主　論　文

The Gangolli estimate for the coefficients of the Harish-Chandra expansions of the Eisenstein integrals and the expressions of the Harish-Chandra C-functions

(アイゼンシュタイン積分のハリッシュ・チャンドラ展開の係数のガンゴリー評価とハリッシュ・チャンドラのC関数の表示)

SHIN KOIZUMI

(小泉　伸)

THE GANGOLLI ESTIMATE FOR THE COEFFICIENTS OF THE HARISH-CHANDRA EXPANSIONS OF THE EISENSTEIN INTEGRALS AND THE EXPRESSIONS OF THE HARISH-CHANDRA C-FUNCTIONS

SHIN KOIZUMI

CONTENTS

1. INTRODUCTION

Let G be a reductive Lie group of class $\mathcal{H}$ and K a maximal compact subgroup of G. One of the main concerns in the harmonic analysis on G has been the characterization of the Fourier transforms of various function spaces on G, such as a space of compactly supported smooth functions, Schwartz space and L^p-type Schwartz space. A number of authors solved these problems for particular classes of groups in their papers (cf. [1,3,5,9,10,14,24,25]). However, even now, the complete answer of these problems dose not seem to be known. One of the most difficult parts of these problems is to show the continuity of the inverse Fourier transform. The inverse Fourier transform in these characterizations is given by integrating a function on the Fourier transform side against the matrix elements of the principal series representations. Thus the Eisenstein integrals on G, that is, the matrix elements of the principal series representations play an essential role in studying these analysis.

One of the fundamental techniques for these analysis is that of approximating the Eisenstein integrals by their constant terms, which was established by Harish-Chandra. For example, Arthur [1] proved the Paley–Wiener type theorem for the

Schwartz spaces on G. He used the leading terms as an approximation of the Eisenstein integrals and estimates of difference between them. For the case of spherical L^p-Schwartz space ($p \neq 2$), taking some terms of the Harish-Chandra expansion of elementary spherical function as an approximation for it, Trombi and Varadarajan [26] gave a uniform estimate for the difference between them. And by virtue of this estimate, they proved the Paley–Wiener type theorem for the L^p-Schwartz spaces of K-biinvariant functions on G. Later Eguchi [5] gave similar estimates for Eisenstein integrals of $(\tau, 1)$-type and proved the Paley–Wiener type theorem for L^p-Schwartz space of K-invariant functions on G. In [25], Trombi showed the Paley–Wiener type theorem for L^p-Schwartz space on semisimple Lie group G of real rank one with the restriction to the K-finite functions. Here in order to describe the contents of this paper, we shall use some notation explained in §2. For $\nu \in \mathfrak{a}_{\mathbf{c}}^*$, the zonal spherical function is defined by

$$\varphi_\nu(x) = \int_K e^{(\nu-\rho)(H(xk))} dk.$$

Harish-Chandra showed that $\varphi_\nu(h)$, $(h \in A^+)$ is expanded as

$$h^{-\rho}\varphi_\nu(h) = \sum_{w \in W(\mathfrak{a})} \sum_{\lambda \in L} c(w\nu)\Gamma_\lambda(w\nu - \rho)h^{w\nu-\lambda}, \ (\nu \in \Upsilon).$$

In [11], Gangolli showed that there exist $d, D > 0$ such that

$$|\Gamma_\lambda(\nu - \rho)| \leq Dm(\lambda)^d, \ (\nu \in \mathcal{R}).$$

And by using this estimate, he completed the Paley–Wiener theorem for compactly supported smooth K-biinvariant functions, which was first proved by Helgason with an assumption. In §5, we get the estimates for the coefficients of the Harish-Chandra expansions of the Eisenstein integrals. In our cases, because singularities of $\Gamma_\lambda(\nu-\rho)$ arise from the double unitary representation of K, we multiply a polynomial $P(\nu)$ that vanish away these singularities to $\Gamma_\lambda(\nu - \rho)$. By using this estimate, Eguchi and Wakayama [10] simplified the Trombi's proof of the Paley–Wiener theorem of L^p-Schwartz space.

In §7 through §11, we get the explicit expression of the Harish-Chandra C-function for $SU(n,1)$. The Harish-Chandra C-functions are given by the leading terms of the Harish-Chandra expansions of the Eisenstein integrals and closely related to the Plancherel measure. The Harish-Chandra C-functions are also obtained by restricting the standard intertwining operators to K-isotypic components of the principal series representation. Therefore the information on the location of the zeros and the singularities of the Harish-Chandra C-function gives the condition for the reducibility of the principal series representations. By the product formula for the Harish-Chandra C-function, the problem of computing the Harish-Chandra C-functions of semisimple Lie groups of general rank is reduced to the real rank one case. For this reason, it is crucial to compute the Harish-Chandra C-function for the semisimple Lie group $SU(n,1)$ of real rank one. For $\tau \in \hat{K}$, the Harish-Chandra C-function is given by

$$C_\tau(\nu) = \int_{\bar{N}} e^{-(\nu+\rho)(H(\bar{n}))} \tau(\kappa(\bar{n}))^{-1} d\bar{n}, \ (\nu \in \mathfrak{a}_{\mathbf{c}}^*).$$

In the case of $SU(n,1)$, because $[\tau : \sigma] \leq 1$ for all $\tau \in \hat{K}$ and $\sigma \in \hat{M}$, there exists a meromorphic function $C_\tau(\sigma : \nu)$ such that

$$TC_\tau(\nu) = C_\tau(\sigma : \nu)T, \ (T \in \mathrm{Hom}_M(V_\tau, H_\sigma)).$$

In this paper we shall obtain the explicit expression of $C_\tau(\sigma : \nu)$ for $SU(n,1)$. This expression gives us the precise information on the zeros and the poles of the Harish-Chandra C-function $C_\tau(\nu)$. On the other hand, Cohn (cf. [4]) showed that for any semisimple Lie group, there exist $p_{i,j}, q_{i,j} \in \mathbf{C}$, $(1 \leq i \leq r, 1 \leq j \leq j_i)$ and $\mu_1, \ldots, \mu_r \in \mathfrak{a}^*$ such that

$$\det C_\tau(\nu) = \prod_{i=1}^{r} \prod_{j=1}^{j_i} \frac{\Gamma\left(\frac{-\langle \nu, \alpha_i \rangle}{2\langle \mu_i, \alpha_i \rangle} + q_{i,j}\right)}{\Gamma\left(\frac{-\langle \nu, \alpha_i \rangle}{2\langle \mu_i, \alpha_i \rangle} + p_{i,j}\right)}.$$

Here $\det C_\tau(\nu)$ means the determinant of the linear endomorphism $C_\tau(\nu)$ of V_τ. In [4], he conjectured that the coefficients $p_{i,j}$ and $q_{i,j}$ appearing in the above expression are rational numbers and depending linearly on the highest weight of τ. By using the expression of $C_\tau(\sigma : \nu)$ together with $V_\tau = \sum_{\sigma \in \hat{M}} [\tau : \sigma] H_\sigma$, we can get the explicit formula for $\det C_\tau(\nu)$ and this shows that Cohn's conjecture is true for $SU(n,1)$.

To compute $C_\tau(\sigma : \nu)$, we use the formula of the infinitesimal operator of the principal series representation for semisimple Lie groups of real rank one. By using this formula, we can get a recursion formula of the standard intertwining operator with respect to the dominant, analytically integral forms on $\mathfrak{t}_\mathbf{c}$. From the relationship between the standard intertwining operator and the Harish-Chandra C-function, this formula leads to the recursion formula of the Harish-Chandra C-function. In our cases, the infinitesimal operator can be written explicitly in terms of the Gel'fand–Tsetlin basis of $\mathfrak{u}(n)$. By using this recursion formula, for getting the expression of the Harish-Chandra C-function, it suffices to consider the case that the dominant, analytically integral form on $\mathfrak{t}_\mathbf{c}$ is minimal in the sense of the betweenness condition of the Gel'fand–Tsetlin basis.

In §12, we show that the information on zeros of the Harish-Chandra C-function can be utilized to get the realizations of discrete series representations of $SU(n,1)$ as subquotients of nonunitary principal series representations. We also give the K-spectra of these representations. We note that these results are already obtained by an another method. However, using the expression of the Harish-Chandra C-function, we can get the explicit expressions of the inner products that make the above subquotients unitary. In §13, by using the results in §12, we get the decompositions of holomorphic and antiholomorphic discrete series when restricted to $U(n-1,1)$, which was proven in [21]. By using the structures of K-spectra of discrete series representations, we can concretely construct the invariant subspaces of the representation spaces of holomorphic and antiholomorphic discrete series.

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2. Notation and Preliminaries

Throughout this paper, we shall use the standard notation $\mathbf{Z}$, $\mathbf{R}$ and $\mathbf{C}$ for the set of integers, real numbers and complex numbers, respectively. For a finite set F, $\operatorname{Card} F$ denotes its cardinal number.

Let G be a reductive Lie group of class $\mathcal{H}$ and K a maximal compact subgroup of G and θ the corresponding Cartan involution. As usual, we shall use lower case German letters to denote the corresponding Lie algebras and upper case German letters their universal enveloping algebras. For any Lie group L, $\hat{L}$ denotes the set of equivalence classes of the irreducible unitary representations of L. If V is a vector space over $\mathbf{R}$, $V_{\mathbf{c}}$, V^* and $V_{\mathbf{c}}^*$ denote its complexification, its real dual and its complex dual, respectively. Let $\langle \cdot, \cdot \rangle$ denote the Killing form on $\mathfrak{g}$. Define the inner product $\langle \cdot, \cdot \rangle_\theta$ on $\mathfrak{g}$ by $\langle X, Y \rangle_\theta = -\langle X, \theta Y \rangle$ and write $\|X\| = \sqrt{\langle X, X \rangle_\theta}$.

Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$ corresponding to θ. Choose a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ and fix an orthonormal basis $\{H_j : 1 \leq j \leq \ell'\}$, $(\ell' = \dim \mathfrak{a})$ of $\mathfrak{a}$. Let $\mathfrak{h}$ be a θ-stable Cartan subalgebra containing $\mathfrak{a}$ and $\mathfrak{h}_{\mathfrak{k}} = \mathfrak{h} \cap \mathfrak{k}$. Let $\mathfrak{t}$ be the Cartan subalgebra of $\mathfrak{k}$ containing $\mathfrak{h}_{\mathfrak{k}}$. Fix an ordering on $\sqrt{-1}\mathfrak{h}_{\mathfrak{k}} + \mathfrak{a}$ that is compatible with the one on $\mathfrak{a}$ and fix the ordering on $\sqrt{-1}\mathfrak{t}$ that is compatible with the one on $\sqrt{-1}\mathfrak{h}_{\mathfrak{k}}$.

Let Δ be the set of all nonzero roots of $\mathfrak{g}_{\mathbf{c}}$ with respect to $\mathfrak{h}_{\mathbf{c}}$ and Δ^+ the subset of Δ consisting of all positive roots. Put $P_+ = \{\alpha \in \Delta^+ : \tilde{\alpha} = \alpha|_{\mathfrak{a}} \neq 0\}$. For $\alpha \in \Delta$, $\mathfrak{g}_{\mathbf{c}}^\alpha$ denotes the corresponding root subspace of $\mathfrak{g}_{\mathbf{c}}$. We put $\mathfrak{n} = (\sum_{\alpha \in P_+} \mathfrak{g}_{\mathbf{c}}^\alpha) \cap \mathfrak{g}$. Let A and N denote the analytic subgroups of G corresponding to $\mathfrak{a}$ and $\mathfrak{n}$, respectively and $\bar{N} = \theta N$. Then $G = KAN$ and $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ are the Iwasawa decompositions of G and $\mathfrak{g}$, respectively. For $g \in G$, g decomposes under $G = KAN$ as $g = \kappa(g) \exp H(g) n(g)$, where $\kappa(g) \in K$, $H(g) \in \mathfrak{a}$ and $n(g) \in N$. For $\alpha \in P_+$, we define $Q_{\tilde{\alpha}} \in \mathfrak{a}$ by $\alpha(H) = \langle Q_{\tilde{\alpha}}, H \rangle$ for $H \in \mathfrak{a}$. For $\alpha \in P_+$, we choose the root vectors $X_{\pm\alpha} \in \mathfrak{g}_{\mathbf{c}}^{\pm\alpha}$ so that $\langle X_\alpha, X_{-\alpha} \rangle = 1$ and write them as $X_{\pm\alpha} = Y_{\pm\alpha} + Z_{\pm\alpha}$, where $Y_{\pm\alpha} \in \mathfrak{k}_{\mathbf{c}}$ and $Z_{\pm\alpha} \in \mathfrak{p}_{\mathbf{c}}$. Let $\mathfrak{a}^+, (\mathfrak{a}^+)^*$ and A^+ be the positive Weyl chambers. We set $\mathcal{R} = \{\nu = \xi + \eta \in \mathfrak{a}_{\mathbf{c}}^* : \xi \in \sqrt{-1}\mathfrak{a}^*, -\eta \in \mathrm{Cl}(\mathfrak{a}^+)^*\}$, where Cl denotes the closure.

Let Σ^+ be the set of all restricted roots of $\mathfrak{g}$ with respect to $\mathfrak{a}$, that is, $\Sigma^+ = \{\tilde{\alpha} : \alpha \in P_+\}$ and $\{\alpha_i : 1 \leq i \leq \ell\}$ the set of all simple restricted roots. For $\alpha \in \Sigma^+$, m_α denotes the multiplicity of α. We denote by M and M' the centralizer and the normalizer of $\mathfrak{a}$ in K respectively. Then $W(\mathfrak{a}) = M'/M$ is the Weyl group of G. For $w \in W(\mathfrak{a})$, $\sigma \in \hat{M}$ and $\nu \in \mathfrak{a}_{\mathbf{c}}^*$, define $w\nu \in \mathfrak{a}_{\mathbf{c}}^*$ and $w\sigma \in \hat{M}$ by $w\nu(H) = \nu(\mathrm{Ad}(w)^{-1}H)$ and $w\sigma(m) = \sigma(w^{-1}mw)$.

Let Δ_K be the set of all roots of $\mathfrak{k}_{\mathbf{c}}$ with respect to $\mathfrak{t}_{\mathbf{c}}$, Δ_K^+ the subset of Δ_K consisting of all positive roots and W_K the Weyl group of $(\mathfrak{k}_{\mathbf{c}}, \mathfrak{t}_{\mathbf{c}})$. As usual, we write $\rho = \frac{1}{2}\sum_{\alpha \in \Sigma^+} m_\alpha \alpha$ and $\delta_K = \frac{1}{2}\sum_{\beta \in \Delta_K^+} \beta$. Let ω and $\omega_{\mathfrak{m}}$ be the Casimir operator of $\mathfrak{G}$ and $\mathfrak{M}$, respectively. For each $D \in \mathfrak{G}$, we denote by $\mathrm{Ⓡ}_A(D)$ the radial component of D.

Let D_K and D_M denote the sets of dominant, analytically integral forms on $\mathfrak{t}_{\mathbf{c}}$ and $\mathfrak{h}_{\mathfrak{k}_{\mathbf{c}}}$, respectively, with respect to the above orderings. If $\lambda \in D_K$ and $\mu \in D_M$, we write $(\tau_\lambda, V_\lambda)$ and (σ_μ, H_μ) for the irreducible unitary representations of K and M whose highest weights are λ and μ, respectively. For $\tau \in \hat{K}$ and $\sigma \in \hat{M}$, $[\tau : \sigma]$

denotes the multiplicity of σ occurring in $\tau|_M$. Let $\hat{K}(\sigma)$ and $\hat{M}(\tau)$ be the subsets of $\hat{K}$ and $\hat{M}$ consisting of the elements satisfying $[\tau : \sigma] \neq 0$, respectively. Similarly $D_K(\mu)$ and $D_M(\lambda)$ denote the subsets of D_K and D_M consisting of the elements satisfying $[\tau_\lambda : \sigma_\mu] \neq 0$, respectively.

Finally let dk and $d\bar{n}$ be the Haar measures on K and $\bar{N}$, respectively, normalized as $\int_K dk = 1$ and $\int_{\bar{N}} \exp\{-2\rho(H(\bar{n}))\} d\bar{n} = 1$.

3. Eisenstein integrals and the Harish-Chandra expansions

We shall first review the compact picture of the principal series to explain the notation and the parametrization. Let $(\sigma, H_\sigma) \in \hat{M}$ and $\nu \in \mathfrak{a}_{\mathbf{C}}^*$. We set

$$C_\sigma^\infty(K) = \left\{\varphi \in C^\infty(K; H_\sigma) : \varphi(km) = \sigma(m)^{-1}\varphi(k)\right\}. \tag{3.1}$$

Let $\mathcal{H}^{\sigma,\nu}$ denote the Hilbert space completion of $C_\sigma^\infty(K)$ relative to the inner product $(f, g) = \int_K (f(k), g(k))_{H_\sigma} dk$. Define the action $\pi_{\sigma,\nu}$ of G on $\mathcal{H}^{\sigma,\nu}$ by

$$(\pi_{\sigma,\nu}(g)\varphi)(k) = e^{-(\nu+\rho)(H(g^{-1}k))}\varphi(\kappa(g^{-1}k)). \tag{3.2}$$

Then $(\pi_{\sigma,\nu}, \mathcal{H}^{\sigma,\nu})$ is called the principal series representation of G. For $(\tau, V_\tau) \in \hat{K}(\sigma)$, let $\mathcal{H}^{\sigma,\nu}(\tau)$ be the τ-isotypic component of $\mathcal{H}^{\sigma,\nu}$. Then Frobenius reciprocity implies the following lemma.

Lemma 3.1. *The correspondence* $T \otimes v \to f_{T\otimes v}(k) = T(\tau(k)^{-1}v)$ *is a* K*-module isomorphism of* $\mathrm{Hom}_M(V_\tau, H_\sigma) \otimes V_\tau$ *onto* $\mathcal{H}^{\sigma,\nu}(\tau)$.

Let (τ_1, V_{τ_1}), $(\tau_2, V_{\tau_2}) \in \hat{K}$. We define a double unitary representation $\tau = (\tau_1, \tau_2)$ of K on $V = \mathrm{Hom}_{\mathbf{C}}(V_{\tau_2}, V_{\tau_1})$ by $\tau(k_1, k_2)(v) = \tau_1(k_1)v\tau_2(k_2)^{-1}$, $(k_1, k_2 \in K, v \in V)$. V_M denotes the subspace of V comprised of all elements $v \in V$ such that $\tau_1(m)v = v\tau_2(m)$ for all $m \in M$. Then the Eisenstein integral on G is defined for $v \in V_M$ and $\nu \in \mathfrak{a}_{\mathbf{C}}^*$ by the following integral:

$$E(v : \nu : x) = \int_K \tau_1(\kappa(xk))v\tau_2(k)^{-1}e^{(\nu-\rho)(H(xk))}dk. \tag{3.3}$$

As well known, all matrix elements of the principal series can be recovered from the Eisenstein integrals. Let $\tau_1, \tau_2 \in \hat{K}(\sigma)$. Then for $T_1 \in \mathrm{Hom}_M(V_{\tau_1}, H_\sigma)$, $T_2 \in \mathrm{Hom}_M(V_{\tau_2}, H_\sigma)$, $v_1 \in V_{\tau_1}$ and $v_2 \in V_{\tau_2}$, it is satisfied that

$$(\pi_{\sigma,\nu}(x)f_{T_1\otimes v_1}, f_{T_2\otimes v_2}) = (E(T_2^*T_1, \nu, x)v_2, v_1)_{V_{\tau_1}}, \tag{3.4}$$

where T_2^* denotes the adjoint operator of T_2 and $(\cdot,\cdot)_{V_{\tau_1}}$ denotes the inner product in V_{τ_1}.

We define an endomorphism γ of $\mathrm{Hom}_{\mathbf{C}}(V_M, V_M)$ by $\gamma(T) = [\tau_2(\omega_{\mathfrak{m}}), T]$. Since the representation τ_2 of K is unitary, all eigenvalues of the transformation $v \mapsto v\tau_2(\omega_{\mathfrak{m}})$ are real. Let $\gamma_1, \cdots, \gamma_t$ be the distinct eigenvalues with multiplicities $m_1, \cdots, m_t$, respectively and suppose that

$$\gamma_1 < \cdots < \gamma_s < 0 \leq \gamma_{s+1} < \cdots < \gamma_t.$$

Let L be the set of $\lambda = \sum_{i=1}^{\ell} n_i\alpha_i$, $(n_i \in \mathbf{Z}_{\geq 0})$ and put $L' = L \setminus \{0\}$. For $\lambda = \sum_{i=1}^{\ell} n_i\alpha_i \in L$, we set $m(\lambda) = \sum_{i=1}^{\ell} n_i$. If $\lambda, \lambda' \in L$ and $\lambda - \lambda' \in L$, we denote $\lambda \gg \lambda'$. For each $\lambda \in L$, the $\mathrm{Hom}_{\mathbf{C}}(V_M, V_M)$-valued functions Γ_λ on $\mathfrak{a}_{\mathbf{C}}^*$ are recursively defined as follows: put $\Gamma_0 = 1$ and for $\lambda \neq 0$,

(3.5)

$$\begin{aligned}(2\lambda - \langle\lambda, \lambda - 2\rho\rangle)\Gamma_\lambda - \gamma(\Gamma_\lambda) &= 2\sum_{\alpha\in P_+}\sum_{n\geq 1}(\tilde{\alpha} - \langle\tilde{\alpha}, \lambda - 2n\tilde{\alpha}\rangle)\Gamma_{\lambda-2n\tilde{\alpha}} \\ &+ 8\sum_{\alpha\in P_+}\sum_{n\geq 1}(2n-1)\tau_1(Y_\alpha)\Gamma_{\lambda-(2n-1)\tilde{\alpha}}\tau_2(Y_{-\alpha}) \\ &- 8\sum_{\alpha\in P_+}\sum_{n\geq 1} n\big\{\tau_1(Y_\alpha Y_{-\alpha})\Gamma_{\lambda-2n\tilde{\alpha}} + \Gamma_{\lambda-2n\tilde{\alpha}}\tau_2(Y_\alpha Y_{-\alpha})\big\}.\end{aligned}$$

Here we put $\Gamma_\lambda = 0$ for $\lambda \notin L$.

