Fuzzy Games in Single and Multiple Objective Systems

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Fuzzy Games
in
Single and Multiple Objective Systems

単一および多目的システムにおける
ファジィゲーム

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Abstract

In this thesis, game theory has been applied for resolution of conflict in competitive systems. We must recognize the existence of ambiguity in decision makers' judgements as well as the imprecision that exists in information in such systems; moreover, decision makers need to be able to accommodate multiple objectives in the solution of the conflict problems. It follows that new solution concepts which take the ambiguity and the multiplicity of the objectives into consideration should be introduced for implementation of game theoretic approach. We present several game representations for the resolution of conflict in competitive systems and demonstrate our computational methods for the proposed solutions. For noncooperative games, two-person games are dealt with and max-min solutions and equilibrium solutions are considered. For cooperative games, lexicographical solutions and a solution based on fuzzy decision rule are examined. The computational methods are based mainly on linear programming techniques, and therefore they have useful practical applications for decision making problems regarding public conflict.

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CHAPTER 1

INTRODUCTION

1.1 Introduction and Historical Remarks

When members of an organization consist of decision makers with conflicting interests, a competition or a partial cooperation among them should be considered as the essential problem. We consider such a situation and call it a competitive system. Game theory has been used as a powerful analytical tool for competitive systems (e.g. von Neumann and Morgenstern 1944, Harsanyi 1977, Owen 1982, Harsanyi and Selten 1988, and Rasmusen 1989). When game theoretic approach is used as a resolution method, solution concepts of games and their computation methods are indispensable because of the choice of strategies they offer decision makers in the conflict problems. Ambiguity in the decision makers' judgements as well as uncertainty and imprecision of information must also be taken into consideration in an analysis for a competitive system. The results of the analysis are not always effective for resolution of a conflict problem in the competitive system when the parameters of a mathematical model for the conflict problem are determined without considering the uncertainty and imprecision likely to occur in the competitive system.

Sensitivity analysis, effective for analyzing problems which have variations of a few parameters, has been used as a conventional method for analysis of such problems. In fuzzy environments, however, it is difficult to analyze problems with sensitivity analysis because variations of many parameters must be considered simultaneously and the analysis becomes too complicated. An analytical device which can effectively cope with such ambiguity, uncertainty and imprecision is a method using fuzzy set theory (Inuiguchi 1991).

With the development of fuzzy set theory (e.g. Zadeh 1965, Dubois and Parade 1980, and Zimmerman 1991), ambiguous events which are not probability events can be represented as fuzzy sets so that, as a result, ambiguity in decision makers' judgements and uncertainty as well as imprecision of information in competitive systems can be treated explicitly in optimization problems with a single decision maker. Studies on game theory which deal with such ambiguity, uncertainty and imprecision, however, are few.

Research on game theory in fuzzy environments has been accumulating since the mid-'70s. In noncooperative fuzzy games, ambiguity for a player's choice of a strategy, vagueness of preference for a payoff and imprecision of payoff representation have been represented as fuzzy sets. In cooperative fuzzy games, games with fuzzy coalitions, which means that players are admitted to participating partially in a coalition, and games with fuzzy payoffs have already been considered.

First, we will review research on noncooperative fuzzy games. Butnariu (1978) was the first to study two-person noncooperative games in fuzzy environments, claiming that all of one player's strategies are not equally possible and the grade of membership of a strategy to the set of his feasible strategies is dependent on the behavior of the opponent. He next considered the case where the set of strategies of one player could be seen as a fuzzy set. Subsequently, he examined n-person noncooperative games in fuzzy environments and presented a concept of equilibrium solutions for such games (1980). Buckley (1984) analyzed behavior of decision makers using two-person fuzzy games similar to Butnariu's (1978). The fuzzy games which he considered involve uncertainty of strategies and multiple fuzzy goals for payoffs, but are essentially decision making problems with a single player because the other player's strategies are given as a prior possibility distribution. Billot (1992) defined the individual relations of preference by a procedure different from Butnariu's preference and has proposed equilibrium solutions of n-person noncooperative games in his recent book.

Ponsard (1986, 1987) studied n-person noncooperative games from a different point of view that supposed imprecise preference for payoffs and occurrences of payoffs that are uncertain, and then generalized the Nash equilibrium concept to the matter of such fuzzy games.

When two-person zero-sum matrix games are applied for analysis of conflict problems, it is difficult to assess matrix entries as a result of the imprecision of information from competitive systems. Campos (1989) examined min-max problems of two-person zero-sum fuzzy matrix games, matrix entries of which were represented as fuzzy numbers, and employed the fuzzy linear programming methods in order to compute the min-max solutions, and recently, Nishizaki and Sakawa (1992b) examined two-person zero-sum matrix games with multiple payoffs represented as fuzzy numbers, i.e., two-person zero-sum multiobjective fuzzy matrix games. Moreover, they have presented a couple of related studies (Sakawa and Nishizaki 1992b, Sakawa and Nishizaki 1993, Nishizaki and Sakawa 1993).

Next, we will move on to research on cooperative fuzzy games. The essence of cooperative games is the formation of coalitions. Therefore, cooperative games are discussed in the n-person case and usually studied, not in the normal form, but in the characteristic function form. Research on cooperative fuzzy games began with considering fuzzy coalitions. Aubin and Butnariu have been studying cooperative fuzzy games independently from about the same time. Aubin investigated the core and the Shapley value (Shapley 1953) for n-person cooperative games with fuzzy coalitions in a book (1979) after he had published some articles in French (1974a, 1974b). Butnariu (1978) has also done some similar work in extending the concept of coalitions in n-person cooperative games and considering the core and the Shapley value (1980); and more recently, he has examined fuzzy games with an infinite number of players (1987). To treat the concepts of the core and the Shapley value in a unified way, Aubin (1984) defined the generalized gradient, which can be regarded as the marginal gains that the players receive when they join the coalition of all players.

Lexicographical solutions such as the nucleolus are considered to be as important as the core and the Shapley value. Sakawa and Nishizaki (1984) first considered such a lexicographical solution in n-person cooperative games with fuzzy coalitions, in which they introduced the concept of a player's excess and proposed a new lexicographical solution for n-person cooperative games with fuzzy coalitions.

In order to take into consideration ambiguity of decision makers' judgements along with uncertainty and imprecision of information of a competitive system in cooperative games as well as in noncooperative games, Nishizaki and Sakawa (1992a) introduced a fuzzy goal for a coalition payoff, which refers to the sum of payoffs of players participating in the coalition, instead of a coalition value, and defined the solution concept based on fuzzy decision rule by Bellman and Zadeh (1970).

The final important factor which should be taken into consideration when competitive systems are analyzed is multiplicity of objectives. Games with a multiplicity of objectives have thus far been studied as games with multiple payoffs and min-max solutions, and equilibrium solutions of the games have been defined through a vector optimization criterion such as Pareto optimality. Several methods for computing such solutions have been developed specifically for two-person multiobjective matrix games.

Studies of games dealing with a multiplicity of objectives date back to the mid-'60s but the few publications examining such games have been mainly limited to games in normal form. Finally, this brings us to a review of the research on multiobjective games.

The first researcher to give attention to multiobjective games was Blackwell, who examined properties of the min-max problems for two-person zero-sum multiobjective matrix games (1956). Shapley (1959) presented the definition of equilibrium solutions using the concept of Pareto optimality and weak Pareto optimality in two-person zero-sum multiobjective matrix games and proved the existence of the solutions by finding the correspondence between the multiobjective game and a single-objective game aggregated by weighting coefficients. Contini, Olivetti and Milano (1966) studied two-person zero-sum multiobjective matrix games where one of the two players was Nature. They considered a single player's decision making problem in terms of the expected payoff maximization and the joint probability maximization whereby Nature would choose a strategy given as a prior probability density. Zeleny (1975) analyzed the min-max values of two-person zero-sum multiobjective matrix games by aggregating multiple payoffs to a single payoff by using parametrically varied weighting coefficients. Cook (1976) introduced a goal for each of the objectives in two-person zero-sum multiobjective matrix games and considered the min-max problems by using the goal programming method.

So far, all of the above mentioned studies are on two-person zero-sum multiobjective matrix games. Bergstresser and Yu (1977) first considered n-person cooperative multiobjective games as a generalization of conventional n-person cooperative games in characteristic function form when they introduced the concept of the multiobjective core and explored its properties. It is regrettable that studies on n-person cooperative multiobjective games in characteristic function form are hardly carried out so far except in this study in spite of the importance of the topic.

Since the early 1980's, n-person multiobjective games in normal form and two-person non-zero-sum multiobjective bimatrix games have also been developed. For the normal form, Nieuwenhuis (1983) presented a generalization of the concepts of min-max, max-min and saddle points for vector valued functions using the concept of Pareto optimality. Recently, the following three papers presented unique investigations in n-person multiobjective games in normal form. Wierzbicki (1990) defined equilibrium solutions based on several concepts of vector optimality such as Pareto optimality, which were defined by order relations in terms of preference cones, in n-person multiobjective games with vector-valued nonlinear payoff functions. Moreover, he analyzed the relation between equilibrium solutions of multiobjective games and equilibrium solutions of the proxy single-objective game so that payoffs would correspond to the scalarizing function values. Charnes, Huang, Rousseau and Wei (1990) considered n-person multiobjective games with cross-constrained strategy sets, which are for more general expressions of games, in normal form and examined equilibrium solutions based on nondominated efficiency. Zhao (1991) incorporated a partition of players in n-person multiobjective games in normal form and generalized equilibrium problems by considering them among coalitions derived from the partition.

In contrast, studies on two-person non-zero-sum multiobjective bimatrix games have presented practical methods for computing solutions. Corley (1985) showed the method for computing equilibrium solutions for two-person non-zero-sum multiobjective matrix games, i.e., two-person multiobjective bimatrix games, by computing equilibrium solutions for single-objective games aggregated by weighting coefficients. The approach adopted by Borm, Tiji and Aarssen (1988) was more or less the same as the one Corley adopted, but they have given the parametric analysis for numerical examples of multiobjective 2×2 bimatrix games. Ghose and Parsad (1989) introduced a new concept, which is called security levels, in two-person zero-sum multiobjective games and proposed a solution concept incorporating not only the concept of Pareto optimality but also the concept of security levels. The concept of security levels is inherent in the definition of min-max points in two-person single-objective games and can also be understood to be one of the desirable properties of solutions for multiobjective games.

As we mentioned in the review of fuzzy games, two-person zero-sum multiobjective matrix games in fuzzy environments were examined by Nishizaki and Sakawa (1992b). They introduced fuzzy goals and considered the min-max problems from a viewpoint of maximization of the degree of minimal goal attainment. So far, we have reviewed both fuzzy games and multiobjective games and have found only a couple of studies on games in fuzzy and multiobjective environments, which are Buckley's and Nishizaki and Sakawa's studies. Buckley (1984), however, considered the game as a single player's decision making problem, so Nishizaki and Sakawa (1992b) is the only study on games in fuzzy and multiobjective environments in the strict sense of the word.

1.2 Outline of the Thesis

In the previous section, we have mentioned that analyses using fuzzy set theory and/or multiobjective optimization are effective when techniques of game theory are applied to a resolution method of conflict problems in competitive systems. Studies on fuzzy games, multiobjective games or fuzzy multiobjective games have never been fully researched.

In this thesis, we intend to apply game theory to resolve a conflict problem in a competitive system. Since solution concepts of games and their computation methods are indispensable for resolution of a conflict problem, the main aim of this thesis is to propose solution concepts and their computation methods for games in fuzzy and/or multiobjective environments. Noncooperative games and cooperative games have been developed nearly independently, however, we are studying noncooperative games and cooperative games because both are effective resolution tools for conflict problems in a competitive system.

Chapters 2, 3 and 4 are devoted to investigating the solution concepts and their computation methods for noncooperative games. Cooperation among some or all of the players is forbidden by the rules of the game in noncooperative games. Therefore, when interests of decision makers in a competitive system are in complete conflict, noncooperative games can be an appropriate analysis tool for such conflict problems. The principal question for noncooperative games is the existence of equilibrium solutions, and in particular, an equilibrium solution of a two-person zero-sum game is represented as a solution of the min-max problem. Since practical methods for computing solutions can be provided in two-person noncooperative matrix games, we deal with only two-person matrix games in this thesis.

Chapter 2 is concerned with two-person zero-sum matrix games in fuzzy and multiobjective environments. We assume that a player has a fuzzy goal for each of the payoffs, which can also be interpreted as a degree of satisfaction for each payoff, and examine a max-min strategy with respect to a degree of attainment for a fuzzy goal, or in other words, a max-min strategy with respect to a satisfaction degree for payoffs. First, we review the solution concept of conventional two-person zero-sum matrix games and the relation between the solution, i.e., a min-max solution, and a linear programming problem. Afterwards, a solution concept of two-person zero-sum matrix games in fuzzy environments is examined. The solution is called a max-min solution with respect to a degree of attainment for a fuzzy goal. Finally, the solution concept is extended to that of games with multiple payoffs. Especially when membership functions of fuzzy goals are linear functions, it is shown that the solution is equivalent to an optimal solution for a linear programming problem.

In Chapter 3, we consider problems which involve not only ambiguity of decision makers' judgements but also imprecision of payoff representation. When competitive systems are modeled as matrix games, entries of the matrix are assessed by utilizing information for the competitive systems. Such information, however, is not always precise, but may involve some ambiguity and imprecision. We represent entries of the matrix as fuzzy numbers in order to express the ambiguity and imprecision of information. Especially when membership functions of fuzzy goals and the shape function of fuzzy numbers are given in linear functions, it is shown that the max-min solution is equivalent to an optimal solution for a mathematical programming problem and also that the solution can be obtained by using an algorithm based on the relaxation procedure, Sakawa's method: this is based on the bisection method and phase one of linear programming, and the variable transformation.

Chapter 4 deals with a two-person non-zero-sum multiobjective bimatrix game with fuzzy goals, which is a generalization of zero-sum games discussed in Chapter 2, and examines equilibrium solutions of the games. Two basic methods, which are an aggregation method by weighting coefficients and an aggregation method by a minimum component, are employed in order to aggregate multiple fuzzy goals. When membership functions are linear functions, methods for computing equilibrium solutions are developed. It is shown that equilibrium solutions are equivalent to optimal solutions for mathematical programming problems in both cases. This means that we can obtain such solutions by solving the mathematical programming problems. Finally, we consider the relation between equilibrium solutions with respect to a degree of attainment for the aggregated fuzzy goal and Pareto optimal equilibrium solutions defined in Corley (1985), Borm, Tiji and Aarssen (1988) or Wierzbicki (1986, 1990).

In Chapters 5 and 6, we intend to consider n-person cooperative games in fuzzy environments. Cooperative games can be applied to competitive systems so that decision makers are not opposed by others but cooperation is permitted. A decision problem in such competitive systems can be interpreted as a decision problem in which all decision makers cooperate but their interests more or less conflict. Cooperative games, as well as the noncooperative games considered in the preceding chapters, should be examined under fuzzy environment.

In Chapter 5, we examine n-person cooperative games with fuzzy coalitions. We have to take coalitions into consideration in cooperative games. A player participates either wholly or not at all in a coalition in conventional cooperative games, but it is more common for a player to participate only partially in a coalition. Such a coalition is called a fuzzy coalition. In this chapter, we define lexicographical solutions with respect to an excess of a player for conventional n-person cooperative games and n-person cooperative games with fuzzy coalitions, and develop methods for computing the solutions.

Chapter 6 is devoted to investigating cooperative games in which the value representing the worth of a coalition cannot be defined clearly and accurately. We introduce fuzzy goals and describe games using fuzzy goals of coalitions instead of the value representing the worth of a coalition, and a coalition's fuzzy goal for a payoff can be interpreted as ambiguity of a coalition's judgements. A solution concept is newly developed for games because the framework of the games differs from that of conventional n-person cooperative games and methods of computing solutions are shown.

Finally, in Chapter 7, we discuss the topics on n-person cooperative multi-objective games in fuzzy environment and mathematical programming problems with multiple decision makers incorporating game theoretic approach as further research directions.

CHAPTER 2

TWO-PERSON ZERO-SUM MULTIOBJECTIVE MATRIX GAMES WITH FUZZY GOALS

2.1 Introduction

In this chapter, we examine two-person zero-sum matrix games in fuzzy and multiobjective environments. Two-person zero-sum games are essentially noncooperative games where the interests of the two players are in total conflict.

The fuzzy environment considered in this chapter is the ambiguity of the players' judgments, which is expressed as the fuzzy goals. We assume that a player has a fuzzy goal for each of the objectives which can also be interpreted as a player's degree of satisfaction for a payoff.

Moreover, we take a multiplicity of objectives into consideration. In general, a decision making problem under conflict involves multiple attributes such as cost, time and productivity. We can make a game theoretic model of a real problem with multiple objectives by making a one-to-one correspondence of each of the objectives for a payoff. In other words, we take an approach to let each of the objectives of the problem correspond to each of the payoffs of the game and they are dealt with in games with multiple payoffs. Therefore, the game with multiple payoffs can be regarded as the multiobjective game.

Since each objective has a different unit of measure, vector optimization must be considered. However, we can reduce vector optimization problems to scalar optimization problems, because we are incorporating fuzzy goals for payoffs, and evaluate alternatives through a degree of attainment of a fuzzy goal. Fuzzy goals express not only ambiguity of the players' judgment but also provide the commensurable unit of measure.

For two-person zero-sum multiobjective matrix non-fuzzy games, Zeleny (1975) introduced a parameter vector, a vector of weighting coefficients, which he varied parametrically to analyze such games. Cook (1976) also introduced a goal vector and formulated such games as goal programming problems.

In this chapter, to accommodate the imprecise nature of human judgments, we suppose that each player has a fuzzy goal for each objective. We introduce concepts of max-min solutions with respect to a degree of attainment of a fuzzy goal and present methods for computing the solutions (Nishizaki and Sakawa 1992b, Sakawa and Nishizaki 1992b).

In section 2.2, we review conventional two-person zero-sum matrix games and a min-max solution is determined by solving a linear programming problem. In section 2.3, we define a new solution concept maximizing a degree of attainment of a fuzzy goal, and it is shown that the problem for calculating the proposed solution can be reduced to a linear programming problem when each membership function is identified as a linear function or a piecewise linear function. Particularly when membership functions of both players are symmetric and linear in a game with a single payoff, it is proved that the equilibrium property of the solution holds. Moreover, the proposed solution is illustrated by the numerical example of Cook (1976).

2.2 Conventional Two-Person Zero-Sum Matrix Games

Let $i \in \{1, 2, \ldots, m\}$ be a pure strategy of Player I and $j \in \{1, 2, \ldots, n\}$ be a pure strategy of Player II.

Definition 2.1 (Zero-sum game) Let f_1 be a payoff function of Player I and f_2 be a payoff of Player II. When Player I chooses a pure strategy i and Player II chooses a pure strategy j, let $f_1(i,j)$ be a payoff of Player I and $f_2(i,j)$ be a payoff of Player II. A game is said to be zero-sum if and only if the payoff functions satisfy

$$f_1(i,j) + f_2(i,j) = 0. \tag{2.1}$$

The normal form of a finite two-person zero-sum game can be reduced to a matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \tag{2.2}$$

with as many rows as Player I and as many columns as Player II have strategies. Matrix A is called the payoff matrix of the game, and games represented by such payoff matrices are called matrix games.

Definition 2.2 (Mixed strategy) A mixed strategy $x = (x_1, x_2, \ldots, x_m)$ for Player I is a probability distribution on the set of his pure strategies and the set of mixed strategies for Player I is represented by

$$X = \left\{ x = (x_1, x_2, \ldots, x_m) \;\middle|\; \sum_{i=1}^{m} x_i = 1, \;\; x_i \geq 0 \right\}. \tag{2.3}$$

Similarly, the set of mixed strategies for Player II is represented by

$$Y = \left\{ y = (y_1, y_2, \ldots, y_n) \;\middle|\; \sum_{j=1}^{n} y_j = 1, \;\; y_j \geq 0 \right\}. \tag{2.4}$$

Definition 2.3 (Expected payoff) When Player I chooses a mixed strategy x and Player II chooses a mixed strategy y, an expected value of the payoff for Player I

$$E(x, y) = \sum_{i=1}^{m} \sum_{j=1}^{n} x_i a_{ij} y_j = xAy^T \tag{2.5}$$

is called an expected payoff of Player I, where y^T is the transposition of y. (We will omit this notation unless a confusion occurs.)

For a matrix game A, when Player I chooses a mixed strategy x, the worst possible expected payoff for Player I is

$$v(x) = \min_{y \in Y} xAy. \tag{2.6}$$

Then Player I should choose x so as to maximize $v(x)$ and obtain the payoff

$$v_I = \max_{x \in X} \min_{y \in Y} xAy. \tag{2.7}$$

Such a strategy x is called Player I's max-min strategy and the payoff v_I is called the value of the game to Player I.

Similarly, Player II's min-max strategy y satisfies

$$v_{II} = \min_{y \in Y} \max_{x \in X} xAy, \tag{2.8}$$

and the payoff v_{II} is called the value of the game to Player II.

Theorem 2.1 (The min-max theorem)

For a matrix game A, it follows that

$$\max_{x\in X}\min_{y\in Y} xAy = \min_{y\in Y}\max_{x\in X} xAy. \tag{2.9}$$

Then a pair of strategies (x^*, y^*) satisfying the above equation is called an equilibrium solution.

Proof This theorem has been proved in many ways. Here we give the proof given by Dantzig (cited in Owen 1982 or Thie 1988) which not only demonstrates the existence of the equilibrium solution but also provides a computational method for the value of the game.

First let $a_{ij} > 0$. Since the value of the game to Player I is attained by a pure strategy,

$$v_I = \max_{x\in X}\min_{y\in Y} xAy = \max_{x\in X}\min_{j} xA_{\cdot j} = \max_{x\in X}\min_{j}\sum_{i=1}^{m} a_{ij}x_i,$$

where $A_{\cdot j}$ is the jth column of the matrix A. Let u be the minimum of $\sum_{i=1}^{m} a_{ij}x_i$ for some x. Then u is a maximum value satisfying the following n inequalities:

$$a_{1j}x_1 + a_{2j}x_2 + \cdots + a_{mj}x_m \geq u, \quad j = 1, 2, \ldots, n.$$

Therefore v_I is the optimal value of the linear programming problem:

$$\begin{array}{ll} \text{maximize} & u \\ \text{subject to} & a_{11}x_1 + a_{21}x_2 + \cdots + a_{m1}x_m \geq u \\ & a_{12}x_1 + a_{22}x_2 + \cdots + a_{m2}x_m \geq u \\ & \vdots \\ & a_{1n}x_1 + a_{2n}x_2 + \cdots + a_{mn}x_m \geq u \\ & x_1 + x_2 + \cdots + x_m = 1 \\ & x_i \geq 0, \qquad i = 1, 2, \ldots, m. \end{array}$$

Let $x'_i = x_i/u$, $i = 1, 2, \ldots, m$. Equivalently, the above linear programming problem can be rewritten as

$$\begin{array}{ll} \text{minimize} & x'_1 + x'_2 + \cdots + x'_m \\ \text{subject to} & a_{11}x'_1 + a_{21}x'_2 + \cdots + a_{m1}x'_m \geq 1 \\ & a_{12}x'_1 + a_{22}x'_2 + \cdots + a_{m2}x'_m \geq 1 \\ & \vdots \\ & a_{1n}x'_1 + a_{2n}x'_2 + \cdots + a_{mn}x'_m \geq 1 \\ & x'_i \geq 0, \qquad i = 1, 2, \ldots, m. \end{array} \tag{2.10}$$

For Player II, we have the following linear programming problem similarly:

$$
\begin{array}{rl}
\text{maximize} & y'_1 + y'_2 + \cdots + y'_n \\
\text{subject to} & a_{11}y'_1 + a_{12}y'_2 + \cdots + a_{1n}y'_n \leq 1 \\
& a_{21}y'_1 + a_{22}y'_2 + \cdots + a_{2n}y'_n \leq 1 \\
& \vdots \\
& a_{m1}y'_1 + a_{m2}y'_2 + \cdots + a_{mn}y'_n \leq 1 \\
& y'_i \geq 0, \qquad i = 1, 2, \ldots, n.
\end{array}
\tag{2.11}
$$

The problems (2.10) and (2.11) are dual linear programming problems. Therefore it follows from the duality theorem that both problems have solutions attaining the same optimal value. Let optimal solutions for (2.10) and (2.11) denote x'^* and y'^*, respectively. From

$$x'^*_1 + x'^*_2 + \cdots + x'^*_m = 1/v_I$$

and

$$y'^*_1 + y'^*_2 + \cdots + y'^*_n = 1/v_{II},$$

we have $v_I = v_{II}$.

Next, suppose some entries a_{ij} are nonpositive. Let r be any constant such that $a_{ij} + r > 0$, $\forall i, j$ and consider a game $A + rE$, where E is the $m \times n$ matrix whereby all of the entries are 1. For the game $A + rE$, when Player I chooses a mixed strategy x and Player II chooses a mixed strategy y, the expected payoff for Player I is $xAy + r$. Since these expected payoffs xAy and $x(A + rE)y$ differ only by the constant r, it follows that the games A and $A + rE$ will have the values of the games differing only by this constant and an equilibrium solution for the game A will be also an equilibrium solution for the game $A + rE$. The result of the previous case can be applied to the game $A + rE$ because all the entries of the matrix $A + rE$ are positive. □

We find that Player I's minimum gain is equal to Player II's maximum loss and can compute the value of the game by solving the linear programming problem (2.10) or (2.11) if mixed strategies are dealt with. The equilibrium solution is also the max-min solution for Player I and the min-max solution for Player II.

