psi} - E\hat{\Psi}$. It is shown by the standard theory obvious that

$$T^{1/2}\hat{\Psi}_* = O_p(1), \qquad \left(T^{1/2}\hat{\Psi}_*\right) = O_p(1),$$

implying that $rank\ E\hat{\Sigma}=k$ if and only if $(\hat{\Sigma})^{-1}=O_p(1)$.

For the case in which $rank\ E\hat{\Sigma}=k$ holds, it is easy to see that $\hat{\Sigma}^{-1}=(E\hat{\Sigma})^{-1}+O_p(T^{-1/2})=Op(1)$. Now, consider the case in which $rank\ E\hat{\Sigma}=k$ does not hold. Then, there must exist full column rank matrices $\bar{\delta}$ and $\bar{\alpha}$, of $k\times(k-\bar{r})$ and $k\times\bar{r}$ respectively, such that $(E\hat{\Sigma})\bar{\delta}$ has full column rank, $(E\hat{\Sigma})\bar{\alpha}=0$, $\bar{\delta}\bar{\delta}'+\bar{\alpha}\bar{\alpha}'=I$ and $k-\bar{r}\geq k-r$ or $\bar{r}\leq r$. In addition, $\bar{\alpha}$ must be expressed as

$$\bar{\alpha} = \alpha\bar{\psi}_1 + \delta\bar{\psi}_2,$$

where $\bar{\psi}_1$ has full column rank and is of $r\times\bar{r}$ and $\bar{\psi}_2$ is of $(k-r)\times\bar{r}$. Noting that $T^{1/2}\hat{\Sigma}\bar{\alpha}=T^{1/2}\hat{\Psi}_*\bar{\alpha}$, it can be shown that

$$\left(\begin{bmatrix} \bar{\delta}' \\ \bar{\alpha}' \end{bmatrix}\hat{\Sigma}\left[\bar{\delta},\ T^{1/2}\bar{\alpha}\right]\right)^{-1} = O_p(1),$$

similar to (A.8) or (A.12). Putting

$$\left[\hat{\Lambda}_{21},\ \hat{\Lambda}_{22}\right] = [0,\ I]\left(\begin{bmatrix} \bar{\delta}' \\ \bar{\alpha}' \end{bmatrix}\hat{\Sigma}\left[\bar{\delta},\ T^{1/2}\bar{\alpha}\right]\right)^{-1},$$

it also follows that

$$\hat{\Sigma}^{-1} = T^{1/2}\bar{\alpha}\left(\hat{\Lambda}_{21}\bar{\delta}' + \hat{\Lambda}_{22}\bar{\alpha}'\right) + O_p(1).$$

Conclusively, it is established that $f_1'\delta' S_{00}^{-1}\delta f_2 = f_1'\delta' \hat{\Sigma}^{-1}\delta f_2 + \hat{\nu}$, and that $T^{1/2}|f_1'\tilde{\theta}_1 f_2| \geq |f_1'\delta'\hat{\Sigma}^{-1}\delta f_2| \geq |f_1'\tilde{\theta}_2 f_2|$, for any $(k-r)$-dimensional nonzero vectors f_i, where $\hat{\nu}$ is a random variable such that $\hat{\nu} O_p(1)$ *if* $T^{-1/2} f_1'\delta'\hat{\Sigma}^{-1}\delta f_2 = O_p(1)$ *and* $\hat{\nu} = O_p(T^{-1/2})$ *otherwise*, and $\tilde{\theta}_i$ are $(k-r)\times(k-r)$ random matrices such that $\tilde{\theta}_i = O_p(1)$ and $\tilde{\theta}_i^{-1} = O_p(1)$.

Combining some of the above results leads to

$$\beta' S_{01} S_{00}^{-1} S_{01}\beta = \tilde{R}_* \alpha' \hat{\Sigma}^{-1}\alpha \tilde{R}_*' + O_p(T^{-1/2}),$$

$\tilde{R}_* \alpha'\hat{\Sigma}^{-1}\alpha \tilde{R}_*' = O_p(1)$ or $O_p(T^{1/2})$, $(\tilde{R}_* \alpha'\hat{\Sigma}^{-1}\alpha \tilde{R}_*')^{-1} = O_p(1)$ if *rank* $E\hat{\Sigma} = k$ and either $|T^{-1/2} b_1'\tilde{R}_* \alpha'\hat{\Sigma}^{-1}\alpha \tilde{R}_*' b_2| \geq |b_1'\hat{\theta} b_2|$ or $|b_1'\tilde{R}_* \alpha'\hat{\Sigma}^{-1}\alpha \tilde{R}_*' b_2| \geq |b_1'\hat{\theta} b_2|$ otherwise, for any r-dimensional nonzero vectors b_i, where $\hat{\theta}$ is a $r\times r$ random matrix such that $\hat{\theta} = O_p(1)$ and $\hat{\theta}^{-1}$.