For each $1 \leq i \leq t$ and $\lambda \in L'$, put

$$\sigma_{\lambda,i} = \big\{\nu \in \mathfrak{a}_{\mathbf{C}}^* : 2\langle\lambda, \nu\rangle = \langle\lambda, \lambda\rangle + \gamma_i\big\}, \tag{3.6}$$

and let Υ and Υ_0 be the complement of the set $\cup_{\lambda\in L'}\cup_{1\leq i\leq t}\sigma_{\lambda,i}$ in $\mathfrak{a}_{\mathbf{C}}^*$ and the subset of $\mathfrak{a}_{\mathbf{C}}^*$ comprised of all $\nu \in \mathfrak{a}_{\mathbf{C}}^*$ such that $w\nu \in \Upsilon$ for all $w \in W(\mathfrak{a})$ respectively.

For $\mu \in \mathfrak{a}^*$ and $h \in A$, we write h^μ for $e^{\mu(\log h)}$. The following theorem has been proved by Harish-Chandra.

Theorem 3.2 (cf. [12,13,14]**).** *Fix a $\nu \in \Upsilon$ and set*

$$\Phi(\nu : h) = \sum_{\lambda\in L}\Gamma_\lambda(\nu - \rho)h^{\nu-\lambda}, \ (h \in A^+).$$

Then the function $h \mapsto \Phi(\nu : h)$ is analytic on A^+ and satisfies the following differential equation:

$$\Phi(\nu : h; e^\rho \circ \mathrm{D}_A(\omega) \circ e^{-\rho}) = \Phi(\nu : h)(\langle\nu, \nu\rangle - \langle\rho, \rho\rangle + \tau_2(\omega_{\mathfrak{m}})). \tag{3.7}$$

Moreover, $h^\rho E(v : \nu : h)$ is expanded as

$$h^\rho E(v : \nu : h) = \sum_{w\in W(\mathfrak{a})}\Phi(w\nu : h)C_\tau(w : \nu)v, \ (v \in V_M, h \in A^+, \nu \in \Upsilon_0), \tag{3.8}$$

where $C_\tau(w : \nu)$ are the Harish-Chandra C-functions.

Remark. The expansion (3.8) is called the Harish-Chandra expansion of the Eisenstein integral.

4. The series expansion of $\Delta(h)^{1/2} \circ \mathrm{D}_A(\omega) \circ \Delta(h)^{-1/2}$

We retain the notation in §3. Let

$$\tilde{\Phi}(\nu : h) = h^{-\rho}\Phi(\nu : h), \quad \Delta(h) = h^{2\rho}\prod_{\alpha\in P_+}(1 - h^{-2\alpha}), \tag{4.1}$$

$$\Psi(\nu : h) = \Delta(h)^{1/2}\tilde{\Phi}(\nu : h).$$

Then from the equation in (3.7), we obtain

$$\tilde{\Phi}(\nu : h; \mathcal{Q}_A(\omega)) = \tilde{\Phi}(\nu : h)(\langle \nu, \nu \rangle - \langle \rho, \rho \rangle + \tau_2(\omega_{\mathfrak{m}})), \tag{4.2}$$
$$\Psi(\nu : h; \Delta(h)^{1/2} \circ \mathcal{Q}_A(\omega) \circ \Delta(h)^{-1/2}) = \Psi(\nu : h)(\langle \nu, \nu \rangle - \langle \rho, \rho \rangle + \tau_2(\omega_{\mathfrak{m}})).$$

We consider the series expansion of $\Delta(h)^{1/2} \circ \mathcal{Q}_A(\omega) \circ \Delta(h)^{-1/2}$. To do this, we need the following lemma.

Lemma 4.1 (cf. [30]). *The radial component $\mathcal{Q}_A(\omega)$ of the Casimir operator ω can be written as follows:*

$$\mathcal{Q}_A(\omega) = \mathcal{Q}_A(\omega_{\mathfrak{m}}) + \delta'(\omega) - 2 \sum_{\alpha \in P_+} (\sinh \alpha)^{-2} (1 \otimes 1 \otimes Y_\alpha Y_{-\alpha} + Y_\alpha Y_{-\alpha} \otimes 1 \otimes 1)$$
$$+ 4 \sum_{\alpha \in P_+} (\sinh \alpha)^{-1} \coth \alpha (Y_\alpha \otimes 1 \otimes Y_{-\alpha}),$$

where $\delta'(\omega) = \sum_{i=1}^{\ell'} H_i^2 + \sum_{\alpha \in P_+} \coth \alpha Q_{\tilde{\alpha}}$.

By using Lemma 4.1, we immediately obtain

$$\Delta(h)^{1/2} \circ \mathcal{Q}_A(\omega) \circ \Delta(h)^{-1/2} = \mathcal{Q}_A(\omega_{\mathfrak{m}}) + \Delta(h)^{1/2} \circ \delta'(\omega) \circ \Delta(h)^{-1/2} \tag{4.3}$$
$$- 2 \sum_{\alpha \in P_+} (\sinh \alpha)^{-2} (1 \otimes 1 \otimes Y_\alpha Y_{-\alpha} + Y_\alpha Y_{-\alpha} \otimes 1 \otimes 1)$$
$$+ 4 \sum_{\alpha \in P_+} (\sinh \alpha)^{-1} \coth \alpha (Y_\alpha \otimes 1 \otimes Y_{-\alpha}).$$

We first compute $\Delta(h)^{1/2} \circ \delta'(\omega) \circ \Delta(h)^{-1/2}$. Since

$$H_i \circ \Delta(h) = \sum_{\alpha \in P_+} \alpha(H_i) \coth \alpha(H_i) \Delta(h) + \Delta(h) \circ H_i,$$

it follows that

$$\delta'(\omega) = \sum_{i=1}^{\ell'} \Delta(h)^{-1} \circ H_i \circ \Delta(h) \circ H_i,$$

and hence

$$\Delta(h)^{1/2} \circ \delta'(\omega) \circ \Delta(h)^{-1/2} = \sum_{i=1}^{\ell'} \Delta(h)^{-1/2} \circ H_i \circ \Delta(h) \circ H_i \circ \Delta(h)^{-1/2}. \tag{4.4}$$

Computing $H_i \circ \Delta(h)^{-1/2}$ and $H_i \circ \Delta(h)^{1/2}$, we see that the expression in (4.3) is

written as follows:

(4.5)

$$\Delta(h)^{1/2} \circ \delta'(\omega) \circ \Delta(h)^{-1/2} = \sum_{i=1}^{\ell'} H_i^2 - \left\{ \frac{1}{2} \sum_{i=1}^{\ell'} \Delta(h)^{-1} \circ (H_i^2 \Delta(h)) \right.$$
$$\left. -\frac{1}{4} \sum_{i=1}^{\ell'} \{\Delta(h)^{-1} \circ (H_i \Delta(h))\}^2 \right\}$$
$$= \sum_{i=1}^{\ell'} H_i^2 - \left\{ \frac{1}{2} \sum_{i=1}^{\ell'} H_i^2 (\log \Delta(h)) + \frac{1}{4} \sum_{i=1}^{\ell'} (H_i \log \Delta(h))^2 \right\}.$$

From the definition of $\Delta(h)$, we have

$$H_i \log \Delta(h) = 2 \left\{ \rho(H_i) + \sum_{\alpha \in P_+} \alpha(H_i) \sum_{j \geq 1} h^{-2j\alpha} \right\},$$
$$H_i^2 \log \Delta(h) = -4 \sum_{\alpha \in P_+} \alpha(H_i)^2 \sum_{j \geq 1} h^{-2j\alpha}.$$

Hence we have

$$(H_i \log \Delta(h))^2 = 4 \left\{ \rho(H_i)^2 + 2 \sum_{\alpha \in P_+} \rho(H_i) \alpha(H_i) \sum_{j \geq 1} h^{-2j\alpha} \right.$$
$$\left. + \sum_{\alpha \in P_+} \alpha(H_i)^2 \sum_{j,k \geq 1} h^{-2(j+k)\alpha} + \sum_{\substack{\alpha, \beta \in P_+ \\ \alpha \neq \beta}} \alpha(H_i) \beta(H_i) \sum_{j,k \geq 1} h^{-2(j\alpha + k\beta)} \right\}.$$

Noting that $\sum_{i=1}^{\ell'} \alpha(H_i) H_i = Q_{\tilde{\alpha}}$ and $\sum_{i=1}^{\ell'} \rho(H_i)^2 = \langle \rho, \rho \rangle$, we have

$$\frac{1}{4} (H_i \log \Delta(h))^2 = \langle \rho, \rho \rangle + \sum_{\alpha \in P_+} \langle \tilde{\alpha}, \tilde{\alpha} \rangle \sum_{j \geq 1} h^{-2j\alpha} + \sum_{\substack{\alpha, \beta \in P_+ \\ \alpha \neq \beta}} \langle \tilde{\alpha}, \tilde{\beta} \rangle \sum_{j \geq 1, k \geq 0} h^{-2(j\alpha + k\beta)}.$$

Substituting these into the expression in (4.5), we get the following.

$$\Delta(h)^{1/2} \circ \delta'(\omega) \circ \Delta(h)^{-1/2} = \sum_{i=1}^{\ell'} H_i^2 - \langle \rho, \rho \rangle + \sum_{\alpha \in P_+} \langle \tilde{\alpha}, \tilde{\alpha} \rangle \sum_{j \geq 1} j h^{-2j\alpha}$$
$$- \sum_{\substack{\alpha, \beta \in P_+ \\ \alpha \neq \beta}} \langle \tilde{\alpha}, \tilde{\beta} \rangle \sum_{j \geq 1, k \geq 0} h^{-2(j\alpha + k\beta)}.$$

Using the above expression and substituting the following series expansions

$$(\sinh\alpha)^{-2} = 4\sum_{i=1}^{\infty} ie^{-2i\alpha},$$

$$\sinh\alpha\coth\alpha = 2\sum_{i=1}^{\infty}(2i-1)e^{-(2i-1)\alpha}$$

into the right-hand side of (4.3), we can immediately obtain the following.

Lemma 4.2. *We have the following expression.*

(4.6)

$$\begin{aligned}\Delta(h)^{1/2}\circ\mathcal{Q}_A(\omega)\circ\Delta(h)^{-1/2} &= \mathcal{Q}_A(\omega_{\mathfrak{m}}) + \sum_{i=1}^{\ell'} H_i^2 - \langle\rho,\rho\rangle \\ &+ \sum_{\alpha\in P_+}\langle\tilde{\alpha},\tilde{\alpha}\rangle\sum_{j\geq 1} jh^{-2j\alpha} - \sum_{\substack{\alpha,\beta\in P_+\\ \alpha\neq\beta}}\langle\tilde{\alpha},\tilde{\beta}\rangle\sum_{j\geq 1,k\geq 0} h^{-2(j\alpha+k\beta)} \\ &- 8\sum_{\alpha\in P_+}\sum_{j\geq 1} jh^{-2j\alpha}(1\otimes 1\otimes Y_\alpha Y_{-\alpha} + Y_\alpha Y_{-\alpha}\otimes 1\otimes 1) \\ &+ 8\sum_{\alpha\in P_+}\sum_{j\geq 1}(2j-1)h^{-(2j-1)\alpha}(Y_\alpha\otimes 1\otimes Y_{-\alpha}).\end{aligned}$$

5. The Estimate of the coefficients of Γ_λ

In this section, applying Lemma 4.2 to the differential equation in (4.2), we shall get the estimate of Γ_λ. We write $h^{-2\rho}\Delta(h)$ by the binomial theorem as

$$h^{-2\rho}\Delta(h) = \prod_{\alpha\in P_+}(1-h^{-2\alpha})^{1/2} = \sum_{\sigma\in L} b_\sigma h^{-\sigma}.$$

By the definition of Ψ, we have

(5.1)
$$\begin{aligned}\Psi(\nu:h) &= \prod_{\alpha\in P_+}(1-h^{-2\alpha})^{1/2}\Phi(\nu:h) \\ &= \left(\sum_{\sigma\in L} b_\sigma h^{-\sigma}\right)\left(h^\nu\sum_{\mu\in L}\Gamma_\mu(\nu-\rho)h^{-\mu}\right) \\ &= h^\nu\sum_{\lambda\in L}\left(\sum_{\substack{\sigma,\mu\in L\\ \sigma+\mu=\lambda}} b_\sigma\Gamma_\mu(\nu-\rho)\right)h^{-\lambda}.\end{aligned}$$

Put $a_\lambda(\nu) = \sum_{\sigma+\mu=\lambda} b_\sigma\Gamma_\mu(\nu-\rho)$. Then the last expression in (5.1) is of the form

(5.2)
$$\Psi(\nu:h) = h^\nu\sum_{\lambda\in L} a_\lambda(\nu)h^{-\lambda}.$$

Conversely, suppose that Ψ is written as in (5.2). By the binomial theorem, we write

$$\prod_{\alpha\in P_+}(1-h^{-2\alpha})^{-1/2}=\sum_{\mu\in L}d_\mu h^{-\mu}, \tag{5.3}$$

Then it is obvious that there exist constants $R_1, R_2 > 0$ such that

$$|d_\mu|\le R_1 m(\mu)^{R_2}. \tag{5.4}$$

By the similar computation as in (5.1), we have

$$\begin{aligned}\Phi(\nu:h)&=\prod_{\alpha\in P_+}(1-h^{-2\alpha})^{-1/2}\Psi(\nu:h)\\&=\Big(\sum_{\mu\in L}d_\mu h^{-\mu}\Big)\Big(h^\nu\sum_{\sigma\in L}a_\sigma(\nu)h^{-\sigma}\Big)\\&=h^\nu\sum_{\lambda\in L}\Big(\sum_{\sigma+\mu=\lambda}d_\mu a_\sigma(\nu)\Big)h^{-\nu}.\end{aligned} \tag{5.5}$$

Thus we obtain

$$\Gamma_\lambda(\nu-\rho)=\sum_{\sigma+\mu=\lambda}d_\mu a_\sigma(\nu). \tag{5.6}$$

Consequently, taking into account (5.4), we see that it is enough to obtain the estimate of a_λ instead of the estimate of Γ_λ.

Let L_1' be the finite set of all $\lambda\in L'$ such that $-\langle\lambda,\lambda\rangle\ge\gamma_1$. For each $\lambda\in L$, we define polynomials of p_λ by

$$\begin{aligned}&p_\lambda(\nu)=1 \text{ if } \lambda\notin L_1',\\&p_\lambda(\nu)=\prod_{\substack{1\le i\le s\\ \langle\lambda,\lambda\rangle+\gamma_i\le 0}}(2\langle\lambda,\nu\rangle-\langle\lambda,\lambda\rangle-\gamma_i)^{m_i} \text{ if } \lambda\in L_1',\end{aligned} \tag{5.7}$$

and set

$$d'(\lambda)=\sum_{\substack{1\le i\le s\\ \langle\lambda,\lambda\rangle+\gamma_i\le 0}}m_i. \tag{5.8}$$

We also put

$$\begin{aligned}P(\nu)&=\prod_{\lambda\in L_1'}p_\lambda(\nu),\quad d=\sum_{\lambda\in L_1'}d'(\lambda),\\P_\lambda(\nu)&=\prod_{\substack{\lambda'\in L_1'\\ \lambda'\ll\lambda}}p_{\lambda'}(\nu),\quad d(\lambda)=\sum_{\substack{\lambda'\in L_1'\\ \lambda'\ll\lambda}}d'(\lambda').\end{aligned} \tag{5.9}$$

Remark. We note that P is of finite degree and thus $d<\infty$.

We shall first show the following proposition.

Proposition 5.1. *There exist constants $D', d_1' > 0$ such that*

$$\|P_\lambda(\nu)a_\lambda(\nu)\| \leq D'(1+\|\nu\|+m(\lambda))^{2d}m(\lambda)^{d_1'}.$$

Proof. We differentiate $\Psi(\nu : h)$ by $\Delta(h)^{1/2} \circ \Omega_A(\omega) \circ \Delta(h)^{-1/2}$ and use Lemma 4.2. Then, comparing the coefficients of $h^{\nu-\lambda}$ in both side, we obtain the following recursive relation:

(5.10)

$$[2\langle\lambda,\nu\rangle - \langle\lambda,\lambda\rangle]a_\lambda(\nu) - \gamma(a_\lambda(\nu)) = \sum_{\alpha\in P_+}[\langle\tilde{\alpha},\tilde{\alpha}\rangle - 8F_\alpha]\sum_{j\geq 1} j a_{\lambda-2j\tilde{\alpha}}(\nu)$$
$$- \sum_{\substack{\alpha,\beta\in P_+\\ \alpha\neq\beta}}\langle\tilde{\alpha},\tilde{\beta}\rangle\sum_{j\geq 1,k\geq 0} a_{\lambda-2j\tilde{\alpha}-2k\tilde{\beta}}(\nu) + 8\sum_{\alpha\in P_+} G_\alpha \sum_{j\geq 1}(2j-1)a_{\lambda-(2j-1)\tilde{\alpha}}(\nu),$$

where $F_\alpha = \tau_1(Y_\alpha Y_{-\alpha}) + \tau_2(Y_\alpha Y_{-\alpha})$, $G_\alpha = \tau_1(Y_\alpha)\circ\tau_2(Y_{-\alpha})$. Since $\{\gamma_1, \cdots, \gamma_s\}$ are the set of distinct negative eigenvalues of γ, if we assume that all $a_{\lambda'}$, $(\lambda' \ll \lambda)$ are defined and regard (5.10) as the defining formula of a_λ, we see that all singularities of a_λ in $\mathcal{R}$ are concentrated into P_λ. We now put

$$Q_\lambda(\nu) = P_\lambda(\nu)(1+\|\nu\|+\|\lambda\|)^{-2d(\lambda)},$$
$$q_\lambda(\nu) = p_\lambda(\nu)(1+\|\nu\|+\|\lambda\|)^{-2d'(\lambda)},$$

and consider (5.10) multiplied by $Q_\lambda(\nu)$ instead of (5.10) itself:

(5.11)

$$[2\langle\lambda,\nu\rangle - \langle\lambda,\lambda\rangle]Q_\lambda(\nu)a_\lambda(\nu) - \gamma(Q_\lambda(\nu)a_\lambda(\nu))$$
$$= \sum_{\alpha\in P_+}[\langle\tilde{\alpha},\tilde{\alpha}\rangle - 8F_\alpha]q_\lambda(\nu)\sum_{j\geq 1} j Q^1_{\lambda,j}(\nu)Q_{\lambda-2j\tilde{\alpha}}(\nu)a_{\lambda-2j\tilde{\alpha}}(\nu)$$
$$- \sum_{\substack{\alpha,\beta\in P_+\\ \alpha\neq\beta}}\langle\tilde{\alpha},\tilde{\beta}\rangle q_\lambda(\nu)\sum_{j\geq 1,k\geq 0} Q_{\lambda,j,k}(\nu)Q_{\lambda-2j\tilde{\alpha}-2k\tilde{\beta}}(\nu)a_{\lambda-2j\tilde{\alpha}-2k\tilde{\beta}}(\nu)$$
$$+ 8\sum_{\alpha\in P_+} G_\alpha q_\lambda(\nu)\sum_{j\geq 1}(2j-1)Q^2_{\lambda,j}(\nu)Q_{\lambda-(2j-1)\tilde{\alpha}}(\nu)a_{\lambda-(2j-1)\tilde{\alpha}}(\nu).$$

Here $Q^1_{\lambda,j}$, $Q_{\lambda,j,k}$ and $Q^2_{\lambda,j}$ are determined by

$$Q_\lambda(\nu)q_\lambda(\nu)^{-1} = Q_{\lambda,j,k}(\nu)Q_{\lambda-2j\tilde{\alpha}-2k\tilde{\beta}}(\nu)$$
$$= Q^2_{\lambda,j}(\nu)Q_{\lambda-(2j-1)\tilde{\alpha}}(\nu) = Q^1_{\lambda,j}(\nu)Q_{\lambda-2j\tilde{\alpha}}(\nu).$$

From the above definition, it is clear that there exists a constant $C_1 > 0$ such that