2.3 Problem Formulation and Solution Concept

Consider two-person zero-sum multiobjective matrix games, which are represented by the multiple payoff matrices:

$$A^1 = \begin{bmatrix} a_{11}^1 \cdots a_{1n}^1 \\ \vdots \ \ddots \ \vdots \\ a_{m1}^1 \cdots a_{mn}^1 \end{bmatrix}, A^2 = \begin{bmatrix} a_{11}^2 \cdots a_{1n}^2 \\ \vdots \ \ddots \ \vdots \\ a_{m1}^2 \cdots a_{mn}^2 \end{bmatrix}, \cdots, A^r = \begin{bmatrix} a_{11}^r \cdots a_{1n}^r \\ \vdots \ \ddots \ \vdots \\ a_{m1}^r \cdots a_{mn}^r \end{bmatrix}, \quad (2.12)$$

where we assume that each of the two players has r objectives. Pure strategies are the rows and the columns of each matrix A^k, $k = 1, 2, \ldots, r$ for Player I and Player II, respectively. Namely, when Player I chooses a pure strategy i and Player II chooses a pure strategy j, Player I receives the payoff vector $(a_{ij}^1, a_{ij}^2, \ldots, a_{ij}^r)$ from Player II.

We assume that a player has a fuzzy goal for each of the objectives, which expresses the player's degree of satisfaction for a payoff.

Definition 2.4 (Fuzzy goal) Let a domain of the kth payoff for Player I be $D^k \in R$. Then the fuzzy goal μ^k with respect to the kth payoff for Player I is a fuzzy set on the set D^k characterized by a membership function

$$\mu^k : D^k \to [0, 1]. \quad (2.13)$$

A membership function value for a fuzzy goal can be interpreted as the degree of attainment of the fuzzy goal for the payoff. Then when a player has two different payoffs, he prefers the payoff possessing the higher membership function value to the other. It means that he is eager to maximize the degree of attainment for the fuzzy goal.

We assume that Player I supposes that Player II will choose a strategy y so as to minimize Player I's degree of attainment of the fuzzy goal $\mu^k(x, y)$; i.e., Player I's degree of attainment of the fuzzy goal, assuming he uses x, will be $e^k(x) = \min_{y \in Y} \mu^k(x, y)$. Hence Player I chooses a strategy so as to maximize his degree of attainment of the fuzzy goal $e^k(x)$. In short, we assume that Player I behaves according to the max-min principle in terms of a degree of attainment of his fuzzy goal.

We usually consider vector optimization for the multiple objectives. However, since each of the units of measure for objectives can be transformed to the unit of measure of the degree of attainment for the fuzzy goal as a commensurable unit of measure, we can consider max-min problems in terms of maximization of the degree of attainment for the aggregated fuzzy goal.

Definition 2.5 (The max-min solution with respect to a degree of attainment for a fuzzy goal) Let the membership function of the aggregated fuzzy goal for Player I be $\mu(x, y)$ when Player I and II choose strategies x and y, respectively. Then Player I's max-min value with respect to a degree of attainment for the fuzzy goal is

$$\max_{x \in X} \min_{y \in Y} \mu(x, y), \tag{2.14}$$

and such a strategy x is called the max-min solution with respect to a degree of attainment of the fuzzy goal. Similarly, Player II's min-max value with respect to a degree of attainment of the fuzzy goal is

$$\min_{y \in Y} \max_{x \in X} \bar{\mu}(x, y), \tag{2.15}$$

and such a strategy y is called the min-max solution with respect to a degree of attainment of the fuzzy goal, where $\bar{\mu}$ is a membership function for Player II.

The max-min solution can be considered to be the solution maximizing the function, which is the minimal value of the function with respect to the opponent's decision variables. We assume that a player has no information about his opponent or the information is not useful for the decision making if he has it. Then a player supposes that his opponent chooses the strategy which makes the player's degree of attainment of the fuzzy goal worst and maximizes his degree of attainment of the fuzzy goal with respect to his decision variables.

2.4 Computational Methods

This section is devoted to developing the methods for computing the max-min solution with respect to a degree of attainment of a fuzzy goal in single-objective games and multiobjective games.

2.4.1 Single-Objective Matrix Games with Fuzzy Goals

Let $A = A^1$ and $\mu = \mu^1$ because we deal with single-objective matrix games here. For any pair of strategies (x, y), a membership function $\mu(x, y)$ of a fuzzy goal, which is a function of an expected payoff xAy, is represented as $\mu(xAy)$.

If the membership function for the fuzzy goal $\mu(xAy)$ is a linear function, it can be represented as

$$\mu(xAy) = \begin{cases} 0 & \text{if } xAy \leq \underline{a} \\ 1 - \dfrac{\overline{a} - xAy}{\overline{a} - \underline{a}} & \text{if } \underline{a} \leq xAy \leq \overline{a} \\ 1 & \text{if } \overline{a} \leq xAy, \end{cases} \quad (2.16)$$

where $\underline{a}$ is the payoff giving the worst degree of satisfaction to Player I and $\overline{a}$ is the payoff giving the best degree of satisfaction to Player I.

For example, we can employ the following payoff indices. The index with respect to the worst degree of satisfaction of Player I is

$$\underline{a} = x^0Ay^0 = \min_x \min_y xAy = \min_i \min_j a_{ij}, \quad (2.17)$$

and the index with respect to the best degree of satisfaction of Player I is

$$\overline{a} = x^1Ay^1 = \max_x \max_y xAy = \max_i \max_j a_{ij}. \quad (2.18)$$

Using these indices, a linear membership function is expressed as follows:

$$\mu(xAy) = \begin{cases} 0 & \text{if } xAy \leq x^0Ay^0 \\ 1 - \dfrac{x^1Ay^1 - xAy}{x^1Ay^1 - x^0Ay^0} & \text{if } x^0Ay^0 \leq xAy \leq x^1Ay^1 \\ 1 & \text{if } x^1Ay^1 \leq xAy. \end{cases} \quad (2.19)$$

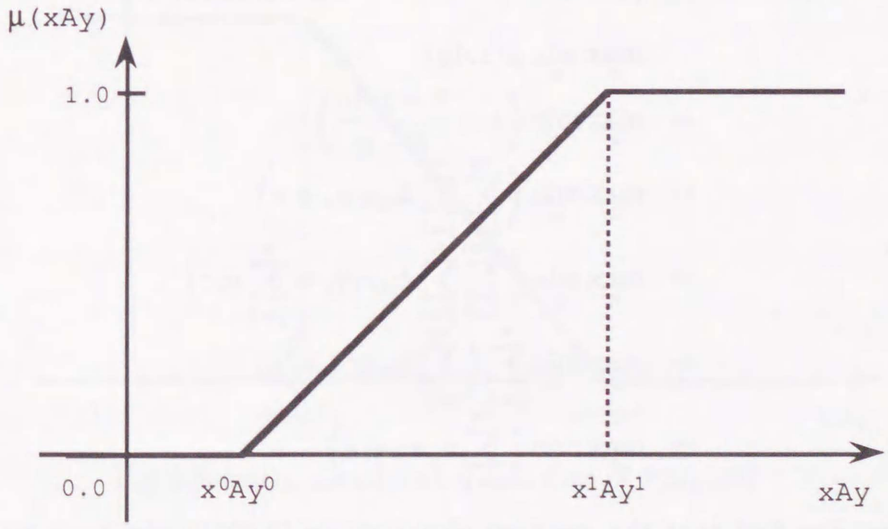

Fig. 2.1 A linear membership function of Player I

The function means that Player I is not satisfied with an expected payoff xAy smaller than x^0Ay^0, but his degree of satisfaction increases linearly as an expected payoff xAy becomes larger than x^0Ay^0, and he is satisfied enough with an expected payoff xAy larger than x^1Ay^1.

Consider a method for computing a max-min solution with respect to a degree of attainment of a fuzzy goal in the case which a membership function of the fuzzy goal is a linear function such as (2.16).

Theorem 2.2

For two-person zero-sum single-objective matrix games, if a membership function of a fuzzy goal is a linear function, Player I's max-min solution with respect to a degree of attainment of the fuzzy goal is equal to an optimal solution of the following linear programming problem:

$$\begin{array}{ll} \text{maximize} & \lambda \\ \text{subject to} & \hat{a}_{11}x_1 + \hat{a}_{21}x_2 + \cdots + \hat{a}_{m1}x_m + c \geq \lambda \\ & \hat{a}_{12}x_1 + \hat{a}_{22}x_2 + \cdots + \hat{a}_{m2}x_m + c \geq \lambda \\ & \vdots \\ & \hat{a}_{1n}x_1 + \hat{a}_{2n}x_2 + \cdots + \hat{a}_{mn}x_m + c \geq \lambda \\ & x_1 + x_2 + \cdots + x_m = 1 \\ & x_i \geq 0, \qquad i = 1, 2, \ldots, m, \end{array} \tag{2.20}$$

where

$$\hat{a}_{ij} = \frac{a_{ij}}{\overline{a} - \underline{a}} \qquad \text{and} \qquad c = -\frac{\underline{a}}{\overline{a} - \underline{a}} \ . \tag{2.21}$$

Proof The max-min problem (2.14) can be transformed into

$$\begin{aligned} & \max_x \min_y \mu(xAy) \\ = \ & \max_x \min_y \left(1 - \frac{\overline{a} - xAy}{\overline{a} - \underline{a}}\right) \\ = \ & \max_x \min_y \left(\sum_{i=1}^m \sum_{j=1}^n \hat{a}_{ij}x_iy_j + c\right) \\ = \ & \max_x \min_y \left(\sum_{i=1}^m \sum_{j=1}^n \hat{a}_{ij}x_iy_j + \sum_{j=1}^n y_jc\right) \\ = \ & \max_x \min_y \sum_{j=1}^n \left(\sum_{i=1}^m \hat{a}_{ij}x_i + c\right)y_j \\ = \ & \max_x \min_j \left(\sum_{i=1}^m \hat{a}_{ij}x_i + c\right). \end{aligned} \tag{2.22}$$

Thus, we can find that the strategy x^* satisfying (2.22) is obtained by solving the linear programming problem (2.20). □

Consider Player II's min-max solution with respect to a degree of attainment of a fuzzy goal. The same kind of the membership function can be used for Player

II. If the membership function for the fuzzy goal $\mu(xAy)$ is a linear function, it can be represented as

$$\mu_1(xAy) = \begin{cases} 1 & \text{if } xAy \le \underline{a} \\ 1 - \dfrac{xAy - \overline{a}}{\overline{a} - \underline{a}} & \text{if } \underline{a} \le xAy \le \overline{a} \\ 0 & \text{if } \overline{a} \le xAy. \end{cases} \tag{2.23}$$

For example, we can employ the following indices in a similar way. The index for the worst degree of satisfaction of Player II is

$$\overline{a} = \max_x \max_y xAy = \max_i \max_j a_{ij} = x^1Ay^1, \tag{2.24}$$

and the index for the best degree of satisfaction of Player II is

$$\underline{a} = \min_x \min_y xAy = \min_i \min_j a_{ij} = x^0Ay^0. \tag{2.25}$$

Using these indices, a linear membership function is expressed as follows:

$$\mu(xAy) = \begin{cases} 1 & \text{if } xAy \le x^0Ay^0 \\ 1 - \dfrac{xAy - x^1Ay^1}{x^1Ay^1 - x^0Ay^0} & \text{if } x^0Ay^0 \le xAy \le x^1Ay^1 \\ 0 & \text{if } x^1Ay^1 \le xAy. \end{cases} \tag{2.26}$$

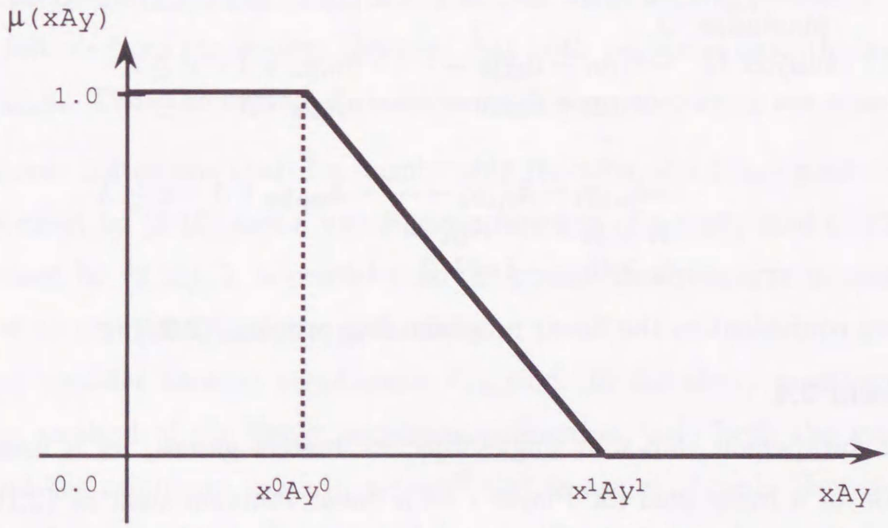

Fig.2 A linear membership function of Player II

We also assume that Player II behaves according to the min-max principle in terms of a degree of attainment for a fuzzy goal.

Theorem 2.3

For two-person zero-sum single-objective matrix games, if a membership function of a fuzzy goal is a linear function, Player II's min-max solution with respect to a degree of attainment of the fuzzy goal is equal to an optimal solution of the following linear programming problem:

$$
\begin{array}{rl}
\text{minimize} & \lambda \\
\text{subject to} & \hat{a}_{11}y_1 + \hat{a}_{12}y_2 + \cdots + \hat{a}_{1n}y_n + c \leq \lambda \\
& \hat{a}_{21}y_1 + \hat{a}_{22}y_2 + \cdots + \hat{a}_{2n}y_n + c \leq \lambda \\
& \vdots \\
& \hat{a}_{m1}y_1 + \hat{a}_{m2}y_2 + \cdots + \hat{a}_{mn}y_n + c \leq \lambda \\
& y_1 + y_2 + \cdots + y_n = 1 \\
& y_j \geq 0, \qquad j = 1, 2, \ldots, n.
\end{array} \tag{2.27}
$$

Proof The min-max problem (2.15) can be transformed into

$$
\begin{aligned}
& \max_y \min_x \left(1 - \frac{xAy - \underline{a}}{\overline{a} - \underline{a}}\right) \\
= \ & \max_y \min_x \left(-\sum_{i=1}^{m}\sum_{j=1}^{n} \hat{a}_{ij}x_i y_j + 1 - c\right) \\
= \ & \max_y \min_i \left(-\sum_{j=1}^{n} \hat{a}_{ij}y_j + 1 - c\right).
\end{aligned} \tag{2.28}
$$

The strategy y^* satisfying (2.28) is obtained by solving the following linear programming problem:

$$
\begin{array}{rl}
\text{maximize} & \lambda \\
\text{subject to} & -\hat{a}_{11}y_1 - \hat{a}_{12}y_2 - \cdots - \hat{a}_{1n}y_n + 1 - c \geq \lambda \\
& -\hat{a}_{21}y_1 - \hat{a}_{22}y_2 - \cdots - \hat{a}_{2n}y_n + 1 - c \geq \lambda \\
& \vdots \\
& -\hat{a}_{m1}y_1 - \hat{a}_{m2}y_2 - \cdots - \hat{a}_{mn}y_n + 1 - c \geq \lambda \\
& y_1 + y_2 + \cdots + y_n = 1 \\
& y_i \geq 0, \qquad i = 1, 2, \ldots, n,
\end{array} \tag{2.29}
$$

which is equivalent to the linear programming problem (2.27). □

Theorem 2.4

For two-person zero-sum single-objective matrix games, let a membership function of a fuzzy goal for Player I be a linear function such as (2.16) and a membership function of a fuzzy goal for Player II be a linear function such as (2.23). Then if both of the players behave according to the max-min or min-max principle in terms of a degree of attainment of a fuzzy goal, Player I's degree of attainment of the fuzzy goal is equal to Player II's degree of attainment of the fuzzy goal.

Proof Set

$$x'_i = \frac{x_i}{\lambda}, \ i = 1, 2, \ldots, m.$$

Then the problem (2.20) can be transformed as follows:

$$\begin{array}{rl} \text{minimize} & x'_1 + x'_2 + \cdots + x'_m \\ \text{subject to} & (\hat{a}_{11} + c)x'_1 + (\hat{a}_{21} + c)x'_2 + \cdots + (\hat{a}_{m1} + c)x'_m \geq 1 \\ & (\hat{a}_{12} + c)x'_1 + (\hat{a}_{22} + c)x'_2 + \cdots + (\hat{a}_{m2} + c)x'_m \geq 1 \\ & \vdots \\ & (\hat{a}_{1n} + c)x'_1 + (\hat{a}_{2n} + c)x'_2 + \cdots + (\hat{a}_{mn} + c)x'_m \geq 1 \\ & x'_i \geq 0, \qquad i = 1, 2, \ldots, m. \end{array} \tag{2.30}$$

Similarly, set

$$y'_j = \frac{y_j}{\lambda}, \ j = 1, 2, \ldots, n.$$

Then the problem (2.27) can be transformed as follows:

$$\begin{array}{rl} \text{maximize} & y'_1 + y'_2 + \cdots + y'_n \\ \text{subject to} & (\hat{a}_{11} + c)y'_1 + (\hat{a}_{12} + c)y'_2 + \cdots + (\hat{a}_{1n} + c)y'_n \leq 1 \\ & (\hat{a}_{21} + c)y'_1 + (\hat{a}_{22} + c)y'_2 + \cdots + (\hat{a}_{2n} + c)y'_n \leq 1 \\ & \vdots \\ & (\hat{a}_{m1} + c)y'_1 + (\hat{a}_{m2} + c)y'_2 + \cdots + (\hat{a}_{mn} + c)y'_n \leq 1 \\ & y'_j \geq 0, \qquad j = 1, 2, \ldots, n. \end{array} \tag{2.31}$$

The problems (2.30) and (2.31) are dual linear programming problems. Therefore, it follows from the duality theorem that both problems have the same optimal values. Thus the degree of attainments of the two players are equal. □

Theorem 2.4 means that if a membership function of a fuzzy goal of Player I is expressed by (2.16) and a membership function of a fuzzy goal of Player II is expressed by (2.23), it is proved that the equilibrium property in terms of a degree of attainment of a fuzzy goal holds.

Next, consider another membership function. In the above mentioned construction method of the linear membership function, only both the maximum value and the minimum value in a payoff matrix are employed. However, in a piecewise linear function, all entries of the payoff matrix can be used.

Put all of the entries a_{ij}, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$ of the payoff matrix A in an ascending order and let this vector be $h = (h_1, h_2, \ldots, h_{mn})$. A piecewise linear membership function can be identified by assessing pairs $\{(h_1, m_1),$

(h_2, m_2), ..., $(h_{mn}, m_{mn})\}$, where m_l, $l = 1, 2, \ldots, mn$ are membership values. Then the membership function is expressed as follows:

$$\mu(xAy) = \sum_{l=2}^{mn-1} \alpha_l |xAy - h_l| + \beta xAy + \gamma, \tag{2.32}$$

where $\alpha_l = (t_{l+1} - t_l)/2$, $l = 2, 3, \ldots, mn-1$, $\beta = (t_{mn} + t_2)/2$, $\gamma = (u_{mn} + u_2)$, and when $h_{v-1} \leq xAy \leq h_v$,

$$\mu(xAy) = t_v xAy + u_v. \tag{2.33}$$

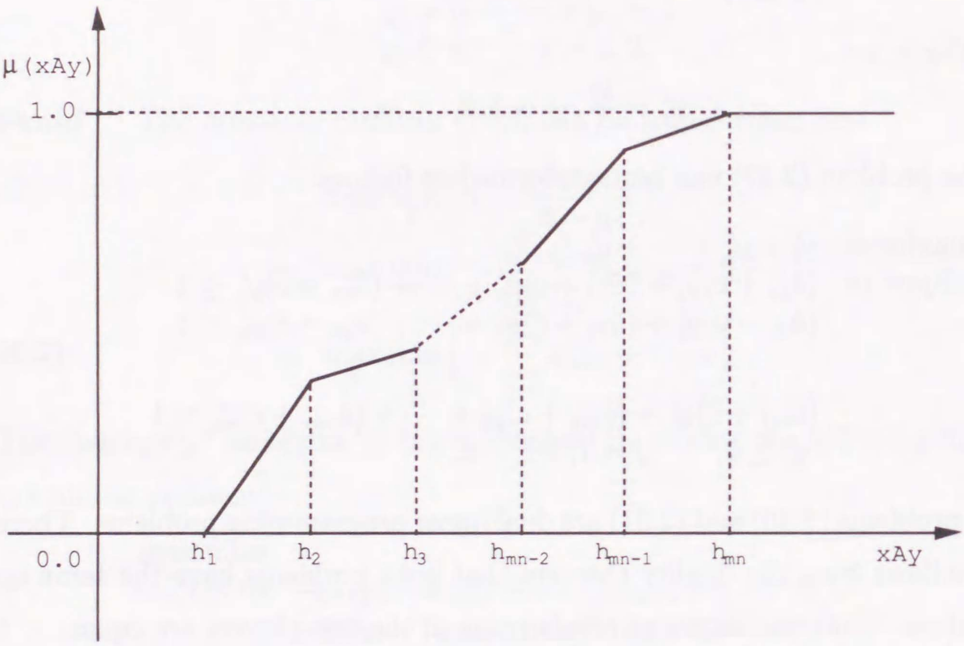

Fig. 2.3 A piecewise linear membership function of Player I

From the properties of the membership function (2.32), the max-min strategy with respect to a degree of attainment of a fuzzy goal, having a piecewise linear function as the membership function, can be obtained by the method in the following theorem.

Lemma 2.5

Let x be an m-dimensional vector, y be an n-dimensional vector and A be an $m \times n$ matrix. We assume that μ is a strictly monotone increasing membership function. Let (σ^*, x^*) be an optimal solution of the following problem:

$$\begin{array}{ll} \text{maximize} & \sigma \\ \text{subject to} & \mu(xAy) \geq \sigma, \quad \forall y \in Y \\ & x_1 + x_2 + \cdots + x_m = 1. \end{array} \tag{2.34}$$

Let the other strictly monotone increasing membership function be

$$\mu'(xAy) = \begin{cases} \mu(xAy) & \text{if } E_0 \leq xAy \leq E_1 \\ \mu'(xAy) & \text{if other.} \end{cases} \tag{2.35}$$

Then, if an optimal solution of the problem (2.34) is

$$E_0 \leq \mu^{-1}(\sigma^*) = x^*Ay \lneq E_1,$$

the optimal solution $\hat{\sigma}$ of the following problem is equal to σ^*.

$$\begin{array}{ll} \text{maximize} & \sigma \\ \text{subject to} & \mu'(xAy) \geq \sigma, \quad \forall y \in Y \\ & x_1 + x_2 + \cdots + x_m = 1. \end{array} \tag{2.36}$$

Proof First we will show that there does not exist an optimal solution for (2.36) when $E_1 < xAy$. Since σ^* is the optimal solution of the problem (2.34), there is no pair of strategies (x, y) such that

$$\mu(xAy) > \sigma^*.$$

Thus $\mu(xAy) \leq \sigma^*$ for any $x \in X$ and $y \in Y$.

On the other hand, since $\mu'(xAy)$ is also a strictly monotone increasing membership function, if $E_0 \leq \mu^{-1}(\sigma^*) = x^*Ay \leq E_1$, there is no pair of strategies (x, y) such that

$$\mu'(xAy) > \sigma^*.$$

Therefore there does not exist an optimal solution for (2.36) when $E_1 < xAy$.

Second, we will show that there does not exist an optimal solution for (2.36) when $xAy < E_0$. Values of the objective function subject to $xAy \geq E_0$ are larger than those subject to $xAy < E_0$. Thus, since the solution (σ^*, x^*) is a feasible solution to (2.36), σ^* is larger than the values of the objective function subject to $xAy < E_0$. Therefore there does not exist an optimal solution for (2.36) when $xAy < E_0$.

Because of the above facts, there exists an optimal solution for (2.36) when $E_0 \leq xAy \leq E_1$. Since the problems (2.34) are equivalent to (2.36) when $E_0 \leq xAy \leq E_1$, it follows that the optimal solution $\hat{\sigma}$ of (2.36) is equal to σ^*, i.e., $\hat{\sigma} = \sigma^*$. □

Theorem 2.6

When a membership function of a fuzzy goal of Player I is a piecewise linear function such as (2.32), the max-min strategy is expressed as the following problem:

$$\max_x \min_y \mu(xAy) = \max_x \min_y \left(\sum_{l=2}^{mn-1} \alpha_l |xAy - h_l| + \beta xAy + \gamma \right). \quad (2.37)$$

Then the max-min strategy satisfying (2.37) can be obtained by solving the following linear programming problem with an index v, $v = 2, 3, \ldots, mn$

$$\begin{array}{ll} \text{maximize} & \lambda \\ \text{subject to} & t_v(\hat{a}_{11}x_1 + \hat{a}_{21}x_2 + \cdots + \hat{a}_{m1}x_m) + u_v \geq \lambda \\ & t_v(\hat{a}_{12}x_1 + \hat{a}_{22}x_2 + \cdots + \hat{a}_{m2}x_m) + u_v \geq \lambda \\ & \vdots \\ & t_v(\hat{a}_{1n}x_1 + \hat{a}_{2n}x_2 + \cdots + \hat{a}_{mn}x_m) + u_v \geq \lambda \\ & x_1 + x_2 + \cdots + x_m = 1 \\ & x_i \geq 0, \qquad i = 1, 2, \ldots, m \end{array} \quad (2.38)$$

and testing whether the optimal solution satisfies the condition with an index v, $v = 2, 3, \ldots, mn$

$$h_{v-1} \leq \frac{\lambda^* - u_v}{t_v} \leq h_v, \quad (2.39)$$

or not at most $mn-1$ times. Namely, there exists the index $\hat{v}$ such that an optimal solution of the linear programming problem (2.38) with an index v satisfies the condition (2.39) with an index v, and the optimal solution of the problem (2.38) with the index $\hat{v}$ is equal to the max-min strategy satisfying (2.37).