On the other hand, put

$$\hat{M} = S_{00}^{-1}\left[I - S_{01}\beta\left(\beta' S_{01} S_{00}^{-1} S_{01}\beta\right)^{-1}\beta' S_{01} S_{00}^{-1}\right],$$

and let $\hat{\eta}_1, \cdots, \hat{\eta}_{k-r}$ denote the eigenvalues corresponding to the equation

$$\det\{T^{-1}\gamma' S_{11}\gamma - \mu\gamma' S_{10}\hat{M} S_{01}\gamma\} = 0,$$

subject to the restriction that $|\hat{\eta}_1| \geq \cdots \geq |\hat{\eta}_{k-r}|$. It is easy to see that $\hat{M} S_{00}\hat{M} = \hat{M}$. It also follows from (A.27) that $\alpha'\hat{M} = O_p(T^{-1/2})$ and $\hat{M}\alpha = O_p(T^{-1/2})$. These results make the form of $\hat{M}$ be such that

$$\hat{M} = \delta\delta' + O_p(T^{-1/2}). \qquad (A.30)$$

We can now see that $(T\hat{\lambda}_1)^{-1}, \cdots, (T\hat{\lambda}_k)^{-1}$ are the eigenvalues corresponding to the equation

$$\det\{\begin{bmatrix} T^{-1}\beta' S_{11}\beta & T^{-1}\beta' S_{11}\gamma \\ T^{-1}\gamma' S_{11}\beta & T^{-1}\gamma' S_{11}\gamma \end{bmatrix} - \mu\begin{bmatrix} \beta' S_{10} S_{00}^{-1} S_{01}\beta & \beta' S_{10} S_{00}^{-1} S_{01}\gamma \\ \gamma' S_{10} S_{00}^{-1} S_{01}\beta & \gamma' S_{10} S_{00}^{-1} S_{01}\gamma \end{bmatrix}\} = 0.$$

Evaluating (A.20) to (A.22), (A.25), the result for $\beta' S_{01} S_{00}^{-1} S_{01}\beta$ and (A.30) in the above equation, it can be shown that

$$T\hat{\lambda}_{r+i} = \hat{\eta}_i + O_p(T^{-1}), \qquad i = 1, \cdots, k-r. \quad (A.31)$$

and that the limiting distribution of $\sum_{i=1}^{k-r}|\hat{\eta}_i|$ is $\check{M}_r$. Thus (i) is established.

(ii) It is easy to see from (A.21), (A.27), (A.29) and the result for $\tilde{R}_* \alpha'\hat{\Sigma}^{-1}\alpha \tilde{R}_*'$ in the proof of (i) that either $\hat{\nu}_i = O_p(1)$ and $\hat{\nu}_i^{-1} = O_p(1)$ or $\hat{\nu}_i = O_p(T^{1/2})$ and $(T^{-1/2}\hat{\nu}_i)^{-1} = O_p(1)$, as stated in (ii) of the theorem. Let $\tilde{\lambda}_1, \cdots, \tilde{\lambda}_r$ be the eigenvalues of the equation

$$\det\{\lambda\beta' S_{11}\beta - \beta' S_{10} S_{00}^{-1} S_{01}\beta\} = 0.$$

By arguments similar to those used above, we have

$$\tilde{\lambda}_i = \hat{\nu}_i + O_p(T^{-1/2}), \qquad i = 1, \cdots, r. \quad (A.32)$$

It is also clearly seen that the eigenvalues corresponding to the equation $\det\{\lambda S_{11} - S_{10} S_{10}^{-1} S_{01}\} = 0$ are equivalent to those of

$$\det\{\lambda\begin{bmatrix} \beta' S_{11}\beta & T^{-1/2}\beta' S_{11}\gamma \\ T^{-1/2}\gamma' S_{11}\beta & T^{-1}\gamma' S_{11}\gamma \end{bmatrix} - \begin{bmatrix} \beta' S_{10} S_{00}^{-1} S_{01}\beta & T^{-1/2}\beta' S_{10} S_{00}^{-1} S_{01}\gamma \\ T^{-1/2}\gamma' S_{10} S_{00}^{-1} S_{01}\beta & T^{-1}\gamma' S_{10} S_{00}^{-1} S_{01}\gamma \end{bmatrix}\} = 0.$$

This, along with arguments similar to those used for (A.31) or (A.32), leads to

$$\hat{\lambda}_i = \tilde{\lambda}_i + O_p(T^{-1/2}) \qquad i = 1, \cdots, r. \quad (A.33)$$

Combining (A.32) and (A.33) with the asymptotics for $\hat{\nu}_i$ derived already immediately establishes (ii).