(5.12) $$|Q^1_{\lambda,j}(\nu)| < C_1,\ |Q_{\lambda,j,k}(\nu)| < C_1,\ |Q^2_{\lambda,j}(\nu)| < C_1,$$

for all $\lambda \in L'$ and $\nu \in \mathfrak{a}_{\mathbf{C}}^*$ and j, k. We define $b_\lambda(\nu)$ by $b_0(\nu) = 1$ and $b_\lambda(\nu) = Q_\lambda(\nu)a_\lambda(\nu)$. For simplicity, we also put $\gamma(\lambda : \nu) = (2\langle\lambda, \nu\rangle - \langle\lambda, \lambda\rangle)I - \gamma$, where I denotes the identity operator of $\mathrm{Hom}_{\mathbf{C}}(V_M, V_M)$. Then (5.11) is written as follows:

(5.13)
$$\begin{aligned}\gamma(\lambda : \nu)b_\lambda(\nu) = &\sum_{\alpha\in P_+}[\langle\tilde{\alpha},\tilde{\alpha}\rangle - 8F_\alpha]q_\lambda(\nu)\sum_{j\geq 1} jQ^1_{\lambda,j}(\nu)b_{\lambda-2j\tilde{\alpha}}(\nu)\\ &-\sum_{\substack{\alpha,\beta\in P_+\\ \alpha\neq\beta}}\langle\tilde{\alpha},\tilde{\beta}\rangle q_\lambda(\nu)\sum_{j\geq 1,k\geq 0} Q_{\lambda,j,k}(\nu)b_{\lambda-2j\tilde{\alpha}-2k\tilde{\beta}}(\nu)\\ &+8\sum_{\alpha\in P_+} G_\alpha q_\lambda(\nu)\sum_{j\geq 1}(2j-1)Q^2_{\lambda,j}(\nu)b_{\lambda-(2j-1)\tilde{\alpha}}(\nu).\end{aligned}$$

Fix an orthonormal basis $\mathcal{B} = \{\phi_1, \cdots, \phi_n\}$ of $\mathrm{Hom}_{\mathbf{C}}(V_M, V_M)$ relative to the Hilbert–Schmidt norm $\|\cdot\|_2$. Let $A_{\gamma(\lambda:\nu)}$ be the matrix of the endomorphism $\gamma(\lambda : \nu)$ relative to $\mathcal{B}$. Since γ is self-adjoint, there exists a unitary matrix B such that

$$BA_{\gamma(\lambda:\nu)}B^{-1} = \mathrm{diag}(a_1, \cdots, a_1, \cdots, a_t, \cdots, a_t),$$

where $a_i = 2\langle\lambda, \nu\rangle - \langle\lambda, \lambda\rangle - \gamma_i$. We then have

$$A^{-1}_{\gamma(\lambda:\nu)} = B^{-1}\,\mathrm{diag}(a_1^{-1}, \cdots, a_1^{-1}, \cdots, a_t^{-1}, \cdots, a_t^{-1})B.$$

Combining this with the fact $\|B\|_2 = \sqrt{n}$, we obtain

$$\begin{aligned}\|p_\lambda(\nu)A^{-1}_{\gamma(\lambda:\nu)}\|_2^2 \leq n\Bigg\{&|p_\lambda(\nu)|^2\sum_{\substack{1\leq i\leq t\\ \|\lambda\|^2+\gamma_i>0}} m_i|a_i|^{-2}\\ &+\sum_{\substack{1\leq i\leq s\\ \|\lambda\|^2+\gamma_i\leq 0}}\left(\prod_{j=1,j\neq i}^{t}|a_j|^{2m_j}\right)m_i|a_i|^{2(m_i-1)}\Bigg\}.\end{aligned}$$

Since we can choose constants $C_2 > 0$ and $C_3 > 0$ so that

$$\|\lambda\|m(\lambda)^{-1} < C_2,\ |p_\lambda(\nu)|^2 < C_3(1 + \|\nu\| + \|\lambda\|)^{4d'(\lambda)},$$

we can find a constant $C_4 > 0$ such that

$$\|p_\lambda(\nu)A^{-1}_{\gamma(\lambda:\nu)}\|_2 < C_4(1 + \|\nu\| + \|\lambda\|)^{2d'(\lambda)}m(\lambda)^{-2}.$$

Hence we have

(5.14)
$$\|q_\lambda(\nu)A^{-1}_{\gamma(\lambda:\nu)}\|_2 < C_5 m(\lambda)^{-2}.$$

Putting $C_6 = C_1C_5 \max\{\|\langle\tilde{\alpha},\tilde{\alpha}\rangle - 8F_\alpha\|, 8\|G_\alpha\|, |\langle\tilde{\alpha},\tilde{\beta}\rangle| : \alpha,\beta \in P_+\}$ and combining (5.14) with (5.13), we obtain the following estimate for b_λ:

$$\begin{aligned}\|b_\lambda(\nu)\| &\le C_6 m(\lambda)^{-2}\left\{\sum_{\alpha\in P_+}\sum_{j\ge 1} 2j\|b_{\lambda-2j\tilde{\alpha}}(\nu)\| + \sum_{\alpha\in P_+}\sum_{j\ge 1}(2j-1)\|b_{\lambda-(2j-1)\tilde{\alpha}}(\nu)\|\right.\\ &\qquad\left. + \sum_{\substack{\alpha,\beta\in P_+\\ \alpha\neq\beta}}\sum_{j\ge 1,k\ge 0}\|b_{\lambda-2j\tilde{\alpha}-2k\tilde{\beta}}(\nu)\|\right\}\\ &= C_6 m(\lambda)^{-2}\left\{\sum_{\alpha\in P_+}\sum_{j\ge 1} j\|b_{\lambda-j\tilde{\alpha}}(\nu)\| + \sum_{\substack{\alpha,\beta\in P_+\\ \alpha\neq\beta}}\sum_{j\ge 1,k\ge 0}\|b_{\lambda-2j\tilde{\alpha}-2k\tilde{\beta}}(\nu)\|\right\}\\ &= m(\lambda)^{-2}\sum_{r=1}^{m(\lambda)-1}(S_1(r)+S_2(r)),\end{aligned}$$

where

$$S_1(r) = C_6\sum_{\alpha\in P_+}\sum_{\substack{m(\lambda-j\tilde{\alpha})=r\\ j\ge 1}} j\|b_{\lambda-j\tilde{\alpha}}(\nu)\|,$$

$$S_2(r) = C_6\sum_{\substack{\alpha,\beta\in P_+\\ \alpha\neq\beta}}\sum_{\substack{m(\lambda-2j\tilde{\alpha}-2j\tilde{\beta})=r\\ j\ge 1,k\ge 0}} j\|b_{\lambda-2j\tilde{\alpha}-2j\tilde{\beta}}(\nu)\|.$$

Put now

$$H_0(\nu) = 1, \quad H_r(\nu) = \sup_{\substack{\mu\in L'\\ m(\mu)=r}}\|b_\mu(\nu)\|.$$

By an argument similar to that as in [11], we see that there exists a constant $C_7 > 0$ such that $S_1(r)$ and $S_2(r)$ are bounded by $C_7H_r(\nu)m(\lambda)$ and thus we can take a constant $C_8 > 0$ so that

$$\|b_\lambda(\nu)\| \le C_8\left\{\sum_{r=1}^{m(\lambda)-1} H_r(\nu)\right\} m(\lambda)^{-1}.$$

Moreover, if we define a series $\{D_r\}$ $(r \in \mathbf{Z}_{\ge 0})$ by

$$D_0 = 1,\ D_r = \frac{1}{r}C_8\sum_{s=0}^{r-1} D_s,$$

then it is easy (cf. [11]) to see that $H_n(\nu) \le D_n$ and that there exists a constant $C_9 > 0$ such that $D_n \le C_9 n^{C_8-1}$ for all $n \in \mathbf{Z}_{>0}$. This shows that

$$\|b_\lambda(\nu)\| \le C_9 m(\lambda)^{C_8}.$$

Because $d(\lambda) \leq d$ for all $\lambda \in L'$, we see from this that we can choose constants $D, d_1 > 0$ so that

$$\|P_\lambda(\nu)a_\lambda(\nu)\| \leq D(1 + \|\nu\| + m(\lambda))^{2d} m(\lambda)^{d_1}.$$

This is the desired estimate for $P_\lambda a_\lambda$. □

By using Proposition 5.1, we immediately obtain the following theorem.

Theorem 5.2. *There exist absolute constants $D, d_1 > 0$ such that*

$$\|P_\lambda(\nu)\Gamma_\lambda(\nu - \rho)\| \leq D(1 + \|\nu\| + m(\lambda))^{2d} m(\lambda)^{d_1}, \ (\nu \in \mathcal{R})$$

for all $\lambda \in L$.

6. Connection with C-function and Intertwining operator

We will first summarize some known results on the relationship between the standard intertwining operator and the Harish-Chandra C-function. The results below are due to Knapp–Stein [17] and Wallach [29].

In the remainder of this paper, we assume that G is of real rank one and has trivial split component. We indicate by α the unique simple restricted root and by w the unique nontrivial element in $W(\mathfrak{a})$. Then $\Sigma^+ = \{\alpha, 2\alpha\}$. In [17], Knapp and Stein constructed the integral expression of the intertwining operator between the principal series representations, which is called the standard intertwining operator. Let $\sigma \in \hat{M}$ and $\nu \in \mathfrak{a}_{\mathbf{c}}^*$ be such that $\mathrm{Re}\langle \nu, \alpha \rangle > 0$. Then the standard intertwining operator is defined as follows:

$$(A(w,\sigma,\nu)\varphi)(k) = \int_{\bar{N}} e^{-(\nu+\rho)(H(\bar{n}))} \varphi(kw\kappa(\bar{n}))d\bar{n}, \ (\varphi \in C_\sigma^\infty(K)). \tag{6.1}$$

Then they proved that for $\varphi \in C_\sigma^\infty(K)$, $A(w,\sigma,\nu)\varphi$, as a function of ν, can be extended to a meromorphic function on $\mathfrak{a}_{\mathbf{c}}^*$. For $\varphi \in C_\sigma^\infty(K)$, it is satisfied that $A(w,\sigma,\nu)\varphi \in C_{w\sigma}^\infty(K)$ and

$$A(w,\sigma,\nu)\pi_{\sigma,\nu}(g)\varphi(k) = \pi_{w\sigma,w\nu}(g)A(w,\sigma,\nu)\varphi(k). \tag{6.2}$$

Let $\tau \in \hat{K}(\sigma)$. Then for $T \otimes v \in \mathrm{Hom}_M(V_\tau, H_\sigma) \otimes V_\tau$, it follows from Wallach (cf. [27, p. 270]) that

$$(A(w,\sigma,\nu)f_{T\otimes v})(k) = T(C_\tau(\nu)\tau(w)^{-1}\tau(k)^{-1}v). \tag{6.3}$$

Here $C_\tau(\nu) = C_\tau(1:\nu)$ is the Harish-Chandra C-function appeared in (3.8). Substituting (6.1) into (6.3), we obtain the following integral expression of the Harish-Chandra C-function:

$$C_\tau(\nu) = \int_{\bar{N}} e^{-(\nu+\rho)(H(\bar{n}))} \tau(\kappa(\bar{n}))^{-1} d\bar{n}. \tag{6.4}$$

Let $(R(w)\varphi)(k) = \varphi(kw)$ for $\varphi \in C_\sigma^\infty(K)$. Define the linear mapping

$$R_\tau(w)\colon \mathrm{Hom}_M(V_\tau, H_\sigma) \otimes V_\tau \to \mathrm{Hom}_M(V_\tau, H_{w\sigma}) \otimes V_\tau \tag{6.5}$$

by $R_\tau(w)(T\otimes v) = T\tau(w)^{-1}\otimes v$. Then it is clear that

$$(R(w)f_{T\otimes v})(k) = f_{R_\tau(w)(T\otimes v)}(k). \tag{6.6}$$

Looking upon $C_\tau(\nu)$ as a linear mapping of $\mathrm{Hom}_M(V_\tau, H_\sigma)$, we write $C_\tau(\sigma:\nu)$ for the determinant of the linear mapping $C_\tau(\nu)$. We call $C_\tau(\sigma:\nu)$ the Harish-Chandra C-function associated with τ and σ. Define the linear mapping

$$T(w,\sigma,\nu)\colon \mathrm{Hom}_M(V_\tau, H_\sigma)\otimes V_\tau \to \mathrm{Hom}_M(V_\tau, H_{w\sigma})\otimes V_\tau \tag{6.7}$$

by $T(w,\sigma,\nu)(T\otimes v) = TC_\tau(\nu)\tau(w)^{-1}\otimes v$. We write $\det T(w,\sigma,\nu)$ for the determinant of $T(w,\sigma,\nu)$ with respect to the bases $\{T_i : 1\le i\le d\}$ of $\mathrm{Hom}_M(V_\tau, H_\sigma)$ and $\{T_i\tau(w)^{-1} : 1\le i\le d\}$ of $\mathrm{Hom}_M(V_\tau, H_{w\sigma})$. Then it follows from (6.3) that

$$\det T(w,\sigma,\nu) = C_\tau(\sigma:\nu)^{\dim V_\tau}. \tag{6.8}$$

Our main concern in this paper is the case that $\dim \mathrm{Hom}_M(V_\tau, H_\sigma) = 1$. It is known that if $G = Spin(n,1)$ or $G = SU(n,1)$ then this assumption holds for all $\tau\in\hat{K}$ and $\sigma\in\hat{M}(\tau)$. Under this assumption, because $TC_\tau(\nu) = C_\tau(\sigma:\nu)T$, we have the following.

Proposition 6.1. *Retain the above notation and assumption. We have*

$$(A(w,\sigma,\nu)f_{T\otimes v})(k) = C_\tau(\sigma:\nu)f_{R_\tau(w)(T\otimes v)}(k).$$

Remark. The function $\det C_\tau(\nu)$ was first introduced by Cohn [4]. Later, Vogan and Wallach [28] studied the function $C_\tau(\sigma:\nu)$ for reductive Lie groups with arbitrary rank. In their paper, they showed that $C_\tau(\sigma:\nu)$, as a function of ν, has a meromorphic extension on $\mathfrak{a}_{\mathbf{c}}^*$ and it can be written as quotients of products of classical Γ functions.

7. Infinitesimal operator of the Principal series

In this section, we shall introduce the formula of the infinitesimal operator of the principal series representation that was shown by Thieleker [23,24]. We shall reform Thieleker's formula for our convenience so that we can get the recursion formula of the Harish-Chandra C-function.

We retain the notation in §6. Let $H\in\mathfrak{a}$ be such that $\alpha(H) = 1$. For $j = 1,2$, we set $P_+^j = \{\lambda\in P_+ : \lambda(H) = j\}$. Fix an orthonormal basis $\{U_j : 1\le j\le m\}$, $(m = \dim\mathfrak{m})$ of $\mathfrak{m}$. For $j = 1,2$, we set $\omega_{j\alpha} = -\sum_{\lambda\in P_+^j} 2Y_\lambda^2/\|X_\lambda\|^2$ and $\omega_{\mathfrak{k}} = -\sum_{i=1}^m U_i^2 - \sum_{j=1}^2 \omega_{j\alpha}$. For $\varphi\in\mathcal{H}^{\sigma,\nu}$, we define the function φ_ν on G by $\varphi_\nu(g) = e^{-(\nu+\rho)(H(g))}\varphi(\kappa(g))$. We set $\phi_Z(k) = \langle \mathrm{Ad}(k)^{-1}Z, H\rangle/\langle H,H\rangle$. We shall first show the following lemma.

Lemma 7.1 (cf. [23, Lemma 1]). *Let $Z\in\mathfrak{p}_{\mathbf{c}}$ and $\varphi\in C_\sigma^\infty(K)$. Then we have*

$$\begin{aligned}(\pi_{\sigma,\nu}(Z)\varphi)(k) = {}& \frac{\langle\nu,\alpha\rangle}{\langle\alpha,\alpha\rangle}(\phi_Z\varphi)(k) + \frac{1}{2\langle\alpha,\alpha\rangle}\big[(\phi_Z\varphi)(k;\omega_{\mathfrak{k}}) - \phi_Z(k)\varphi(k;\omega_{\mathfrak{k}})\big]\\ & - \frac{1}{4\langle\alpha,\alpha\rangle}\big[(\phi_Z\varphi)(k;\omega_{2\alpha}) - \phi_Z(k)\varphi(k;\omega_{2\alpha})\big].\end{aligned}$$

Proof. We first note that

$$\text{(7.1)} \qquad \mathrm{Ad}(k)^{-1}Z = \frac{\langle \mathrm{Ad}(k)^{-1}Z, H\rangle H}{\langle H, H\rangle} + \sum_{j=1}^{2}\sum_{\lambda\in P_+^j} \frac{\langle \mathrm{Ad}(k)^{-1}Z, Z_\lambda\rangle Z_\lambda}{\langle Z_\lambda, Z_\lambda\rangle}.$$

It follows from the definition of $\varphi_\nu(g)$ that

$$\text{(7.2)} \qquad \varphi_\nu(k;H) = -(\nu+\rho)(H)\varphi_\nu(k) \text{ for } H\in\mathfrak{a},\ k\in K,$$

$$\text{(7.3)} \qquad \varphi_\nu(k;X) = 0 \text{ for } X\in\mathfrak{n},\ k\in K.$$

Noting $Z_\lambda = -Y_\lambda + X_\lambda,\ (\lambda\in P_+)$, we have

$$\text{(7.4)} \qquad \varphi_\nu(k;Z_\lambda) = -\varphi_\nu(k;Y_\lambda) = -\varphi(k;Y_\lambda).$$

Taking into account (7.1) and (7.4), we obtain

(7.5)

$$\begin{aligned}(\pi_{\sigma,\nu}(Z)\varphi)(k) &= \varphi_\nu(-Z;k) = -\varphi_\nu(k;\mathrm{Ad}(k)^{-1}Z)\\ &= \langle\nu+\rho,\alpha\rangle\langle \mathrm{Ad}(k)^{-1}Z, H\rangle\varphi(k) + \sum_{j=1}^{2}\sum_{\lambda\in P_+^j}\frac{\langle \mathrm{Ad}(k)^{-1}Z, Z_\lambda\rangle\varphi(k;Y_\lambda)}{\langle Z_\lambda, Z_\lambda\rangle}.\end{aligned}$$

A simple calculation yields that for $\lambda\in P_+^j$,

$$\text{(7.6)} \qquad [H,Y_\lambda] = jZ_\lambda, \qquad [Y_\lambda, Z_\lambda] = j\langle\alpha,\alpha\rangle\langle Z_\lambda, Z_\lambda\rangle H.$$

From (7.6), we have for $\lambda\in P_+^j$ that

$$\text{(7.7)} \qquad \begin{aligned}\phi_Z(k;Y_\lambda) &= \frac{\langle \mathrm{ad}(-Y_\lambda)\,\mathrm{Ad}(k)^{-1}Z, H\rangle}{\langle H,H\rangle}\\ &= \frac{-j\langle \mathrm{Ad}(k)^{-1}Z, Z_\lambda\rangle}{\langle H,H\rangle}.\end{aligned}$$

Therefore, substituting (7.7) into (7.5), we obtain

(7.8)

$$\begin{aligned}(\pi_{\sigma,\nu}(Z)\varphi)(k) = \frac{\langle\nu+\rho,\alpha\rangle}{\langle\alpha,\alpha\rangle}(\phi_Z\varphi)(k) &- \frac{1}{\langle\alpha,\alpha\rangle}\sum_{\lambda\in P_+^1}\frac{\phi_Z(k;Y_\lambda)\varphi(k;Y_\lambda)}{\langle Z_\lambda,Z_\lambda\rangle}\\ &- \frac{1}{2\langle\alpha,\alpha\rangle}\sum_{\lambda\in P_+^2}\frac{\phi_Z(k;Y_\lambda)\varphi(k;Y_\lambda)}{\langle Z_\lambda,Z_\lambda\rangle}.\end{aligned}$$

A simple calculation using (7.6) gives that

$$\text{(7.9)} \qquad \phi_Z(k;U_i) = 0, \qquad \phi_Z(k;Y_\lambda^2) = -j^2\langle\alpha,\alpha\rangle\langle Z_\lambda,Z_\lambda\rangle\phi_Z(k),\ (\lambda\in P_+^j).$$

Noting $\langle Z_\lambda, Z_\lambda \rangle = \|X_\lambda\|^2/2$, we have

$$\begin{aligned} \phi_Z(k;\omega_{j\alpha}) &= j^2 m_{j\alpha}\langle\alpha,\alpha\rangle\phi_Z(k), \\ \phi_Z(k;\omega_{\mathfrak{k}}) &= (m_\alpha + 4m_{2\alpha})\langle\alpha,\alpha\rangle\phi_Z(k). \end{aligned} \tag{7.10}$$

By using Leibniz's formula, we have for $j = 1, 2$ that

(7.11)

$$\begin{aligned} (\phi_Z\varphi)(k;\omega_{j\alpha}) &= \phi_Z(k)\varphi(k;\omega_{j\alpha}) + \phi_Z(k;\omega_{j\alpha})\varphi(k) - 2\sum_{\lambda\in P_+^j}\phi_Z(k;Y_\lambda)\varphi(k;Y_\lambda) \\ &= \phi_Z(k)\varphi(k;\omega_{j\alpha}) + j^2 m_{j\alpha}\langle\alpha,\alpha\rangle\phi_Z(k)\varphi(k) - 2\sum_{\lambda\in P_+^j}\phi_Z(k;Y_\lambda)\varphi(k;Y_\lambda). \end{aligned}$$