Proof When $h_{v-1} \leq xAy \leq h_v$, the problem (2.37) can be reduced to

$$\max_x \min_y \mu(xAy) = \max_x \min_y \left(t_v xAy + u_v \right). \quad (2.40)$$

If the condition $h_{v-1} \leq xAy \leq h_v$ is taken off, the max-min strategy, i.e., the strategy x^* satisfying (2.40), can be obtained by solving the linear programming problem (2.38).

A membership function of a fuzzy goal of Player I such as (2.32) has $mn - 1$ segments of straight lines. Since $0 \leq \lambda \leq 1$, we can find the linear programming problem (2.38) with the index $\hat{v}$ such that the optimal solution (x^*, λ^*) satisfies the condition (2.39). Set

$$\mu(xAy) = t_v xAy + u_v,$$

and

$$\mu'(xAy) = \sum_{l=2}^{mn-1} \alpha_v |xAy - h_v| + \beta xAy + \gamma.$$

Then, from the Lemma 2.5, the optimal solution x^* of the linear programming problem (2.38) satisfying the condition (2.39) is the max-min strategy satisfying (2.37). Therefore the max-min strategy satisfying (2.37) can be obtained by solving the linear programming problem (2.38) at most $mn-1$ times. □

We can also obtain the min-max strategy for Player II by solving the following linear programming problem in a similar way:

$$\begin{array}{ll} \text{maximize} & \lambda \\ \text{subject to} & t_v(\hat{a}_{11}y_1 + \hat{a}_{12}y_2 + \cdots + \hat{a}_{1n}y_n) + u_v \geq \lambda \\ & t_v(\hat{a}_{21}y_1 + \hat{a}_{22}y_2 + \cdots + \hat{a}_{2n}y_n) + u_v \geq \lambda \\ & \vdots \\ & t_v(\hat{a}_{m1}y_1 + \hat{a}_{m2}y_2 + \cdots + \hat{a}_{mn}y_n) + u_v \geq \lambda \\ & y_1 + y_2 + \cdots + y_n = 1 \\ & y_j \geq 0, \qquad j = 1, 2, \ldots, n. \end{array} \tag{2.41}$$

2.4.2 Multiobjective Matrix Games with Fuzzy Goals

Consider a two-person zero-sum multiobjective matrix game, i.e., a two-person zero-sum game with multiple payoff matrices A^k, $k = 1, 2, \ldots, r$. We assume that a player has a fuzzy goal for each of the objectives which expresses the player's degree of satisfaction for a payoff. Let Player I's membership function of the fuzzy goal for the kth objective be $\mu^k(xA^ky)$ for any pair of mixed strategies (x, y).

If the membership function $\mu^k(xA^ky)$ for the fuzzy goal is a linear function, it can be represented as

$$\mu^k(xA^ky) = \begin{cases} 0 & \text{if } xA^ky \leq \underline{a}^k \\ 1 - \dfrac{\overline{a}^k - xA^ky}{\overline{a}^k - \underline{a}^k} & \text{if } \underline{a}^k \leq xA^ky \leq \overline{a}^k \\ 1 & \text{if } \overline{a}^k \leq xA^ky, \end{cases} \tag{2.42}$$

where $\underline{a}^k$ is the payoff giving the worst degree of satisfaction for Player I with respect to the kth objective and $\overline{a}^k$ is the payoff giving the best degree of satisfaction for Player I with respect to the kth objective.

For example, in a manner similar to the single objective case,

$$\underline{a}^k = x_k^0 A^k y_k^0 = \min_x \min_y xA^ky = \min_i \min_j a_{ij}^k \tag{2.43}$$

can be employed as a payoff index for the worst degree of satisfaction of Player I with respect to the kth objective and

$$\overline{a}^k = x_k^1 A^k y_k^1 = \max_x \max_y xA^k y = \max_i \max_j a_{ij}^k \tag{2.44}$$

can also be employed as a payoff index for the best degree of satisfaction of Player I with respect to the kth objective. Using these indices, the linear membership function is expressed as

$$\mu^k(xA^k y) = \begin{cases} 0 & \text{if } \; xA^k y \leq x_k^0 A^k y_k^0 \\ 1 - \dfrac{x_k^1 A^k y_k^1 - xA^k y}{x_k^1 A^k y_k^1 - x_k^0 A^k y_k^0} & \text{if } \; x_k^0 A^k y_k^0 \leq xA^k y \leq x_k^1 A^k y_k^1 \\ 1 & \text{if } \; x_k^1 A^k y_k^1 \leq xA^k y. \end{cases} \tag{2.45}$$

In multiple objective cases, the preferable expected payoffs for Player I such as $\max_y xA^k y$, which are chosen by Player II and expected to be at a disadvantage for the kth objective for Player II, are more meaningful. In other words, it is more possible that Player II would choose the strategy by which his opponent, i.e., Player I, could receive a lot of payoff with respect to the kth objective because of a trade-off between the kth objective and the other objectives.

We employ the fuzzy decision rule by Bellman and Zadeh (1970), which is often used in decision making problems under fuzzy environment, as an aggregation rule for multiple fuzzy goals. Then the membership function of the aggregated fuzzy goal is expressed as

$$\mu(x, y) = \min_k \mu^k(xA^k y). \tag{2.46}$$

If the membership function is a linear function such as (2.42), it is also expressed as

$$\begin{aligned} \mu(x, y) &= \min_k \left(1 - \frac{\overline{a}^k - xA^k y}{\overline{a}^k - \underline{a}^k}\right) \\ &= \min_k \left(\sum_{i=1}^m \sum_{j=1}^n \frac{a_{ij}^k}{\overline{a}^k - \underline{a}^k} x_i y_j - \frac{\underline{a}^k}{\overline{a}^k - \underline{a}^k}\right) \\ &= \min_k \left(\sum_{i=1}^m \sum_{j=1}^n \hat{a}_{ij}^k x_i y_j + c^k\right), \end{aligned} \tag{2.47}$$

where

$$\hat{a}_{ij}^k = \frac{a_{ij}^k}{\overline{a}^k - \underline{a}^k} \qquad \text{and} \qquad c^k = -\frac{\underline{a}^k}{\overline{a}^k - \underline{a}^k}. \tag{2.48}$$

Consider a method for computing a max-min solution with respect to a degree of attainment of the aggregated fuzzy goal in two-person zero-sum multiobjective games.

Theorem 2.7

For two-person zero-sum multiobjective matrix games, if membership functions of fuzzy goals are linear functions such as (2.42) and fuzzy goals are aggregated by the fuzzy decision rule, Player I's max-min solution with respect to a degree of attainment of the aggregated fuzzy goal is equal to an optimal solution of the following linear programming problem:

$$\begin{array}{ll} \text{maximize} & \lambda \\ \text{subject to} & \hat{a}^1_{11}x_1 + \hat{a}^1_{21}x_2 + \cdots + \hat{a}^1_{m1}x_m + c^1 \geq \lambda \\ & \hat{a}^1_{12}x_1 + \hat{a}^1_{22}x_2 + \cdots + \hat{a}^1_{m2}x_m + c^1 \geq \lambda \\ & \vdots \\ & \hat{a}^1_{1n}x_1 + \hat{a}^1_{2n}x_2 + \cdots + \hat{a}^1_{mn}x_m + c^1 \geq \lambda \\ & \vdots \\ & \vdots \\ & \hat{a}^r_{11}x_1 + \hat{a}^r_{21}x_2 + \cdots + \hat{a}^r_{m1}x_m + c^r \geq \lambda \\ & \hat{a}^r_{12}x_1 + \hat{a}^r_{22}x_2 + \cdots + \hat{a}^r_{m2}x_m + c^r \geq \lambda \\ & \vdots \\ & \hat{a}^r_{1n}x_1 + \hat{a}^r_{2n}x_2 + \cdots + \hat{a}^r_{mn}x_m + c^r \geq \lambda \\ & x_1 + x_2 + \cdots + x_m = 1 \\ & x_i \geq 0, \qquad i = 1, 2, \ldots, m. \end{array} \tag{2.49}$$

Proof When the fuzzy decision rule is used as an aggregation rule, a max-min problem (2.14) can be expressed as

$$\max_x \min_y \mu(x, y) = \max_x \min_y \min_k \mu^k(xA^k y). \tag{2.50}$$

From (2.47),

$$\max_x \min_y \mu(x, y) = \max_x \min_y \min_k \left(\sum_{i=1}^{m} \sum_{j=1}^{n} \hat{a}^k_{ij} x_i y_j + c^k \right). \tag{2.51}$$

By introducing a new variable $z = (z_1, z_2, \ldots, z_r)$, $\sum_{k=1}^{r} z_k = 1$, the problem (2.51) can be expressed as

$$\begin{aligned} \max_x \min_y \mu(x, y) &= \max_x \min_y \min_z \left(\sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{r} \hat{a}^k_{ij} x_i y_j z_k + \sum_{k=1}^{r} c^k z_k \right) \\ &= \max_x \min_y \min_z \left(\sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{r} \hat{a}^k_{ij} x_i y_j z_k + \sum_{j=1}^{n} y_j \sum_{k=1}^{r} c^k z_k \right) \\ &= \max_x \min_y \min_z \left(\sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{r} \hat{a}^k_{ij} x_i y_j z_k + \sum_{j=1}^{n} \sum_{k=1}^{r} c^k y_j z_k \right) \\ &= \max_x \min_y \min_z \sum_{j=1}^{n} \sum_{k=1}^{r} \left(\sum_{i=1}^{m} \hat{a}^k_{ij} x_i y_j z_k + c^k y_j z_k \right). \end{aligned} \tag{2.52}$$

Furthermore, if we make the transformation

$$\alpha_{jk} = y_j z_k, \quad \sum_{jk=1}^{nr} \alpha_{jk} = 1, \tag{2.53}$$

then the problem (2.52) is reduced to the following formulation:

$$\begin{aligned} \max_{x} \min_{y} \mu(x, y) &= \max_{x} \min_{\alpha} \sum_{jk=1}^{nr} \Big(\sum_{i=1}^{m} \hat{a}_{ij}^k x_i \alpha_{jk} + c^k \alpha_{jk} \Big) \\ &= \max_{x} \min_{\alpha} \sum_{jk=1}^{nr} \Big(\sum_{i=1}^{m} \hat{a}_{ij}^k x_i + c^k \Big) \alpha_{jk} \, . \end{aligned} \tag{2.54}$$

Thus, we can find that the strategy x^* satisfying (2.54) is obtained by solving the linear programming problem (2.49). □

We can also obtain the min-max strategy for Player II by solving the following linear programming problem in a similar way:

$$\begin{array}{ll} \text{minimize} & \lambda \\ \text{subject to} & \hat{a}_{11}^1 y_1 + \hat{a}_{12}^1 y_2 + \cdots + \hat{a}_{1n}^1 y_n + c^1 \leq \lambda \\ & \hat{a}_{21}^1 y_1 + \hat{a}_{22}^1 y_2 + \cdots + \hat{a}_{2n}^1 y_n + c^1 \leq \lambda \\ & \vdots \\ & \hat{a}_{m1}^1 y_1 + \hat{a}_{m2}^1 y_2 + \cdots + \hat{a}_{mn}^1 y_n + c^1 \leq \lambda \\ & \vdots \\ & \vdots \\ & \hat{a}_{11}^r x_1 + \hat{a}_{12}^r y_2 + \cdots + \hat{a}_{1n}^r y_n + c^r \leq \lambda \\ & \hat{a}_{21}^r y_1 + \hat{a}_{22}^r y_2 + \cdots + \hat{a}_{2n}^r y_n + c^r \leq \lambda \\ & \vdots \\ & \hat{a}_{m1}^r y_1 + \hat{a}_{m2}^r y_2 + \cdots + \hat{a}_{mn}^r y_n + c^r \leq \lambda \\ & y_1 + y_2 + \cdots + y_n = 1 \\ & y_j \geq 0, \qquad j = 1, 2, \ldots, n. \end{array} \tag{2.55}$$

Let us consider a piecewise linear function as we do in a single objective case. Put all the entries a_{ij}^k, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, $k = 1, 2, \ldots, r$ of the r payoff matrices in an ascending order and let this vector be $h^k = (h_l^k, h_2^k, h_{mn}^k)$, $k = 1, 2, \ldots, r$. A piecewise linear membership function can be identified by assessing pairs $\{(h_1^k, m_1^k), (h_2^k, m_2^k), \ldots, (h_{mn}^k, m_{mn}^k)\}$, $k = 1, 2, \ldots, r$, where m_l^k, $l = 1, 2, \ldots, mn$, $k = 1, 2, \ldots, r$ are membership values. Then the kth membership function is expressed as

$$\mu^k(xA^k y) = \sum_{l=2}^{mn-1} \alpha_l^k |xA^k y - h_l^k| + \beta^k x A^k y + \gamma^k, \tag{2.56}$$

where $\alpha_l^k = (t_{l+1}^k - t_l^k)/2$, $l = 2, 3, \ldots, mn-1$, $\beta^k = (t_{mn}^k + t_2^k)/2$, $\gamma^k = (u_{mn}^k + u_2^k)$; and when $h_{v-1}^k \le xA^k y \le h_v^k$,

$$\mu^k(xA^k y) = t_v^k xA^k y + u_v^k. \tag{2.57}$$

When membership functions are piecewise linear functions, the max-min strategy with respect to a degree of attainment of the aggregated fuzzy goal can be obtained by the method in the following theorem.

Theorem 2.8

When membership functions of fuzzy goals of Player I are piecewise linear functions such as (2.56) and the fuzzy decision rule is employed as an aggregation method for multiple fuzzy goals, the max-min strategy with respect to a degree of attainment of the aggregated fuzzy goal is expressed as the following problem:

$$\begin{aligned}\max_x \min_y \mu(xAy) &= \max_x \min_y \min_k \mu^k(xAy) \\ &= \max_x \min_y \min_k \left(\sum_{l=2}^{mn-1} \alpha_l^k |xA^k y - h_l^k| + \beta^k xA^k y + \gamma^k \right).\end{aligned} \tag{2.58}$$

Then the max-min strategy satisfying (2.58) can be obtained by solving the following linear programming problem with an index v, $v = 2, 3, \ldots, mn$,

$$\begin{array}{ll} \text{maximize} & \lambda \\ \text{subject to} & t_v^1(\hat{a}_{11}^1 x_1 + \hat{a}_{21}^1 x_2 + \cdots + \hat{a}_{m1}^1 x_m) + u_v^1 \ge \lambda \\ & t_v^1(\hat{a}_{12}^1 x_1 + \hat{a}_{22}^1 x_2 + \cdots + \hat{a}_{m2}^1 x_m) + u_v^1 \ge \lambda \\ & \vdots \\ & t_v^1(\hat{a}_{1n}^1 x_1 + \hat{a}_{2n}^1 x_2 + \cdots + \hat{a}_{mn}^1 x_m) + u_v^1 \ge \lambda \\ & \vdots \\ & \vdots \\ & t_v^r(\hat{a}_{11}^r x_1 + \hat{a}_{21}^r x_2 + \cdots + \hat{a}_{m1}^r x_m) + u_v^r \ge \lambda \\ & t_v^r(\hat{a}_{12}^r x_1 + \hat{a}_{22}^r x_2 + \cdots + \hat{a}_{m2}^r x_m) + u_v^r \ge \lambda \\ & \vdots \\ & t_v^r(\hat{a}_{1n}^r x_1 + \hat{a}_{2n}^r x_2 + \cdots + \hat{a}_{mn}^r x_m) + u_v^r \ge \lambda \\ & x_1 + x_2 + \cdots + x_m = 1 \\ & x_i \ge 0, \qquad i = 1, 2, \ldots, m, \end{array} \tag{2.59}$$

and testing whether the optimal solution satisfies the condition with a pair of indices v, $v = 2, 3, \ldots, mn$ and k, $k = 1, 2, \ldots, r$,

$$h_{v-1}^k \le \frac{\lambda^* - u_v^k}{t_v^k} \le h_v^k, \tag{2.60}$$

or not at most $(mn-1)^r$ times. Namely, there exists the pair of indices $\hat{v}$ and $\hat{k}$ such that an optimal solution of the linear programming problem (2.59) with an index v satisfies the condition (2.60) with a pair of indices v and k; and the optimal solution of the problem (2.59) with the index $\hat{v}$ satisfying the condition with the pair of indices $\hat{v}$ and $\hat{k}$ is equal to the max-min strategy satisfying (2.58).

Proof The theorem can be proved by a procedure similar to Theorem 2.6. □

We can also obtain the min-max strategy for Player II by solving the following linear programming problem with a similar manner:

$$
\begin{array}{ll}
\text{maximize} & \lambda \\
\text{subject to} & t_v^1(\hat{a}_{11}y_1 + \hat{a}_{12}y_2 + \cdots + \hat{a}_{1n}y_n) + u_v^1 \geq \lambda \\
& t_v^1(\hat{a}_{21}y_1 + \hat{a}_{22}y_2 + \cdots + \hat{a}_{2n}y_n) + u_v^1 \geq \lambda \\
& \vdots \\
& t_v^1(\hat{a}_{m1}y_1 + \hat{a}_{m2}y_2 + \cdots + \hat{a}_{mn}y_n) + u_v^1 \geq \lambda \\
& \vdots \\
& \vdots \\
& t_v^r(\hat{a}_{11}y_1 + \hat{a}_{12}y_2 + \cdots + \hat{a}_{1n}y_n) + u_v^r \geq \lambda \\
& t_v^r(\hat{a}_{21}y_1 + \hat{a}_{22}y_2 + \cdots + \hat{a}_{2n}y_n) + u_v^r \geq \lambda \\
& \vdots \\
& t_v^r(\hat{a}_{m1}y_1 + \hat{a}_{m2}y_2 + \cdots + \hat{a}_{mn}y_n) + u_v^r \geq \lambda \\
& y_1 + y_2 + \cdots + y_n = 1 \\
& y_j \geq 0, \qquad j = 1, 2, \ldots, n.
\end{array}
\tag{2.61}
$$

Example 2.1

We cite the numerical example by Cook (1976). The multiobjective game has three kinds of payoffs and each player has three pure strategies. The three payoff matrices are

$$
A^1 = \begin{bmatrix} 2 & 5 & 1 \\ -1 & -2 & 6 \\ 0 & 3 & -1 \end{bmatrix}, \quad A^2 = \begin{bmatrix} -3 & 7 & 2 \\ 0 & -2 & 0 \\ 3 & -1 & -6 \end{bmatrix}, \text{and } A^3 = \begin{bmatrix} 8 & -2 & 3 \\ -5 & 6 & 0 \\ -3 & 1 & 6 \end{bmatrix}. \tag{2.62}
$$

We can interpret, for example, that A^1 means cost, A^2 means time and A^3 means productivity.

By identifying the membership functions as (2.45) and (2.46), we have the linear programming problem such that

$$
\begin{array}{llllllll}
\text{maximize} & \lambda & & & & & & \\
\text{subject to} & 2/8x_1 & -1/8x_2 & & +1/4 & \geq & \lambda & \\
& 5/8x_1 & -2/8x_2 & +3/8x_3 & +1/4 & \geq & \lambda & \\
& 1/8x_1 & +6/8x_2 & -1/8x_3 & +1/4 & \geq & \lambda & \\
& -3/13x_1 & & +3/13x_3 & +6/13 & \geq & \lambda & \\
& 7/13x_1 & -2/13x_2 & -1/13x_3 & +6/13 & \geq & \lambda & (2.63) \\
& 2/13x_1 & & -6/13x_3 & +6/13 & \geq & \lambda & \\
& 8/13x_1 & -2/13x_2 & +3/13x_3 & +5/13 & \geq & \lambda & \\
& -2/13x_1 & +6/13x_2 & +1/13x_3 & +5/13 & \geq & \lambda & \\
& 3/13x_1 & & +6/13x_3 & +5/13 & \geq & \lambda & \\
& x_1 & +x_2 & +x_3 & & = & 1. &
\end{array}
$$

The optimal solution of the problem, which is the max-min strategy of Player I, is

$$x_1 = 0.59928, \quad x_2 = 0.15027, \quad \text{and} \quad x_3 = 0.25045. \tag{2.64}$$

In this case, the worst degree of attainment of the fuzzy goal for Player I is 0.38104. On the other hand, the min-max strategy of Player II is

$$y_1 = 0.38462, \quad y_2 = 0.38462, \quad \text{and} \quad y_3 = 0.23077, \tag{2.65}$$

and his/her worst degree of attainment of the fuzzy goal is 0.38462.

Let us compare our solution with the solution of Cook who supposed that the goal for the matrix A^1 was 4, the goal for the matrix A^2 was 1, and the goal for the matrix A^3 was 2; and the weights of objectives were 1, 2, and 2.5. Then the max-min strategy of Player I was

$$x_1 = 0.636024, \quad x_2 = 0.157764, \quad \text{and} \quad x_3 = 0.206211, \tag{2.66}$$

and the min-max strategy of Player II was

$$y_1 = 0.0, \quad y_2 = 1.0, \quad \text{and} \quad y_3 = 0.0. \tag{2.67}$$

We calculated the degree of attainment of the fuzzy goal for Cook's solution. The worst degree of attainment of Player I was 0.36235 and of Player II, 0.0. We found that the degree of attainment of Cook's solution was smaller than ours by 0.01869 for Player I, and there was at least one objective with which Player II was never satisfied.

2.5 Conclusion

In this chapter, we have reviewed conventional two-person zero-sum matrix games, proposed a new solution concept for two-person zero-sum multiobjective matrix games incorporating fuzzy goals and developed the methods for computing the proposed solutions.

The chapter can be summarized by the following conclusions.

1) Fuzzy goals have been employed to consider the imprecise nature of human judgment in decision making problems under conflict and the problem has been expressed in two-person zero-sum multiobjective matrix games with fuzzy goals.

2) The concepts of the max-min solution and the min-max solution with respect to a degree of attainment of a fuzzy goal have been introduced in two-person zero-sum multiobjective matrix games.

3) When membership functions of fuzzy goals can be constructed as linear functions or piecewise linear functions, the methods for computing their solutions, formulated as linear programming problems, have been developed.

4) The identification methods of linear membership functions and piecewise linear functions have been proposed by using entries of multiple payoff matrices.

5) Especially, if membership functions of both players are symmetric and linear in a game with a single payoff, it has been proved that the equilibrium property holds.

In general, the max-min value with respect to a degree of attainment of a fuzzy goal is not equal to the min-max value with respect to a degree of attainment of a fuzzy goal in two-person zero-sum multiobjective matrix games. Namely, the max-min solution and the min-max solution are not equilibrium solutions. However, when a player has no information about his opponent or the information is not useful for the decision making when it is available, the behavior based on the min-max principle is one of the most important behavior criteria. For equilibrium solutions, we will examine such solutions for two-person multiobjective matrix games in more general cases in a later chapter.

CHAPTER 3

TWO-PERSON ZERO-SUM MULTIOBJECTIVE FUZZY MATRIX GAMES WITH FUZZY GOALS

3.1 Introduction

Chapter 3 is concerned with two-person zero-sum fuzzy matrix games with fuzzy goals. We will consider problems which involve not only ambiguity of decision makers' judgments but also imprecision of information in the decision problem. When a competitive system is modeled as a matrix game, entries of a payoff matrix are assessed by utilizing information available on the competitive system; however, since such information is not always accurate, we represent entries of the payoff matrix as fuzzy numbers (Dubois and Parade 1980) in order to express the ambiguity and imprecision in the information (Sakawa and Nishizaki 1993).

Two-person zero-sum multiobjective fuzzy matrix games with fuzzy goals and conventional two-person zero-sum matrix games differ by the following three points. First, each player has a fuzzy goal for a payoff in order to incorporate ambiguity of human judgment. A typical goal is often set for an objective in the real world. When a goal for an objective is characterized by a one-point value, the difference between the goal value and an achievement value can be interpreted as an under-attainment or an over-attainment, which decision makers will try to minimize. On the other hand, a fuzzy goal is characterized by a membership function mapping a domain of payoffs into the range of the degree of attainment of the fuzzy goal, i.e., [0, 1], whereby a player tries to maximize his degree of attainment for the fuzzy goal. The fuzzy goal can also be interpreted as a degree of satisfaction for a payoff.

Second, multiple payoffs are considered in games because a decision making problem under conflict involves multiple objectives or attributes such as cost, time and productivity. Moreover, we correspond each of the objectives of the problem to each of the payoffs of the game and deal with them as games with multiple payoffs.