Thus the proof is completed.

Proof of Theorem 2 The proof of (i) is trivial. (A.28) as $h = 0$ is the same as

$$S_{00;0} = \hat{\Sigma}_* + O_p(T^{-1/2}), \qquad (A.34)$$

which establishes that $\tilde{\nu}_i = O_p(1)$, $\tilde{\nu}_i^{-1} = O_p(1)$. The remaining part of the proof of (ii) can be shown to be as trivial as that of (i).

TABLE 1

Relative frequency of determining r correctly: The first group

DGP\Test K_T	$\hat{Q}_j$	$\hat{Q}_{*;j}$	LR	$P(k,j)$		$P_*(k,j)$	
				4	8	4	8
(i): $q=1$, $d_1=d_2=d_3=d_4=0.6$ and $d_5=d_6=d_7=d_8=d_9=0$							
				$T=200$			
$r=0$	91.84	91.84	89.17	98.96	99.78	100.0	100.0
$r=1$	93.27	93.26	90.87	98.02	2.03	97.36	0.0
$r=2$	93.12	93.11	90.11	48.93	33.65	0.0	0.0
				$T=500$			
$r=0$	93.86	93.86	90.53	97.65	98.05	100.0	100.0
$r=1$	94.21	94.21	91.67	97.09	94.86	100.0	85.83
$r=2$	94.29	94.29	91.58	99.32	99.1	0.0	0.0
(ii): $q=1$, $d_1=d_2=d_3=d_4=-0.6$ and $d_5=d_6=d_7=d_8=d_9=0$							
				$T=200$			
$r=0$	29.79	98.31	4.79	22.35	100.0	99.58	100.0
$r=1$	52.78	99.8	20.4	22.62	0.0	99.29	1.07
$r=2$	74.1	99.65	58.13	82.78	2.49	4.32	0.0
				$T=500$			
$r=0$	38.24	99.76	2.29	1.15	96.69	97.58	100.0
$r=1$	60.14	100.0	15.15	7.49	15.79	97.90	100.0
$r=2$	80.82	99.86	56.78	69.81	97.63	99.36	0.0
(iii): $q=2$, $d_1=d_2=d_3=d_4=1.5$, $d_5=0$ and $d_6=d_7=d_8=d_9=0.56$							
				$T=200$			
$r=0$	88.73	89.07	69.76	99.76	99.71	100.0	100.0
$r=1$	87.04	87.23	78.73	99.26	5.0	70.88	0.0
$r=2$	84.62	84.93	83.62	33.99	32.34	0.0	0.0
				$T=500$			
$r=0$	93.42	93.53	74.15	99.39	98.63	100.0	100.0
$r=1$	92.56	92.5	80.37	98.7	97.68	100.0	2.34
$r=2$	91.75	91.69	85.07	99.62	99.16	0.0	0.0

In all the tables, figures indicate percentiles, and K_T is required only for $P(k,j)$ and $P_*(k,j)$.

TABLE 1 (continued)