Therefore

(7.12)

$$\begin{aligned} &-\sum_{\lambda\in P_+^j}\phi_Z(k;Y_\lambda)\varphi(k;Y_\lambda) \\ &\qquad = \frac{1}{2}\big[(\phi_Z\varphi)(k;\omega_{j\alpha}) - \phi_Z(k)\varphi(k;\omega_{j\alpha}) - j^2 m_{j\alpha}\langle\alpha,\alpha\rangle(\phi_Z\varphi)(k)\big]. \end{aligned}$$

Substituting (7.12) into (7.8), we obtain

(7.13)

$$\begin{aligned} (\pi_{\sigma,\nu}(Z)\phi)(k) &= \frac{\langle\nu+\rho,\alpha\rangle}{\langle\alpha,\alpha\rangle}(\phi_Z\varphi)(k) \\ &\quad + \frac{1}{2\langle\alpha,\alpha\rangle}\big[(\phi_Z\varphi)(k;\omega_\alpha) - \phi_Z(k)\varphi(k;\omega_\alpha) - m_\alpha\langle\alpha,\alpha\rangle(\phi_Z\varphi)(k)\big] \\ &\quad + \frac{1}{4\langle\alpha,\alpha\rangle}\big[(\phi_Z\varphi)(k;\omega_{2\alpha}) - \phi_Z(k)\varphi(k;\omega_{2\alpha}) - 4m_{2\alpha}\langle\alpha,\alpha\rangle(\phi_Z\varphi)(k)\big] \\ &= \frac{\langle\nu,\alpha\rangle}{\langle\alpha,\alpha\rangle}(\phi_Z\varphi)(k) + \frac{1}{2\langle\alpha,\alpha\rangle}\big[(\phi_Z\varphi)(k;\omega_\alpha+\omega_{2\alpha}) - \phi_Z(k)\varphi(k;\omega_\alpha+\omega_{2\alpha})\big] \\ &\qquad - \frac{1}{4\langle\alpha,\alpha\rangle}\big[(\phi_Z\varphi)(k;\omega_{2\alpha}) - \phi_Z(k)\varphi(k;\omega_{2\alpha})\big]. \end{aligned}$$

Noting (7.9), and using Leibniz's formula, we immediately obtain

$$(\phi_Z\varphi)(k;\omega_\alpha+\omega_{2\alpha}) - \phi_Z(k)\varphi(k;\omega_\alpha+\omega_{2\alpha}) = (\phi_Z\varphi)(k;\omega_{\mathfrak{k}}) - \phi_Z(k)\varphi(k;\omega_{\mathfrak{k}}). \tag{7.14}$$

Substituting (7.14) into the last expression in (7.13), we get the assertion. □

In the remainder of this section, we assume that the unitary representation $(\mathrm{Ad}, \mathfrak{p}_{\mathbf{c}})$ of K has no multiple weights and $\dim \mathrm{Hom}_M(V_\lambda, H_\mu) \leq 1$ for all $\lambda \in D_K$ and $\mu \in D_M$. Under these assumptions, we shall precisely write the formula in Lemma 7.1. Let $\Delta_{\mathfrak{p}}$ be the set of all weights of $(\mathrm{Ad}, \mathfrak{p}_{\mathbf{c}})$ with respect to $\mathfrak{t}_{\mathbf{c}}$. Then the following lemma is valid.

Lemma 7.2 (cf. [16, p. 111]). *Let $\lambda \in D_K$. Then*

$$\mathrm{Ad} \otimes \tau_\lambda = \sum_{\beta \in \Delta_\mathfrak{p}} \mathrm{sgn}(\lambda + \beta - \delta_K) \tau_{(\lambda+\beta-\delta_K)^\vee + \delta_K}.$$

Here for any integral form λ' on $\sqrt{-1}\mathfrak{t}$, we denote by $\mathrm{sgn}\,\lambda'$ the sign of $w \in W_K$ such that $w\lambda'$ is dominant and put $(\lambda')^\vee = w\lambda'$. For simplicity we write $\lambda(\beta)$ for $(\lambda+\beta-\delta_K)^\vee + \delta_K$. Let $E_{\lambda+\beta}$ denote the canonical projection of $\mathfrak{p}_\mathbf{c} \otimes V_\lambda$ into $V_{\lambda(\beta)}$ given by the decomposition in Lemma 7.2 satisfying $E_{\lambda+\beta}E_{\lambda+\beta}^* = I_{\lambda+\beta}$, where $E_{\lambda+\beta}^*$ and $I_{\lambda+\beta}$ denote the adjoint operator of $E_{\lambda+\beta}$ and the identity operator on $V_{\lambda(\beta)}$, respectively.

Let $\lambda \in D_K$ and $\mu \in D_M(\lambda)$. For $T \in \mathrm{Hom}_M(V_\lambda, H_\mu)$, define $\tilde{T} \in \mathrm{Hom}_M(\mathfrak{p}_\mathbf{c} \otimes V_\lambda, H_\mu)$ by

$$\tilde{T}(Z \otimes v) = \frac{\langle Z, H \rangle}{\langle H, H \rangle} T(v). \tag{7.15}$$

Define the linear mapping

$$\mathcal{M}_\mu(Z; \lambda+\beta, \lambda) \colon \mathrm{Hom}_M(V_\lambda, H_\mu) \otimes V_\lambda \to \mathrm{Hom}_M(V_{\lambda(\beta)}, H_\mu) \otimes V_{\lambda(\beta)} \tag{7.16}$$

by

$$\mathcal{M}_\mu(Z; \lambda+\beta, \lambda)(T \otimes v) = \tilde{T}E_{\lambda+\beta}^* \otimes E_{\lambda+\beta}(Z \otimes v). \tag{7.17}$$

Lemma 7.3. *Retain the above notation and assumption. We have*

$$(\phi_Z f_{T\otimes v})(k) = \sum_{\beta \in \Delta_\mathfrak{p}} \mathrm{sgn}(\lambda + \beta - \delta_K) f_{\mathcal{M}_\mu(Z;\lambda+\beta,\lambda)(T\otimes v)}(k).$$

Proof. We compute

$$\begin{aligned}
(\phi_Z f_{T\otimes v})(k) &= \frac{\langle \mathrm{Ad}(k)^{-1}Z, H \rangle}{\langle H, H \rangle} T(\tau_\lambda(k)^{-1}v) \\
&= \tilde{T}((\mathrm{Ad} \otimes \tau_\lambda)(k)^{-1}(Z \otimes v)) \\
&= \tilde{T}\left((\mathrm{Ad} \otimes \tau_\lambda)(k)^{-1} \sum_{\beta \in \Delta_\mathfrak{p}} E_{\lambda+\beta}^* E_{\lambda+\beta}(Z \otimes v) \right) \\
&= \sum_{\beta \in \Delta_\mathfrak{p}} \mathrm{sgn}(\lambda + \beta - \delta_K) \tilde{T}(E_{\lambda+\beta}^* \tau_{\lambda(\beta)}(k)^{-1} E_{\lambda+\beta}(Z \otimes v)) \\
&= \sum_{\beta \in \Delta_\mathfrak{p}} \mathrm{sgn}(\lambda + \beta - \delta_K) f_{\tilde{T}E_{\lambda+\beta}^* \otimes E_{\lambda+\beta}(Z\otimes v)}(k).
\end{aligned}$$

Therefore the assertion holds. □

For $\mu \in D_M$ and $w \in W(\mathfrak{a})$, define $w\mu \in D_M$ by $w\sigma_\mu = \sigma_{w\mu}$. In the following discussion, R_λ is an abbreviation of R_{τ_λ} and when there is no possibility of confusion, we shall use similar abbreviations. The next lemma is immediately obtained.

Lemma 7.4. *If* $\mathrm{sgn}(\lambda+\beta-\delta_K)\neq 0$ *then it follows that*

$$R_{\lambda+\beta}(w)\mathcal{M}_\mu(Z;\lambda+\beta,\lambda)=-\mathcal{M}_{w\mu}(Z;\lambda+\beta,\lambda)R_\lambda(w).$$

Proof. We compute

(7.18)

$$\begin{aligned}(R(w)(\phi_Z f_{T\otimes v}))(k)&=\frac{\langle \mathrm{Ad}(kw)^{-1}Z,H\rangle}{\langle H,H\rangle}T(\tau_\lambda(kw)^{-1}v)\\&=\frac{\langle \mathrm{Ad}(k)^{-1}Z,\mathrm{Ad}(w)H\rangle}{\langle H,H\rangle}T\tau_\lambda(w)^{-1}(\tau_\lambda(k)^{-1}v)\\&=-(\phi_Z f_{R_\lambda(w)(T\otimes v)})(k).\end{aligned}$$

Noting

$$f_{\mathcal{M}_\mu(Z;\lambda+\beta,\lambda)(T\otimes v)}\in\mathcal{H}^{\sigma_\mu,\nu}(\tau_{\lambda(\beta)}),\ f_{R_\lambda(w)(T\otimes v)}\in\mathcal{H}^{w\sigma_\mu,w\nu}(\tau_\lambda),$$

we see that

(7.19)

$$\begin{aligned}(R(w)(\phi_Z f_{T\otimes v}))(k)&=\sum_{\beta\in\Delta_\mathfrak{p}}\mathrm{sgn}(\lambda+\beta-\delta_K)f_{R_{\lambda+\beta}(w)\mathcal{M}_\mu(Z;\lambda+\beta,\lambda)(T\otimes v)}(k),\\(\phi_Z f_{R_\lambda(w)(T\otimes v)})(k)&=\sum_{\beta\in\Delta_\mathfrak{p}}\mathrm{sgn}(\lambda+\beta-\delta_K)f_{\mathcal{M}_{w\mu}(Z;\lambda+\beta,\lambda)R_\lambda(w)(T\otimes v)}(k).\end{aligned}$$

Substituting (7.19) into (7.18) and comparing side by side, we obtain the assertion. □

Combining Lemma 7.1 and Lemma 7.3, we have the following theorem.

Proposition 7.5. *Let* $\mu\in D_M$ *and* $\lambda\in D_K(\mu)$. *Then there exists* $\eta^\mu_\lambda(\omega_{2\alpha})\in\mathbf{C}$ *such that*

$$\begin{aligned}&(\pi_{\sigma_\mu,\nu}(Z)f_{T\otimes v})(k)\\&\quad=\sum_{\beta\in\Delta_\mathfrak{p}}\left\{\frac{\langle\nu,\alpha\rangle}{\langle\alpha,\alpha\rangle}+\frac{\langle 2\lambda+2\delta_K+\beta,\beta\rangle}{2\langle\alpha,\alpha\rangle}-\frac{\eta^\mu_{\lambda+\beta}(\omega_{2\alpha})-\eta^\mu_\lambda(\omega_{2\alpha})}{4\langle\alpha,\alpha\rangle}\right\}\\&\qquad\times\mathrm{sgn}(\lambda+\beta-\delta_K)f_{\mathcal{M}_\mu(Z;\lambda+\beta,\lambda)(T\otimes v)}(k),\end{aligned}$$

$$\begin{aligned}&(\pi_{w\sigma_\mu,w\nu}(Z)f_{R_\lambda(w)(T\otimes v)})(k)\\&\quad=\sum_{\beta\in\Delta_\mathfrak{p}}\left\{\frac{\langle\nu,\alpha\rangle}{\langle\alpha,\alpha\rangle}-\frac{\langle 2\lambda+2\delta_K+\beta,\beta\rangle}{2\langle\alpha,\alpha\rangle}+\frac{\eta^\mu_{\lambda+\beta}(\omega_{2\alpha})-\eta^\mu_\lambda(\omega_{2\alpha})}{4\langle\alpha,\alpha\rangle}\right\}\\&\qquad\times\mathrm{sgn}(\lambda+\beta-\delta_K)f_{R_{\lambda+\beta}(w)\mathcal{M}_\mu(Z;\lambda+\beta,\lambda)(T\otimes v)}(k).\end{aligned}$$

Proof. By using Lemma 7.3, we have

$$\begin{aligned}(\phi_Z f_{T\otimes v})(k;\omega_{\mathfrak{k}}) &= \sum_{\beta\in\Delta_{\mathfrak{p}}} \operatorname{sgn}(\lambda+\beta-\delta_K) f_{\mathcal{M}_\mu(Z;\lambda+\beta,\lambda)(T\otimes v)}(k;\omega_{\mathfrak{k}}) \\ &= \sum_{\beta\in\Delta_{\mathfrak{p}}} (\langle\lambda+\beta+\delta_K,\lambda+\beta+\delta_K\rangle - \langle\delta_K,\delta_K\rangle) \\ &\quad \times \operatorname{sgn}(\lambda+\beta-\delta_K) f_{\mathcal{M}_\mu(Z;\lambda+\beta,\lambda)(T\otimes v)}(k),\end{aligned}$$

$$\begin{aligned}&\phi_Z(k) f_{T\otimes v}(k;\omega_{\mathfrak{k}}) = (\langle\lambda+\delta_K,\lambda+\delta_K\rangle - \langle\delta_K,\delta_K\rangle)(\phi_Z f_{T\otimes v})(k) \\ &= \sum_{\beta\in\Delta_{\mathfrak{p}}} (\langle\lambda+\delta_K,\lambda+\delta_K\rangle - \langle\delta_K,\delta_K\rangle) \operatorname{sgn}(\lambda+\beta-\delta_K) f_{\mathcal{M}_\mu(Z;\lambda+\beta,\lambda)(T\otimes v)}(k).\end{aligned}$$

Hence

$$\begin{aligned}&(\phi_Z f_{T\otimes v})(k;\omega_{\mathfrak{k}}) - \phi_Z(k) f_{T\otimes v}(k;\omega_{\mathfrak{k}}) \\ &\qquad = \sum_{\beta\in\Delta_{\mathfrak{p}}} \langle 2\lambda+2\delta_K+\beta,\beta\rangle \operatorname{sgn}(\lambda+\beta-\delta_K) f_{\mathcal{M}_\mu(Z;\lambda+\beta,\lambda)(T\otimes v)}(k).\end{aligned}$$

On the other side, under the assumption that $\dim \operatorname{Hom}_M(V_\lambda, H_\mu) = 1$, there exists $\eta_\lambda^\mu(\omega_{2\alpha}) \in \mathbf{C}$ such that

$$T\tau_\lambda(\omega_{2\alpha}) = \eta_\lambda^\mu(\omega_{2\alpha}) T,$$

and hence

$$f_{T\otimes v}(k;\omega_{2\alpha}) = T\tau_\lambda(\omega_{2\alpha})(\tau_\lambda(k)^{-1}v) = \eta_\lambda^\mu(\omega_{2\alpha}) f_{T\otimes v}(k).$$

Likewise we have

$$f_{\mathcal{M}_\mu(Z;\lambda+\beta,\lambda)(T\otimes v)}(k;\omega_{2\alpha}) = \eta_{\lambda+\beta}^\mu(\omega_{2\alpha}) f_{\mathcal{M}_\mu(Z;\lambda+\beta,\lambda)(T\otimes v)}(k).$$

Consequently we have

$$\begin{aligned}&(\phi_Z f_{T\otimes v})(k;\omega_{2\alpha}) - \phi_Z(k) f_{T\otimes v}(k;\omega_{2\alpha}) \\ &\qquad = \sum_{\beta\in\Delta_{\mathfrak{p}}} (\eta_{\lambda+\beta}^\mu(\omega_{2\alpha}) - \eta_\lambda^\mu(\omega_{2\alpha})) \operatorname{sgn}(\lambda+\beta-\delta_K) f_{\mathcal{M}_\mu(Z;\lambda+\beta,\lambda)(T\otimes v)}(k).\end{aligned}$$

Noting

$$\begin{aligned}f_{T\tau_\lambda(w)^{-1}\otimes v}(k;\omega_{2\alpha}) &= T\tau_\lambda(w)^{-1}\tau_\lambda(\omega_{2\alpha})(\tau_\lambda(k)^{-1}v) \\ &= T\tau_\lambda(\omega_{2\alpha})\tau_\lambda(w)^{-1}(\tau_\lambda(k)^{-1}v) \\ &= \eta_\lambda^\mu(\omega_{2\alpha}) f_{T\tau_\lambda(w)^{-1}\otimes v}(k),\end{aligned}$$

and taking into account Lemma 7.4, we can get immediately the second equation in Proposition 7.5. □

8. Representations of K and M

In the remainder of this paper, we shall confine our attention to the case of $SU(n,1)$. In this case, because $K = U(n)$, any irreducible unitary representations of K and M can be constructed in terms of the Gel'fand–Tsetlin basis of $\mathfrak{u}(n)$. Later, this realizations are utilized for getting the matrix elements of the Harish-Chandra C-functions with respect to the highest weight vector. We will borrow the notation concerning the Gel'fand–Testlin basis from the Vilenkin–Klimyk's book [27, pp. 361–365].

Let $E_{p,q}$ be the matrix unit whose (k,l)-component is equal to $\delta_{pk}\delta_{ql}$. Put $H = E_{n,n+1} + E_{n+1,n}$ and $\mathfrak{a} = \mathbf{R}H$. Then we have

$$K = \left\{ \begin{pmatrix} X & & \\ & u & \\ & & \end{pmatrix} : X \in U(n), u \in U(1), u \det X = 1 \right\}, \tag{8.1}$$

$$A = \left\{ \begin{pmatrix} I_{n-1} & & \\ & \cosh t & \sinh t \\ & \sinh t & \cosh t \end{pmatrix} : t \in \mathbf{R} \right\}, \tag{8.2}$$

$$M = \left\{ \begin{pmatrix} X & & \\ & u & \\ & & u \end{pmatrix} : X \in U(n-1), u \in U(1), u^2 \det X = 1 \right\}, \tag{8.3}$$

$$N = \left\{ n(z,u) = \begin{pmatrix} I_{n-1} & z & -z \\ -z^* & 1-\omega/2 & \omega/2 \\ -z^* & -\omega/2 & 1+\omega/2 \end{pmatrix} : \begin{matrix} z \in \mathbf{C}^{n-1}, u \in \sqrt{-1}\mathbf{R} \\ \omega = |z|^2 - 2u \end{matrix} \right\}, \tag{8.4}$$

$$\bar{N} = \left\{ \bar{n}(z,u) = \begin{pmatrix} I_{n-1} & -z & -z \\ z^* & 1-\omega/2 & -\omega/2 \\ -z^* & \omega/2 & 1+\omega/2 \end{pmatrix} : \begin{matrix} z \in \mathbf{C}^{n-1}, u \in \sqrt{-1}\mathbf{R} \\ \omega = |z|^2 - 2u \end{matrix} \right\}. \tag{8.5}$$

The lemma below is easy to obtain and hence we omit the proof.

Lemma 8.1. *Let $\bar{n}(z,u)$ be as above. Then we have*

$$H(\bar{n}(z,u)) = \log|1+\omega| H,$$

$$\kappa(\bar{n}(z,u)) = \begin{pmatrix} I_{n-1} - \frac{2zz^*}{1+\omega} & \frac{-2z}{|1+\omega|} & 0 \\ \frac{2z^*}{1+\omega} & \frac{1-\omega}{|1+\omega|} & 0 \\ 0 & 0 & \frac{1+\omega}{|1+\omega|} \end{pmatrix}.$$

We will now compute the second term $\langle 2\lambda + 2\delta_K + \beta, \beta\rangle / 2\langle \alpha, \alpha\rangle$ appeared in Proposition 7.5. Put $\mathbf{R}^p_> = \{x \in \mathbf{R}^p : x_j - x_{j+1} \in \mathbf{Z}_{\geq 0},\ (1 \leq j \leq p-1)\}$, where x_i denotes the i-component of $x \in \mathbf{R}^p$. For $x \in \mathbf{R}^{p+1}_>$ and $y \in \mathbf{R}^p_>$, $x > y$ means $x_j - y_j \in \mathbf{Z}_{\geq 0}$ and $y_j - x_{j+1} \in \mathbf{Z}_{\geq 0}$. For $x \in \mathbf{R}^p_>$ and $1 \leq q \leq p$, we set $x(q) = (x_1, \cdots, x_q) \in \mathbf{R}^q_>$ and $x[q] = (x_q, \cdots, x_p) \in \mathbf{R}^{p-q+1}_>$. Referring to [19,20], we have the following.