These two points have already been examined in the games in the preceding chapter, but a point newly introduced in this chapter is where a payoff is represented as a fuzzy number. A payoff matrix with entries represented as fuzzy numbers is called a fuzzy payoff matrix. For any pair of strategies, a player receives a payoff represented as a fuzzy number, i.e., the strategy itself is not fuzzy but the payoffs are fuzzy. For example, when a payoff matrix of a game is constructed by information from a competitive system, entries of the payoff matrix must be ambiguous if imprecision or vagueness exists in the information.

Recently, Campos (1989) has explored zero-sum fuzzy matrix games. The problem treated by Campos was a game with a single payoff, and the min-max problem was formulated using the fuzzy mathematical programming method. However, no studies have yet been attempted for zero-sum multiobjective fuzzy matrix games, which will be examined in this chapter.

The problem with non-fuzzy multiple payoffs in the previous chapter is extended to a problem with fuzzy multiple payoffs, and the max-min problem will be examined in terms of a degree of attainment of a fuzzy goal. In section 3.2, a fuzzy expected payoff is defined, and a degree of attainment of a fuzzy goal is considered in games with fuzzy payoff matrices. The max-min solution with respect to a degree of attainment of a fuzzy goal is also defined. In section 3.3, the methods for computing the solution of a single-objective game and of a multiobjective game are proposed when membership functions of fuzzy goals and a shape function of L-R fuzzy numbers for fuzzy payoffs are linear. An original problem for computing the max-min solution is formulated as a nonlinear programming problem, but it can be transformed to a linear programming problem by making use of the bisection method and phase one of the simplex method (Sakawa 1983), the variable transformation (Charnes and Cooper 1962) and the relaxation procedure (Shimizu and Aiyoshi 1980).

3.2 Problem Formulation and Solution Concepts

Definition 3.1 (Zero-sum fuzzy matrix game) When Player I chooses a pure strategy i and Player II chooses a pure strategy j, let $\tilde{a}_{ij}$ be a fuzzy payoff for Player I and $-\tilde{a}_{ij}$ be a fuzzy payoff for Player II. The fuzzy payoffs are represented by the L-R fuzzy numbers, i.e.,

$$\tilde{a}_{ij} = (a_{ij},\ \alpha_{ij},\ \beta_{ij})_{LR}, \tag{3.1}$$

where a_{ij} is a mean value, α_{ij} is a right spread and β_{ij} is a left spread. The two-person zero-sum fuzzy matrix game can be represented as a fuzzy payoff matrix:

$$\tilde{A} = \begin{bmatrix} \tilde{a}_{11} & \tilde{a}_{12} & \cdots & \tilde{a}_{1n} \\ \tilde{a}_{21} & \tilde{a}_{22} & \cdots & \tilde{a}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{a}_{m1} & \tilde{a}_{m2} & \cdots & \tilde{a}_{mn} \end{bmatrix}. \tag{3.2}$$

Games defined by (3.2) are called two-person zero-sum fuzzy matrix games.

When each of the players chooses a strategy, a payoff for each of them is represented as a fuzzy number, but their outcome has a zero-sum structure such that, when one player receives a gain, the other player suffers an equal loss.

Two-person zero-sum multiobjective fuzzy matrix games can also be represented by multiple fuzzy payoff matrices

$$\tilde{A}^1 = \begin{bmatrix} \tilde{a}^1_{11} \cdots \tilde{a}^1_{1n} \\ \vdots \quad \ddots \quad \vdots \\ \tilde{a}^1_{m1} \cdots \tilde{a}^1_{mn} \end{bmatrix}, \tilde{A}^2 = \begin{bmatrix} \tilde{a}^2_{11} \cdots \tilde{a}^2_{1n} \\ \vdots \quad \ddots \quad \vdots \\ \tilde{a}^2_{m1} \cdots \tilde{a}^2_{mn} \end{bmatrix}, \cdots, \tilde{A}^r = \begin{bmatrix} \tilde{a}^r_{11} \cdots \tilde{a}^r_{1n} \\ \vdots \quad \ddots \quad \vdots \\ \tilde{a}^r_{m1} \cdots \tilde{a}^r_{mn} \end{bmatrix}, \tag{3.3}$$

where we assume that each of the two players has r objectives. Then a fuzzy expected payoff can be represented by an L-R fuzzy number. A fuzzy payoff can be extended to a fuzzy expected payoff by using mixed strategies in a procedure similar to the extension from a payoff to an expected payoff.

Definition 3.2 (Fuzzy expected payoff) For any pair of mixed strategies $x \in X$ and $y \in Y$, the kth fuzzy expected payoff of Player I is defined as the fuzzy number

$$\tilde{E}^k(x,y) = \left(\sum_{i=1}^{m}\sum_{j=1}^{n} a^k_{ij} x_i y_j,\ \sum_{i=1}^{m}\sum_{j=1}^{n} \alpha^k_{ij} x_i y_j,\ \sum_{i=1}^{m}\sum_{j=1}^{n} \beta^k_{ij} x_i y_j \right)_{LR} \tag{3.4}$$

characterized by the membership function

$$\mu_{\tilde{E}^k(x,y)} : D^k \to [0,1], \tag{3.5}$$

where $D^k \in R$ is the domain of the kth payoff for Player I.

Addition and scalar multiplication on L-R fuzzy numbers are used in the definition of a fuzzy expected payoff (3.4).

Definition 3.3 (Fuzzy goal) Let the domain of the kth payoff for Player I be denoted $D^k \in R$. Then the fuzzy goal $\tilde{G}^k$ with respect to the kth payoff for Player I is defined as the fuzzy set on the set D^k characterized by the membership function

$$\mu_{\tilde{G}^k} : D^k \rightarrow [0,1]. \tag{3.6}$$

A membership function value of a fuzzy goal can be interpreted as a degree of attainment of the fuzzy goal. Then we assume that, for any pair of payoffs, a player prefers the payoff having the greater degree of attainment of the fuzzy goal to the other payoff.

Definition 3.4 (A degree of attainment of a fuzzy goal) For any pair of mixed strategies (x, y), let the kth fuzzy expected payoff for Player I be denoted $\tilde{E}^k(x, y)$ and let the kth fuzzy goal for Player I be denoted $\tilde{G}^k$. Then a fuzzy set expressing an attainment state of the fuzzy goal is represented by the intersection of the fuzzy expected payoff $\tilde{E}^k(x, y)$ and the fuzzy goal $\tilde{G}^k$. The membership function of the fuzzy set is represented as

$$\mu^k_{a(x,y)}(p) = \min\ (\ \mu_{\tilde{E}^k(x,y)}(p),\ \mu_{\tilde{G}^k}(p)\ ), \tag{3.7}$$

where $p \in D^k$ is a payoff for Player I. A degree of attainment of the kth fuzzy goal is defined as the maximum of the membership function (3.7), i.e.,

$$\begin{aligned} \hat{\mu}^k_{a(x,y)}(p^*) &= \max_p \mu^k_{a(x,y)}(p) \\ &= \max_p\{\min(\ \mu_{\tilde{E}^k(x,y)}(p),\ \mu_{\tilde{G}^k}(p)\ )\}. \end{aligned} \tag{3.8}$$

A degree of attainment of a fuzzy goal can be considered to be a concept similar to a degree of satisfaction of the fuzzy decision by Bellman and Zadeh (1970) when the fuzzy constraint can be replaced by the fuzzy expected payoff. When Players I and II choose strategies $\hat{x}$ and $\hat{y}$, respectively, the degree of attainment of the fuzzy goal $\hat{\mu}^k_{a(\hat{x},\hat{y})}(p^*)$ is determined by (3.8).

We assume that Player I supposes that Player II chooses a strategy y so as to minimize Player I's degree of attainment of the fuzzy goal $\hat{\mu}^k_{a(\hat{x},\hat{y})}(p^*)$, i.e., Player I's degree of attainment of the fuzzy goal, assuming he uses x, will be

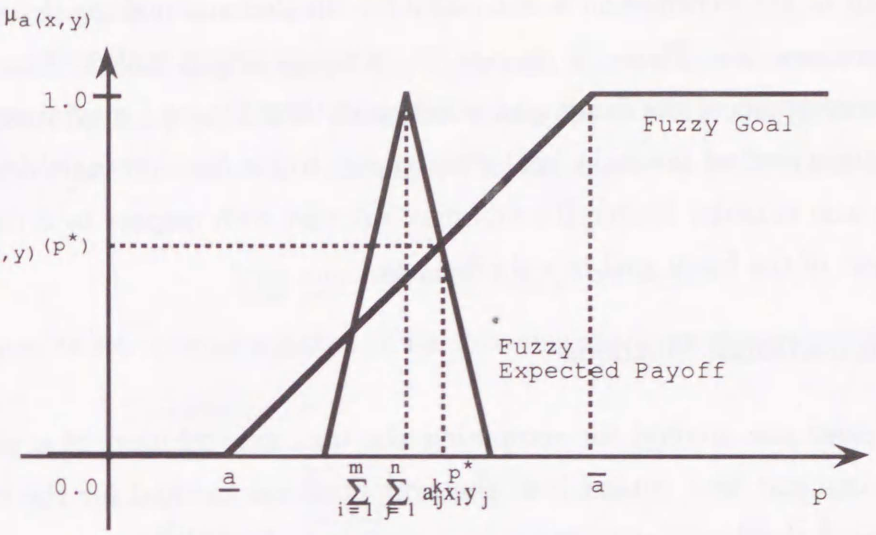

Fig. 3.1 A degree of attainment of a fuzzy goal

$e^k(x) = \min_{y \in Y} \hat{\mu}^k_{a(\hat{x},\hat{y})}(p^*)$. Hence, Player I chooses a strategy so as to maximize his degree of attainment of the fuzzy goal $e^k(x)$. In short, we assume that Player I behaves according to the max-min principle in terms of a degree of attainment of his fuzzy goal.

We usually consider the vector optimization for the multiple objectives, but each of the measures for objectives can be transformed to the measure of the degree of attainment of the fuzzy goal as a commensurable measure. Thus, we can consider max-min problems in terms of maximization and minimization of the degree of attainment of the aggregated fuzzy goal.

Definition 3.5 (A max-min solution with respect to a degree of attainment of a fuzzy goal) For any pair of mixed strategies (x, y), let the aggregated degree of attainment of the fuzzy goal for Player I be denoted $\hat{\mu}_{a(x,y)}(p^*)$. Then Player I's max-min value with respect to a degree of attainment of the fuzzy goal is

$$\max_{x \in X} \min_{y \in Y} \hat{\mu}_{a(x,y)}(p^*), \tag{3.9}$$

and such a strategy x is called the max-min solution with respect to the degree of attainment of the fuzzy goal.

The max-min solution can be considered to be the solution maximizing the function, which is the minimal value of the function with respect to the opponent's decision variables. We assume that a player has no information about his opponent or the information is not useful for the decision making if he has. Player I supposes that Player II chooses the strategy which makes Player I's degree of attainment of the fuzzy goal worst, and then Player I maximizes his degree of attainment of the fuzzy goal with respect to his decision variables.

We can also consider Player II's min-max solution with respect to a degree of attainment of the fuzzy goal in a similar way.

3.3 Computational Methods

We consider the method for computing the max-min solution of a single objective game and then extend it to the computational method for the max-min solution of a multiobjective game.

3.3.1 Single-Objective Fuzzy Matrix Games with Fuzzy Goals

Let $\tilde{A} = \tilde{A}^1$, $\tilde{G} = \tilde{G}^1$ and $\tilde{E}(x, y) = \tilde{E}^1(x, y)$. We assume that membership functions of fuzzy goals and a shape function for fuzzy numbers representing fuzzy payoffs are linear. A membership function of Player I's fuzzy goal is represented as

$$\mu_{\tilde{G}}(p) = \begin{cases} 0 & \text{if } p \leq \underline{a} \\ (p - \underline{a})/(\overline{a} - \underline{a}) & \text{if } \underline{a} \leq p \leq \overline{a} \\ 1 & \text{if } \underline{a} \leq p, \end{cases} \tag{3.10}$$

where $\underline{a}$ is the payoff giving the worst degree of satisfaction to Player I and $\overline{a}$ is the payoff giving the best degree of satisfaction to Player I. Namely, Player I is not satisfied by a payoff less than $\underline{a}$ but is fully satisfied by a payoff greater than $\overline{a}$. Let a shape function for fuzzy numbers be

$$L(p) = R(p) = \max(0,\ 1 - |p|). \tag{3.11}$$

When Players I and II choose pure strategies i and j, respectively, a payoff for Player I is represented as the fuzzy number $\tilde{a}_{ij} = (a_{ij}, \alpha_{ij}, \beta_{ij})_{LR}$ characterized by a membership function

$$\mu_{\tilde{a}_{ij}}(p) = \begin{cases} 0 & \text{if } p \leq a_{ij} - \alpha_{ij} \\ (p - a_{ij} + \alpha_{ij})/\alpha_{ij} & \text{if } a_{ij} - \alpha_{ij} \leq p \leq a_{ij} \\ (a_{ij} + \beta_{ij} - p)/\beta_{ij} & \text{if } a_{ij} \leq p \leq a_{ij} + \beta_{ij} \\ 1 & \text{if } a_{ij} + \beta_{ij} \leq p. \end{cases} \tag{3.12}$$

Theorem 3.1

Let a membership function of a fuzzy goal and a shape function of L-R fuzzy numbers for fuzzy payoffs be linear functions such as (3.10) and (3.12). A solution for the max-min problem with respect to the degree of attainment of the fuzzy goal

$$\max_{x\in X}\min_{y\in Y}\max_{p}\min(\ \mu_{\tilde{E}(x,y)}(p),\ \mu_{\tilde{G}}(p)\ ) \tag{3.13}$$

is equal to an optimal solution of the following nonlinear programming problem:

$$\begin{aligned} &\underset{(x,\sigma)}{\text{maximize}} \quad \sigma \\ &\text{subject to} \quad \frac{\sum_{i=1}^{m}\sum_{j=1}^{n}(a_{ij}+\beta_{ij})x_i y_j - \underline{a}}{\sum_{i=1}^{m}\sum_{j=1}^{n}\beta_{ij}x_i y_j + \overline{a} - \underline{a}} \geq \sigma, \quad \forall y \in Y \\ &\qquad\qquad \sum_{i=1}^{m} x_i = 1, \end{aligned} \tag{3.14}$$

when the optimal solution σ^* satisfies $0 \leq \sigma^* \leq 1$. The problem (3.14) is a nonlinear programming problem which has decision variables $x_i, i = 1, 2, \ldots, m$ and σ, and has an infinite number of inequality constraints and one equality constraint.

Proof For any pair of mixed strategies x and y, Player I's degree of attainment of the fuzzy goal is represented as

$$\begin{aligned} \mu_{a(x,y)}(p^*) &= \max_{p}\min(\ \mu_{\tilde{E}(x,y)}(p),\ \mu_{\tilde{G}}(p)\ ) \\ &= \frac{\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}x_i y_j + \sum_{i=1}^{m}\sum_{j=1}^{n}\beta_{ij}x_i y_j - \underline{a}}{\overline{a} - \underline{a} + \sum_{i=1}^{m}\sum_{j=1}^{n}\beta_{ij}x_i y_j}, \end{aligned} \tag{3.15}$$

and the payoff corresponding to the degree of attainment becomes a function of x and y, i.e.,

$$p^*(x,y) = \frac{\sum_{i=1}^{m}\sum_{j=1}^{n}((\overline{a} - \underline{a})a_{ij} + \overline{a}\beta_{ij}x_i y_j)}{\sum_{i=1}^{m}\sum_{j=1}^{n}\beta_{ij}x_i y_j + \overline{a} - \underline{a}}. \tag{3.16}$$

Therefore, the max-min problem with respect to the degree of attainment of the fuzzy goal is represented as

$$\begin{aligned}
&\max_{x\in X}\min_{y\in Y}\max_{p}\min(\ \mu_{\tilde{E}(x,y)}(p), \mu_{\tilde{G}}(p)\ ) \\
=\ &\max_{x\in X}\min_{y\in Y} \frac{\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}x_iy_j + \sum_{i=1}^{m}\sum_{j=1}^{n} \beta_{ij}x_iy_j - \underline{a}}{\overline{a} - \underline{a} + \sum_{i=1}^{m}\sum_{j=1}^{n} \beta_{ij}x_iy_j}.
\end{aligned} \tag{3.17}$$

Since the constraints of maximizing decision variable x and the minimizing decision variable y in the problem (3.17) are separated each other, the max-min solution can be determined by solving the following mathematical programming problem by introducing an auxiliary variable σ:

$$\begin{aligned}
&\underset{(x,\sigma)}{\text{maximize}} \quad \sigma \\
&\text{subject to} \quad \min_{y\in Y} \frac{\sum_{i=1}^{m}\sum_{j=1}^{n}(a_{ij}+\beta_{ij})x_iy_j - \underline{a}}{\sum_{i=1}^{m}\sum_{j=1}^{n} \beta_{ij}x_iy_j + \overline{a} - \underline{a}} \geq \sigma \\
&\qquad\qquad\quad \sum_{i=1}^{m} x_i = 1.
\end{aligned} \tag{3.18}$$

Since the condition

$$\min_{y\in Y} \frac{\sum_{i=1}^{m}\sum_{j=1}^{n}(a_{ij}+\beta_{ij})x_iy_j - \underline{a}}{\sum_{i=1}^{m}\sum_{j=1}^{n} \beta_{ij}x_iy_j + \overline{a} - \underline{a}} \geq \sigma \tag{3.19}$$

in (3.18) is equivalent to the following condition

$$\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}(a_{ij}+\beta_{ij})x_iy_j - \underline{a}}{\sum_{i=1}^{m}\sum_{j=1}^{n} \beta_{ij}x_iy_j + \overline{a} - \underline{a}} \geq \sigma, \quad \forall y \in Y, \tag{3.20}$$

the problem (3.18) is equivalent to the problem (3.14). □

If $\sigma < 0$, the max-min value becomes 0, and, if $\sigma > 1$, it becomes 1. From the following inequalities

$$\sum_{i=1}^{m}\sum_{j=1}^{n}(a_{ij}+\beta_{ij})x_iy_j \geq \min_{(i,j)}(a_{ij}+\beta_{ij}) \triangleq m \tag{3.21}$$

and

$$\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}x_iy_j \leq \max_{(i,j)} a_{ij} \triangleq M, \tag{3.22}$$

sufficient conditions such that the optimal solution σ^* of the problem (3.14) satisfies $0 \leq \sigma^* \leq 1$ are

$$m \geq \underline{a} \tag{3.23}$$

and

$$M \leq \overline{a}. \tag{3.24}$$

Since the constraints of maximizing decision variable x and the minimizing decision variable y in the problem (3.17) are separated each other, we can calculate the max-min solution defined in the previous section by applying the method based on the relaxation procedure by Shimizu and Aiyoshi (1980).

Consider the following relaxed problem for the problem (3.14) by taking L points $y_j^l, l = 1, 2, \ldots, L$ satisfying $\sum_{j=1}^{n} y_j^l = 1$.

$$\begin{aligned} &\underset{(x,\sigma)}{\text{maximize}} \quad \sigma \\ &\text{subject to} \quad \frac{\sum_{i=1}^{m}\sum_{j=1}^{n}(a_{ij}+\beta_{ij})x_iy_j^l - \underline{a}}{\sum_{i=1}^{m}\sum_{j=1}^{n}\beta_{ij}x_iy_j^l + \overline{a} - \underline{a}} \geq \sigma, \quad l = 1, 2, \ldots, L \\ &\qquad\qquad \sum_{i=1}^{m} x_i = 1. \end{aligned} \tag{3.25}$$

Let an optimal solution of the relaxed problem (3.25) be denoted (x^L, σ^L). If (x^L, σ^L) is feasible for the original problem (3.14), it must be optimal for (3.14). The test for feasibility (i.e., whether the optimal solution (x^L, σ^L) of the relaxed problem (3.25) is feasible for the original problem (3.14) or not) and the generation of the most violated constraint can be accomplished by solving the following minimization problem:

$$\begin{aligned} &\underset{y}{\text{minimize}} \quad \frac{\sum_{i=1}^{m}\sum_{j=1}^{n}(a_{ij}+\beta_{ij})x_i^Ly_j - \underline{a}}{\sum_{i=1}^{m}\sum_{j=1}^{n}\beta_{ij}x_i^Ly_j + \overline{a} - \underline{a}} \\ &\text{subject to} \quad \sum_{j=1}^{n} y_j = 1. \end{aligned} \tag{3.26}$$

Let an optimal solution of the minimization problem (3.26) be denoted $y^{L+1} \triangleq \hat{y}(x^L)$. If $(x^L, \hat{y}(x^L), \sigma^L)$ satisfies the constraints of the original problem (3.14), it must be optimal for (3.14). If it does not satisfy them, add the constraint

$$\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}(a_{ij}+\beta_{ij})x_i y_j^{L+1} - \underline{a}}{\sum_{i=1}^{m}\sum_{j=1}^{n}\beta_{ij}x_i y_j^{L+1} + \overline{a} - \underline{a}} \geq \sigma \tag{3.27}$$

to the relaxed problem (3.25) and solve it again. The constraint (3.27) violates the constraint of the original problem (3.14) to the greatest extent. The optimal solution of the original problem (3.14) can be obtained by repeating this procedure in a finite number of iterations (Shimizu and Aiyoshi 1980), but it is supposed that solving the relaxed problem (3.25) is difficult because it has nonlinear constraints.

However, we can reduce the relaxed problem (3.25), which is a linear fractional programming problem, to a linear programming problem by using Sakawa's method (1983). His method is based on the bisection method, which searches for a solution by repeatedly dividing the range of the variable into two parts, and phase one of the simplex method, which tests feasibility of a linear programming problem and finds a feasible solution if it is feasible.

The variable σ in the relaxed problem (3.25) satisfies the condition $0 \leq \sigma \leq 1$ because the variable σ corresponds to the max-min value with respect to a degree of attainment of a fuzzy goal. Let $\sigma = \hat{\sigma}$, where $\hat{\sigma}$ is a constant value in $[0, 1]$. Then the constraints of the relaxed problem (3.25) become as follows:

$$\begin{aligned} &\sum_{i=1}^{m}\sum_{j=1}^{n}(a_{ij}+\beta_{ij})x_i y_j^{l} - \underline{a} \geq \hat{\sigma}\Big(\sum_{i=1}^{m}\sum_{j=1}^{n}\beta_{ij}x_i y_j^{l} + \overline{a} - \underline{a}\Big), \quad l = 1, 2, \ldots, L \\ &\sum_{i=1}^{m} x_i = 1. \end{aligned} \tag{3.28}$$

The test for feasibility (i.e., whether the problem with the constraints (3.28) is feasible or not) can be accomplished by using phase one of the simplex method. If it is feasible, renew the constant value $\hat{\sigma}$ as follows:

$$\hat{\sigma} \leftarrow \hat{\sigma} + \frac{1}{2}\hat{\sigma}. \tag{3.29}$$

If it is not feasible, renew the constant value $\hat{\sigma}$ as follows:

$$\hat{\sigma} \leftarrow \hat{\sigma} - \frac{1}{2}\hat{\sigma}. \tag{3.30}$$

Then the test for feasibility is executed again after renewing the constant value $\hat{\sigma}$. We can find the maximal constant value $\hat{\sigma}$ by repeating this procedure in a finite number of iterations. Then the feasible solution x^* and the maximal constant value $\hat{\sigma}$ must be the optimal solution $(x^*, \sigma^* = \hat{\sigma})$ of the relaxed problem (3.25).

For this method, when $\sigma = 0$ and the problem (3.25) is not feasible, i.e., an optimal value of the problem (3.25) is less than 0, the max-min strategy cannot be determined. However, by resetting the lower limit of the domain of σ to a value smaller than 0, the max-min strategy for which the degree of attainment of the fuzzy goal is 0 can be determined.

The minimization problem (3.26), which generates the most violated constraint, can be reduced to a linear programming problem by using the variable transformation by Charnes and Cooper (1962). Set

$$1 \Big/ \left(\sum_{i=1}^{m} \sum_{j=1}^{n} \beta_{ij} x_i^L y_j + \overline{a} - \underline{a} \right) = t, \tag{3.31}$$

and

$$y_j t = z_j. \tag{3.32}$$

The minimization problem (3.26) can be rewritten as follows:

$$\begin{array}{ll} \underset{(z,t)}{\text{minimize}} & \sum_{i=1}^{m} \sum_{j=1}^{n} (a_{ij} + \beta_{ij}) x_i^L z_j - \underline{a}\, t \\ \text{subject to} & \sum_{j=1}^{n} z_j = t \\ & \sum_{i=1}^{m} \sum_{j=1}^{n} \beta_{ij} x_i^L z_j + (\overline{a} - \underline{a})\, t = 1. \end{array} \tag{3.33}$$

The problem (3.33) is a linear programming problem which has decision variables $z_j, j = 1, 2, \ldots, n$ and t, and has two equality constraints.

Thus we can obtain the max-min solution with respect to a degree of attainment of a fuzzy goal by repeating the following procedure. i) Compute an optimal solution of the relaxed problem (3.25) by the combined use of the bisection method and phase one of the simplex method and ii) solve the linear programming problems to which the minimization problem (3.26) is reduced by the variable transformation.

The algorithm for computing the max-min solution of fuzzy single-objective matrix games can be summarized in the following steps.

Algorithm 3.1

[**Step 1**]

Identify the fuzzy goal for the payoff. Choose any initial point $y^1 \in Y$ and set $l = 1$. Then formulate a relaxed problem (3.25), which is a linear fractional programming problem.

[**Step 2**]

Formulate the constraints (3.28) by setting $\sigma = \hat{\sigma}$ in the constraints of the relaxed problem (3.25) . Compute an optimal solution (x^*, σ^*) by making use of the bisection method and phase one of the simplex method. Then set $x^L = x^*$.