Relative frequency of determining r correctly: The first group

DGP\Test K_T	$\hat{Q}_j$	$\hat{Q}_{*;j}$	LR	$P(k,j)$		$P_*(k,j)$	
				4	8	4	8
(iv): $q=2$, $d_1=d_2=d_3=d_4=0.2$, $d_5=0$ and $d_6=d_7=d_8=d_9=-0.48$							
			$T=200$				
$r=0$	60.16	99.24	26.45	19.19	99.45	100.0	100.0
$r=1$	73.52	93.87	43.29	33.98	0.7	99.98	0.02
$r=2$	75.57	94.17	72.67	85.05	37.82	85.05	37.82
			$T=500$				
$r=0$	66.26	99.93	20.6	7.79	81.04	99.98	100.0
$r=1$	77.88	98.97	36.31	21.22	48.67	99.91	100.0
$r=2$	79.06	98.23	72.73	81.39	95.86	89.86	0.0
(v): $q=2$, $d_1=1.5$, $d_2=d_4=0.9$, $d_3=0.4$, $d_5=0.5$, $d_6=0.56$, $d_7=d_9=0.2$ and $d_8=0.04$							
			$T=200$				
$r=0$	86.47	89.69	82.77	99.26	99.74	100.0	100.0
$r=1$	87.01	88.85	87.35	99.02	3.38	88.83	0.0
$r=2$	85.97	88.21	84.73	35.08	33.92	0.0	0.0
			$T=500$				
$r=0$	91.93	93.24	85.7	98.46	98.49	100.0	100.0
$r=1$	93.21	93.65	88.32	98.47	97.15	100.0	5.92
$r=2$	91.55	92.97	85.85	99.43	99.17	0.0	0.0
(vi): $q=2$, $d_1=1.5$, $d_2=0.9$, $d_3=0.4$, $d_4=-0.4$, $d_5=0.5$, $d_6=0.56$, $d_7=0.2$ and $d_8=d_9=0$							
			$T=200$				
$r=0$	76.06	86.69	80.42	96.38	99.93	100.0	100.0
$r=1$	82.51	88.91	86.24	96.45	1.48	96.76	0.0
$r=2$	83.39	85.59	87.53	40.81	34.69	0.0	0.0
			$T=500$				
$r=0$	85.34	92.34	82.27	89.57	97.88	100.0	100.0
$r=1$	89.74	93.3	86.96	91.55	96.01	100.0	12.28
$r=2$	92.11	94.07	88.31	97.75	98.9	0.0	0.0

TABLE 2

Relative frequency of determining r correctly: The second group

DGP\Test K_T	$\hat{Q}_j$	$\hat{Q}_{*;j}$	LR	$P(k,j)$		$P_*(k,j)$	
				4	8	4	8
(*i*): $q=1$, $d_1=d_2=d_3=d_4=0.6$ and $d_5=d_6=d_7=d_8=d_9=0$							
				$T=200$			
$r=0$	89.27	89.34	84.73	97.71	99.95	100.0	100.0
$r=1$	92.24	92.11	89.52	97.63	4.55	1.58	0.0
$r=2$	96.43	97.01	89.2	82.32	0.0	81.01	0.0
				$T=500$			
$r=0$	92.95	92.97	87.17	92.0	98.57	100.0	100.0
$r=1$	94.34	94.44	89.67	94.89	97.53	100.0	0.0
$r=2$	98.02	98.12	90.13	82.33	5.36	99.89	0.0
(*ii*): $q=1$, $d_1=d_2=d_3=d_4=-0.6$ and $d_5=d_6=d_7=d_8=d_9=0$							
				$T=200$			
$r=0$	2.38	65.51	0.48	73.3	100.0	36.18	99.98
$r=1$	36.66	82.28	12.95	50.18	0.0	91.1	1.2
$r=2$	53.26	88.46	41.56	71.33	0.0	45.12	1.69
				$T=500$			
$r=0$	1.95	82.05	0.1	1.19	99.99	21.88	100.0
$r=1$	50.12	93.94	8.82	4.57	0.72	82.21	99.97
$r=2$	59.67	97.74	37.92	94.33	0.0	40.97	99.83
(*iii*): $q=2$, $d_1=d_2=d_3=d_4=1.5$, $d_5=0$ and $d_6=d_7=d_8=d_9=0.56$							
				$T=200$			
$r=0$	87.76	89.12	75.77	99.13	99.8	100.0	100.0
$r=1$	81.26	80.84	78.59	99.01	26.48	0.01	0.0
$r=2$	72.31	61.45	85.35	90.22	1.06	20.79	0.0
				$T=500$			
$r=0$	93.77	93.98	77.94	97.67	98.09	100.0	100.0
$r=1$	92.37	92.54	80.0	98.11	97.76	100.0	0.0
$r=2$	92.1	88.66	86.43	85.52	98.01	100.0	0.0

TABLE 2 (continued)