Let $H_p = \sqrt{-1}E_{p,p}$ for $1 \leq p \leq n+1$. Then

$$\mathfrak{t} = \left\{ \sum_{p=1}^{n+1} h_p H_p : h_p \in \mathbf{R},\ \sum_{p=1}^{n+1} h_p = 0 \right\}, \tag{8.6}$$

$$\mathfrak{h}_{\mathfrak{k}} = \left\{ \sum_{p=1}^{n+1} h_p H_p : h_p \in \mathbf{R}, h_n = h_{n+1},\ \sum_{p=1}^{n+1} h_p = 0 \right\}.$$

Let $\{\varepsilon_j\}$ denote the dual basis of $\mathfrak{t}_{\mathbf{c}}^*$ with respect to $\{H_j\}$. Then

$$\Delta_K^+ = \{\varepsilon_i - \varepsilon_j,\ (1 \le i < j \le n)\}, \tag{8.7}$$

$$\Delta_{\mathfrak{p}} = \{\beta_j = \varepsilon_j - \varepsilon_{n+1}, -\beta_j,\ (1 \le j \le n)\}, \tag{8.8}$$

$$D_K = \left(\frac{1}{n+1}\mathbf{Z}\right)_>^n, \quad D_M = \left(\frac{1}{n+1}\mathbf{Z}\right)_>^{n-1}. \tag{8.9}$$

It follows from $\langle X, Y\rangle = 2(n+1)\operatorname{tr} XY$ that for $\lambda \in D_K$,

$$\frac{\langle 2\lambda + 2\delta_K + \beta_j, \beta_j\rangle}{2\langle\alpha,\alpha\rangle} = 2\lambda_j + 2|\tau_\lambda| + n - 2j + 3,\ (1 \le j \le n), \tag{8.10}$$

where $|\tau_\lambda| = \sum_{p=1}^n \lambda_p$.

We shall next compute $\eta_\lambda^\mu(\omega_{2\alpha})$. Let $\mathbf{M} = (m_n, \dots, m_1)$ be a sequence such that

$$m_p = (m_{1,p}, \dots, m_{p,p}) \in \left(\frac{1}{n+1}\mathbf{Z}\right)_>^p. \tag{8.11}$$

Then the preceding sequence $\mathbf{M}$ is called a Gel'fand–Tsetlin data if $m_{p+1} > m_p$ for all $1 \le p \le n$. For the Gel'fand–Tsetlin data $\mathbf{M}$, we write $v(\mathbf{M})$ for the corresponding Gel'fand–Tsetlin basis. For $\lambda \in D_K$, we denote by V_λ the Hilbert space generated by the orthonormal basis $v(\mathbf{M})$ with $m_n = \lambda$. We put $X_p = E_{p,p+1}$, $Y_p = E_{p+1,p}$, $H_p = \sqrt{-1}(E_{p,p} - E_{p+1,p+1})$ and $H_0 = \sqrt{-1}\operatorname{diag}(-1, \dots, -1, n)$. Then there exists an irreducible unitary representation $(\tau_\lambda, V_\lambda)$ of K satisfying the following condition:

$$\tau_\lambda(X_p)v(\mathbf{M}) = \sum_{j=1}^p A_p^j(\mathbf{M})v(\mathbf{M}_p^{+j}), \tag{8.12}$$

$$\tau_\lambda(Y_p)v(\mathbf{M}) = \sum_{j=1}^p B_p^j(\mathbf{M})v(\mathbf{M}_p^{-j}), \tag{8.13}$$

$$\tau_\lambda(H_p)v(\mathbf{M}) = \left\{2\sum_{j=1}^p m_{j,p} - \sum_{j=1}^{p-1} m_{j,p-1} - \sum_{j=1}^{p+1} m_{j,p+1}\right\}\sqrt{-1}v(\mathbf{M}), \tag{8.14}$$

$$\tau_\lambda(H_0)v(\mathbf{M}) = -(n+1)\sum_{j=1}^n m_{j,n}\sqrt{-1}v(\mathbf{M}), \tag{8.15}$$

where $\mathbf{M}_p^{\pm j}$ is the Gel'fand–Tsetlin data obtained by replacing $m_{j,p}$ with $m_{j,p} \pm 1$ in m_p of $\mathbf{M}$. For the explicit forms of $A_p^j(\mathbf{M})$ and $B_p^j(\mathbf{M})$, see [27, p. 363].

Let $\lambda \in D_K$ and $\mu \in D_M(\lambda)$. By $V_\lambda(\mu)$, we indicate the subspace of V_λ consisting of the Gel'fand–Testlin basis $v(\mathbf{M})$ satisfying $m_n = \lambda$ and $m_{n-1} = \mu$. We put $\mathbf{M}_{\lambda,\mu} = (\lambda, \mu, \mu(n-2), \cdots, \mu(1))$. Because

$$Y_{2\alpha} = \frac{\sqrt{-1}}{2\sqrt{n+1}} \operatorname{diag}(\overbrace{0, \ldots, 0}^{n-1}, 1, -1) \tag{8.16}$$
$$= \frac{-1}{2n\sqrt{n+1}}\{H_0 + H_1 + 2H_2 + \cdots + (n-1)H_{n-1}\},$$

we have from (8.14) and (8.15) that

$$\tau_\lambda(Y_{2\alpha})v(\mathbf{M}_{\lambda,\mu}) = \frac{1}{2\sqrt{n+1}}(2|\tau_\lambda| - |\sigma_\mu|)\sqrt{-1}v(\mathbf{M}_{\lambda,\mu}). \tag{8.17}$$

Here we write $|\tau_\lambda| = \sum_{p=1}^{n} \lambda_p$ and $|\sigma_\mu| = \sum_{p=1}^{n-1} \mu_p$. Since $\omega_{2\alpha} = -Y_{2\alpha}^2$, it follows

$$\tau_\lambda(\omega_{2\alpha})v(\mathbf{M}_{\lambda,\mu}) = \frac{1}{4(n+1)}(2|\tau_\lambda| - |\sigma_\mu|)^2 v(\mathbf{M}_{\lambda,\mu}). \tag{8.18}$$

Taking into account $T\tau_\lambda(\omega_{2\alpha}) = \eta_\lambda^\mu(\omega_{2\alpha})T$ with $T \in \operatorname{Hom}_M(V_\lambda, H_\mu)$, we have

$$T\tau_\lambda(\omega_{2\alpha})v(\mathbf{M}_{\lambda,\mu}) = \eta_\lambda^\mu(\omega_{2\alpha})Tv(\mathbf{M}_{\lambda,\mu}). \tag{8.19}$$

Therefore, it follows from $Tv(\lambda, \mu) \neq 0$ that

$$\eta_\lambda^\mu(\omega_{2\alpha}) = \frac{1}{4(n+1)}(2|\tau_\lambda| - |\sigma_\mu|)^2. \tag{8.20}$$

Noting $|\tau_{\lambda+\beta_j}| = |\tau_\lambda| + 1$, we get the following.

$$\frac{\eta_{\lambda+\beta_j}^\mu(\omega_{2\alpha}) - \eta_\lambda^\mu(\omega_{2\alpha})}{4\langle\alpha,\alpha\rangle} = 2|\tau_\lambda| - |\sigma_\mu| + 1. \tag{8.21}$$

Using these results, we shall write down the expressions in Proposition 7.5. Let $\lambda \in D_K$ and $\mu \in D_M(\lambda)$. In the case of $SU(n,1)$, because all noncompact roots have same length, we see that $\operatorname{sgn}(\lambda + \beta_j + \delta_K) = 1$ iff $[\tau_{\lambda+\beta_j} : \sigma_\mu] = 1$. We simply write ν for $\langle\nu,\alpha\rangle/\langle\alpha,\alpha\rangle$. Substituting (8.10) and (8.21) into the expressions in Proposition 7.5, we have the following lemma.

Lemma 8.2. *Let $T \otimes v \in \operatorname{Hom}_M(V_\lambda, H_\mu) \otimes V_\lambda$ and $Z \in \mathfrak{p}_{\mathbf{c}}$. Then we have*

$$\begin{aligned}
&\pi_{\sigma_\mu,\nu}(Z)f_{T\otimes v} \\
&= \sum_{j=1}^{n-1}(\nu + 2\lambda_j + |\sigma_\mu| + n - 2j + 2)[\tau_{\lambda+\beta_j} : \sigma_\mu] f_{\mathcal{M}_\mu(Z;\lambda+\beta_j,\lambda)(T\otimes v)} \\
&+ \sum_{j=1}^{n-1}(\nu - 2\lambda_j - |\sigma_\mu| - n + 2j)[\tau_{\lambda-\beta_j} : \sigma_\mu] f_{\mathcal{M}_\mu(Z;\lambda-\beta_j,\lambda)(T\otimes v)}, \\
&\pi_{w\sigma_\mu,w\nu}(Z)f_{R_\lambda(w)(T\otimes v)} \\
&= \sum_{j=1}^{n-1}(\nu - 2\lambda_j - |\sigma_\mu| - n + 2j - 2)[\tau_{\lambda+\beta_j} : \sigma_\mu] f_{R_{\lambda+\beta_j}(w)\mathcal{M}_\mu(Z;\lambda+\beta_j,\lambda)(T\otimes v)} \\
&+ \sum_{j=1}^{n-1}(\nu + 2\lambda_j + |\sigma_\mu| + n - 2j)[\tau_{\lambda-\beta_j} : \sigma_\mu] f_{R_{\lambda-\beta_j}(w)\mathcal{M}_\mu(Z;\lambda-\beta_j,\lambda)(T\otimes v)}.
\end{aligned}$$

9. The Recursion Formula for the C-function

In this section we shall give the recursion formula of the Harish-Chandra C-function for $SU(n,1)$. Let $\mu \in D_M$ and $\lambda \in D_K(\mu)$. We first recall that

(9.1)

$$A(w,\sigma,\nu)\pi_{\sigma,\nu}(Z)f_{T\otimes v} = \pi_{w\sigma,w\nu}(Z)A(w,\sigma,\nu)f_{T\otimes v},\ (T\otimes v \in \mathrm{Hom}_M(V_\lambda, H_\mu)\otimes V_\lambda).$$

Applying Proposition 6.1 and Lemma 8.1 to (9.1), we have

$$\begin{aligned}\text{the right-hand side of (9.1)} &= C_{\tau_\lambda}(\sigma_\mu:\nu)\pi_{w\sigma_\mu,w\nu}(Z)f_{R_\lambda(w)(T\otimes v)}\\ &= \sum_{j=1}^{n-1}(\nu-2\lambda_j-|\sigma_\mu|-n+2j-2)[\tau_{\lambda+\beta_j}:\sigma_\mu]C_{\tau_\lambda}(\sigma_\mu:\nu)f_{\lambda+\beta_j}\\ &+ \sum_{j=1}^{n-1}(\nu+2\lambda_j+|\sigma_\mu|+n-2j)[\tau_{\lambda-\beta_j}:\sigma_\mu]C_{\tau_\lambda}(\sigma_\mu:\nu)f_{\lambda-\beta_j}.\end{aligned}$$

Here $f_{\lambda\pm\beta_j} = f_{R_{\lambda\pm\beta_j}(w)\mathcal{M}_\mu(Z;\lambda\pm\beta_j,\lambda)(T\otimes v)}$. Similarly we have that

$$\begin{aligned}&\text{the left-hand side of (9.1)}\\ &= A(w,\sigma_\mu,\nu)\left[\sum_{j=1}^{n-1}(\nu+2\lambda_j+|\sigma_\mu|+n-2j+2)[\tau_{\lambda+\beta_j}:\sigma_\mu]f_{\mathcal{M}_\mu(Z;\lambda+\beta_j,\lambda)(T\otimes v)}\right.\\ &\left.+\sum_{j=1}^{n-1}(\nu-2\lambda_j-|\sigma_\mu|-n+2j)[\tau_{\lambda-\beta_j}:\sigma_\mu]f_{\mathcal{M}_\mu(Z;\lambda-\beta_j,\lambda)(T\otimes v)}\right]\\ &= \sum_{j=1}^{n-1}(\nu+2\lambda_j+|\sigma_\mu|+n-2j+2)[\tau_{\lambda+\beta_j}:\sigma_\mu]C_{\tau_{\lambda+\beta_j}}(\sigma_\mu:\nu)f_{\lambda+\beta_j}\\ &+\sum_{j=1}^{n-1}(\nu-2\lambda_j-|\sigma_\mu|-n+2j)[\tau_{\lambda-\beta_j}:\sigma_\mu]C_{\tau_{\lambda-\beta_j}}(\sigma_\mu:\nu)f_{\lambda-\beta_j}.\end{aligned}$$

Comparing side by side, we obtain the following recursion formulae.
If $[\tau_{\lambda+\beta_j}:\sigma_\mu]=1$, then

$$(\nu-2\lambda_j-|\sigma_\mu|-n+2j-2)C_{\tau_\lambda}(\sigma_\mu:\nu) = (\nu+2\lambda_j+|\sigma_\mu|+n-2j+2)C_{\tau_{\lambda+\beta_j}}(\sigma_\mu:\nu).$$

If $[\tau_{\lambda-\beta_j}:\sigma_\mu]=1$, then

$$(\nu+2\lambda_j+|\sigma_\mu|+n-2j)C_{\tau_\lambda}(\sigma_\mu:\nu) = (\nu-2\lambda_j-|\sigma_\mu|-n+2j)C_{\tau_{\lambda-\beta_j}}(\sigma_\mu:\nu).$$

We set $\lambda(\mu) = (\mu_1,\ldots,\mu_{n-1},\mu_{n-1}) \in D_K(\mu)$. Thus, using the preceding recursion formulae and shifting the parameters as $\mu_p \to \lambda_p$, $(1\le p\le n-1)$ and $\mu_{n-1}\to\lambda_n$, we can find the following theorem.

Theorem 9.1. *Retain the above notation. We have*

$$C_{\tau_\lambda}(\sigma_\mu : \nu) = \prod_{j=1}^{n-1} \frac{\left(\frac{\nu-n-|\sigma_\mu|}{2} - \mu_j + j - 1\right)_{\lambda_j-\mu_j}}{\left(\frac{\nu+n+|\sigma_\mu|}{2} + \mu_j - j + 1\right)_{\lambda_j-\mu_j}} \frac{\left(\frac{\nu-n+|\sigma_\mu|}{2} + \mu_{n-1}\right)_{\mu_{n-1}-\lambda_n}}{\left(\frac{\nu+n-|\sigma_\mu|}{2} - \mu_{n-1}\right)_{\mu_{n-1}-\lambda_n}} \times C_{\tau_{\lambda(\mu)}}(\sigma_\mu : \nu).$$

Here $(a)_n = \Gamma(a+n)/\Gamma(a)$.

Theorem 9.1 says that for getting the expression of the Harish-Chandra C-function, it suffices to calculate $C_{\tau_{\lambda(\mu)}}(\sigma_\mu : \nu)$. To do this, we shall get the integral expression of $C_{\tau_{\lambda(\mu)}}(\sigma_\mu : \nu)$. Let $\mu \in D_M$. Define the Gel'fand–Tsetlin data $\mathrm{M}_{\lambda(\mu)} = (m_1, \cdots, m_n)$ by $m_1 = (\mu_1, \cdots, \mu_{n-1}, \mu_{n-1})$ and $m_j = (\mu_1, \cdots, m_{n-j})$ for $1 \leq j \leq n-1$. Then it is obvious that the Gel'fand–Testlin basis $v(\mathrm{M}_{\lambda(\mu)})$ becomes a highest weight vector of both $(\tau_{\lambda(\mu)}|_M, V_{\lambda(\mu)}(\mu))$ and $(\tau_{\lambda(\mu)}, V_{\lambda(\mu)})$. Choosing $T \in \mathrm{Hom}_M(V_{\lambda(\mu)}, V_{\lambda(\mu)}(\mu))$ as a canonical projection, we have

(9.2)

$$\begin{aligned} C_{\tau_{\lambda(\mu)}}(\sigma_\mu : \nu) &= \langle TC_{\tau_{\lambda(\mu)}}(\nu)v(\mathrm{M}_{\lambda(\mu)}), v(\mathrm{M}_{\lambda(\mu)})\rangle \\ &= \langle C_{\tau_{\lambda(\mu)}}(\nu)v(\mathrm{M}_{\lambda(\mu)}), v(\mathrm{M}_{\lambda(\mu)})\rangle \\ &= \int_{\bar{N}} e^{-(\nu+\rho)(H(\bar{n}))}\langle \tau_{\lambda(\mu)}(\kappa(\bar{n})^{-1})v(\mathrm{M}_{\lambda(\mu)}), v(\mathrm{M}_{\lambda(\mu)})\rangle d\bar{n}. \end{aligned}$$

Putting $\phi_{\lambda(\mu)}(k) = \langle \tau_{\lambda(\mu)}(k)v(\mathrm{M}_{\lambda(\mu)}), v(\mathrm{M}_{\lambda(\mu)})\rangle$, we obtain the following proposition.

Proposition 9.2. *Retain the above notation. We have*

$$C_{\tau_{\lambda(\mu)}}(\sigma_\mu : \nu) = \int_{\bar{N}} e^{-(\nu+\rho)(H(\bar{n}))}\phi_{\lambda(\mu)}(\kappa(\bar{n})^{-1})d\bar{n}.$$

10. Fundamental representations of K

In order to compute $\phi_{\lambda(\mu)}(\kappa(\bar{n})^{-1})$, it suffices to compute $\phi_\Lambda(\kappa(\bar{n})^{-1})$ in the case of the fundamental representation Λ. Since the fundamental representations can be constructed as alternating tensor products of the usual representations of K, we can concretely write the matrix elements of $\tau_\Lambda(\kappa(\bar{n})^{-1})$. We note that since K is connected reductive compact, τ_λ can be extended to a holomorphic representation on $K_{\mathbf{c}}$, which is an analytic subgroup of matrices whose Lie algebra is $\mathfrak{k}_{\mathbf{c}}$.

We know that the fundamental representations are listed as follows:

(10.1) $$\omega_p = \varepsilon_1 + \cdots + \varepsilon_p - p\varepsilon_{n+1},\ (1 \leq p \leq n-1),\ \omega_0 = -\varepsilon_{n+1}.$$

Let $(\Phi, \mathbf{C}^n)$ be the usual representation of K, that is, for $k = \begin{pmatrix} X & \\ & u \end{pmatrix} \in K$ and $z \in \mathbf{C}^n$, $\Phi(k)z = u^{-1}Xz$ and $(\Phi_r, \wedge^r\mathbf{C}^n)$ be the alternating tensor representation of Φ. We denote by $(\Phi_0, \mathbf{C})$ the representation of K defined by $\Phi_0(k)z = u^{-1}z$.

Then Φ_r, $(1 \leq r \leq n-1)$ and Φ_0 are irreducible with highest weights ω_r and ω_0, respectively. An easy computation yields

$$\langle \Phi(\kappa(\bar{n}(z,u))^{-1})e_p, e_q \rangle = \frac{1+\omega}{|1+\omega|}\left(\delta_{pq} - \frac{2z_q\bar{z}_p}{1+\bar{\omega}}\right). \tag{10.2}$$

Therefore

$$\phi_{\omega_r}(\kappa(\bar{n}(z,u))^{-1}) = \left(\frac{1+\omega}{|1+\omega|}\right)^r \left(1 - \frac{2\sum_{p=1}^r |z_p|^2}{1+\bar{\omega}}\right), \tag{10.3}$$

$$\phi_{\omega_0}(\kappa(\bar{n}(z,u))^{-1}) = \left(\frac{1+\bar{\omega}}{|1+\omega|}\right)^{-1}.$$

Let $\mu \in D_M$. Then $\lambda(\mu) = \sum_{p=1}^{n-2}(\mu_p - \mu_{p+1})\omega_p - (n+1)\mu_n\varepsilon_{n+1}$. We write $\mathfrak{k}_+$ (resp. $\mathfrak{k}_-$) for the sum of all positive root subspaces (resp. negative root subspaces) with respect to $(\mathfrak{k}_{\mathbf{c}}, \mathfrak{t}_{\mathbf{c}})$. Let K_+ and K_- denote the analytic subgroups of $K_{\mathbf{c}}$ corresponding to $\mathfrak{k}_+$ and $\mathfrak{k}_-$, respectively. By the definition of ϕ_λ, $(\lambda \in D_K)$, it follows that

$$\phi_\lambda(k_1 \exp H k_2) = \phi_\lambda(\exp H) = e^{\lambda(H)}, \ (k_1 \in K_+, k_2 \in K_-, H \in \mathfrak{t}_{\mathbf{c}}), \tag{10.4}$$

and thus

$$\phi_{\lambda(\mu)}(k_1 \exp H k_2) = \prod_{p=1}^{n-2} \phi_{\omega_p}(k_1 \exp H k_2)^{\mu_p - \mu_{p+1}} \phi_{\omega_0}(k_1 \exp H k_2)^{n+1}. \tag{10.5}$$

Noting $K_1 \exp \mathfrak{t}_{\mathbf{c}} K_2$ is dense in $K_{\mathbf{c}}$ and ϕ_λ is holomorphic, we have for any $k \in K_{\mathbf{c}}$ that

$$\phi_{\lambda(\mu)}(k) = \prod_{p=1}^{n-2} \phi_{\omega_p}(k)^{\mu_p - \mu_{p+1}} \phi_{\omega_0}(k)^{n+1}. \tag{10.6}$$

Thus substituting (10.3) into (10.6), we obtain the following lemma.