[**Step 3**]

Formulate the minimization linear programming problem (3.33) with x^L.

[**Step 4**]

Solve the problem (3.33) and obtain an optimal solution (z^*, t^*). Let the objective function value be denoted $\phi(z^*, t^*)$.

[**Step 5**]

If $\phi(z^*, t^*) \geq \sigma^* + \varepsilon$, terminate, where ε is the predetermined constant. Then x^L is a max-min solution with respect to a degree of attainment of a fuzzy goal. Otherwise, i.e., if $\phi(z^*, t^*) < \sigma^* + \varepsilon$, set $l = l + 1$ and go back to [Step 2].

Theorem 3.2

For any given $\varepsilon > 0$, the above algorithm for the max-min problem (3.17) terminates in a finite number of iterations.

Proof The theorem can be proved by a procedure similar to the theorem (Shimizu and Aiyoshi) in the Appendix. □

We can also obtain Player II's min-max solution with respect to a degree of attainment of a fuzzy goal in a similar way.

The computational method for the max-min solution has been given by Algorithm 3.1, which utilizes Sakawa's method, Shimizu and Aiyoshi's relaxation procedure, and Charnes and Cooper's variable transformation. We now present the other method for computing the solution. We observe that the constraints of the linear programming problem (3.33) consist of two equalities. This means that two decision variables become basic variables and the rest of $n - 1$ decision variables are non-basic variables, i.e., $n - 1$ decision variables of an optimal solution become 0. Moreover, from the constraints of the problem (3.33), it follows that $t \neq 0$. If $t = 0$, we have

$$\sum_{i=1}^{m}\sum_{j=1}^{n} \beta_{ij} x_i^L z_j = 1, \tag{3.34}$$

and at least one z_j becomes positive. This contradicts the first constraint. Thus, for a certain $\hat{j}$, $z_{\hat{j}} \neq 0$, and, for the rest of $j \neq \hat{j}$, $z_j = 0$. Therefore, an optimal solution of the problem (3.33) is restricted by n possible cases, and it is found that the solution can be obtained by at most n iterations. This is another proof of Theorem 3.2.

Furthermore, from the variable transformation (3.32) and $z_{\hat{j}} = t$, we have

$$y_j = \begin{cases} 1 & \text{if } j = \hat{j} \\ 0 & \text{other.} \end{cases} \tag{3.35}$$

Therefore, since the first constraint of the original problem (3.14) can be replaced with n inequities, the problem (3.14) is equivalent to the following problem:

$$\begin{aligned} & \underset{(x,\sigma)}{\text{maximize}} \quad \sigma \\ & \text{subject to} \quad \frac{\sum_{i=1}^{m}(a_{i1} + \beta_{i1})x_i - E_0}{\sum_{i=1}^{m} \beta_{i1} x_i + E_1 - E_0} \geq \sigma \\ & \qquad\qquad\quad \frac{\sum_{i=1}^{m}(a_{i2} + \beta_{i2})x_i - E_0}{\sum_{i=1}^{m} \beta_{i2} x_i + E_1 - E_0} \geq \sigma \\ & \qquad\qquad\qquad \vdots \\ & \qquad\qquad\quad \frac{\sum_{i=1}^{m}(a_{in} + \beta_{in})x_i - E_0}{\sum_{i=1}^{m} \beta_{in} x_i + E_1 - E_0} \geq \sigma \\ & \qquad\qquad\quad \sum_{i=1}^{m} x_i = 1. \end{aligned} \tag{3.36}$$

From the above examination, an optimal solution of the problem (3.36) can be obtained by utilizing only Sakawa's method. Since the number of constraints of the problem (3.36) is determined by the number n of Player II's strategies, the above method is more efficient than the computational method utilizing the relaxation procedure, Algorithm 3.1, when n is not large.

3.3.2 Multiobjective Fuzzy Matrix Games with Fuzzy Goals

Consider two-person zero-sum multiobjective fuzzy matrix games, i.e., two-person zero-sum games with multiple fuzzy payoff matrices $\tilde{A}^k$, $k = 1, 2, \ldots, r$. We assume that a player has a fuzzy goal for each of the objectives, which expresses the player's degree of satisfaction for a payoff. Let Player I's membership function of the fuzzy goal for the kth objective be denoted $\mu_{\tilde{G}^k}(p^k)$ for any payoff p^k.

When the membership function $\mu_{\tilde{G}^k}(p^k)$ of the fuzzy goal is a linear function, it can be represented as

$$\mu_{\tilde{G}^k}(p^k) = \begin{cases} 0 & \text{if } \ p^k \leq \underline{a}^k \\ 1 - \dfrac{\overline{a}^k - p^k}{\overline{a}^k - \underline{a}^k} & \text{if } \ \underline{a}^k \leq p^k \leq \overline{a}^k \\ 1 & \text{if } \ \overline{a}^k \leq p^k, \end{cases} \tag{3.37}$$

where, for the kth objective, $\underline{a}^k$ is the payoff giving the worst degree of satisfaction for Player I and $\overline{a}^k$ is the payoff giving the best degree of satisfaction for PlayerI.

Moreover, when the membership function $\mu_{\tilde{a}^k_{ij}}(p^k)$ of the entry $\tilde{a}^k_{ij}$, which is a fuzzy number, of the fuzzy payoff matrix $\tilde{A}^k$ for the kth objective is a linear function, it can be represented as

$$\mu_{\tilde{a}^k_{ij}}(p^k) = \begin{cases} 0 & \text{if } \ p^k \leq a^k_{ij} - \alpha^k_{ij} \\ (p^k - a^k_{ij} + \alpha^k_{ij})/\alpha^k_{ij} & \text{if } \ a^k_{ij} - \alpha^k_{ij} \leq p^k \leq a^k_{ij} \\ (a^k_{ij} + \beta^k_{ij} - p^k)/\beta^k_{ij} & \text{if } \ a^k_{ij} \leq p^k \leq a^k_{ij} + \beta^k_{ij} \\ 1 & \text{if } \ a^k_{ij} + \beta^k_{ij} \leq p^k. \end{cases} \tag{3.38}$$

In general, the degree of attainment of the fuzzy goal can be represented as the following vector expression:

$$\begin{pmatrix} \max\limits_{p^1} \min(\ \mu_{\tilde{E}^1(x,y)}(p^1),\ \mu_{\tilde{G}^1}(p^1)\ ) \\ \max\limits_{p^2} \min(\ \mu_{\tilde{E}^2(x,y)}(p^2),\ \mu_{\tilde{G}^2}(p^2)\ ) \\ \vdots \\ \max\limits_{p^r} \min(\ \mu_{\tilde{E}^r(x,y)}(p^r),\ \mu_{\tilde{G}^r}(p^r)\ ) \end{pmatrix}. \tag{3.39}$$

For such a problem, we employ the fuzzy decision rule by Bellman and Zadeh (1970), which is often used in decision making problems in fuzzy environments, as an aggregation rule of multiple fuzzy goals. Then the membership function of the aggregated fuzzy goal is expressed as

$$\hat{\mu}_{a(x,y)}(p^*) = \min_k \max_{p^k} \min(\ \mu_{\tilde{E}^k(x,y)}(p^k),\ \mu_{\tilde{G}^k}(p^k)\ ), \tag{3.40}$$

where $p^* = (p^{*1}, p^{*2}, \ldots, p^{*r})$.

Theorem 3.3

Let membership functions of fuzzy goals and a shape function of L-R fuzzy numbers for fuzzy payoffs be linear functions such as (3.37) and (3.38). A solution for the max-min problem with respect to the degree of attainment of the fuzzy goal

$$\max_{x\in X}\min_{y\in Y}\min_{k}\max_{p^k}\min(\ \mu_{\tilde{E}^k(x,y)}(p^k),\ \mu_{\tilde{G}^k}(p^k)\ ) \tag{3.41}$$

is equal to an optimal solution of the following nonlinear programming problem:

$$\begin{array}{ll} \underset{(x,\sigma)}{\text{maximize}} & \sigma \\ \text{subject to} & \dfrac{\sum_{i=1}^{m}\sum_{j=1}^{n}(a_{ij}^k + \beta_{ij}^k)x_i y_j - \underline{a}^k}{\sum_{i=1}^{m}\sum_{j=1}^{n}\beta_{ij}^k x_i y_j + \overline{a}^k - \underline{a}^k} \geq \sigma,\ \forall y \in Y,\ k = 1, 2, \ldots, r \\ & \sum_{i=1}^{m} x_i = 1, \end{array} \tag{3.42}$$

when the optimal solution σ^* satisfies $0 \leq \sigma^* \leq 1$. The problem (3.42) is a nonlinear programming problem which has decision variables $x_i, i = 1, 2, \ldots, m$ and σ, and has an infinite number of inequality constraints and one equality constraint.

Proof For any pair of mixed strategies x and y, Player I's degree of attainment of the fuzzy goal is represented as

$$\begin{aligned} \mu_{a(x,y)}(p^*) &= \min_{k}\max_{p^k}\min(\ \mu_{\tilde{E}^k(x,y)}(p^k),\ \mu_{\tilde{G}^k}(p^k)\ ) \\ &= \min_{k}\frac{\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}^k x_i y_j + \sum_{i=1}^{m}\sum_{j=1}^{n}\beta_{ij}^k x_i y_j - \underline{a}^k}{\overline{a}^k - \underline{a}^k + \sum_{i=1}^{m}\sum_{j=1}^{n}\beta_{ij}^k x_i y_j}. \end{aligned} \tag{3.43}$$

Therefore, the max-min problem with respect to the degree of attainment of the fuzzy goal is represented as

$$\begin{aligned} &\max_{x\in X}\min_{y\in Y}\min_{k}\max_{p^k}\min(\ \mu_{\tilde{E}^k(x,y)}(p^k),\ \mu_{\tilde{G}^k}(p^k)\ ) \\ &= \max_{x\in X}\min_{y\in Y}\min_{k}\frac{\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}^k x_i y_j + \sum_{i=1}^{m}\sum_{j=1}^{n}\beta_{ij}^k x_i y_j - \underline{a}^k}{\overline{a}^k - \underline{a}^k + \sum_{i=1}^{m}\sum_{j=1}^{n}\beta_{ij}^k x_i y_j}. \end{aligned} \tag{3.44}$$

Since the constraints of maximizing decision variable x and the minimizing decision variable y in the problem (3.44) are separated each other, the max-min solution can be determined by solving the following mathematical programming problem by introducing an auxiliary variable σ:

$$\begin{aligned} \underset{(x,\sigma)}{\text{maximize}} \quad & \sigma \\ \text{subject to} \quad & \min_{y\in Y} \min_{k} \frac{\sum_{i=1}^{m}\sum_{j=1}^{n}(a_{ij}^k + \beta_{ij}^k)x_i y_j - \underline{a}^k}{\sum_{i=1}^{m}\sum_{j=1}^{n}\beta_{ij}^k x_i y_j + \overline{a}^k - \underline{a}^k} \geq \sigma \\ & \sum_{i=1}^{m} x_i = 1. \end{aligned} \tag{3.45}$$

Since the condition

$$\min_{y\in Y} \min_{k} \frac{\sum_{i=1}^{m}\sum_{j=1}^{n}(a_{ij}^k + \beta_{ij}^k)x_i y_j - \underline{a}^k}{\sum_{i=1}^{m}\sum_{j=1}^{n}\beta_{ij}^k x_i y_j + \overline{a}^k - \underline{a}^k} \geq \sigma \tag{3.46}$$

in (3.45) is equivalent to the following conditions:

$$\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}(a_{ij} + \beta_{ij})x_i y_j - \underline{a}}{\sum_{i=1}^{m}\sum_{j=1}^{n}\beta_{ij} x_i y_j + \overline{a} - \underline{a}} \geq \sigma, \qquad \forall y \in Y,\ k = 1, 2, \ldots, r, \tag{3.47}$$

the problem (3.45) is equivalent to the problem (3.42). □

Determined by a method similar to a single-objective game, sufficient conditions such that the optimal solution σ^* of the problem (3.42) satisfies $0 \leq \sigma^* \leq 1$ are

$$\min_{(i,j)}(a_{ij}^k + \beta_{ij}^k) \triangleq m^k \geq \underline{a}^k, \quad k = 1, 2, \ldots, r \tag{3.48}$$

and

$$\max_{(i,j)} a_{ij}^k \triangleq M^k \leq \overline{a}^k, \quad k = 1, 2, \ldots, r. \tag{3.49}$$

The constraints of maximizing decision variable x and the minimizing decision variable y in the problem (3.44) are separated each other, so we can calculate the max-min solution with respect to a degree of attainment of a fuzzy goal by applying the method based on the relaxation procedure by Shimizu and Aiyoshi

(1980) in a process similar to a single-objective case. However, although the problem (3.44) is still a max-min problem, it has an extra min operator. Thus we have to revise the algorithm by the relaxation procedure.

Consider the following relaxed problem for the original problem (3.42) by taking L points $y_j^l, l = 1, 2, \ldots, L$, satisfying $\sum_{j=1}^n y_j^l = 1$:

$$\begin{array}{ll} \underset{(x,\sigma)}{\text{maximize}} & \sigma \\ \text{subject to} & \dfrac{\sum_{i=1}^m \sum_{j=1}^n (a_{ij}^k + \beta_{ij}^k) x_i y_j^l - \underline{a}^k}{\sum_{i=1}^m \sum_{j=1}^n \beta_{ij}^k x_i y_j^l + \overline{a}^k - \underline{a}^k} \geq \sigma, \quad l = 1, 2, \ldots, L, \ k = 1, 2, \ldots, r \\ & \sum_{i=1}^m x_i = 1. \end{array} \tag{3.50}$$

Let $\sigma = \hat{\sigma}$, where $\hat{\sigma}$ is a constant value in $[0, 1]$. Then the constraints of the relaxed problem (3.50) become as follows:

$$\begin{array}{ll} \sum_{i=1}^m \sum_{j=1}^n (a_{ij}^k + \beta_{ij}^k) x_i y_j^l - \underline{a}^k & \geq \hat{\sigma}\Big(\sum_{i=1}^m \sum_{j=1}^n \beta_{ij}^k x_i y_j^l + \overline{a}^k - \underline{a}^k\Big), \\ & l = 1, 2, \ldots, L, \ k = 1, 2, \ldots, r \\ \sum_{i=1}^m x_i = 1. & \end{array} \tag{3.51}$$

We can find the maximal constant value $\hat{\sigma}$ satisfying the constraints (3.51), so it follows that the pair of the feasible solution x^* and the maximal constant value $\hat{\sigma}$ must be an optimal solution $(x^*, \sigma^* = \hat{\sigma})$ of the relaxed problem (3.50).

The r minimization problems for the test of feasibility and the generation of the most violated constraint are represented as follows:

$$\left.\begin{array}{ll} \underset{y}{\text{minimize}} & \dfrac{\sum_{i=1}^m \sum_{j=1}^n (a_{ij}^k + \beta_{ij}^k) x_i^L y_j - \underline{a}^k}{\sum_{i=1}^m \sum_{j=1}^n \beta_{ij}^k x_i^L y_j + \overline{a}^k - \underline{a}^k} \\ \text{subject to} & \sum_{j=1}^n y_j = 1 \end{array}\right\}, \quad k = 1, 2, \ldots, r. \tag{3.52}$$

The above minimization problems (3.52) can be reduced to linear programming problems by using the following variable transformations. Set

$$1 \Big/ \Big(\sum_{i=1}^m \sum_{j=1}^n \beta_{ij}^k x_i^L y_j + \overline{a}^k - \underline{a}^k\Big) = t^k, \ k = 1, 2, \ldots, r, \tag{3.53}$$

and

$$y_j t^k = z_j^k, \;\; k = 1, 2, \ldots, r. \tag{3.54}$$

The minimization problem can be represented as the following r linear programming problems:

$$\left.\begin{array}{ll} \underset{(z^k, t^k)}{\text{minimize}} & \displaystyle\sum_{i=1}^{m}\sum_{j=1}^{n}(a_{ij}^k + \beta_{ij}^k)x_i^L z_j^k - \underline{a}^k \; t^k \\ \text{subject to} & \displaystyle\sum_{j=1}^{n} z_j^k = t^k \\ & \displaystyle\sum_{i=1}^{m}\sum_{j=1}^{n} \beta_{ij}^k x_i^L z_j^k + (\overline{a} - \underline{a}) \; t^k = 1. \end{array}\right\}, \quad k = 1, 2, \ldots, r. \tag{3.55}$$

The kth problem in (3.55) is a linear programming problem which has decision variables $z_j^k, j = 1, 2, \ldots, n$ and t^k, and has two equality constraints. Since there are r problems, the test for feasibility for the original problem and the generation of the most violated constraint can be accomplished by solving the r linear programming problems and finding the problem having the smallest optimal value.

The algorithm for computing the max-min solution of fuzzy multiobjective matrix games can be summarized in the following steps.

Algorithm 3.2

[**Step 1**]

Identify r fuzzy goals for payoffs. Choose any initial point $y^1 \in Y$ and set $l = 1$. Then formulate a relaxed problem (3.50), which is a linear fractional programming problem.

[**Step 2**]

Formulate the constraints (3.51) by setting $\sigma = \hat{\sigma}$ in the constraints of the relaxed problem (3.50). Compute an optimal solution (x^*, σ^*) by making use of the bisection method and phase one of the simplex method. Then set $x^L = x^*$.

[**Step 3**]

Formulate r minimization linear programming problems (3.55) with x^L.

[**Step 4**]

Solve r problems (3.55) and obtain r optimal solutions (z^{k^*}, t^{k^*}), $k = 1, 2, \ldots, r$. Let each of the minimal objective function values be denoted $\phi^k(z^{k^*}, t^{k^*})$, $k = 1, 2, \ldots, r$ and then let $\phi^{\hat{k}}(z^{\hat{k}^*}, t^{\hat{k}^*}) = \min_k \phi^k(z^{k^*}, t^{k^*})$.

[**Step 5**]

If $\phi^{\hat{k}}(z^{\hat{k}^*}, t^{\hat{k}^*}) \geq \sigma^* + \varepsilon$, terminate, where ε is the predetermined constant. Then x^L is a max-min solution with respect to a degree of attainment of a fuzzy goal. Otherwise, i.e., if $\phi^{\hat{k}}(z^{\hat{k}^*}, t^{\hat{k}^*}) < \sigma^* + \varepsilon$, set $l = l + 1$ and go back to [Step 2].

Theorem 3.4

For any given $\varepsilon > 0$, the above algorithm for the max-min problem (3.41) terminates in a finite number of iterations.

Proof The theorem can be proved by a procedure similar to the theorem (Shimizu and Aiyoshi) in the Appendix. □

We can also obtain Player II's min-max solution with respect to a degree of attainment of a fuzzy goal in a similar way.

Along the lines of the single-objective game, from the property of the constraints of the linear programming problem (3.55), the problem (3.42) is equivalent to the following problem:

$$
\begin{aligned}
&\underset{(x,\sigma)}{\text{maximize}} \quad \sigma \\
&\text{subject to} \quad \frac{\sum_{i=1}^{m}(a_{i1}^k + \beta_{i1}^k)x_i - \underline{a}^k}{\sum_{i=1}^{m}\beta_{i1}^k x_i + \overline{a}^k - \underline{a}^k} \geq \sigma, \ k = 1, 2, \ldots, r \\
&\qquad\qquad\quad \frac{\sum_{i=1}^{m}(a_{i2}^k + \beta_{i2}^k)x_i - \underline{a}^k}{\sum_{i=1}^{m}\beta_{i2}^k x_i + \overline{a}^k - \underline{a}^k} \geq \sigma, \ k = 1, 2, \ldots, r \\
&\qquad\qquad\quad \vdots \\
&\qquad\qquad\quad \frac{\sum_{i=1}^{m}(a_{in}^k + \beta_{in}^k)x_i - \underline{a}^k}{\sum_{i=1}^{m}\beta_{in}^k x_i + \overline{a}^k - \underline{a}^k} \geq \sigma, \ k = 1, 2, \ldots, r \\
&\qquad\qquad\quad \sum_{i=1}^{m} x_i = 1.
\end{aligned}
\tag{3.56}
$$

The number of the constraints of the problem (3.56) is $nr + 1$, which becomes larger as the numbers of Player II's strategies and objectives increase. Therefore, the method that includes the relaxation procedure, Algorithm 3.2, is considered to be efficient when the numbers of Player II's strategies and objectives are large.

Example 3.1

Consider a numerical example based on Cook's example (1976). Let each player have three pure strategies and three objectives, and let a two-person zero-sum multiobjective game be represented by

$$\tilde{A}^1 = \begin{bmatrix} (2,0.2,0.2), & (5,0.5,0.5), & (1,0.8,0.8) \\ (-1,0.8,0.8), & (-2,0.4,0.4), & (6,0.1,0.1) \\ (0,0.1,0.1), & (3,0.5,0.5), & (-1,0.8,0.8) \end{bmatrix},$$

$$\tilde{A}^2 = \begin{bmatrix} (-3,0.8,0.8), & (7,0.3,0.3), & (2,0.4,0.4) \\ (0,0.5,0.5), & (-2,0.2,0.2), & (0,0.7,0.7) \\ (3,0.4,0.4), & (-1,0.8,0.8), & (-6,0.5,0.5) \end{bmatrix},$$

and

$$\tilde{A}^3 = \begin{bmatrix} (8,0.1,0.1), & (-2,0.5,0.5), & (3,0.7,0.7) \\ (-5,0.5,0.5), & (6,0.4,0.4), & (0,0.6,0.6) \\ (-3,0.8,0.8), & (1,0.6,0.6), & (6,0.1,0.1) \end{bmatrix}.$$

Let fuzzy goals G^1, G^2 and G^3 of Player I for the three objectives be represented by the following linear membership functions:

$$\mu_{G^1}(p^1) = \begin{cases} 0 & \text{if } \; p^1 \leq -1 \\ (p^1+1)/7.5 & \text{if } \; -1 \leq p^1 \leq 6.5 \\ 1 & \text{if } \; 6.5 \leq p^1, \end{cases}$$

$$\mu_{G^2}(p^2) = \begin{cases} 0 & \text{if } \; p^2 \leq -2 \\ (p^2+2)/7.5 & \text{if } \; -2 \leq p^2 \leq 5.5 \\ 1 & \text{if } \; 5.5 \leq p^2, \end{cases}$$

and

$$\mu_{G^3}(p^3) = \begin{cases} 0 & \text{if } \; p^3 \leq -1 \\ (p^3+1)/6.8 & \text{if } \; -1 \leq p^3 \leq 5.8 \\ 1 & \text{if } \; 5.8 \leq p^3. \end{cases}$$

We computed the max-min solution by two methods, which were Algorithm 3.2 and the method directly solving the problem (3.56) by Sakawa's method, and obtained the same solution:

$$x_1 = 0.51976, \quad x_2 = 0.23447, \quad \text{and} \quad x_3 = 0.24577.$$

The degree of attainment of the fuzzy goal for the max-min solution was 0.22252. In Algorithm 3.2, we set the initial value at $y_1 = 0$, $y_2 = 1$, $y_3 = 0$, and the number of iterations was three times.

3.4 Conclusion

In this chapter, we treated games with fuzzy payoffs in the framework examined in Chapter 2 and developed the computational methods for the solutions.

To conclude, the results of this chapter are summarized as follows.

1) In two-person zero-sum multiobjective matrix games, we have represented entries of payoff matrices as fuzzy numbers in order to express ambiguity and imprecision of information about decision making problems under conflict.

2) To consider the imprecise nature of human judgment, we have employed fuzzy goals, as we did in the previous chapter, and have expressed competitive systems as two-person zero-sum multiobjective fuzzy matrix games with fuzzy goals.

3) When membership functions of fuzzy goals and a shape function of fuzzy number entries in a fuzzy payoff matrix can be constructed as linear functions, a method that utilizes three techniques for computing the solutions has been developed. The first technique is Sakawa's method, which is based on the bisection method and phase one of the simplex method for solving nonlinear problems which have a variable with a closed admissible interval in nonlinear terms. The second technique is the variable transformation by Charnes and Cooper, which is used to transform linear fractional terms to linear ones. The third technique is the relaxation procedure for min-max problems by Shimizu and Aiyoshi.

Chapters 2 and 3 have been devoted to examining two-person zero-sum games with single and multiple payoffs. We have considered max-min solutions of such games and have developed the computational methods for their solutions. The next chapter will deal with more general two-person games; i.e., two-person non-zero-sum games with single and multiple payoffs for which equilibrium solutions of the games will be considered.

CHAPTER 4

TWO-PERSON NON-ZERO-SUM MULTIOBJECTIVE BIMATRIX GAMES WITH FUZZY GOALS

4.1 Introduction

So far we have examined only zero-sum games. In this chapter, we consider a two-person non-zero-sum bimatrix game with single and multiple payoffs, which is a generalization of the games discussed in the previous chapters. Such a game would be called a non-zero-sum or general-sum game, which includes the zero-sum game as a special case, and is also referred to as a bimatrix game because it can be expressed as a pair of payoff matrices. Cooperation between the players can be seen in such situations, but in this chapter the noncooperative case will be treated. A max-min solution is also an equilibrium solution in a conventional two-person zero-sum game, but a max-min solution with respect to a degree of attainment of a fuzzy goal does not always possess the equilibrium property. The max-min solution is considered to be more conservative than equilibrium solutions (Nash 1951).