Relative frequency of determining r correctly: The second group

DGP\Test K_T	$\hat{Q}_j$	$\hat{Q}_{*;j}$	LR	$P(k,j)$ 4	$P(k,j)$ 8	$P_*(k,j)$ 4	$P_*(k,j)$ 8
(iv): $q=2$, $d_1=d_2=d_3=d_4=0.2$, $d_5=0$ and $d_6=d_7=d_8=d_9=-0.48$							
			$T=200$				
$r=0$	43.64	99.52	44.96	14.13	100.0	98.31	100.0
$r=1$	65.2	84.57	56.18	25.86	0.21	99.64	0.0
$r=2$	62.87	78.29	81.0	92.55	0.0	80.5	0.0
			$T=500$				
$r=0$	39.35	99.9	34.08	0.63	95.28	94.39	100.0
$r=1$	75.57	97.01	49.95	12.88	57.06	98.94	36.13
$r=2$	77.92	97.08	80.87	26.21	0.02	73.98	64.5
(v): $q=2$, $d_1=1.5$, $d_2=d_4=0.9$, $d_3=0.4$, $d_5=0.5$, $d_6=0.56$, $d_7=d_9=0.2$ and $d_8=0.04$							
			$T=200$				
$r=0$	87.23	90.11	80.49	98.71	99.88	100.0	100.0
$r=1$	76.06	70.5	86.12	98.8	24.01	0.03	0.0
$r=2$	83.66	90.81	86.06	81.44	0.5	19.71	0.0
			$T=500$				
$r=0$	93.7	94.44	83.41	96.94	98.55	100.0	100.0
$r=1$	89.81	88.93	87.14	97.21	97.83	100.0	0.0
$r=2$	93.14	97.87	86.12	88.52	48.29	99.97	0.0
(vi): $q=2$, $d_1=1.5$, $d_2=0.9$, $d_3=0.4$, $d_4=-0.4$, $d_5=0.5$, $d_6=0.56$, $d_7=0.2$ and $d_8=d_9=0$							
			$T=200$				
$r=0$	70.91	82.83	62.13	98.51	100.0	100.0	100.0
$r=1$	74.0	75.7	79.22	93.21	11.95	0.24	0.0
$r=2$	65.08	87.89	69.47	59.95	0.03	65.49	0.0
			$T=500$				
$r=0$	81.42	90.32	62.22	83.21	99.46	100.0	100.0
$r=1$	88.16	94.05	78.17	82.84	97.6	100.0	0.0
$r=2$	76.14	95.77	69.66	16.96	89.68	72.19	0.0

TABLE 3

Relative frequency of determining r correctly: The third group

DGP\Test K_T	$\hat{Q}_j$	$\hat{Q}_{*;j}$	LR	$P(k,j)$		$P_*(k,j)$	
				4	8	4	8
(*i*): $q=1$, $d_1=d_2=d_3=d_4=0.6$ and $d_5=d_6=d_7=d_8=d_9=0$							
			$T=200$				
$r=1$	93.74	93.74	92.15	87.35	9.02	81.26	0.0
			$T=500$				
$r=1$	94.72	94.72	93.39	80.82	94.49	100.0	8.91
(*ii*): $q=1$, $d_1=d_2=d_3=d_4=-0.6$ and $d_5=d_6=d_7=d_8=d_9=0$							
			$T=200$				
$r=1$	27.61	91.93	48.3	2.99	0.01	86.79	0.45
			$T=500$				
$r=1$	28.77	96.33	41.07	0.6	36.18	72.76	100.0
(*iii*): $q=2$, $d_1=d_2=d_3=d_4=1.5$, $d_5=0$ and $d_6=d_7=d_8=d_9=0.56$							
			$T=200$				
$r=1$	90.25	90.48	86.76	94.03	20.62	31.23	0.0
			$T=500$				
$r=1$	94.47	94.55	89.04	91.43	95.51	100.0	1.5
(*iv*): $q=2$, $d_1=d_2=d_3=d_4=0.2$, $d_5=0$ and $d_6=d_7=d_8=d_9=-0.48$,							
			$T=200$				
$r=1$	62.59	94.57	82.5	7.19	3.13	98.13	0.02
			$T=500$				
$r=1$	71.03	99.14	80.21	3.28	66.32	93.81	100.0
(*v*): $q=2$, $d_1=1.5$, $d_2=d_4=0.9$, $d_3=0.4$, $d_5=0.5$, $d_6=0.56$, $d_7=d_9=0.2$ and $d_8=0.04$							
			$T=200$				
$r=1$	89.43	91.08	89.44	94.44	15.56	47.24	0.0
			$T=500$				
$r=1$	93.54	94.05	91.14	91.42	95.54	100.0	2.92

TABLE 3 (Continued)

Relative frequency of determining r correctly: The third group

DGP\Test K_T	$\hat{Q}_j$	$\hat{Q}_{*;j}$	LR	$P(k,j)$		$P_*(k,j)$	
				4	8	4	8
(vi): $q=2$, $d_1=1.5$, $d_2=0.9$, $d_3=0.4$, $d_4=-0.4$, $d_5=0.5$, $d_6=0.56$, $d_7=0.2$ and $d_8=d_9=0$							
$T=200$							
$r=1$	84.15	89.42	90.22	82.63	12.06	64.68	0.0
$T=500$							
$r=1$	92.11	93.84	91.66	73.99	96.16	99.99	5.58