Lemma 10.1. *Retain the above notation. Then we have*

$$\phi_{\lambda(\mu)}(\kappa(\bar{n}(z,u))^{-1})$$

$$= (1+\omega)^{(|\sigma_\mu|+2\mu_{n-1})/2}(1+\bar{\omega})^{-(|\sigma_\mu|+2\mu_1)/2} \prod_{p=1}^{n-2} \left(1 + \bar{\omega} - 2\sum_{j=1}^{p} |z_j|^2\right)^{\mu_p - \mu_{p+1}}.$$

11. Expressions of the Harish-Chandra C-functions

Using Proposition 9.2 and Lemma 10.1 and carrying out the integration, we can get the explicit expression of $C_{\tau_{\lambda(\mu)}}(\sigma_\mu : \nu)$. Combining this with Theorem 9.1, we can get the explicit expression of the Harish-Chandra C-function for $SU(n,1)$. In the case of $SU(n,1)$, we see that $\rho = n$, $\bar{N} = \mathbf{C}^{n-1} \times \mathbf{R}$ and

$$\int_{\mathbf{C}^{n-1}\times\mathbf{R}} |1+\omega|^{-2n} dz d\bar{z} du = \frac{\pi^n}{2^n (n-1)!} (= c_n, \text{ say}). \tag{11.1}$$

From Proposition 9.1 and Lemma 10.1, we have

(11.2)

$$\begin{aligned} c_n C_{\tau_{\lambda(\mu)}}(\sigma_\mu : \nu) = \int_{\mathbf{C}^{n-1}\times\mathbf{R}} & (1+\omega)^{-(\nu+n-|\sigma_\mu|-2\mu_{n-1})/2} (1+\bar{\omega})^{-(\nu+n+|\sigma_\mu|+2\mu_1)/2} \\ & \times \prod_{p=1}^{n-2} \left(1+\bar{\omega} - 2\sum_{j=1}^{p} |z_j|^2 \right)^{\mu_p - \mu_{p+1}} dz d\bar{z} du. \end{aligned}$$

In order to compute the integral in (11.2), we need the following lemma.

Lemma 11.1 (cf. [8,22]). *Let $n \geq 1$, $\lambda \in \mathbf{C}$, $\ell \in \mathbf{Z}$, $q_j \in \mathbf{Z}_{\geq 0}$, $(1 \leq j \leq n-1)$, and $F = 1 + \frac{1}{2}(|z_1|^2 + \cdots + |z_{n-1}|^2) + \sqrt{-1}u$. Then*

$$\begin{aligned} & \int_{\mathbf{C}^{n-1}\times\mathbf{R}} \bar{F}^{(\lambda+\ell)/2} F^{(\lambda-\ell)/2} \prod_{p=1}^{n-1} \left(\bar{F} - \sum_{j=1}^{p} |z_j|^2 \right)^{q_p} dz d\bar{z} du = \\ & \frac{(2\pi)^n 2^{\lambda+n+q_1+\cdots+q_{n-1}} \Gamma(-\lambda-n-q_1-\cdots-q_{n-1})}{\prod_{j=1}^{n-1} \left(-\frac{\lambda+\ell}{2} - q_1 - \cdots - q_{j-1} - j\right) \Gamma\left(-\frac{\lambda+\ell}{2} - q_1 - \cdots - q_{n-1} - n + 1\right) \Gamma\left(-\frac{\lambda-\ell}{2}\right)}. \end{aligned}$$

Taking into account (11.1), we obtain from Lemma 11.1 that

$$\begin{aligned} & C_{\tau_{\lambda(\mu)}}(\sigma_\mu : \nu) \\ & = \frac{(n-1)! 2^{-\nu+n} \Gamma(\nu)}{\prod_{j=1}^{n-1} \left(\frac{\nu+n+|\sigma_\mu|}{2} - j + \mu_j\right) \Gamma\left(\frac{\nu+n-|\sigma_\mu|}{2} - \mu_{n-1}\right) \Gamma\left(\frac{\nu-n+|\sigma_\mu|}{2} + 1 + \mu_{n-1}\right)}. \end{aligned} \tag{11.3}$$

Combining Theorem 9.1 with the above expressions, we can get the following expression of the Harish-Chandra C-function for $SU(n,1)$.

Theorem 11.2. *The Harish-Chandra C-function $C_{\tau_\lambda}(\sigma_\mu : \nu)$ for $SU(n,1)$ associated with $\tau_\lambda \in \hat{K}$ and $\sigma_\mu \in \hat{M}(\tau_\lambda)$ is given as follows:*

$$\begin{aligned} & C_{\tau_\lambda}(\sigma_\mu : \nu) \\ & = \frac{(n-1)! 2^{-\nu+n} \Gamma(\nu) \prod_{j=1}^{n-1} \Gamma\left(\frac{\nu-n-|\sigma_\mu|}{2} + j - \mu_j\right) \prod_{j=1}^{n-1} \Gamma\left(\frac{\nu+n+|\sigma_\mu|}{2} - j + \mu_j\right)}{\prod_{j=1}^{n} \Gamma\left(\frac{\nu-n-|\sigma_\mu|}{2} + j - \lambda_j\right) \prod_{j=1}^{n} \Gamma\left(\frac{\nu+n+|\sigma_\mu|}{2} - j + 1 + \lambda_j\right)}. \end{aligned}$$

Remark. If $n = 1$, putting $|\sigma_\mu| = 0$, we can get the expression of the Harish-Chandra C-function for $SU(1,1)$.

We write $\det C_{\tau_\lambda}(\nu)$ for the determinant of the linear mapping $C_{\tau_\lambda}(\nu)$ of V_λ. Taking into account $V_\lambda = \sum_{\mu \in D_M(\lambda)} H_\mu$, we see that

$$\det C_{\tau_\lambda}(\nu) = \prod_{\mu \in D_M(\lambda)} C_{\tau_\lambda}(\sigma_\mu : \nu)^{\dim H_\mu}. \tag{11.4}$$

Thus, substituting the expression in Theorem 11.2 into (11.4), we obtain the explicit formula of $\det C_{\tau_\lambda}(\nu)$. On the other hand, in [4], Cohn obtained the expression of $\det C_\tau(\nu)$ for any semisimple Lie group. He showed that there exist $p_{i,j}, q_{i,j} \in \mathbf{C}$, $(1 \leq i \leq r, 1 \leq j \leq j_i)$ and $\mu_1, \ldots, \mu_r \in \mathfrak{a}^*$ such that

$$\det C_\tau(\nu) = \prod_{i=1}^{r} \prod_{j=1}^{j_i} \frac{\Gamma\left(\frac{-\langle \nu, \alpha_i \rangle}{2\langle \mu_i, \alpha_i \rangle} + q_{i,j}\right)}{\Gamma\left(\frac{-\langle \nu, \alpha_i \rangle}{2\langle \mu_i, \alpha_i \rangle} + p_{i,j}\right)}. \tag{11.5}$$

He conjectured in his paper [4] that the coefficients $p_{i,j}$ and $q_{i,j}$ appearing in the above expression are rational numbers and depending linearly on the highest weight of τ. We can now concretely write the values of $p_{i,j}$ and $q_{i,j}$ and thus we obtain the following corollary.

Corollary 11.3. *Cohn's conjecture is true for* $SU(n,1)$.

12. Composition series and unitarizability of $SU(n,1)$

In this section, we shall write down the composition series of the nonunitary principal series representations and determine which parts of the composition series are unitarizable, which was shown by Kraljević [20]. By virtue of the expression of the Harish-Chandra C-function, we can get the explicit forms of the inner products that make the subquotients unitary.

Let $\mu \in D_M$ and $\lambda \in D_K(\mu)$. Suppose $\nu \in \mathbf{R}$ and $\nu > 0$. For $1 \leq j \leq n-1$, we set $h_j = (\nu - n - |\sigma_\mu|)/2 + j - \mu_j$ and $k_j = (\nu + n + |\sigma_\mu|)/2 - j + \mu_j$ and assume $h_j \in \mathbf{Z}$ and $k_j \in \mathbf{Z}$. We choose $0 \leq a, b \leq n-1$ satisfying the following conditions:

$$\begin{aligned} &h_1 < \cdots < h_a \leq 0 < h_{a+1} < \cdots < h_{n-1}, \\ &k_1 > \cdots > k_b > 0 \geq k_{b+1} > \cdots > k_{n-1}. \end{aligned} \tag{12.1}$$

In the following, we set $\mu_0 = \infty$ and $\mu_n = -\infty$. By using the expression of the Harish-Chandra C-function, we see that the zeros of $C_{\tau_\lambda}(\sigma_\mu : \nu)$ coincide with the ones of the following function:

$$\frac{1}{\Gamma\left(\frac{\nu - n - |\sigma_\mu|}{2} + a + 1 - \lambda_{a+1}\right)\Gamma\left(\frac{\nu + n + |\sigma_\mu|}{2} - b + \lambda_{b+1}\right)}.$$

We set

(12.2)

$$S^{\sigma_\mu}_{\nu,+}(a) = \left\{\lambda \in D_K(\mu) : \frac{\nu - n - |\sigma_\mu|}{2} + a + 1 \le \lambda_{a+1} \le \mu_a\right\},$$
$$S^{\sigma_\mu}_{\nu,-}(b) = \left\{\lambda \in D_K(\mu) : \mu_{b+1} \le \lambda_{b+1} \le -\frac{\nu + n + |\sigma_\mu|}{2} + b\right\},$$
$$\mathcal{H}^{\sigma_\mu}_{\nu,+}(a) = \sum_{\lambda \in S^{\sigma_\mu}_{\nu,+}(a)} V_\lambda,$$
$$\mathcal{H}^{\sigma_\mu}_{\nu,-}(b) = \sum_{\lambda \in S^{\sigma_\mu}_{\nu,-}(b)} V_\lambda.$$

In addition, if $a \neq b$, we set

(12.3)

$$S^{\sigma_\mu}_{\nu}(a,b) = S^{\sigma_\mu}_{\nu,+}(a) \cap S^{\sigma_\mu}_{\nu,-}(b),$$
$$\mathcal{H}^{\sigma_\mu}_{\nu}(a,b) = \sum_{\lambda \in S^{\sigma_\mu}_{\nu}(a,b)} V_\lambda.$$

Then we have the following results.

Theorem 12.1 (cf. [20]). *$\pi_{\sigma_\mu,\nu}$ is reducible iff $(\nu - n - |\sigma_\mu|)/2 + j - \mu_j \in \mathbf{Z}\backslash\{0\}$ or $(\nu + n + |\sigma_\mu|)/2 - j + \mu_j \in \mathbf{Z}\backslash\{0\}$ for $1 \le j \le n-1$.*

Theorem 12.2 (cf. [20]). *Assume $\pi_{\sigma_\mu,\nu}$ is reducible. Choose $0 \le a \le b \le n-1$ satisfying the relations in (12.1). Then the composition series of $\pi_{\sigma_\mu,\nu}$ are given as follows:*

(1) *If $h_a = 0$ and $k_{b+1} \neq 0$, then*

$$\mathcal{H}^{\sigma_\mu,\nu} \supset \mathcal{H}^{\sigma_\mu}_{\nu,-}(b) \supset \{0\}.$$

(2) *If $h_a \neq 0$ and $k_{b+1} = 0$, then*

$$\mathcal{H}^{\sigma_\mu,\nu} \supset \mathcal{H}^{\sigma_\mu}_{\nu,+}(a) \supset \{0\}.$$

(3) *If $h_a \neq 0$ and $k_{b+1} \neq 0$ and $a = b$, then*

$$\mathcal{H}^{\sigma_\mu,\nu} \supset \mathcal{H}^{\sigma_\mu}_{\nu,+}(a) + \mathcal{H}^{\sigma_\mu}_{\nu,-}(a) \supset \mathcal{H}^{\sigma_\mu}_{\nu,+}(a) \supset \{0\},$$
$$\mathcal{H}^{\sigma_\mu,\nu} \supset \mathcal{H}^{\sigma_\mu}_{\nu,+}(a) + \mathcal{H}^{\sigma_\mu}_{\nu,-}(a) \supset \mathcal{H}^{\sigma_\mu}_{\nu,-}(a) \supset \{0\}.$$

(4) *If $h_a \neq 0$ and $k_{b+1} \neq 0$ and $a < b$, then*

$$\mathcal{H}^{\sigma_\mu,\nu} \supset \mathcal{H}^{\sigma_\mu}_{\nu,+}(a) + \mathcal{H}^{\sigma_\mu}_{\nu,-}(b) \supset \mathcal{H}^{\sigma_\mu}_{\nu,+}(a) \supset \mathcal{H}^{\sigma_\mu}_{\nu}(a,b) \supset \{0\},$$
$$\mathcal{H}^{\sigma_\mu,\nu} \supset \mathcal{H}^{\sigma_\mu}_{\nu,+}(a) + \mathcal{H}^{\sigma_\mu}_{\nu,-}(b) \supset \mathcal{H}^{\sigma_\mu}_{\nu,-}(b) \supset \mathcal{H}^{\sigma_\mu}_{\nu}(a,b) \supset \{0\}.$$

Theorem 12.3 (cf. [20]).

(1) *$\pi_{\sigma_\mu,0}$ is reducible iff $(-n-|\sigma_\mu|)/2+j-\mu_j \in \mathbf{Z}\backslash\{0\}$ for $1 \le j \le n-1$.*

(2) *Assume that $\pi_{\sigma_\mu,0}$ is reducible. Then the composition series of $\pi_{\sigma_\mu,0}$ can be written as follows:*

$$\mathcal{H}^{\sigma_\mu,0} \supset \mathcal{H}^{\sigma_\mu}_{0,+}(a) \supset \{0\},$$
$$\mathcal{H}^{\sigma_\mu,0} \supset \mathcal{H}^{\sigma_\mu}_{0,-}(a) \supset \{0\}.$$

Let $\mathcal{H}^{\sigma,\nu}(K)$ be the set of K-finite elements in $C^\infty_\sigma(K)$. Let $\mathcal{A}(w,\sigma,\nu)$ be the normalized intertwining operator (cf. [17]). In case $G=SU(n,1)$, because $\sigma \cong w\sigma$ for any $\sigma \in \hat{M}$, it is possible to define $\sigma(w)$. Following Knapp–Stein [17], we determine the ambiguous sign of $\sigma(w)$ so that $\sigma(w)\mathcal{A}(w,\sigma,0)$ coincides with the identity operator on $\mathcal{H}^{\sigma,0}$. Then Theorem 8.1(3) implies the following lemma.

Lemma 12.4. *Let $\mu \in D_M$ and $\lambda \in D_K(\mu)$. Then we have*

$$\sigma_\mu(w)\mathcal{A}(w,\sigma_\mu,\nu)|_{\mathcal{H}^{\sigma_\mu,\nu}(\tau_\lambda)} = \prod_{j=1}^{n-1} \frac{(-h_j+1)_{\lambda_j-\mu_j}}{(k_j+1)_{\lambda_j-\mu_j}} \frac{(-k_{n-1}+1)_{\mu_{n-1}-\lambda_n}}{(h_{n-1}+1)_{\mu_{n-1}-\lambda_n}} I_\lambda,$$

where I_λ denotes the identity operator on $\mathcal{H}^{\sigma_\mu,\nu}(\tau_\lambda)$.

We define the sesquilinear Hermitian form on $\mathcal{H}^{\sigma_\mu,\nu}(K)$ by

$$\langle f,g\rangle_i = (f,\sigma_\mu(w)\mathcal{A}(w,\sigma_\mu,\nu)g), \text{ for } f,g \in \mathcal{H}^{\sigma_\mu,\nu}(K),$$

where $(\cdot,\cdot)$ denotes the sesquilinear pairing on $\mathcal{H}^{\sigma_\mu,\nu} \times \mathcal{H}^{w\sigma_\mu,w\nu}$. Using Lemma 12.4 and investigating the sign of $\sigma_\mu(w)\mathcal{A}(w,\sigma_\mu,\nu)|_{\mathcal{H}^{\sigma_\mu,\nu}(\tau_\lambda)}$, we immediately obtain the condition for that $\langle\cdot,\cdot\rangle_i$ is positive definite.

Theorem 12.5 (cf. [20]). *We have the following.*

(1) *Suppose that $a=b$.*
If $a=n-1$, $\langle\cdot,\cdot\rangle_i$ induces a positive definite Hermitian form on $\mathcal{H}^{\sigma_\mu}_{\nu,+}(a)$.
If $a<n-1$, $(-1)^\nu\langle\cdot,\cdot\rangle_i$ induces a positive definite Hermitian form on $\mathcal{H}^{\sigma_\mu}_{\nu,+}(a)$.
If $a=n-1$, $(-1)^\nu\langle\cdot,\cdot\rangle$ induces a positive definite Hermitian form on $\mathcal{H}^{\sigma_\mu}_{\nu,-}(a)$.
If $a<n-1$, $\langle\cdot,\cdot\rangle_i$ induces a positive definite Hermitian form on $\mathcal{H}^{\sigma_\mu}_{\nu,-}(a)$.

(2) *Suppose that $a<b$ and $\mu_{a+1}=\cdots=\mu_b$.*
If $b=n-1$, $\Gamma(-h_{a+1}+1)\Gamma(-k_{n-1}+1)\langle\cdot,\cdot\rangle_i$ induces a positive definite Hermitian form on $\mathcal{H}^{\sigma_\mu}_\nu(a,b)$.
If $b<n-1$, $\Gamma(-h_{a+1}+1)\langle\cdot,\cdot\rangle_i$ induces a positive definite Hermitian form on $\mathcal{H}^{\sigma_\mu}_\nu(a,b)$.
If $b=n-1$ and $k_b=1$, $\Gamma(-h_{a+1}+1)\langle\cdot,\cdot\rangle_i$ induces a positive definite Hermitian form on $\mathcal{H}^{\sigma_\mu}_{\nu,+}(a)/\mathcal{H}^{\sigma_\mu}_\nu(a,b)$.
If $b<n-1$ and $k_b=1$, $(-1)^\nu\langle\cdot,\cdot\rangle_i$ induces a positive definite Hermitian form on $\mathcal{H}^{\sigma_\mu}_{\nu,+}(a)/\mathcal{H}^{\sigma_\mu}_\nu(a,b)$.
If $b=n-1$ and $h_{a+1}=1$, $\Gamma(-k_{n-1}+1)\langle\cdot,\cdot\rangle_i$ induces a positive definite Hermitian form on $\mathcal{H}^{\sigma_\mu}_{\nu,-}(a)/\mathcal{H}^{\sigma_\mu}_\nu(a,b)$.
If $b<n-1$ and $h_{a+1}=1$, $\langle\cdot,\cdot\rangle_i$ induces a positive definite Hermitian form on $\mathcal{H}^{\sigma_\mu}_{\nu,-}(a)/\mathcal{H}^{\sigma_\mu}_\nu(a,b)$.

Remark. Applying Nelson's theorem, we can construct unitary representations of G on the Hilbert space completions of the above subquotients relative to the respective positive definite Hermitian forms. We denote these unitary representations by the same symbols.

Applying Theorem 6(i) and Theorem 7(ii) in [20], we can find which subquotients appeared in Theorem 12.5 belong to the discrete series representations. Because these computations can be carried out without any difficulty, we shall only write the conclusions.

Theorem 12.6 (cf. [18]). *The discrete series representations of $SU(n,1)$ are listed as follows:*

(1) *The holomorphic discrete series. $\mathcal{H}^{\sigma_\mu}_{\nu,+}(n-1)$ with $h_{n-1}<0$ and $k_{n-1}>0$.*
minimal K-type: $\lambda_m = \left(\mu_1,\dots,\mu_{n-1},\frac{\nu+n-|\sigma_\mu|}{2}\right)$.
Harish-Chandra parameter:
$$\Lambda = \sum_{p=1}^{n-1}\left(\mu_p+\tfrac{n}{2}-p\right)\varepsilon_p + \tfrac{\nu-|\sigma_\mu|}{2}\varepsilon_n - \tfrac{\nu+|\sigma_\mu|}{2}\varepsilon_{n+1}.$$
K-spectrum: $\Gamma(\Lambda) = \left\{\tau_\lambda\in\hat{K} : \lambda < (\infty,\lambda_m)\right\}$.

(2) *The antiholomorphic discrete series. $\mathcal{H}^{\sigma_\mu}_{\nu,-}(0)$ with $h_1>0$ and $k_1<0$.*
minimal K-type: $\lambda_m = \left(-\frac{\nu+n+|\sigma_\mu|}{2},\mu_1,\dots,\mu_{n-1}\right)$.
Harish-Chandra parameter:
$$\Lambda = -\tfrac{\nu+|\sigma_\mu|}{2}\varepsilon_1 + \sum_{p=1}^{n-1}\left(\mu_p+\tfrac{n}{2}-p\right)\varepsilon_{p+1} + \tfrac{\nu-|\sigma_\mu|}{2}\varepsilon_{n+1}.$$
K-spectrum: $\Gamma(\Lambda) = \left\{\tau_\lambda\in\hat{K} : \lambda < (\lambda_m,-\infty)\right\}$.