For studies on equilibrium solutions of multiobjective games, Wierzbicki (1990) defined equilibrium solutions based on order relations, using several preference cones and optimality criteria such as Pareto optimality for noncooperative multiobjective n-person games with nonlinear payoff functions. Furthermore, he theoretically analyzed relations between equilibrium solutions for multiobjective games and equilibrium solutions for single-objective proxy games with payoffs equal to scalarizing functions. Corley (1985) defined equilibrium solutions for multiobjective bimatrix games by using $R^n_+\backslash\{0\}$ as a preference cone and developed a method for computing the solutions. Borm, Tijs and van den Aarssen (1988) defined a proxy single-objective game with payoffs equal to a scalarizing function with weighting coefficients in multiobjective bimatrix games and discussed the existence of equilibrium solutions for the original multiobjective bimatrix games through the existence of the equilibrium solutions for the single objective proxy game. No studies, however, have ever been tried for performing multiobjective games in fuzzy environments.

We will examine equilibrium solutions in terms of a degree of attainment of a fuzzy goal for games in fuzzy and multiobjective environments (Nishizaki and Sakawa 1993). First, we introduce a fuzzy goal for a payoff in order to incorporate ambiguity of human judgments and assume that a player tries to maximize his degree of attainment of the fuzzy goal as we did in the previous chapters.

In section 4.2, a fuzzy goal for a payoff and the equilibrium solution with respect to a degree of attainment of the fuzzy goal are defined. In section 4.3, two basic methods, one by weighting coefficients and the other by a minimum component, are employed to aggregate multiple fuzzy goals. When membership functions are linear functions, the computational methods for the equilibrium solutions are developed. It is shown that the equilibrium solutions are equal to optimal solutions of mathematical programming problems in both cases. This means that we can obtain the equilibrium solutions by solving the mathematical programming problems. In section 4.4, we consider the relation between equilibrium solutions for multiobjective bimatrix games incorporating fuzzy goals and the Pareto optimal equilibrium solutions defined in Borm, Tijs and Aarssen (1988) or Wierzbicki (1990). The set of the Pareto optimal equilibrium solutions in such games often contains sets of continuum power; we can, however, select restricted and reasonable solutions on the assumption that a player has fuzzy goals and tries to maximize the degrees of attainment for the fuzzy goals.

4.2 Problem Formulation and Solution Concepts

Two-person non-zero-sum games can be expressed as a pair of $m \times n$ matrices,

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix}. \quad (4.1)$$

When Players I and II choose their ith and jth pure strategies, respectively, a_{ij} is the payoff for Player I and b_{ij} is the payoff for Player II.

Definition 4.1 (Equilibrium solution) For a bimatrix game (A, B), an equilibrium solution is a pair of strategies m-dimensional vector x^* and n-

dimensional vector y^* if, for any other mixed strategies x and y,

$$\begin{aligned} x^*Ay^* &\geq xAy^* \\ x^*By^* &\geq x^*By. \end{aligned} \tag{4.2}$$

It has been proved that every bimatrix game (A, B) has at least one equilibrium solution (Nash 1951).

Two-person non-zero-sum multiobjective games can be also expressed as multiple $m \times n$ matrices,

$$A^1 = \begin{bmatrix} a_{11}^1 \cdots a_{1n}^1 \\ \vdots \ \ddots \ \vdots \\ a_{m1}^1 \cdots a_{mn}^1 \end{bmatrix}, A^2 = \begin{bmatrix} a_{11}^2 \cdots a_{1n}^2 \\ \vdots \ \ddots \ \vdots \\ a_{m1}^2 \cdots a_{mn}^2 \end{bmatrix}, \cdots, A^r = \begin{bmatrix} a_{11}^r \cdots a_{1n}^r \\ \vdots \ \ddots \ \vdots \\ a_{m1}^r \cdots a_{mn}^r \end{bmatrix},$$

$$B^1 = \begin{bmatrix} b_{11}^1 \cdots b_{1n}^1 \\ \vdots \ \ddots \ \vdots \\ b_{m1}^1 \cdots b_{mn}^1 \end{bmatrix}, B^2 = \begin{bmatrix} b_{11}^2 \cdots b_{1n}^2 \\ \vdots \ \ddots \ \vdots \\ b_{m1}^2 \cdots b_{mn}^2 \end{bmatrix}, \cdots, B^s = \begin{bmatrix} b_{11}^s \cdots b_{1n}^s \\ \vdots \ \ddots \ \vdots \\ b_{m1}^s \cdots b_{mn}^s \end{bmatrix}, \tag{4.3}$$

where Player I has r objectives and Player II has s objectives.

Equilibrium problems for single-objective games are considered in terms of expected payoffs such as (4.2). On the other hand, since each of the payoffs represents an objective or an attribute and has a different unit of measure in multiobjective games, vector optimization is usually considered; however, we will employ another approach. Namely, we incorporate fuzzy goals for objectives and consider the equilibrium problems in terms of maximization of the degree of attainment for the aggregated fuzzy goal. Each of the measures for objectives can be transformed to the degree of attainment of the fuzzy goal as a commensurable measure.

Definition 4.2 (Fuzzy goal) Let Player I's payoff and Player II's payoff be denoted $p_1 = (p_1^1, \ldots, p_1^k, \ldots, p_1^r) \in D_1$ and $p_2 = (p_2^1, \ldots, p_2^l, \ldots, p_2^s) \in D_2$, respectively, where $D_1 = D_1^1 \times \cdots \times D_1^r \subseteq R^r$ is the set of Player I's payoffs and $D_2 = D_2^1 \times \cdots \times D_2^s \subseteq R^s$ is the set of Player II's payoffs. Then Player I's fuzzy goal $\tilde{G}_1^k$ for the kth payoff is a fuzzy set on the set D_1^k characterized by the membership function

$$\mu_1^k : D_1^k \to [0, 1]. \tag{4.4}$$

Player II's fuzzy goal $\tilde{G}_2^l$ for the lth payoff is also a fuzzy set characterized by the membership function

$$\mu_2^l : D_2^l \to [0, 1]. \tag{4.5}$$

In bimatrix games, when Player I chooses a mixed strategy x and Player II chooses a mixed strategy y, the kth payoff of Player I is represented by an expected payoff, i.e., $p_1^k = xA^k y$, and the lth payoff of Player II is $p_2^l = xB^l y$. For Player I, the membership function value $\mu_1^k(xA^k y)$ of the fuzzy goals can be interpreted as a degree of attainment of the fuzzy goal for the payoff $xA^k y$ or a degree of satisfaction with respect to the payoff $xA^k y$. That of Player II can be interpreted similarly.

Equilibrium conditions in multiobjective decision making must be examined under the partial order relation if each of the objectives has incommensurable measures. It is supposed that equilibrium solutions under such a formulation often exist infinitely.

In this chapter we assume that a player has a fuzzy goal for each of the objectives and employ the degree of attainment of the fuzzy goal as a commensurable measure. We do not treat multiple payoffs directly but treat the single value, which is the degree of attainment, by aggregating multiple fuzzy goals. Then we consider equilibrium problems with respect to the degree of attainment of the aggregated fuzzy goal.

Definition 4.3 (An equilibrium solution with respect to the degree of attainment of the aggregated fuzzy goal) When Player I chooses a mixed strategy x and Player II chooses a mixed strategy y in a multiobjective bimatrix game (A^k, B^l), $k = 1, 2, \dots, r$, $l = 1, 2, \dots, s$, let Player I's membership function for the aggregated fuzzy goal be $\mu_1(x, y)$ and Player II's membership function be $\mu_2(x, y)$. Then a pair of strategies x^* and y^* is said to be an equilibrium solution with respect to the degree of attainment of the aggregated fuzzy goal if, for any other mixed strategies x and y,

$$\begin{aligned} \mu_1(x^*, y^*) &\geq \mu_1(x, y^*) \\ \mu_2(x^*, y^*) &\geq \mu_2(x^*, y). \end{aligned} \tag{4.6}$$

If the membership function μ_1 in the above definition can be regarded as Player I's payoff function and the membership function μ_2 can be regarded as Player II's payoff function, the game (A^k, B^l) can be reduced to an ordinary two-person bimatrix game. If the function μ_1 and μ_2 are convex and continuous functions, it can be proved by the existence theorem of the equilibrium solutions (Rosen 1965) that there exists an equilibrium solution with respect to the degree of attainment of the aggregated fuzzy goal in the game (A^k, B^l).

4.3 Computational Methods

This section is devoted to developing computational methods for equilibrium solutions with respect to a degree of attainment of the fuzzy goal in single objective games and multiobjective games.

4.3.1 Single-Objective Bimatrix Games with Fuzzy Goals

Let $A = A^1$ and $B = B^1$. When Player I chooses a strategy x and Player II chooses a strategy y, the membership functions $\mu_1(x, y)$ and $\mu_2(x, y)$ of the fuzzy goals are functions of expected payoffs xAy and xBy, i.e., they are represented as

$$\begin{aligned} \mu_1(x, y) &= \mu_1(xAy) \\ \mu_2(x, y) &= \mu_2(xBy). \end{aligned} \tag{4.7}$$

Then a pair of strategies x^* and y^* is an equilibrium solution with respect to a degree of attainment of the fuzzy goal in a single-objective bimatrix game (A, B) if, for any other mixed strategies x and y,

$$\begin{aligned} \mu_1(x^*Ay^*) &\geq \mu_1(xAy^*) \\ \mu_2(x^*By^*) &\geq \mu_2(x^*By). \end{aligned} \tag{4.8}$$

If the membership functions of the fuzzy goals $\mu_1(xAy)$ and $\mu_2(xBy)$ are linear functions, they can be represented as

$$\mu_1(xAy) = \begin{cases} 0 & \text{if } xAy \leq \underline{a} \\ 1 - \dfrac{\overline{a} - xAy}{\overline{a} - \underline{a}} & \text{if } \underline{a} \leq xAy \leq \overline{a} \\ 1 & \text{if } \overline{a} \leq xAy \end{cases} \tag{4.9}$$

and

$$\mu_2(xBy) = \begin{cases} 0 & \text{if } xBy \leq \underline{b} \\ 1 - \dfrac{\overline{b} - xBy}{\overline{b} - \underline{b}} & \text{if } \underline{b} \leq xBy \leq \overline{b} \\ 1 & \text{if } \overline{b} \leq xBy, \end{cases} \tag{4.10}$$

respectively.

Let $\hat{A}$ and $\hat{B}$ denote the $m \times n$ matrices transformed by using the following equations (4.11) and (4.12), respectively.

$$\hat{a}_{ij} = \frac{a_{ij}}{\overline{a} - \underline{a}}, \quad c_1 = -\frac{\underline{a}}{\overline{a} - \underline{a}} \tag{4.11}$$

and

$$\hat{b}_{ij} = \frac{b_{ij}}{\overline{b} - \underline{b}}, \quad c_2 = -\frac{\underline{b}}{\overline{b} - \underline{b}}. \tag{4.12}$$

Equivalently, the membership functions (4.9) and (4.10) are represented as

$$\mu_1(xAy) = \begin{cases} 0 & \text{if} \quad xAy \leq \underline{a} \\ x\hat{A}y - c_1 & \text{if} \quad \underline{a} \leq xAy \leq \overline{a} \\ 1 & \text{if} \quad \overline{a} \leq xAy \end{cases} \tag{4.13}$$

and

$$\mu_2(xBy) = \begin{cases} 0 & \text{if} \quad xBy \leq \underline{b} \\ x\hat{B}y - c_2 & \text{if} \quad \underline{b} \leq xBy \leq \overline{b} \\ 1 & \text{if} \quad \overline{b} \leq xBy, \end{cases} \tag{4.14}$$

respectively.

Then equilibrium solutions with respect to a degree of attainment of the fuzzy goal possess the properties described in the following theorem.

Theorem 4.1

Let $\hat{A}$ and $\hat{B}$ denote matrices transformed by using the equations (4.11) and (4.12). If a pair of strategies (x^*, y^*) satisfies the conditions

$$\begin{aligned} x^*\hat{A}y^* &\geq x\hat{A}y^* \\ x^*\hat{B}y^* &\geq x^*\hat{B}y \end{aligned} \tag{4.15}$$

for any other mixed strategies x and y, then (x^*, y^*) also satisfies the following conditions:

$$\begin{aligned} x^*Ay^* &\geq xAy^* \\ x^*By^* &\geq x^*By. \end{aligned} \tag{4.16}$$

Furthermore, when the membership functions of the fuzzy goals are linear functions such as (4.9) and (4.10), (x^*, y^*) satisfies the following conditions:

$$\begin{aligned} \mu_1(x^*Ay^*) &\geq \mu_1(xAy^*) \\ \mu_2(x^*By^*) &\geq \mu_2(x^*By) \end{aligned} \tag{4.17}$$

for any other mixed strategies x and y.

Proof First we will prove that a pair of strategies (x^*, y^*) which satisfies the conditions (4.15) satisfies the conditions (4.16). We can transform the first condition of (4.15) into the following:

$$\sum_{i=1}^{m}\sum_{j=1}^{n} x_i^* \hat{a}_{ij} y_j^* \geq \sum_{i=1}^{m}\sum_{j=1}^{n} x_i \hat{a}_{ij} y_j^*,$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n} x_i^* \frac{a_{ij}}{\overline{a}-\underline{a}} y_j^* \geq \sum_{i=1}^{m}\sum_{j=1}^{n} x_i \frac{a_{ij}}{\overline{a}-\underline{a}} y_j^*,$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n} x_i^* a_{ij} y_j^* \geq \sum_{i=1}^{m}\sum_{j=1}^{n} x_i a_{ij} y_j^*.$$

Thus we have

$$x^*Ay^* \geq xAy^*,$$

and, similarly, we have

$$x^*By^* \geq x^*By.$$

i) If $xAy^* \leq x^*Ay^* \leq \underline{a}$,

$$\mu_1(x^*Ay^*) = \mu_1(xAy^*) = 0.$$

ii) If $xAy^* \leq \underline{a} \leq x^*Ay^*$,

$$\mu_1(x^*Ay^*) \geq \mu_1(xAy^*) = 0.$$

iii) If $\underline{a} \leq xAy^* \leq x^*Ay^* \leq \overline{a}$,

$$\mu_1(xAy^*) = x\hat{A}y^* + c_1$$

and

$$\mu_1(x^*Ay^*) = x^*\hat{A}y^* + c_1.$$

From (4.15), since $x^*\hat{A}y^* \geq x\hat{A}y^*$,

$$\mu_1(x^*Ay^*) \geq \mu_1(xAy^*).$$

iv) If $\underline{a} \leq xAy^* \leq \overline{a} \leq x^*Ay^*$, since

$$\mu_1(xAy^*) \leq 1 \quad \text{and} \quad \mu_1(x^*Ay^*) = 1,$$

$$1 = \mu_1(x^*Ay^*) \geq \mu_1(xAy^*).$$

v) If $\overline{a} \leq xAy^* \leq x^*Ay^*$,

$$\mu_1(x^*Ay^*) = \mu_1(xAy^*) = 1.$$

We also have similar statements from i) to v) for the function μ_2. Thus, from the statements from i) to v) for the functions μ_1 and μ_2, the second part of the theorem has been proved. □

The theorem means that if a pair of strategies (x^*, y^*) is a conventional equilibrium solution for a single-objective bimatrix game $(\hat{A}, \hat{B})$ generated by using the transformations (4.11) and (4.12), (x^*, y^*) is also a conventional equilibrium solution for a single-objective bimatrix game (A, B). Moreover, it is also an equilibrium solution with respect to a degree of attainment of the fuzzy goal for a single-objective bimatrix game (A, B) when the fuzzy goals are represented by linear membership functions such as (4.9) and (4.10).

The equilibrium conditions for the bimatrix game $(\hat{A}, \hat{B})$ can be expressed in the following form of mathematical programming problems:

$$\begin{aligned} x^*\hat{A}y^* \;=\; & \underset{x}{\text{maximize}} \;\; x\hat{A}y^* \\ & \text{subject to} \;\; \sum_{i=1}^{m} x_i = 1, \end{aligned} \tag{4.18}$$

and

$$\begin{aligned} x^*\hat{B}y^* \;=\; & \underset{y}{\text{maximize}} \;\; x^*\hat{B}y \\ & \text{subject to} \;\; \sum_{j=1}^{n} y_j = 1. \end{aligned} \tag{4.19}$$

From Theorem 4.1, an optimal solution (x^*, y^*) to the above two linear programming problems is an equilibrium solution with respect to a degree of attainment of the fuzzy goal with a linear membership function in a single-objective bimatrix game (A, B).

Since the constraints of the problems (4.18) and (4.19) are separated each other on the decision variables x and y, the two problems (4.18) and (4.19) become the following single mathematical programming problem:

$$\begin{aligned} x^*\hat{A}y^* + x^*\hat{B}y^* \;=\; & \underset{(x,y)}{\text{maximize}} \;\; \{x\hat{A}y^* + x^*\hat{B}y\} \\ & \text{subject to} \;\; \sum_{i=1}^{m} x_i = 1 \\ & \qquad\qquad\quad \sum_{j=1}^{n} y_j = 1. \end{aligned} \tag{4.20}$$

Theorem 4.2

If all of the membership functions of the fuzzy goals are linear functions such as (4.9) and (4.10), an optimal solution of the following quadratic programming problem is equal to the equilibrium solution with respect to a degree of attainment of the fuzzy goal for the single-objective bimatrix game (A, B).

$$\begin{aligned} \underset{(x,y,p,q)}{\text{maximize}} \quad & x\hat{A}y + x\hat{B}y - p - q \\ \text{subject to} \quad & \hat{A}y \leq pe^m \\ & \hat{B}x \leq qe^n \\ & \sum_{i=1}^{m} x_i = 1 \\ & \sum_{j=1}^{n} y_j = 1, \end{aligned} \tag{4.21}$$

where e^m and e^n are m and n dimensional column vectors for which each of the entries is 1, respectively, i.e.,

$$e^m = (\underbrace{1, 1, \ldots, 1}_{m})^T \quad \text{and} \quad e^n = (\underbrace{1, 1, \ldots, 1}_{n})^T.$$

Proof From Theorem 4.1, the theorem can be proved in a way similar to the theorem by Parthasarathy and Raghavan (1971), which is shown in the Appendix. □

From Theorem 4.2, we can obtain the equilibrium solution with respect to a degree of attainment of the fuzzy goal for the single-objective bimatrix game by solving the quadratic programming problem (4.21). Some algorithms for solving the quadratic programming problem have been developed. Especially, Lemke and Howson's method was developed as the computational method for equilibrium solutions of bimatrix games (1964) and, needless to say, it is also used for regular quadratic programming problems (Lemke 1965).

4.3.2 Multiobjective Bimatrix Games with Fuzzy Goals

We develop the methods for computing the proposed equilibrium solution in multiobjective games. Two aggregation rules are employed for multiple fuzzy goals in multiobjective bimatrix games (A^k, B^l), $k = 1, 2, \ldots, r$, $l = 1, 2, \ldots, s$. The first is the aggregation rule by weighting coefficients and the other is the aggregation rule by a minimum component. Both aggregation rules are popular for scalarizing methods in multiobjective programming problems.

Let Player I's fuzzy goals be denoted $\mu_1^k(xA^ky)$, $k = 1, 2, \ldots, r$ and Player II's fuzzy goals be denoted $\mu_2^l(xB^ly)$, $l = 1, 2, \ldots, s$.

a) An equilibrium solution with respect to a degree of attainment of the fuzzy goal aggregated by weighting coefficients

Let Player I's weighting coefficients for fuzzy goals be $v \in \{v \mid \sum_{k=1}^r v_k = 1, v_k \geq 0, k = 1, 2, \ldots, r\}$ and Player II's weighting coefficients for fuzzy goals be $w \in \{w \mid \sum_{l=1}^s w_l = 1, w_l \geq 0, l = 1, 2, \ldots, s\}$. Then Player I's aggregated fuzzy goals and Player II's aggregated fuzzy goals are represented by

$$\mu_1(x, y) = \sum_{k=1}^{r} v_k \mu_1^k(xA^ky) \tag{4.22}$$

and

$$\mu_2(x, y) = \sum_{l=1}^{s} w_l \mu_2^l(xB^ly). \tag{4.23}$$

A pair of strategies (x^*, y^*) is an equilibrium solution with respect to a degree of attainment of the fuzzy goal aggregated by weighting coefficients in a multiobjective game (A^k, B^l) if, for any other mixed strategies x and y,

$$\begin{aligned} \sum_{k=1}^{r} v_k \mu_1^k(x^*A^ky^*) &\geq \sum_{k=1}^{r} v_k \mu_1^k(xA^ky^*) \\ \sum_{l=1}^{s} w_l \mu_2^l(x^*B^ly^*) &\geq \sum_{l=1}^{s} w_l \mu_2^l(x^*B^ly). \end{aligned} \tag{4.24}$$

The equilibrium solution is equal to an optimal solution to the following two mathematical programming problems:

$$\begin{aligned} \sum_{k=1}^{r} v_k \mu_1^k(x^*A^ky^*) \;=\; & \underset{x}{\text{maximize}} \; \sum_{k=1}^{r} v_k \mu_1^k(xA^ky^*) \\ & \text{subject to} \; \sum_{i=1}^{m} x_i = 1, \end{aligned} \tag{4.25}$$

and

$$\begin{aligned} \sum_{l=1}^{s} w_l \mu_2^l(x^*B^ly^*) \;=\; & \underset{y}{\text{maximize}} \; \sum_{l=1}^{s} w_l \mu_2^l(x^*B^ly) \\ & \text{subject to} \; \sum_{j=1}^{n} y_j = 1. \end{aligned} \tag{4.26}$$

Since the constraints of the above two problems are separated each other, the above two problems become the following single mathematical programming problem:

$$\begin{aligned}
&\sum_{k=1}^{r} v_k \mu_1^k(x^* A^k y^*) + \sum_{l=1}^{s} w_l \mu_2^l(x^* B^l y^*) \\
&\qquad = \underset{(x,y)}{\text{maximize}} \ \Big\{ \sum_{k=1}^{r} v_k \mu_1^k(x A^k y^*) + \sum_{l=1}^{s} w_l \mu_2^l(x^* B^l y) \Big\} \\
&\qquad \text{subject to} \ \sum_{i=1}^{m} x_i = 1 \\
&\qquad\qquad\qquad\quad \sum_{j=1}^{n} y_j = 1.
\end{aligned} \tag{4.27}$$

Theorem 4.3

If all of the membership functions of the fuzzy goals are linear functions such as (4.9) and (4.10), an equilibrium solution with respect to a degree of attainment of the fuzzy goal aggregated by weighting coefficients for a multiobjective game (A^k, B^l) is equal to an optimal solution of the quadratic programming problem

$$\begin{aligned}
\underset{(x,y,p,q)}{\text{maximize}} \quad & x\hat{A}(v)y + x\hat{B}(w)y - p - q \\
\text{subject to} \quad & \hat{A}(v)y \le pe^m \\
& \hat{B}(w)x \le qe^n \\
& \sum_{i=1}^{m} x_i = 1 \\
& \sum_{j=1}^{n} y_j = 1,
\end{aligned} \tag{4.28}$$

where

$$\hat{A}(v) = \sum_{k=1}^{r} v_k \hat{A}^k \tag{4.29}$$

and

$$\hat{B}(w) = \sum_{l=1}^{s} w_l \hat{B}^l. \tag{4.30}$$

Proof If all of the membership functions of the fuzzy goals are linear functions such as (4.9) and (4.10), the membership functions (4.22) and (4.23) of the aggregated fuzzy goal can be transformed into the following by using the transformations (4.11) and (4.12):

$$\begin{aligned}
\mu_1(x,y) &= \sum_{k=1}^{r} v_k \mu_1^k(x A^k y) \\
&= x\hat{A}(v)y + c_1,
\end{aligned} \tag{4.31}$$

and

$$\mu_2(x,y) = \sum_{l=1}^{s} w_l \mu_2^l(xB^k y) = x\hat{B}(w)y + c_2. \quad (4.32)$$

Thus, from Theorem 4.1 and the theorem by Parthasarathy and Raghavan, the equilibrium solution is equal to an optimal solution of the quadratic programming problem (4.28). □

Since the problem (4.28) is a quadratic programming problem, the equilibrium solution with respect to a degree of attainment of the fuzzy goal aggregated by weighting coefficients for multiobjective games can be obtained by Lemke and Howson's method in a procedure similar to that for single-objective games.

When we regard this problem as an optimization problem for Player I's decision making, Player I must assess his and the opponent's weighting coefficients. It is especially difficult to assess the opponent's weighting coefficients. When partial information about the opponent's preference for objectives can be derived from the opponent's previous behavior, Barron and Schmidt's method (1988), which is an entropy-based procedure, is efficient and practical.

In their method, if there is no information, all weighting coefficients are equal in the sense of maximizing entropy. If there is partial information, the information is incorporated in the constraint of the maximizing entropy problem and weighting coefficients can be obtained by solving the maximizing entropy problem with the constraint. For example, if Player II prefers the first objective to the second one, the inequality $w_1 > w_2$ becomes a component of the constraint of the problem.

b) An equilibrium solution with respect to a degree of attainment of the fuzzy goal aggregated by a minimum component

Consider an equilibrium problem with respect to a degree of attainment of the fuzzy goal aggregated by a minimum component. The aggregation rule is often adopted in a multiple criteria decision making problem. Particularly in fuzzy decision making, the aggregation corresponds to the intersection of all of the fuzzy sets and a solution is determined by maximizing the membership degree of the intersection, and this decision rule is called Bellman and Zadeh's fuzzy decision rule.