(3) *The nonholomorphic discrete series. $\mathcal{H}^{\sigma_\mu}_{\nu}(a,a+1)$ $(0\le a\le n-2)$ with $h_a<0<h_{a+1}$ and $k_{a+1}>0>k_{a+2}$.*
minimal K-type:
$$\lambda_m = \left(\mu_1,\dots,\mu_a,\tfrac{\nu-n-|\sigma_\mu|}{2}+a+1,-\tfrac{\nu+n+|\sigma_\mu|}{2}+a+1,\mu_{a+2},\dots,\mu_{n-1}\right).$$
Harish-Chandra parameter:
$$\Lambda = \sum_{p=1}^{a}\left(\mu_p+\tfrac{n}{2}-p\right)\varepsilon_p + \tfrac{\nu-|\sigma_\mu|}{2}\varepsilon_{a+1} - \tfrac{\nu+|\sigma_\mu|}{2}\varepsilon_{a+2} + \sum_{p=a+2}^{n-1}\left(\mu_p+\tfrac{n}{2}-p\right)\varepsilon_{p+1} + \left(\mu_{a+1}+\tfrac{n}{2}-a-1\right)\varepsilon_{n+1}.$$
K-spectrum:
$$\Gamma(\Lambda) = \left\{\tau_\lambda\in\hat{K} : \begin{array}{c}(\lambda_1,\dots,\lambda_{a+1})<\left(\infty,\mu_1,\dots,\mu_a,\frac{\nu-n-|\sigma_\mu|}{2}+a+1\right)\\ (\lambda_{a+2},\dots,\lambda_n)<\left(-\frac{\nu+n+|\sigma_\mu|}{2}+a+1,\mu_{a+2},\dots,\mu_{n-1},-\infty\right)\end{array}\right\}.$$

Remark. In the next section, we write the discrete series representation with the Harish-Chandra parameter Λ as (π_Λ, V_Λ).

13. Restriction of Discrete series

Let us embed $G_1 = U(n-1,1)$ into $G = SU(n,1)$ by $g = \begin{pmatrix} X & v\\ w^* & u\end{pmatrix} \mapsto \begin{pmatrix} X & 0 & v\\ 0 & a & 0\\ w^* & 0 & u\end{pmatrix}$,

where $a = (\det g)^{-1} \in U(1)$. Let

(13.1)
$$K_1 = \left\{ \begin{pmatrix} X & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & u \end{pmatrix} : X \in U(n-1), a, u \in U(1), au \det X = 1 \right\},$$
$$Z = \{ z(h) = \operatorname{diag}(\overbrace{h, \dots, h}^{n-1}, h^{-n}, h) : h \in U(1) \},$$
$$A_1 = \exp \mathbf{R} H_1,$$

where $H_1 = E_{n-1,n+1} + E_{n+1,n-1}$. Then $G_1 = K_1 A_1 K_1$ is a Cartan decomposition of G_1. For $\ell \in \mathbf{Z}$ and $\mu \in D_M$, define the unitary representation $(\chi_{(\ell,\mu)}, H_\mu)$ of $Z \times M$ by $\chi_{(\ell,\mu)}(z(h), m)v = h^\ell \sigma_\mu(m)v$, $(m \in M, v \in H_\mu)$. Since $K_1 = MZ$, $\chi_{(\ell,\mu)} \in \hat{K}_1$ iff $\ell + (n+1)\mu_{n-1} \in (n+1)\mathbf{Z}$. For $x \in \mathbf{R}^p_>$, we write $|x| = \sum_{j=1}^p x_j$. For $\alpha \in D_K$, it follows that

(13.2)
$$\tau_\alpha|_{K_1} = \sum_{\beta < \alpha} \chi_{(-(n+1)(|\alpha|-|\beta|),\beta)}, \quad V_\alpha = \sum_{\beta < \alpha} V_\alpha(\beta).$$

So when we look upon $V_\alpha(\beta)$ as a representation space of K_1, we write this representation space as $V_{(-(n+1)(|\alpha|-|\beta|),\beta)}$. In this section, we shall give the irreducible decompositions of holomorphic or antiholomorphic discrete series when they are restricted to G_1, which was proven in [21] in general case. By virtue of the embedding of discrete series into nonunitary principal series, we can concretely construct the G_1-invariant subspaces of the representation spaces of the discrete series in terms of the Gel'fand–Testlin basis.

We shall first rewrite the results in Proposition 8.2 in terms of the Clebsch–Gordan coefficients. Fix an orthonormal basis $\{E_i = E_{n+1,i}/\sqrt{2(n+1)},\ F_i = E_{i,n+1}/\sqrt{2(n+1)},\ (1 \le i \le n)\}$ of $\mathfrak{p}_\mathbf{c}$. Then E_i and F_i correspond to the Gel'fand–Tsetlin basis with data $(\bar{\mathbf{1}}_n, \dots, \bar{\mathbf{1}}_i, \mathbf{0}_{i-1}, \dots, \mathbf{0}_1)$ and $(\mathbf{1}_n, \dots, \mathbf{1}_i, \mathbf{0}_{i-1}, \dots, \mathbf{0}_1)$ respectively. Here $\mathbf{0}_i = (\overbrace{0, \dots, 0}^{i})$, $\mathbf{1}_i = (1, \mathbf{0}_{i-1})$ and $\bar{\mathbf{1}}_i = (\mathbf{0}_{i-1}, -1)$. Let $(\cdot, \cdot|\cdot)$ denote the Clebsch–Gordan coefficients relative to the decomposition $V_\lambda \otimes V_{\lambda'} = \sum_{\lambda'' \in D_K} V_{\lambda''}$, that is, for $v(\mathrm{M}) \in V_\lambda, v(\mathrm{M}') \in V_{\lambda'}$ and $v(\mathrm{M}'') \in V_{\lambda''}$,

$$\big(v(\mathrm{M}), v(\mathrm{M}')|v(\mathrm{M}'')\big) = \langle E_{\lambda''}(v(\mathrm{M}) \otimes v(\mathrm{M}')), v(\mathrm{M}'') \rangle,$$

where $E_{\lambda''}$ denotes the canonical projection of $V_\lambda \otimes V_{\lambda'}$ to $V_{\lambda''}$. In this section we use the following fact concerning the Clebsch–Gordan coefficients of $U(n)$.

Lemma 13.1 (cf. [27, p. 385]). *For any Gel'fand–Testlin data* $\mathrm{M} = (m_n, \dots, m_1), \mathrm{M}' = (m'_n, \dots, m'_1), \mathrm{M}'' = (m''_n, \dots, m''_1)$ *and* $2 \le j \le n$, *there exists constants* $\begin{pmatrix} m_j & m'_j & | & m''_j \\ m_{j-1} & m'_{j-1} & | & m''_{j-1} \end{pmatrix}$ *such that the Clebsch–Gordan coefficient* $\big(v(\mathrm{M}), v(\mathrm{M}')|v(\mathrm{M}'')\big)$ *can be expressed as follows:*

$$\big(v(\mathrm{M}), v(\mathrm{M}')|v(\mathrm{M}'')\big) = \prod_{j=2}^{n} \begin{pmatrix} m_j & m'_j & | & m''_j \\ m_{j-1} & m'_{j-1} & | & m''_{j-1} \end{pmatrix}.$$

Moreover, $\left(\begin{array}{cc|c} m_j & m'_j & m''_j \\ m_{j-1} & m'_{j-1} & m''_{j-1} \end{array}\right)$ *has the following properties.*

(1) *If* $|m_j| + |m'_j| \neq |m''_j|$ *or* $|m_{j-1}| + |m'_{j-1}| \neq |m''_{j-1}|$, *then*

$$\left(\begin{array}{cc|c} m_j & m'_j & m''_j \\ m_{j-1} & m'_{j-1} & m''_{j-1} \end{array}\right) = 0.$$

(2) $$\left(\begin{array}{cc|c} 1_j & m_j & m'_j \\ 1_{j-1} & m_j(j-1) & m'_j(j-1) \end{array}\right) = \left(\begin{array}{cc|c} \bar{1}_j & m_j & m'_j \\ \bar{1}_{j-1} & m_j[2] & m'_j[2] \end{array}\right) = 1.$$

Remark. $\left(\begin{array}{cc|c} m_j & m'_j & m''_j \\ m_{j-1} & m'_{j-1} & m''_{j-1} \end{array}\right)$ are called the scalar factors of the Clebsch–Gordan coefficients.

For each $\alpha \in D_K(\mu)$ and $\beta \in D_M(\alpha)$, let T_α^β be the canonical projection of V_α into $V_\alpha(\beta)$ and write $P_\alpha^\beta = \sqrt{\dim V_\alpha / \dim V_\alpha(\beta)}\, T_\alpha^\beta$. Throughout this section we shall identify $\mathcal{H}^{\sigma_\mu,\nu}(\tau_\alpha)$ with V_α and simply write v instead of $f_{P_\alpha^\beta \otimes v}$. A simple calculation implies that

$$\tilde{P}_\alpha^\beta E^*_{\alpha^{+j}} = \frac{1}{2\sqrt{2(n+1)}} \sqrt{\frac{\dim V_{\alpha^{+j}}}{\dim V_\alpha}} \left(\begin{array}{cc|c} 1_n & \alpha & \alpha^{+j} \\ 0_{n-1} & \mu & \mu \end{array}\right) P^\beta_{\alpha^{+j}}, \tag{13.3}$$

$$\tilde{P}_\alpha^\beta E^*_{\alpha^{-j}} = \frac{1}{2\sqrt{2(n+1)}} \sqrt{\frac{\dim V_{\alpha^{-j}}}{\dim V_\alpha}} \left(\begin{array}{cc|c} \bar{1}_n & \alpha & \alpha^{-j} \\ 0_{n-1} & \mu & \mu \end{array}\right) P^\beta_{\alpha^{-j}},$$

where $\alpha^{\pm j} = (\alpha_1, \ldots, \alpha_{j-1}, \alpha_j \pm 1, \alpha_{j+1}, \ldots, \alpha_n)$. For $\alpha \in D_K(\mu)$ and $\beta \in D_M(\alpha)$, we set $\mathrm{M}_{\alpha,\beta} = (\alpha, \beta, \beta[2], \ldots, \beta[n-1])$ and $\tilde{\mathrm{M}}_{\alpha,\beta} = (\alpha, \beta, \beta(n-2), \ldots, \beta(1))$. Then for $1 \leq i \leq n-1$, we have from Lemma 13.1 that

(13.4)

$$F_i \otimes v(\tilde{\mathrm{M}}_{\alpha,\beta}) = \sum_{j=1}^{n} \sum_{k=1}^{i} \left(\begin{array}{cc|c} 1_n & \alpha & \alpha^{+j} \\ 1_{n-1} & \beta & \beta^{+k} \end{array}\right) \left(\begin{array}{cc|c} 1_i & \beta(i) & \beta(i)^{+k} \\ 0_{i-1} & \beta(i-1) & \beta(i-1) \end{array}\right) v(\tilde{\mathrm{M}}^{j,k}_{\alpha,\beta}),$$

$$E_i \otimes v(\mathrm{M}_{\alpha,\beta}) = \sum_{j=1}^{n} \sum_{k=n-i}^{n-1} \left(\begin{array}{cc|c} \bar{1}_n & \alpha & \alpha^{-j} \\ 0_{n-1} & \beta & \beta^{-k} \end{array}\right) \left(\begin{array}{cc|c} \bar{1}_i & \beta[n-i] & \beta[n-i]^{-k+n-i-1} \\ 0_{i-1} & \beta[n-i+1] & \beta[n-i+1] \end{array}\right) v(\mathrm{M}^{j,k}_{\alpha,\beta}),$$

where

$$\mathrm{M}^{j,k}_{\alpha,\beta} = (\alpha^{-j}, \beta^{-k}, \beta[2]^{-k+1}, \ldots, \beta[n-i]^{-k+n-i-1}, \beta[n-i+1], \ldots, \beta[n-1]),$$
$$\tilde{\mathrm{M}}^{j,k}_{\alpha,\beta} = (\alpha^{+j}, \beta^{+k}, \beta(n-2)^{-k}, \ldots, \beta(i)^{-k}, \beta(i-1), \ldots, \beta(1)).$$

Remark. For the explicit forms of the scalar factors appeared in (13.3) and (13.4), see [27, p. 385].

Substituting (13.3) and (13.4) into the expressions in Theorem 8.4, we obtain

(13.5)

$$\pi_{\sigma_\mu,\nu}(E_{i,n+1})v(\tilde{\mathbf{M}}_{\alpha,\beta}) = \sum_{j=1}^{n}\sum_{k=1}^{i}\sqrt{\frac{\dim V_{\alpha^{+j}}}{\dim V_\alpha}}k_j(\alpha)\begin{pmatrix} 1_n & \alpha & \alpha^{+j} \\ 0_{n-1} & \mu & \mu \end{pmatrix}\begin{pmatrix} 1_n & \alpha & \alpha^{+j} \\ 1_{n-1} & \beta & \beta^{+k} \end{pmatrix}$$
$$\times \begin{pmatrix} 1_i & \beta(i) & \beta(i)^{+k} \\ 0_{i-1} & \beta(i-1) & \beta(i-1) \end{pmatrix} v(\tilde{\mathbf{M}}^{j,k}_{\alpha,\beta}),$$

$$\pi_{\sigma_\mu,\nu}(E_{n+1,i})v(\mathbf{M}_{\alpha,\beta}) = \sum_{j=1}^{n}\sum_{k=n-i}^{n-1}\sqrt{\frac{\dim V_{\alpha^{-j}}}{\dim V_\alpha}}h_j(\alpha)\begin{pmatrix} \bar{1}_n & \alpha & \alpha^{-j} \\ 0_{n-1} & \mu & \mu \end{pmatrix}\begin{pmatrix} \bar{1}_n & \alpha & \alpha^{-j} \\ \bar{1}_{n-1} & \beta & \beta^{-k} \end{pmatrix}$$
$$\times \begin{pmatrix} \bar{1}_i & \beta[n-i] & \beta[n-i]^{-k+n-i-1} \\ 0_{i-1} & \beta[n-i+1] & \beta[n-i+1] \end{pmatrix} v(\mathbf{M}^{j,k}_{\alpha,\beta}),$$

where $h_j(\alpha) = (\nu - n - |\sigma_\mu|)/2 + j - \alpha_j$ and $k_j(\alpha) = (\nu + n + |\sigma_\mu|)/2 - j + \alpha_j + 1$. For $1 \le i < j \le n-1$, it follows from (8.12) and (8.13) that

$$\pi_{\sigma_\mu,\nu}(E_{j,i})v(\mathbf{M}_{\alpha,\beta}) = \pi_{\sigma_\mu,\nu}(E_{i,j})v(\tilde{\mathbf{M}}_{\alpha,\beta}) = 0. \tag{13.6}$$

Let ω_1 be the Casimir operator of G_1, that is

(13.7)

$$\omega_1 = \frac{1}{2(n+1)}\sum_{\substack{1\le i\le n+1\\ i\ne n}} E_{i,i}^2 + \frac{1}{n+1}\sum_{1\le i<j\le n-1}(E_{j,i}E_{i,j} + E_{i,j}E_{j,i})$$
$$+ 2\sum_{j=1}^{n-1}(F_jE_j + E_jF_j).$$

We shall first consider the case of holomorphic discrete series. Fix $\mu \in D_M$ and $\nu \in \mathfrak{a}^*$ so that the condition indicated in Theorem 12.6(1) is fulfilled. Then the holomorphic discrete series is realized as $\big(\pi_{\sigma_\mu,\nu}, \mathcal{H}^{\sigma_\mu}_{\nu,+}(n-1)\big)$ with the inner product $\langle\cdot,\cdot\rangle_i$. For simplicity we set $\mu_n = (\nu + n - |\sigma_\mu|)/2$. Let Λ be the Harish-Chandra parameter of the above representation and write $(\pi_\Lambda, V_\Lambda) = \big(\pi_{\sigma_\mu,\nu}, \mathcal{H}^{\sigma_\mu}_{\nu,+}(n-1)\big)$. Let $V_\Lambda(K)$ be the set of the K-finite elements in V_Λ. Then it follows from Theorem 12.6(1) that

(13.8)

$$V_\Lambda(K) = \sum_{\alpha<(\infty,\lambda_m)} V_\alpha = \sum_{\alpha<(\infty,\lambda_m)}\sum_{\beta<\alpha} V_\alpha(\beta) = \sum_{\substack{\ell-\mu_n\in\mathbf{Z}_{\ge0}\\ \beta\in S}}\ \sum_{\substack{\beta<\alpha<(\infty,\lambda_m)\\ |\alpha|-|\beta|=\ell}} V_\alpha(\beta),$$

where $S = \big\{\beta \in D_M : \beta_1 \ge \mu_2, \mu_{j-1} \ge \beta_j \ge \mu_{j+1}, (2 \le j \le n-1)\big\}$. For our convenience, we introduce the following notation:

$$S_m = \big\{\beta \in S : \beta < \lambda_m\big\}, \quad S_c = \left\{\ell \in \frac{1}{n+1}\mathbf{Z} : \ell - \mu_n \in \mathbf{Z}_{\ge0}\right\},$$
$$S_c^+ = \big\{\ell \in S_c : \ell - \mu_1 \in \mathbf{Z}_{\ge0}\big\}, \quad S_c^- = S_c \setminus S_c^+.$$

For $\alpha \in D_K$ and $\beta \in D_M(\alpha)$, let

$$c(\alpha, j, k) = \sqrt{\frac{\dim V_{\alpha^{-j}}}{\dim V_\alpha}} h_j(\alpha) \left(\begin{array}{cc|c} \bar{1}_n & \alpha & \alpha^{-j} \\ 0_{n-1} & \mu & \mu \end{array}\right) \left(\begin{array}{cc|c} \bar{1}_n & \alpha & \alpha^{-j} \\ \bar{1}_{n-1} & \beta & \beta^{-k} \end{array}\right),$$

$$d(k) = \left(\begin{array}{cc|c} \bar{1}_i & \beta[n-i] & \beta[n-i]^{-k+n-i-1} \\ 0_{i-1} & \beta[n-i+1] & \beta[n-i+1] \end{array}\right).$$

For $\beta \in S_m$, let

$$\mathbf{Z}(\beta) = \{k \in \mathbf{Z}_{>0} : 1 \leq k \leq n-1, \beta^{-k} \in S_m\},$$
$$N_\ell(\beta) = \{\alpha \in D_K : \beta < \alpha < (\infty, \lambda_m), |\alpha| - |\beta| = \ell\}, \quad m(\ell, \beta) = \operatorname{Card} N_\ell(\beta),$$
$$V_\ell(\beta) = \sum_{\alpha \in N_\ell(\beta)} V_\alpha(\beta).$$

For $\beta \in S$, let

$$\beta_f = (\max(\beta_2, \mu_2), \ldots, \max(\beta_{n-1}, \mu_{n-1}), \mu_n),$$
$$\beta_\ell = (\min(\beta_1, \mu_1), \ldots, \min(\beta_{n-1}, \mu_{n-1})).$$

Taking into account $\beta_\ell, \beta_f \in S_m, \beta_f < (\beta_\ell, -\infty)$ and $\beta_\ell < (\beta, -\infty)$, we see that $N_\ell(\beta)$ can be written as

(13.9)
$$N_\ell(\beta) = \{\alpha \in D_K : \alpha[2] \in S_m, \beta_f < (\alpha[2], -\infty), \alpha[2] < (\beta_\ell, -\infty), \\ |\alpha[2]| \leq |\beta| + \ell - \max(\beta_1, \mu_1)\}.$$

Taking into account $(\beta_\ell)_\ell = \beta_\ell$ and $(\beta_\ell)_f = \lambda_m[2]$, we can easily see that $m(\ell, \beta) \leq m(\ell, \beta_\ell)$. For this reason, we write (13.8) as the following form:

(13.10)
$$V_\Lambda(K) = \sum_{\substack{\ell \in S_c^+ \\ \beta \in S_m}} \sum_{\alpha \in N_\ell(\beta)} V_\alpha(\beta) + \sum_{\substack{\ell \in S_c^+ \\ \beta \notin S_m}} \sum_{\alpha \in N_\ell(\beta)} V_\alpha(\beta) \\ + \sum_{\substack{\ell \in S_c^- \\ \beta \in S_m}} \sum_{\alpha \in N_\ell(\beta)} V_\alpha(\beta) + \sum_{\substack{\ell \in S_c^- \\ \beta \notin S_m}} \sum_{\alpha \in N_\ell(\beta)} V_\alpha(\beta).$$

We shall here get the expression of the eigenvector of ω_1. Assume that an eigenvector v is represented as $v = \sum_{\alpha \in N_\ell(\beta)} c_\alpha v(\mathrm{M}_{\alpha,\beta})$, $(c_\alpha \in \mathbf{C})$. Then it follows from (13.6) that $\pi_\Lambda(E_{j,i})v = 0$ for $1 \leq i < j \leq n-1$. Thus for v being an eigenvector, it suffices to determine c_α such that $\pi_\Lambda(E_{n+1,i})v = 0$ $(1 \leq i \leq n-1)$ are satisfied. It follows from (13.5) that

(13.11)
$$\pi_\Lambda(E_{n+1,i})v = \sum_{\alpha \in N_\ell(\beta)} \left\{ c_\alpha \sum_{j=1}^{n} \sum_{k=n-i}^{n-1} c(\alpha, j, k) d(k) v(\mathrm{M}_{\alpha,\beta}^{j,k}) \right\}.$$

Using the fact that $h_j(\alpha) < 0$, $(1 \leq j < n)$, $h_n(\alpha) \leq 0$ and $h_n(\alpha) = 0$ iff $\alpha_n = \mu_n$, we see that $c(\alpha, j, k) \neq 0$ iff $\beta^{-k} \in S_m$ and $\alpha^{-j} \in N_\ell(\beta^{-k})$. Letting $\mathbf{Z}(\beta, i) = \mathbf{Z}(\beta) \cap \{n-i, \cdots, n-1\}$ and rewriting α^{-j} as α, we have from (13.11) that

$$\pi_\Lambda(E_{n+1,i})v = \sum_{k \in \mathbf{Z}(\beta,i)} \sum_{\alpha \in N_\ell(\beta^{-k})} \sum_{\alpha^{+j} \in N_\ell(\beta)} c_{\alpha^{+j}} c(\alpha^{+j}, j, k) d(k) v(\mathrm{M}^{0,k}_{\alpha,\beta}). \tag{13.12}$$

Therefore $\pi_\Lambda(E_{n+1,i})v = 0$ implies that we have for $k \in \mathbf{Z}(\beta, i)$ and $\alpha \in N_\ell(\beta^{-k})$ that

$$\sum_{\alpha^{+j} \in N_\ell(\beta)} c_{\alpha^{+j}} c(\alpha^{+j}, j, k) = 0. \tag{13.13}$$

From this, we see that it suffices to determine c_α such that $\pi_\Lambda(E_{n+1,n-1})v = 0$ is satisfied. To determine c_α, we use similar arguments as in [31, Theorem 3.1].

Lemma 13.2. *Let $\ell \in S_c$ and $\beta \in S_m$.*

(1) *If $\ell \in S_c^+$, then there exists $v = \sum_{\alpha \in N_\ell(\beta)} c_\alpha v(\mathrm{M}_{\alpha,\beta}) \in V_\ell(\beta)$ such that $\pi_\Lambda(E_{n+1,n-1})v = 0$. Moreover, such a v is unique up to a scalar multiple.*

(2) *If $\ell \in S_c^-$ and $|\beta| \geq |\lambda_m| - \ell$, then there exists $v = \sum_{\alpha \in N_\ell(\beta)} c_\alpha v(\mathrm{M}_{\alpha,\beta}) \in V_\ell(\beta)$ such that $\pi_\Lambda(E_{n+1,n-1})v = 0$. Moreover, such a v is unique up to a scalar multiple.*

Proof. (1) We obtain from (13.9) that

$$N_\ell(\beta) = \{\alpha \in D_K : \alpha[2] \in S_m, \beta < \alpha\}. \tag{13.14}$$

Then setting $N_\ell(\beta, p) = \{\alpha \in N_\ell(\beta) : \alpha_1 = p\}$, we have

$$N_\ell(\beta) = \cup_{\mu_1 + \ell + |\beta| - |\lambda_m| \leq p \leq \ell} N_\ell(\beta, p).$$

We first remark the following fact. For $\lambda \in N_\ell(\beta, p)$, we put

$$\mathbf{Z}(\lambda, \beta) = \{k \in \mathbf{Z}_{>0} : k \in \mathbf{Z}(\beta), \lambda^{-1} \in N_\ell(\beta^{-k})\}.$$

Then setting $\alpha = \lambda^{-1}$, we have from (13.13) that

$$c_\lambda c(\lambda, 1, k) + \sum_{j \in \mathbf{Z}(\lambda,\beta)} c_{\alpha^{+j+1}} c(\alpha^{+j+1}, j+1, k) = 0, \ (k \in \mathbf{Z}(\lambda, \beta)). \tag{13.15}$$

By the orthogonality relations of the Clebsch–Gordan coefficients, it is easy to check that $c(\alpha^{+j+1}, j+1, k)$ are linearly independent and thus we can get $c_{\alpha^{+j+1}}$, $(j \in \mathbf{Z}(\lambda, \beta))$ from the above simultaneous equations.

We can find the constants c_α, $(\alpha \in N_\ell(\beta))$ by induction on α_1. Let $\alpha_f = (\mu_1 + \ell + |\beta| - |\lambda_m|, \mu_2, \ldots, \mu_n)$. We first choose c_{α_f} as an arbitrary nonzero real number. Suppose that c_α are determined for all $\alpha \in N_\ell(\beta, p)$. For $\alpha \in N_\ell(\beta, p-1)$, we pick $k \in \mathbf{Z}_{>0}$ so that $\alpha[2]^{-k} \in S_m$. Then setting $\lambda = (\alpha^{+1})^{-k} \in N_\ell(\beta, p)$, we can get c_α from the simultaneous equations (13.15). By the orthogonality relations

of the Clebsch–Gordan coefficients, it is easy to check that c_α is independent of the choice of k.
(2) Because $\beta \in S_m$ and $\ell \in S_c^-$, we have from (13.9) that

$$N_\ell(\beta) = \{\alpha \in D_K : \alpha[2] \in S_m, \beta < \alpha, |\alpha[2]| \le |\beta| + \ell - \mu_1\}. \tag{13.16}$$

Thus $N_\ell(\beta) = \emptyset$ if $|\beta| < |\lambda_m| - \ell$. By a similar way as in (1), we can also determine the constants c_α satisfying $\mu_1 + \ell + |\beta| - |\lambda_m| \le \alpha_1 \le \mu_1$. □

For $\ell \in S_c$ and $\beta \in S_m$, we choose v as in Lemma 13.2. We denote by $\mathcal{V}(\ell, \beta)(K)$ the $\pi_\Lambda(K_1)$-invariant subspace of $V_\Lambda(K)$ containing $\{\pi_\Lambda(F_{n-1})^j v : j \in \mathbf{Z}_{\ge 0}\}$. Then $\pi_\Lambda(E_{n-1})v = 0$ implies $\mathcal{V}(\ell, \beta)(K) = \sum_{\beta' < (\beta, -\infty)} V_{(-(n+1)\ell, \beta')}$. Taking into account $m(\ell, \beta) \le m(\ell, \beta_\ell)$ for $\beta \notin S_m$, we obtain from (13.10) that

$$V_\Lambda(K) = \sum_{\substack{\ell \in S_c^+ \\ \beta \in S_m}} \mathcal{V}(\ell, \beta)(K) + \sum_{\substack{\ell \in S_c^- \\ \beta \in S_m \\ |\beta| \ge |\lambda_m| - \ell}} \mathcal{V}(\ell, \beta)(K). \tag{13.17}$$

We shall next consider the case of antiholomorphic discrete series. Fix $\mu \in D_M$ and $\nu \in \mathfrak{a}^*$ so that the condition indicated in Theorem 12.6(2) is fulfilled. Then the antiholomorphic discrete series is realized as $(\pi_{\sigma_\mu, \nu}, \mathcal{H}^{\sigma_\mu}_{\nu,-}(0))$ with the inner product $\langle \cdot, \cdot \rangle_i$. For simplicity we set $\mu_0 = -(\nu + n + |\sigma_\mu|)/2$. In this case, if $\alpha \in \Gamma(\Lambda)$, then $k_j(\alpha) < 0$, $(1 < j \le n)$, $h_j(\alpha) > 0$, $(1 \le j \le n)$ and $k_1(\alpha) \le 0$. Moreover $k_1(\alpha) = 0$ iff $\alpha_1 = \mu_0$. Let Λ be the Harish-Chandra parameter of the above representation and write $(\pi_\Lambda, V_\Lambda) = (\pi_{\sigma_\mu, \nu}, \mathcal{H}^{\sigma_\mu}_{\nu,+}(n-1))$. We have from Theorem 12.6(2) that

$$V_\Lambda(K) = \sum_{\alpha < (\lambda_m, -\infty)} V_\alpha = \sum_{\alpha < (\lambda_m, -\infty)} \sum_{\beta < \alpha} V_\alpha(\beta) = \sum_{\substack{\ell \in \tilde{S}_c \\ \beta \in \tilde{S}}} \sum_{\substack{\beta < \alpha < (\lambda_m, -\infty) \\ |\alpha| - |\beta| = \ell}} V_\alpha(\beta), \tag{13.18}$$

where

$$\tilde{S} = \{\beta \in D_M : \mu_{j-1} \ge \beta_j \ge \mu_{j+1},\ (1 \le j \le n-2), \mu_{n-2} \ge \beta_{n-1}\},$$
$$\tilde{S}_c = \left\{\ell \in \frac{1}{n+1}\mathbf{Z}_{\ge 0} : \mu_0 - \ell \in \mathbf{Z}_{\ge 0}\right\}.$$

For our convenience, we introduce the following notation:

$$\tilde{S}_m = \{\beta \in \tilde{S} : \beta < \lambda_m\},$$
$$\tilde{S}_c^+ = \{\ell \in \tilde{S}_c : \ell - \mu_1 \in \mathbf{Z}_{\ge 0}\}, \quad \tilde{S}_c^- = \tilde{S}_c \setminus \tilde{S}_c^+.$$

For $\beta \in \tilde{S}_m$, let

$$\tilde{\mathbf{Z}}(\beta) = \{k \in \mathbf{Z}_{>0} : 1 \le k \le n-1, \beta^{-k} \in \tilde{S}_m\},$$
$$\tilde{N}_\ell(\beta) = \{\alpha \in D_K : \beta < \alpha < (\lambda_m, -\infty), |\alpha| - |\beta| = \ell\}, \quad m(\ell, \beta) = \operatorname{Card} \tilde{N}_\ell(\beta),$$
$$W_\ell(\beta) = \sum_{\alpha \in \tilde{N}_\ell(\beta)} V_\alpha(\beta).$$

For $\beta \in \tilde{S}$, let

$$\beta_f = (\max(\beta_1, \mu_1), \ldots, \max(\beta_{n-1}, \mu_{n-1})),$$
$$\beta_\ell = (\mu_0, \min(\beta_1, \mu_1), \ldots, \min(\beta_{n-2}, \mu_{n-2})).$$

By the same reason as in the case of holomorphic discrete series, we have

(13.19)
$$\tilde{N}_\ell(\beta) = \big\{\alpha \in D_K : \alpha(n-2) \in \tilde{S}_m, \beta_f < (\alpha(n-2), -\infty), \alpha(n-2) < (\beta_\ell, -\infty), \\ |\alpha(n-2)| \le |\beta| + \ell - \min(\beta_{n-1}, \mu_{n-1})\big\}.$$

and thus $m(\ell, \beta) \le m(\ell, \beta_\ell)$. For this reason, we write (13.18) as follows:

$$V_\Lambda(K) = \sum_{\substack{\ell \in \tilde{S}_c^+ \\ \beta \in \tilde{S}_m}} W_\ell(\beta) + \sum_{\substack{\ell \in \tilde{S}_c^+ \\ \beta \notin \tilde{S}_m}} W_\ell(\beta) + \sum_{\substack{\ell \in \tilde{S}_c^- \\ \beta \in \tilde{S}_m}} W_\ell(\beta) + \sum_{\substack{\ell \in \tilde{S}_c^- \\ \beta \notin \tilde{S}_m}} W_\ell(\beta). \tag{13.20}$$

Assume that an eigenvector v is represented as $v = \sum_{\alpha \in M_\ell(\beta)} c_\alpha v(\tilde{\mathbf{M}}_{\alpha,\beta})$, $(c_\alpha \in \mathbf{C})$. Then by the similar arguments as in the case of holomorphic discrete series, for v being an eigenvector, it suffices to determine c_α such that $\pi_\Lambda(E_{n-1,n+1})v = 0$.

Lemma 13.3. *Let $\ell \in \tilde{S}_c$ and $\beta \in \tilde{S}_m$.*

(1) *If $\ell \in \tilde{S}_c^+$, then there exists $v = \sum_{\alpha \in \tilde{N}_\ell(\beta)} c_\alpha v(\tilde{\mathbf{M}}_{\alpha,\beta}) \in W_\ell(\beta)$ such that $\pi_\Lambda(E_{n-1,n+1})v = 0$. Moreover such a v is unique up to a scalar multiple.*

(2) *If $\ell \in \tilde{S}_c^-$ and $|\beta| \ge |\lambda_m| - \ell$, then there exists $v = \sum_{\alpha \in \tilde{N}_\ell(\beta)} c_\alpha v(\tilde{\mathbf{M}}_{\alpha,\beta}) \in W_\ell(\beta)$ such that $\pi_\Lambda(E_{n-1,n+1})v = 0$. Moreover such a v is unique up to a scalar multiple.*

For $\ell \in \tilde{S}_c$ and $\beta \in \tilde{S}_m$, we choose v as in Lemma 13.3. We denote by $\mathcal{W}(\ell, \beta)(K)$ the $\pi_\Lambda(K_1)$-invariant subspace of $V_\Lambda(K)$ containing $\{\pi_\Lambda(E_{n-1})^j v : j \in \mathbf{Z}_{\ge 0}\}$. Then $\pi_\Lambda(F_{n-1})v = 0$ implies $\mathcal{W}(\ell, \beta)(K) = \sum_{\beta < (\beta', -\infty)} V_{(-(n+1)\ell, \beta')}$. Therefore we obtain from (13.19) that

$$V_\Lambda(K) = \sum_{\substack{\ell \in \tilde{S}_c^+ \\ \beta \in \tilde{S}_m}} \mathcal{W}(\ell, \beta)(K) + \sum_{\substack{\ell \in \tilde{S}_m^- \\ \beta \in \tilde{S}_m \\ |\beta| \ge |\lambda_m| - \ell}} \mathcal{W}(\ell, \beta)(K). \tag{13.21}$$

Summarizing these, we obtain the following theorem.

Theorem 13.4. *Let $\mathcal{V}(\ell, \beta)$ and $\mathcal{W}(\ell, \beta)$ be the completion of $\mathcal{V}(\ell, \beta)(K)$ and $\mathcal{W}(\ell, \beta)(K)$ relative to $\langle \cdot, \cdot \rangle_i$ respectively.*

(1) *The holomorphic discrete series (π_Λ, V_Λ) is decomposed with no multiplicity as follows:*

$$V_\Lambda = \sum_{\substack{\ell \in S_c^+ \\ \beta \in S_m}} \mathcal{V}(\ell, \beta) + \sum_{\substack{\ell \in S_c^- \\ \beta \in S_m \\ |\beta| \ge |\lambda_m| - \ell}} \mathcal{V}(\ell, \beta).$$

The Blattner parameter of $\mathcal{V}(\ell,\beta)$ *is* $(-(n+1)\ell,\beta)$.

(2) *The antiholomorphic discrete series* (π_Λ, V_Λ) *is decomposed with no multiplicity as follows:*

$$V_\Lambda = \sum_{\substack{\ell\in\tilde{S}_c^+ \\ \beta\in\tilde{S}_m}} \mathcal{W}(\ell,\beta) + \sum_{\substack{\ell\in\tilde{S}_c^- \\ \beta\in\tilde{S}_m \\ |\beta|\geq|\lambda_m|-\ell}} \mathcal{W}(\ell,\beta).$$

The Blattner parameter of $\mathcal{W}(\ell,\beta)$ *is* $(-(n+1)\ell,\beta)$.

References

1. J. G. Arthur, *Harmonic Analysis of the Schwartz space on reductive Lie group* I,II (preprint).
2. T. Branson, G. Ólafsson and B. Ørsted, *Spectrum Generating Operators and Intertwining Operators Induced from a Maximal Parabolic Subgroup*, J. Funct. Anal. **135** (1996), 163–205.
3. O. Campoli, *The Complex Fourier Transform for Rank One Semisimple Lie Groups*, Ph. D. thesis, Rutgers University (1977).
4. L. Cohn, *Analytic Theory of the Harish-Chandra C-function*, Lecture Notes in Mathematics 429, Springer-Verlag, Berlin-Heidelberg-New York, 1974.
5. M. Eguchi, *Asymptotic Expansions of Eisenstein Integrals and Fourier Transform on Symmetric Spaces*, J. Funct. Anal. **34** (1979), 167–216.
6. M. Eguchi and S. Koizumi, *Determination of the Harish-Chandra C-function for* $SU(n,1)$ *and its application to the construction of the composition series*, Proc. Japan Acad. Ser. A **72** (1996), 121–125.
7. M. Eguchi and S. Koizumi, *The Explicit Formula for the Harish-Chandra C-function of* $SU(n,1)$ *for Arbitrary Irreducible Representations of* K *which Contain One Dimensional* M*-types*, Proc. Japan Acad. Ser. A **70** (1994), 256–259.
8. M. Eguchi, S. Koizumi, M. Miyamoto and R. Wada, *The explicit expression of the Harish-Chandra C-function of* $SU(n,1)$ *associated with the fundamental representations of* K, Hiroshima Math. Jour. **26** (1996), 151–164.
9. M. Eguchi, M. Hashizume and S. Koizumi, *The Gangolli estimates for the coefficients of the Harish-Chandra expansions of the Eisenstein integrals on real reductive Lie groups*, Hiroshima Math. Jour. **17** (1987), 457–469.
10. M. Eguchi and M. Wakayama, *An elementary proof of the Trombi theorem for the Fourier transform of* $\mathcal{C}^p(G:F)$, Hiroshima Math. Jour. **7** (1987), 471–487.
11. R. Gangolli, On the Plancherel formula and the Paley–Wiener theorem for spherical functions on semisimple Lie groups, Ann. of Math. **93** (1971), 150–165.
12. Harish-Chandra, *Harmonic analysis on real reductive Lie groups* I, J. Funct. Anal. **19** (1975), 104–204.
13. ———, *Harmonic analysis on real reductive Lie groups* II, Invent. Math. **36** (1976), 1–55.
14. ———, *Harmonic analysis on real reductive Lie groups* III, Ann. of Math. **104** (1976), 117–201.
15. S. Helgason, *The surjectivity of invariant differential operators on symmetric spaces* I, Ann. of Math. **98** (1973), 451–479.
16. A. W. Knapp, *Representation Theory of Semisimple Groups, An Overview Based on Examples*, Princeton University Press, Princeton, New Jersey, 1986.
17. A. W. Knapp and E. M. Stein, *Intertwining operators for semisimple groups*, Ann. of Math. **93** (1971), 489–578.
18. Y. Konno, *Multiplicity formulas for discrete series of* $Spin(1,2m)$ *and* $SU(1,n)$, Osaka J. Math. **18** (1981), 583–619.
19. K. Kraljević, *Representations of the universal covering group of the group* $SU(n,1)$, Glasnik. Mat. Ser. III **8(28)** (1973), 23–72.
20. ———, *On representations of the group* $SU(n,1)$, Trans. Amer. Math. Soc. **221** (1976), 433–448.

21. S. Martens, *The characters of the holomorphic discrete series*, Proc. Nat. Acad. Sci. U.S.A. **72**, no. 9, 3275–3276.
22. J. Sekiguchi, *On Harish-Chandra's C-function*, Seminar Reports of Unitary Representation (1981), no. 1, 68–114. (Japanese)
23. E. Thieleker, *On the quasi-simple irreducible representations of the Lorentz groups*, Trans. Amer. Math. Soc. **179** (1973), 465–505.
24. ———, *The unitary representations of the generalized Lorentz groups*, Trans. Amer. Math. Soc. **199** (1974), 327–367.
25. P. C. Trombi, *Harmonic Analysis of* $C^p(G : F)$, J. Funct. Anal. **40** (1981), 84–125.
26. P. C. Trombi and V. S. Varadarajan, *Spherical transforms on semisimple Lie groups*, Ann. of Math. **94** (1971).
27. N. Ja. Vilenkin and A. U. Klimyk, *Representation of Lie Groups and Special Functions*, Volume 3, Kluwer Academic Publishers, 1993.
28. D. A. Vogan and N. R. Wallach, *Intertwining Operators for Reductive Groups*, Adv. in Math. **82** (1990), 203–243.
29. N. R. Wallach, *Harmonic Analysis on Homogeneous Spaces*, Marcel Dekker, New York, 1973.
30. G. Warner, *Harmonic Analysis on Semisimple Lie Groups* I,II, Springer–Verlag, Berlin–Heidelberg–New York, 1972.
31. J. Xie, *Restriction of Discrete Series of* $SU(2,1)$ *to* $S(U(1) \times U(1,1))$, J. Funct. Anal. **122** (1994), 478–518.

公 表 論 文

The Gangolli estimates for the coefficients of the Harish-Chandra expansions of the Eisenstein integrals on real reductive Lie groups
(簡約可能リー群のアイゼンシュタイン積分のハリッシュ・チャンドラ展開の係数のガンゴリー評価)
(Masaaki Eguchi, Shin Koizumi and Michihiko Hashizume)
Hiroshima Mathematical Journal, Vol. 17, No. 3, (1987), pp. 457–469

Determination of the Harish-Chandra C-function for $SU(n,1)$ and its application to the construction of the composition series
($SU(n,1)$ のハリッシュ・チャンドラのC関数の決定と組成列の構成への応用)
(Masaaki Eguchi and Shin Koizumi)
Proceedings of the Japan Academy, Ser. A, Vol. 72, No. 6, (1996), pp. 121–125

参 考 論 文

The Explicit Formula for the Harish-Chandra C-function of $SU(n,1)$ for Arbitrary Irreducible Representations of K which Contain One Dimensional M-types
（1次元 Mタイプを含む Kの任意の既約表現に関する $SU(n,1)$ のハリッシュ・チャンドラのC関数の具体的な公式）
(Masaaki Eguchi and Shin Koizumi)

The explicit expression of the Harish-Chandra C-function of $SU(n,1)$ associated with the fundamental representations of K
（基本表現に付随する $SU(n,1)$ のハリッシュ・チャンドラのC関数の具体的な表示）
(Masaaki Eguchi, Shin Koizumi, Mari Miyamoto, Ryoko Wada)