Player I's and II's fuzzy goals aggregated by a minimum component are represented as

$$\mu_1(x, y) = \min_k \mu_1^k(xA^k y) \tag{4.33}$$

and

$$\mu_2(x, y) = \min_l \mu_2^l(xB^l y), \tag{4.34}$$

respectively.

Then a pair of strategies (x^*, y^*) is an equilibrium solution with respect to a degree of attainment of the fuzzy goal aggregated by a minimum component for a multiobjective game (A^k, B^l), $k = 1, 2, \ldots, s$, $l = 1, 2, \ldots, r$ if, for any other mixed strategies x and y,

$$\begin{aligned} \min_k \mu_1^k(x^* A^k y^*) &\geq \min_k \mu_1^k(xA^k y^*) \\ \min_l \mu_2^l(x^* B^l y^*) &\geq \min_l \mu_2^l(x^* B^l y). \end{aligned} \tag{4.35}$$

The equilibrium solution is equal to an optimal solution for the following two mathematical programming problems:

$$\begin{aligned} \min_k \mu_1^k(x^* A^k y^*) \;=\; & \underset{x}{\text{maximize}} \;\; \min_k \mu_1^k(xA^k y^*) \\ & \text{subject to} \;\; \sum_{i=1}^{m} x_i = 1, \end{aligned} \tag{4.36}$$

and

$$\begin{aligned} \min_l \mu_2^l(x^* B^l y^*) \;=\; & \underset{y}{\text{maximize}} \;\; \min_l \mu_2^l(x^* B^l y) \\ & \text{subject to} \;\; \sum_{j=1}^{n} y_j = 1. \end{aligned} \tag{4.37}$$

Since the constraints of the above two problems are separated each other, the above two problems become the following single mathematical programming problem:

$$\begin{aligned} & \min_k \mu_1^k(x^* A^k y^*) + \min_l \mu_2^l(x^* B^l y^*) \\ & \quad = \underset{(x,y)}{\text{maximize}} \;\; \left\{ \min_k \mu_1^k(xA^k y^*) + \min_l \mu_2^l(x^* B^l y) \right\} \\ & \quad \text{subject to} \;\; \sum_{i=1}^{m} x_i = 1 \\ & \qquad\qquad\qquad \sum_{j=1}^{n} y_j = 1. \end{aligned} \tag{4.38}$$

If the membership functions μ_1^k and μ_2^l are linear functions such as (4.9) and (4.10), it can be proved by the following theorem that the above mathematical programming problem (4.38) becomes a nonlinear programming problem with quadratic inequality constraints.

Lemma 4.4

Transform the $m \times n$ payoff matrices A^k, $k = 1, 2, \ldots, s$ and B^l, $l = 1, 2, \ldots, r$ into $\hat{A}^k$, $k = 1, 2, \ldots, s$ and $\hat{B}^l$, $l = 1, 2, \ldots, r$ by using the equalities (4.11) and (4.12). If a pair of strategies (x^*, y^*) satisfies the conditions

$$\begin{aligned} \min_k (x^* \hat{A}^k y^* + c_1^k) &\geq \min_k (x \hat{A}^k y^* + c_1^k) \\ \min_l (x^* \hat{B}^l y^* + c_2^l) &\geq \min_l (x^* \hat{B}^l y + c_2^l) \end{aligned} \tag{4.39}$$

for any other mixed strategies x and y, then (x^*, y^*) also satisfies the conditions (4.35), i.e., (x^*, y^*) is an equilibrium solution with respect to a degree of attainment of the fuzzy goal aggregated by a minimum component for a multiobjective game (A^k, B^l).

Proof First we prove that the first condition of (4.35) implies the first condition of (4.39).

i) If $\min_k (x \hat{A}^k y^* + c_1^k) \leq \min_k (x^* \hat{A}^k y^* + c_1^k) \leq 0$,

$$\min_k \mu_1^k (x A^k y^*) = \min_k \mu_1^k (x^* A^k y^*) = 0.$$

ii) If $\min_k (x \hat{A}^k y^* + c_1^k) \leq 0 \leq \min_k (x^* \hat{A}^k y^* + c_1^k)$,

$$\min_k \mu_1^k (x A^k y^*) = 0 \leq \min_k \mu_1^k (x^* A^k y^*).$$

iii) If $0 \leq \min_k (x \hat{A}^k y^* + c_1^k) \leq \min_k (x^* \hat{A}^k y^* + c_1^k) \leq 1$,

$$\min_k \mu_1^k (x A^k y^*) \leq \min_k \mu_1^k (x^* A^k y^*).$$

iv) If $0 \leq \min_k (x \hat{A}^k y^* + c_1^k) \leq 1 \leq \min_k (x^* \hat{A}^k y^* + c_1^k)$,

$$\min_k \mu_1^k (x A^k y^*) \leq 1 = \min_k \mu_1^k (x^* A^k y^*).$$

v) If $1 \leq \min_k (x \hat{A}^k y^* + c_1^k) \leq \min_k (x^* \hat{A}^k y^* + c_1^k)$,

$$\min_k \mu_1^k (x A^k y^*) = \min_k \mu_1^k (x^* A^k y^*) = 1.$$

From i) to v), the first part of the Lemma 4.4 have been proved. Similarly, we can prove the second part of the Lemma 4.4. □

Theorem 4.5

If all membership functions of fuzzy goals are linear functions, an equilibrium solution with respect to a degree of attainment of the fuzzy goal aggregated by a minimum component is equal to an optimal solution of the following nonlinear programming problem:

$$
\begin{array}{ll}
\underset{(x,y,p,q,\sigma_1,\sigma_2)}{\text{maximize}} & \sigma_1 + \sigma_2 - p - q \\
\text{subject to} & \hat{A}^k y + c_1^k e^m \leq p e^m, \ k = 1, 2, \ldots, r \\
& \hat{B^l}^T x + c_2^l e^n \leq q e^n, \ l = 1, 2, \ldots, s \\
& x\hat{A}^k y + c_1^k \geq \sigma_1, \ k = 1, 2, \ldots, r \\
& x\hat{B}^l y + c_2^l \geq \sigma_2, \ l = 1, 2, \ldots, s \\
& \sum_{i=1}^{m} x_i = 1 \\
& \sum_{j=1}^{n} y_j = 1.
\end{array}
\tag{4.40}
$$

Proof From the inequalities $\hat{A}^k y + c_1^k e^m \leq p e^m, k = 1, 2, \ldots, r$ in the constraints of the problem (4.40),

$$x\hat{A}^k y + c_1^k \geq p, \ k = 1, 2, \ldots, r.$$

Similarly,

$$x\hat{B}^l y + c_2^l \geq q, \ l = 1, 2, \ldots, s.$$

Furthermore, from $x\hat{A}^k y + c_1^k \geq \sigma_1, k = 1, 2, \ldots, s$,

$$\min_k (x\hat{A}^k y + c_1^k) \geq \sigma_1.$$

Similarly,

$$\min_l (x\hat{B}^l y + c_2^l) \geq \sigma_2.$$

Thus, since

$$
\begin{aligned}
\sigma_1 + \sigma_2 - p - q &\leq \min_k (x\hat{A}^k y + c_1^k) + \min_l (x\hat{B}^l y + c_2^l) - p - q \\
&\leq \min_k (x\hat{A}^k y + c_1^k) + \min_l (x\hat{B}^l y + c_2^l) \\
&\quad -(x\hat{A}^k y + c_1^k) - (x\hat{B}^l y + c_2^l) \leq 0,
\end{aligned}
$$

the maximal objective function value is 0.

From Lemma 4.4, a pair of strategies (x^*, y^*) satisfying the following conditions for any other x and y is an equilibrium solution with respect to a degree of attainment of the fuzzy goal aggregated by a minimum component:

$$\begin{aligned}\min_k(x^*\hat{A}^k y^* + c_1^k) &\geq \min_k(x\hat{A}^k y^* + c_1^k)\\ \min_l(x^*\hat{B}^l y^* + c_2^l) &\geq \min_l(x^*\hat{B}^l y + c_2^l).\end{aligned}$$

Set

$$\begin{aligned}p &= \min_k(x^*\hat{A}^k y^* + c_1^k)\\ q &= \min_l(x^*\hat{B}^l y^* + c_2^l),\end{aligned}$$

then (x^*, y^*) becomes an optimal solution of the problem (4.40).

On the other hand, let an optimal solution of the problem (4.40) be (x^*, y^*, p, q). Then

$$\min_k(x^*\hat{A}^k y^* + c_1^k) + \min_l(x^*\hat{B}^l y^* + c_2^l) - p - q = 0. \tag{4.41}$$

From the inequalities $\hat{A}^k y^* + c_1^k e^m \leq p e^m$, $k = 1, 2, \ldots, r$ in the constraints of the problem (4.40),

$$x\hat{A}^k y^* + c_1^k \leq p,\ k = 1, 2, \ldots, r. \tag{4.42}$$

Let $x = x^*$ and $y = y^*$. Then we have

$$x^*\hat{A}^k y^* + c_1^k \leq p,\ k = 1, 2, \ldots, r. \tag{4.43}$$

Similarly,

$$x^*\hat{B}^l y^* + c_2^l \leq q,\ l = 1, 2, \ldots, r. \tag{4.44}$$

From (4.41), (4.43) and (4.44),

$$\begin{aligned}\min_k(x^*\hat{A}^k y^* + c_1^k) &= p\\ \min_l(x^*\hat{B}^l y^* + c_2^l) &= q.\end{aligned}$$

From (4.42),

$$x\hat{A}^k y^* + c_1^k \leq \min_k(x^*\hat{A}^k y^* + c_1^k), \quad k = 1, 2, \ldots, r,$$

and then

$$\min_k(x\hat{A}^k y^* + c_1^k) \leq \min_k(x^*\hat{A}^k y^* + c_1^k).$$

Similarly,

$$\min_l(x^*\hat{B}^l y + c_2^l) \leq \min_l(x^*\hat{B}^l y^* + c_2^l).$$

Therefore, from Lemma 4.4, (x^*, y^*) is an equilibrium solution with respect to a degree of attainment of the fuzzy goal aggregated by a minimum component for a multiobjective bimatrix game (A^k, B^l), $k = 1, 2, \ldots, r$, $l = 1, 2, \ldots, s$. □

From Theorem 4.5, we can obtain an equilibrium solution with respect to a degree of attainment of the fuzzy goal aggregated by a minimum component by solving the special nonlinear programming problem which consists of a linear objective function and constraints with quadratic inequalities, linear equalities and linear inequalities.

Example 4.1

Consider equilibrium solutions with respect to a degree of attainment of the fuzzy goal aggregated by weighting coefficients. Let Player I have three pure strategies and three objectives, and let Player II have four pure strategies and three objectives. Then a two-person non-zero-sum multiobjective game can be represented by

$$A^1 = \begin{bmatrix} 2 & 6 & 5 & 7 \\ 2 & 5 & 5 & 4 \\ 4 & 7 & 6 & 9 \end{bmatrix}, \; A^2 = \begin{bmatrix} 3 & 6 & 8 & 2 \\ 6 & 2 & 2 & 8 \\ 2 & 9 & 7 & 4 \end{bmatrix}, \text{ and } \; A^3 = \begin{bmatrix} 1 & 4 & 7 & 2 \\ 3 & 6 & 1 & 8 \\ 2 & 5 & 3 & 9 \end{bmatrix},$$

and

$$B^1 = \begin{bmatrix} 1 & 6 & 7 & 1 \\ 8 & 2 & 3 & 4 \\ 4 & 9 & 3 & 5 \end{bmatrix}, \; B^2 = \begin{bmatrix} 8 & 2 & 2 & 8 \\ 1 & 9 & 7 & 6 \\ 5 & 2 & 8 & 5 \end{bmatrix}, \text{ and } \; B^3 = \begin{bmatrix} 5 & 1 & 2 & 4 \\ 3 & 4 & 8 & 3 \\ 1 & 8 & 1 & 2 \end{bmatrix}.$$

Let fuzzy goals G_1^1, G_1^2, and G_1^3 of Player I for the three objectives be represented by the following linear membership functions:

$$\mu_1^1(x^T A^1 y) = \begin{cases} 0 & \text{if } \; x^T A^1 y \leq 0 \\ x^T A^1 y/9 & \text{if } \; 0 \leq x^T A^1 y \leq 9 \\ 1 & \text{if } \; 9 \leq x^T A^1 y, \end{cases}$$

$$\mu_1^2(x^T A^2 y) = \begin{cases} 0 & \text{if } \; x^T A^2 y \leq 0 \\ x^T A^2 y/8 & \text{if } \; 0 \leq x^T A^2 y \leq 8 \\ 1 & \text{if } \; 8 \leq x^T A^2 y, \end{cases}$$

and

$$\mu_1^3(x^T A^3 y) = \begin{cases} 0 & \text{if } \; x^T A^3 y \leq 1 \\ (x^T A^3 y - 1)/6 & \text{if } \; 1 \leq x^T A^3 y \leq 7 \\ 1 & \text{if } \; 7 \leq x^T A^3 y. \end{cases}$$

Let fuzzy goals G_2^1, G_2^2, and G_2^3 of Player II for the three objectives be represented by the following linear membership functions:

$$\mu_2^1(x^T B^1 y) = \begin{cases} 0 & \text{if } \; x^T B^1 y \leq 1 \\ (x^T B^1 y - 1)/8 & \text{if } \; 1 \leq x^T B^1 y \leq 9 \\ 1 & \text{if } \; 9 \leq x^T B^1 y, \end{cases}$$

$$\mu_2^2(x^TB^2y) = \begin{cases} 0 & \text{if} \;\; x^TB^2y \leq 0 \\ x^TB^2y/5 & \text{if} \;\; 0 \leq x^TB^2y \leq 5 \\ 1 & \text{if} \;\; 5 \leq x^TB^2y, \end{cases}$$

and

$$\mu_2^3(x^TB^3y) = \begin{cases} 0 & \text{if} \;\; x^TB^3y \leq 1 \\ (x^TB^3y-1)/7 & \text{if} \;\; 1 \leq x^TB^3y \leq 8 \\ 1 & \text{if} \;\; 8 \leq x^TB^3y. \end{cases}$$

Let weighting coefficients for the three objectives of Player I be

$$v_1 = 0.05, \quad v_2 = 0.85, \text{ and } \quad v_3 = 0.1,$$

and let weighting coefficients for the three objectives of Player II be

$$w_1 = 0.8, \quad w_2 = 0.1, \text{ and } \quad w_3 = 0.1.$$

The equilibrium solutions with respect to a degree of attainment of the fuzzy goal aggregated by the weighting coefficients were obtained by using Lemke and Howson's method (1964). There exist three equilibrium solutions, and the results are shown in Table 4.1.

Table 4.1 The equilibrium solutions

x_1	x_2	x_3	y_1	y_2	y_3	y_4
0.422	0.578	0.000	0.672	0.000	0.328	0.000
0.000	1.000	0.000	1.000	0.000	0.000	0.000
0.000	0.457	0.543	0.595	0.405	0.000	0.000

4.4 Related Properties for Pareto Equilibrium Solutions

Research on equilibrium solutions for multiobjective games was started from defining the best reply strategies in terms of vector optimality concepts such as Pareto optimality. So far we have discussed equilibrium solutions for multiobjective games without such concepts. Namely, we have introduced the degree of attainment of the fuzzy goal as a commensurable measure and have considered equilibrium solutions in terms of the degree of attainment of the fuzzy goal. In this section, we consider Pareto optimality of the proposed equilibrium solutions for multiobjective bimatrix games.

Pareto optimality and related concepts have been discussed for multiple criteria decision making (Geoffrion 1968; Chankong and Haimes 1983; Sawaragi, Nakayama and Tanino 1985; Wierzbicki 1986; Seo and Sakawa 1988; Sakawa 1993). An optimal point in single-objective decision making problems can be defined under the total order but an optimal point in multiobjective decision making problems cannot be defined in the same way because the order relation among vector alternatives is a partial order. Therefore, an optimal point under the partial order is defined as a point being not inferior to each of the other feasible solutions. In a mathematical description, the sets of such optimal points are often defined in terms of preference cones. The best reply strategies are defined as follows by using the concept of Pareto optimality in multiobjective games.

Definition 4.4 (A set of the Pareto best reply strategies) Let a payoff of Player I be denoted $p_1(x, y) \in R^r$ when Player I chooses a strategy x and Player II chooses a strategy y. Player I's preference cone is defined by

$$C_1 = \{p_1(x, y) \in R^r \mid p_1^k(x, y) \geq 0, k = 1, 2, \ldots, r\} = R_+^r. \quad (4.45)$$

Then, given Player II's strategy $\hat{y}$, the set of payoffs for the Pareto best reply strategies is defined by

$$P_1(\hat{y}) = \{p_1(x, \hat{y}) \in Z_1(\hat{y}) \mid Z_1(\hat{y}) \cap (p_1(x, \hat{y}) + \tilde{C}_1) = \phi\}, \quad (4.46)$$

where $Z_1(\hat{y})$ is a set of attainable payoffs, ϕ is the empty set and $\tilde{C}_1 = C_1 \backslash \{0\}$. Similarly, let Player II's payoff be denoted $p_2(x, y) \in R^s$ and Player II's preference cone be

$$C_2 = \{p_2(x, y) \in R^s \mid p_2^l(x, y) \geq 0, l = 1, 2, \ldots, s\} = R_+^s. \quad (4.47)$$

Then, given Player I's strategy $\hat{x}$, the set of payoffs for Pareto best reply strategies is defined by

$$P_2(\hat{x}) = \{p_2(\hat{x}, y) \in Z_2(\hat{x}) \mid Z_2(\hat{x}) \cap (p_2(\hat{x}, y) + \tilde{C}_2) = \phi\}. \quad (4.48)$$

Especially, for a multiobjective bimatrix game (A^k, B^l), $k = 1, 2, \ldots, r$, $l = 1, 2, \ldots, s$, $Z_1(y)$ and $Z_2(x)$ become the following convex polyhedrons spanned by vertices $\{u_1^{1T} Ay^T, \ldots, u_1^{mT} Ay^T\}$ and $\{xBu_2^1, \ldots, xBu_2^n\}$, respectively.

$$CH_1(y) = \text{conv } \{u_1^{1T} Ay^T, \ldots, u_1^{mT} Ay^T\} \quad (4.49)$$

and

$$CH_2(x) = \text{conv } \{xBu_2^1, \ldots, xBu_2^n\}, \tag{4.50}$$

where u_1^i is an m-dimensional column vector such that the ith entry is 1 and the other entries are 0, and u_2^i is a similar n-dimensional column vector (Borm, Tijs and Aarssen 1988).

Definition 4.5 (Pareto optimal equilibrium solution) Let a payoff of Player I and a payoff of Player II be $p_1(x, y) = (p_1^1(x, y), p_1^2(x, y), \ldots, p_1^r(x, y))$ and $p_2(x, y) = (p_2^1(x, y), p_2^2(x, y), \ldots, p_2^s(x, y))$, respectively. For any pair of strategies x and y, let Player I's set of payoffs for the Pareto best reply strategies and Player II's set of payoffs for the Pareto best reply strategies be denoted $P^1(y)$ and $P^2(x)$, respectively. Then the set of the Pareto optimal equilibrium solutions is defined by

$$PE = \{(x^*, y^*) \mid p_1^*(x^*, y^*) \in P_1(y^*), p_2^*(x^*, y^*) \in P_2(x^*)\}. \tag{4.51}$$

Wierzbicki (1990) explored the relation between scalarizing functions and Pareto optimal equilibrium solutions in detail. We briefly refer to his work and then examine the properties on Pareto optimality of the proposed solutions.

Theorem (Wierzbicki)

Suppose that scalarizing functions $s_i(p_i(x, y), w_i), i = 1, 2$ such that

$$\text{Arg} \max_{p_1(x,y) \in Z_1(y)} s_1(p_1(x, y), w_1) \subset P_1(y), \forall w_1 \in W_1, \forall y \in Y \tag{4.52}$$

and

$$\text{Arg} \max_{p_2(x,y) \in Z_2(x)} s_2(p_2(x, y), w_2) \subset P_2(x), \forall w_2 \in W_2, \forall x \in X \tag{4.53}$$

are used for an aggregation of all objectives in a noncooperative two-person multiobjective game in normal form. Then an equilibrium solution of the single-objective game with payoffs $s_i(p_i(x, y), w_i)$, $i = 1, 2$ for any w_i is a Pareto optimal equilibrium solution of the multiobjective game, where w_i is a parameter of the scalarizing function and W_i is a set of the parameters.

Wierzbicki proved the theorem in an n-person version and the proof of the theorem is shown in the Appendix. The theorem is interpreted to means that an equilibrium solution of the proxy single-objective game whose payoffs correspond to the scalarizing function values is a Pareto optimal equilibrium solution of an original multiobjective game if scalarizing functions satisfy the conditions (4.52) and (4.53). The conditions (4.52) and (4.53) stipulate that scalarizing functions are strictly monotone, i.e., if $p_i''(x,y) - p_i'(x,y) \in \tilde{C}_i$, $s_i(p_i''(x,y), w_i) > s_i(p_i'(x,y), w_i)$.

Consider relations between the equilibrium solutions with respect to a degree of attainment of the fuzzy goal and the Pareto optimal equilibrium solutions. Player I's scalarizing function for the aggregation by weighting coefficients is represented by

$$s_1(p_1(x,y), w_1) = \sum_{k=1}^{r} v_k \mu_1^k(xA^k y); \quad w_1 = (v, \underline{a}, \overline{a}), \tag{4.54}$$

and it is not always strictly monotone. Similarly, Player II's scalarizing function can be represented as

$$s_2(p_2(x,y), w_2) = \sum_{l=1}^{s} w_l \mu_2^l(xB^l y); \quad w_2 = (w, \underline{b}, \overline{b}). \tag{4.55}$$

However, the functions (4.54) and (4.55) are strictly monotone in the intervals $\underline{a}^k \le xA^k y \le \overline{a}^k$, $k = 1, 2, \ldots, r$ and $\underline{b}^l \le xB^l y \le \overline{b}^l$, $l = 1, 2, \ldots, s$. Therefore, if a player assesses $\underline{a}^k$ or $\underline{b}^l$ sufficiently small and assesses $\overline{a}^k$ or $\overline{b}^l$ sufficiently large, an equilibrium solution with respect to a degree of attainment of the fuzzy goal aggregated by weighting coefficients is also a Pareto optimal equilibrium solution. For example, Nishizaki and Sakawa (1992) suggest setting the minimal entry and the maximal entry of the kth payoff matrix for Player I to the values of $\underline{a}^k$ and $\overline{a}^k$, respectively, and setting the minimal entry and the maximal entry of the lth payoff matrix for Player II to the values of $\underline{b}^l$ and $\overline{b}^l$, respectively. If

$$\underline{a}^k \le \min_{i,j} a_{ij}^k, \quad k = 1, 2, \ldots, r \tag{4.56}$$

and

$$\overline{a}^k \ge \max_{i,j} a_{ij}^k, \quad k = 1, 2, \ldots, r, \tag{4.57}$$

the scalarizing function (4.54) becomes strictly monotone. Similarly, if

$$\underline{b}^l \le \min_{i,j} b_{ij}^l, \quad l = 1, 2, \ldots, s \tag{4.58}$$

and

$$\overline{b}^l \geq \max_{i,j} b_{ij}^l, \quad l = 1, 2, \ldots, s, \tag{4.59}$$

the scalarizing function (4.55) becomes strictly monotone. Thus, if fuzzy goals are identified so as to satisfy the conditions from (4.56) to (4.59), an equilibrium solution with respect to a degree of attainment of the fuzzy goal aggregated by weighting coefficients is also a Pareto optimal equilibrium solution.

Player I's scalarizing function for the aggregation by a minimum component is represented by

$$s_1(p_1(x,y), w_1) = \min_k \mu_1^k(xA^k y); \quad w_1 = (\underline{a}, \overline{a}), \tag{4.60}$$

and it is not always strictly monotone. Similarly, Player II's scalarizing function can be represented as

$$s_2(p_2(x,y), w_2) = \min_l \mu_2^l(xB^l y); \quad w_2 = (\underline{b}, \overline{b}). \tag{4.61}$$

If a player assesses $\underline{a}^k$ or $\underline{b}^l$ sufficiently small and assesses $\overline{a}^k$ or $\overline{b}^l$ sufficiently large such as from (4.56) to (4.59), an equilibrium solution with respect to a degree of attainment of the fuzzy goal aggregated by a minimum component is a weak Pareto optimal equilibrium solution which can be defined by exchanging $\tilde{C}_1 = C_1 \backslash \{0\}$ for $\tilde{C}_1 = \text{int } C_1$ in the definition of Pareto optimality.

4.5 Conclusion

In this chapter we have considered equilibrium solutions for multiobjective bimatrix games incorporating fuzzy goals and established the computational methods.

To conclude, the results of this chapter are summarized as follows.

1) To treat more general cases than in the previous chapters, we have examined two-person non-zero-sum multiobjective bimatrix games with fuzzy goals.

2) We have defined the equilibrium solutions in terms of a degree of attainment of a fuzzy goal for games in fuzzy and multiobjective environments.

3) The computational method for the equilibrium solution with respect to a degree of attainment of the fuzzy goal for two-person non-zero-sum single-objective games has been developed and the solution can be obtained by solving a quadratic programming problem.

4) For multiobjective games, the methods by weighting coefficients and by a minimum component have been adopted to aggregate multiple fuzzy goals. The methods for computing equilibrium solutions have been presented when membership functions of fuzzy goals are linear functions. The solution can be obtained by solving the quadratic programming problem when weighting coefficients are used as an aggregation rule. The solution can also be obtained by solving the special nonlinear programming problem, which consists of a linear objective function and constraints with quadratic inequalities, linear equalities and linear inequalities, when a minimum component is used as an aggregation rule.

5) We have considered the relation between the proposed equilibrium solutions for multiobjective bimatrix games incorporating fuzzy goals and the Pareto optimal equilibrium solutions and have shown the conditions that the proposed solutions belong to the set of the Pareto optimal equilibrium solutions.

So far we have examined noncooperative games in fuzzy and multiobjective environments and have developed the computational methods for the max-min solutions and the equilibrium solutions. The following two chapters will deal with a fuzzy game in which cooperation is permitted.

CHAPTER 5

N-PERSON COOPERATIVE GAMES WITH FUZZY COALITIONS

5.1 Introduction

We intend to examine n-person cooperative games in fuzzy environments in this chapter. In noncooperative games, there is not much difference between two-person games and n-person games; however, in cooperative games, coalitions are organized by group agreement among some or all of the players. There is only one possible coalition in the two-person case, but in the n-person case, many coalitions are possible. For conventional n-person cooperative games, a coalition is defined as any nonempty subset of the set of all players, making the number of possible coalitions at most $2^n - 1$, which includes one-person coalitions. Any player participating in a coalition must accept completely the decisions of the coalition; that is, a coalition behaves like an individual decision maker.

To ease this binding regulation, the concept of a fuzzy coalition was introduced, whereby players participating in a fuzzy coalition do not transfer all of their decisional rights to the fuzzy coalition. For n-person cooperative games with fuzzy coalitions, a fuzzy coalition is defined as any nonempty fuzzy subset of the set of the all players. By incorporating fuzzy coalitions, we can take ambiguity in the formation of an organization into consideration.

In a nonfuzzy game, the lexicographical framework (devised by Davis and Maschler 1965; Schmeidler 1969; Spinetto 1974; Littelchild and Vaidya 1976; Machler, Peleg and Shapley 1979; Michener, Yuen and Sakuari 1981; Shubik 1982; Sakawa, Tada and Nishizaki 1983; Sakawa 1985; and Sakawa and Nakao 1985; Nishizaki and Sakawa 1992c) can accommodate a wide variety of properties to be minimized, and it has given rise to an entire class of solution concepts. Aubin (1979, 1981) and Butnariu (1978, 1980) proposed the solution concepts in games with fuzzy coalitions, such as the core, the Shapley value and others. In such a game, however, lexicographical solution concepts have not been proposed.

In this chapter, new lexicographical solution concepts for conventional n-person cooperative games and n-person cooperative games with fuzzy coalitions are proposed by making use of an excess of a player with respect to a payoff vector (Sakawa and Nishizaki 1984, 1991).

In section 5.2, we review some popular solution concepts such as the core and the nucleolus in n-person cooperative games. Next, to express conflict among players directly, we newly introduce the concept of an excess of a player, which is defined by summing up all the excesses of coalitions to which he belongs, and propose the lexicographical solution concepts based on these excesses in a conventional n-person cooperative game. Moreover, we consider other solutions related to the proposed solution concept and examine relationships among them. For each solution concept, we present a computational method.

In section 5.3, we define an excess of a player in games with fuzzy coalitions and consider the solution concepts and their computational methods in these games as well as in a game without fuzzy coalitions. The extensions from a game without fuzzy coalitions to a game with fuzzy coalitions are considered. Usually, such extensions are represented as mappings by extension operators. We provide some extension operators such as Owen's extension (Owen 1972, 1982), Cornet's extension (cited in Aubin 1979) and so forth. A numerical example for games extended by the extension operators is shown and the solutions are computed.

5.2 n-Person Cooperative Games

This section deals with the lexicographical solutions in conventional n-person cooperative games. We review basic concepts of n-person cooperative games and propose the solution concepts based on the lexicographical framework.

5.2.1 Problem Formulation and Solution Concepts

Let us define some basic concepts of n-person cooperative games.

Definition 5.1 (Coalition) For an n-person game, let the set of all players be denoted $N = \{1, 2, \ldots, n\}$. Any nonempty subset of N (including N itself and all one-element subsets) is called a coalition.

Definition 5.2 (Characteristic function) The function v, called a characteristic function of a game, is a real-valued function which associates any coalition S with its real value $v(S)$.

With side-payments, the cooperative possibilities of the game can be described by the characteristic function v. $v(S)$ is called the value of coalition S and it represents the total amount of a side-payment (transferable utility) that a member of S could earn without any help from the players outside of S. In any characteristic function, we always let

$$v(\phi) = 0, \tag{5.1}$$

where ϕ denotes the empty set. Therefore, the game is described by the pair (N, v).

Once a representation of the game has been specified, we can try to predict the outcome of bargaining among the players. Such an analysis is usually based on the assumption that the players will form the grand coalition and divide the value $v(N)$ among themselves. It is clear that no player will accept less than the minimum which he can attain for himself.

Definition 5.3 (Imputation) For a game (N, v), an imputation is a payoff vector $x = (x_1, x_2, \ldots, x_n)$ satisfying

$$\sum_{i \in N} x_i = v(N) \tag{5.2}$$

and

$$x_i \geq v(\{i\}), \qquad i = 1, 2, \ldots, n. \tag{5.3}$$

Let all imputations of the game (N, v) be denoted $X(N, v)$.

We introduce the preference relation between two imputations.

Definition 5.4 (Domination relation) For a game (N, v), let x and y be two imputations and let S be a coalition. We say x dominates y through the coalition S (notation: $x \text{ dom}_S\, y$) if

$$x_i > y_i, \qquad \forall i \in S \tag{5.4}$$

and

$$\sum_{i \in S} x_i \leq v(S). \tag{5.5}$$

We also say x dominates y if there is any coalition such that $x \text{ dom}_S\ y$.

The condition (5.4) means that all of the members of S prefer x to y; the condition (5.5) means that they can obtain what x gives them.

Consider a solution concept in terms of the domination relation.

Definition 5.5 (Core) The set of all undominated imputations for a game (N, v) is called the core $C(N, v)$. Equivalently, $C(N, v)$ is defined as the set of all payoff vectors x satisfying the following conditions:

$$\sum_{i \in S} x_i \geq v(S), \qquad \forall S \subseteq N, \tag{5.6}$$

and

$$\sum_{i \in N} x_i = v(N). \tag{5.7}$$

Next, we present the concept of the nucleolus defined by Schmeidler (1969). A kind of lexicographical solution, the nucleolus is related to the bargaining set, which is obtained by considering the discussion that may take place during a play of the game. The nucleolus is based on the idea of the excess and, in its definition, the order relation which is named the lexicographical order is used.

Definition 5.6 (Excess) For the game (N, v), let S be a coalition and let x be a payoff vector. Then the excess of the coalition S with respect to x is

$$e(S, x) = v(S) - \sum_{i \in S} x_i. \tag{5.8}$$

Definition 5.7 (Lexicographical order) Let $r(x)$ be a vector arranged in order of decreasing magnitude, i.e., if $i < j$, $r_i(x) \geq r_j(x)$. Then, for any pair of payoff vectors x and y, if $x = y$ or, for the first entry h in which they differ,

$$r_h(x) < r_h(y), \tag{5.9}$$

x is smaller than y in the lexicographical order. Let the lexicographical order be denoted $\leq_L$.

Definition 5.8 (Nucleolus) Let $H_{2^n} : R^{2^n} \to R^{2^n}$ be a mapping which arranges entries of a 2^n-dimensional vector in order of decreasing magnitude.

Then, for a game (N, v), the solution minimizing the vector of the excesses $H_{2^n}(e(S, x))$ in the lexicographical order is defined as

$$\begin{aligned} N(N,v) &= \{x \mid H_{2^n}(e(S_1,x),\cdots,e(S_{2^n},x)) \\ &\leq_L H_{2^n}(e(S_1,y),\cdots,e(S_{2^n},y)), \forall y \in X(N,v)\}. \end{aligned} \tag{5.10}$$

The set $N(N, v)$ is called the nucleolus.

By considering the discussion that may take place during a play of the game, the concept of the nucleolus was defined. It is supposed that the rule of the minimization of the maximum excess of the coalition is adopted as a decision criterion by the players. In this case, excesses of coalitions are thought of as an evaluation of payoff vectors in terms of coalitions. To evaluate payoff vectors in terms of players, we now define an excess of a player.

Definition 5.9 (Excess of a player) For a game (N, v), let $e(S, x)$ be an excess of a coalition with respect to a payoff vector x. Then, an excess of a player i with respect to a payoff vector x is defined as

$$\begin{aligned} w(i,x) &= \sum_{\substack{S\subseteq N \\ S\ni i}} e(S,x) = \sum_{\substack{S\subseteq N \\ S\ni i}} \left(v(S) - \sum_{i\in S} x_i\right) \\ &= \sum_{\substack{S\subseteq N \\ S\ni i}} v(S) - \left(2\cdot 2^{n-1}x_i + \sum_{i\neq j} 2^{n-2}x_j\right). \end{aligned} \tag{5.11}$$

Consider a new solution concept using the excess of a player and the lexicographical order.

Definition 5.10 (Lexicographical solution using an excess of a player) Let $H_n : R^n \to R^n$ be a mapping which arranges entries of an n-dimensional vector in order of decreasing magnitude. Then, for a game (N, v), the solution minimizing the vector of the excesses of a player $H_n(w(i, x))$ in the lexicographical order is defined as

$$\begin{aligned} LS(N,v) &= \{x \mid H_n(w(1,x),\cdots,w(n,x)) \\ &\leq_L H_n(w(1,y),\cdots,w(n,y)), \forall y \in X(N,v)\}. \end{aligned} \tag{5.12}$$

5.2.2 Computational Method

We present the computational methods for the nucleolus and the lexicographical solution proposed in the previous subsection and examine the relation between the proposed solution and related solutions which will be defined in this subsection.

The algorithm for computing the nucleolus of a game (N, v) can be summarized in the following steps (Kopelowitz 1967). We assume the game (N, v) is zero-normalized[†].

Algorithm 5.1

[**Step 1**]

Formulate the following linear programming problem:

$$\begin{aligned}
&\text{minimize} && \varepsilon \\
&\text{subject to} && e(S,x) = v(S) - \sum_{i \in S} x_i \leq \varepsilon, \qquad \forall S \neq \phi, N \\
& && \sum_{i \in N} x_i = v(N) \\
& && x_i \geq 0, \qquad i = 1, 2, \ldots, n,
\end{aligned} \tag{5.13}$$

and solve the problem (5.13). Let the optimal solution of (5.13) be ε_1 and the set of coalitions for the active inequality constraints be denoted $\mathcal{T}_1$.

[**Step 2**]

By fixing the active inequality constraints in Step 1 as $\varepsilon = \varepsilon_1$, the linear programming problem

$$\begin{aligned}
&\text{minimize} && \varepsilon \\
&\text{subject to} && v(S) - \sum_{i \in S} x_i = \varepsilon_1, \qquad \forall S \in \mathcal{T}_1 \\
& && v(S) - \sum_{i \in S} x_i \leq \varepsilon, \qquad \forall S \notin \mathcal{T}_1, S \neq \phi, N \\
& && \sum_{i \in N} x_i = v(N) \\
& && x_i \geq 0, \qquad i = 1, 2, \ldots, n,
\end{aligned} \tag{5.14}$$

can be obtained. Let the optimal solution of (5.14) be ε_2 and the set of coalitions for the new active inequality constraints be denoted $\mathcal{T}_2$.

[**Step 3**]

In a procedure similar to [Step 2], the linear programming problem can be obtained by fixing the active inequality constraints in the previous step. Then let the optimal solution of the problem be ε_3 and the set of coalitions for the new active inequality constraints be denoted $\mathcal{T}_3$.

⋮

[**Step t**]

[†]Given a game (N, v), let $v'(S) = v(S) - \sum_{i \in S} v(\{i\})$, $\forall S \subseteq N$. Then the game (N, v') is said to be zero-normalized.

Let the optimal solution at Step $t-1$ be ε_{t-1}. By fixing $\varepsilon = \varepsilon_{t-1}$, the active inequality constraints containing ε are converted into equality constraints. The obtained linear programming problem

$$
\begin{array}{ll}
\text{minimize} & \varepsilon \\
\text{subject to} & v(S) - \sum_{i \in S} x_i = \varepsilon_1, \qquad \forall S \in \mathcal{T}_1 \\
& v(S) - \sum_{i \in S} x_i = \varepsilon_2, \qquad \forall S \in \mathcal{T}_2 \\
& \vdots \\
& v(S) - \sum_{i \in S} x_i = \varepsilon_{t-1}, \qquad \forall S \in \mathcal{T}_{t-1} \\
& v(S) - \sum_{i \in S} x_i \le \varepsilon, \qquad \forall S \notin \mathcal{T}_1 \cup \mathcal{T}_2 \cup \cdots \cup \mathcal{T}_{t-1}, S \neq \phi, N \\
& \sum_{i \in N} x_i = v(N) \\
& x_i \ge 0, \qquad i = 1, 2, \ldots, n,
\end{array}
\tag{5.15}
$$

is solved. Let the optimal solution of (5.15) be ε_t.

The following lemma on the convergence of Algorithm 5.1 is important.

Lemma 5.1

The unique payoff vector x^* minimizing ε can always be determined by at most $t = n$ steps in Algorithm 5.1.

Proof It is obvious that the linear programming problem (5.13) is feasible. Since the objective function is lower bounded, there exists an optimal solution. Let the minimum value be ε and let the optimal extreme point solutions be $(\hat{x}^1, \varepsilon_1), \ldots, (\hat{x}^{m_1}, \varepsilon_1)$. Let $\mathcal{T}_1$ be defined as follows:

$$\mathcal{T}_1 = \{S \mid S \neq \phi, N, e(S, \hat{x}^k) = \varepsilon_1, k = 1, 2, \ldots, m_1\}.$$

Then $\mathcal{T}_1 \neq \phi$. This is proved as follows:

If $\mathcal{T}_1 = \phi$, for any S,

$$e(S, \hat{x}^k) \le \varepsilon_1, \qquad k = 1, 2, \ldots, m_1$$

and

$$e(S, \hat{x}^l) < \varepsilon_1$$

for a certain l. Let

$$z = \frac{1}{m_1} \sum_{k=1}^{m_1} \hat{x}^k;$$

z is an imputation and, for any S,

$$e(S, z) < \varepsilon_1.$$

This contradicts the minimal property of ε_1. Thus $\mathcal{T}_1 \neq \phi$.

Then consider Step 2. As in Step 1, the linear programming problem (5.14) is feasible and has an optimal solution. Let the optimal solution be ε_2 and let the optimal extreme point solutions be $(\hat{y}^1, \varepsilon_2)$, ..., $(\hat{y}^{m_1}, \varepsilon_2)$. Let $\mathcal{T}_2$ be defined as follows:

$$\mathcal{T}_2 = \{S \mid S \neq \phi, N, e(S, \hat{y}^k) = \varepsilon_2, k = 1, 2, \ldots, m_2\}.$$

As in Step 1, it is concluded that $\mathcal{T}_2 \neq \phi$. It is obvious that by repeating the steps, $\varepsilon_1 \geq \varepsilon_2 \geq \cdots$.

At each step, at least one inequality constraint becomes active. And at least $t-1$ inequality constraints become active at Step t. Since n equality constraints including $v(N) = \sum_{i \in N} x_i$ determine the optimal solution uniquely, by Algorithm 5.1, the payoff vector x^* can be uniquely determined by at most $t = n$ steps. □

This lemma allows us to prove the following theorem.

Theorem 5.2

The solution obtained by Algorithm 5.1 is the nucleolus for the game (N, v).

Proof Let the solution obtained by Algorithm 5.1 be denoted x^*. Assume that x^* is not the nucleolus for the game (N, v). For the payoff vector x^*, let $r(x^*)$ be the vector of all excesses, entries of which are arranged in a descending order. Then, there exists a certain imputation y such that, for a certain h,

$$r_i(x^*) = r_i(y), \qquad i = 1, 2, \ldots, h-1$$

and

$$r_h(x^*) > r_h(y).$$

However, since $r_h(x^*) = \varepsilon_t$ for a certain Step $t \geq 1$, y gives an objective function value which is less than the minimum value ε_t. This is a contradiction. Thus, from Lemma 5.1, x^* is the nucleolus for the game (N, v). □

We present the algorithm for computing the lexicographical solution using an excess of a player $LS(N, v)$ for a game (N, v) in a way similar to the algorithm for the nucleolus. The algorithm for computing the solution $LS(N, v)$ can be summarized in the following steps.

Algorithm 5.2

[**Step 1**]

Formulate the following linear programming problem:

$$\begin{array}{ll} \text{minimize} & \varepsilon \\ \text{subject to} & w(i,x) \leq \varepsilon, \qquad i = 1,2,\ldots,n \\ & \sum_{i \in N} x_i = v(N) \\ & x_i \geq 0, \qquad i = 1,2,\ldots,n, \end{array} \tag{5.16}$$

and solve the problem (5.16). Let the optimal solution of (5.16) be ε_1 and the set of players for the active inequality constraints be denoted $\mathcal{T}_1$.

[**Step 2**]

By fixing the active inequality constraints of Step 1 as $\varepsilon = \varepsilon_1$, the linear programming problem

$$\begin{array}{ll} \text{minimize} & \varepsilon \\ \text{subject to} & w(i,x) = \varepsilon_1, \qquad \forall i \in \mathcal{T}_1 \\ & w(i,x) \leq \varepsilon, \qquad \forall i \in N,\ i \notin \mathcal{T}_1 \\ & \sum_{i \in N} x_i = v(N) \\ & x_i \geq 0, \qquad i = 1,2,\ldots,n, \end{array} \tag{5.17}$$

can be obtained. Let the optimal solution of (5.17) be ε_2 and the set of players for the new active inequality constraints be denoted $\mathcal{T}_2$.

[**Step 3**]

In a procedure similar to [Step 2], the linear programming problem can be obtained by fixing the active inequality constraints in the previous step. Then let the optimal solution of the problem be ε_3 and the set of coalitions for the new active inequality constraints be denoted $\mathcal{T}_3$.

⋮

[**Step t**]

Let the optimal solution at Step $t-1$ be ε_{t-1}. By fixing $\varepsilon = \varepsilon_{t-1}$, the active inequality constraints containing ε are converted into equality constraints. The obtained linear programming problem

$$
\begin{array}{ll}
\text{minimize} & \varepsilon \\
\text{subject to} & w(i,x) = \varepsilon_1, \qquad \forall i \in \mathcal{T}_1 \\
& w(i,x) = \varepsilon_2, \qquad \forall i \in \mathcal{T}_2 \\
& \vdots \\
& w(i,x) = \varepsilon_{t-1}, \qquad \forall i \in \mathcal{T}_{t-1} \\
& w(i,x) \leq \varepsilon, \qquad \forall i \in N,\ i \notin \mathcal{T}_1 \cup \mathcal{T}_2 \cup \cdots \cup \mathcal{T}_{t-1} \\
& \sum_{i \in N} x_i = v(N) \\
& x_i \geq 0, \qquad i = 1, 2, \ldots, n,
\end{array}
\tag{5.18}
$$

is solved. Let the optimal solution of (5.18) be ε_t.

Theorem 5.3

The solution obtained by Algorithm 5.2 is the lexicographical solution using an excess of a player $LS(N,v)$ for the game (N,v) and the solution $LS(N,v)$ is always unique.

Proof We can prove the theorem by a procedure similar to the proof of Theorem 5.2. □

We now define solution concepts related to the lexicographical solution $LS(N,v)$ using an excess of a player. To note the equity on excesses of players, we define the difference in the excesses between two players.

Definition 5.11 (Excess difference between two players) For a game (N,v), let $w(i,x)$ be an excess of a player with respect to a payoff vector x. Then, an excess difference between two players i and j with respect to a payoff vector x is defined as

$$dw(i,j,x) = w(i,x) - w(j,x), \qquad i \neq j. \tag{5.19}$$

Definition 5.12 (Lexicographical solution using an excess difference between two players) Let $H_{n(n-1)} : R^{n(n-1)} \to R^{n(n-1)}$ be a mapping which arranges entries of an $n(n-1)$-dimensional vector in order of decreasing magnitude. Then, for a game (N,v), the solution minimizing the vector of the excess differences between two players $H_{n(n-1)}(w(i,j,x))$ in the lexicographical order is defined as follows:

$$
\begin{aligned}
LSD(N,v) \;=\; & \{x \mid H_{n(n-1)}(w(1,2,x), \cdots, w(n-1,n,x)) \\
\leq_L \; & H_{n(n-1)}(w(1,2,y), \cdots, w(n-1,n,y)), \forall y \in X(N,v)\}.
\end{aligned}
\tag{5.20}
$$

For the lexicographical solution $LS(N, v)$ using an excess of a player, all excesses of players are compared simultaneously in the sense of the minimization of the maximum excess of the player. On the other hand, the lexicographical solution $LSD(N, v)$ using an excess difference between two players is defined in order to reduce the difference between excesses of two arbitrary players.

We can present the algorithm for computing the solution $LSD(N, v)$ for a game (N, v) in a method similar to devising the algorithm for the nucleolus or the solution $LS(N, v)$. It can also be proved that there always exists uniquely a solution $LSD(N, v)$.

To examine the relation between the two solutions $LS(N, v)$ and $LSD(N, v)$, consider a solution concept which equates excesses of all the players.

Definition 5.13 (Solution equating excesses of all the players) For a game (N, v), the solution which equates excesses of all the players can be defined as follows:

$$ES(N, v) = \{x \mid w(1, x) = w(2, x) = \cdots = w(n, x) \quad \forall x \in X(N, v)\}. \tag{5.21}$$

When players receive the payoffs given by the solution $ES(N, v)$, all of the players are in equilibrium in terms of an excess of each player. Especially, when the solution $LS(N, v)$ can be obtained by solving the linear programming problem only one time, the imputation is such that the excesses of all players become the same value. Therefore the solution $ES(N, v)$ can be interpreted as the solution $LS(N, v)$, which has been redefined restrictively from this point of view.

The solution $ES(N, v)$ can be obtained by solving the following simultaneous linear equations:

$$\left.\begin{array}{l} w(1, x) = \varepsilon \\ w(2, x) = \varepsilon \\ \quad\vdots \\ w(n, x) = \varepsilon \\ x_1 + x_2 + \cdots + x_n = v(N). \end{array}\right\} \tag{5.22}$$

From (5.11), the above simultaneous equations are rewritten as follows:

$$\left.\begin{array}{l} 2\cdot 2^{n-2}x_1 + 2^{n-2}x_2 + \cdots + 2^{n-2}x_n + \varepsilon = \sum_{S\ni 1} v(S) \\ 2^{n-2}x_1 + 2\cdot 2^{n-2}x_2 + \cdots + 2^{n-2}x_n + \varepsilon = \sum_{S\ni 2} v(S) \\ \vdots \\ 2^{n-2}x_1 + 2^{n-2}x_2 + \cdots + 2\cdot 2^{n-2}x_n + \varepsilon = \sum_{S\ni n} v(S) \\ x_1 + x_2 + \cdots + x_n = v(N). \end{array}\right\} \tag{5.23}$$

The solution of (5.23) is

$$\begin{aligned} x_i^* &= \frac{1}{2^{n-2}}\left\{\frac{2^{n-2}}{n}v(N) + \sum_{i\in S} v(S) - \frac{1}{n}\sum_{j\in N}\Big(\sum_{j\in S} v(S)\Big)\right\},\ \forall i \in N \\ \varepsilon^* &= \frac{1}{n}\left\{\sum_{j\in N}\sum_{j\in S} v(S) - (n+1)2^{n-2}v(N)\right\}. \end{aligned} \tag{5.24}$$

If x^* belongs to the set of imputation $X(N,v)$, it is the solution $ES(N,v)$.

The condition such that $x^* \in X(N,v)$ can be expressed as the following n inequalities:

$$2^{n-2}v(N) + n\sum_{S\ni i} v(S) - \sum_{\substack{j\in N\\ j\neq i}}\sum_{\substack{S\ni j\\ j\neq i}} v(S) \geq 0,\ i = 1,2,\ldots,n. \tag{5.25}$$

The following theorem shows the relation among the three solutions $LS(N,v)$, $LSD(N,v)$ and $ES(N,v)$.

Theorem 5.4

For a game (N,v), if the solution $ES(N,v)$ exists, it is equal to both the solution $LS(N,v)$ and the solution $LSD(N,v)$.

Proof First, we will prove the first part of the theorem. Let (x^*,ε^*) be the solution of the simultaneous linear equations (5.22). Then if the game (N,v) satisfies the condition (5.25), x^* is the solution $ES(N,v)$. Let (x',ε') be the solution obtained by Algorithm 5.2, i.e., the solution $LS(N,v)$.

If (x^*,ε^*) is not equal to (x',ε'), $\varepsilon^* > \varepsilon'$; and for a certain j, there exists $\rho_j > 0$ such that

$$w(i,x') + \rho_j = \varepsilon'.$$

We have

$$\sum_{i\in N} w(i,x') + \sum_j \rho_j = n\varepsilon',$$

and

$$\sum_{i \in N} w(i, x^*) = n\varepsilon^*.$$

Since, for any x,

$$\sum_{i \in N} w(i, x) = \sum_{i \in N} \sum_{\substack{S \subseteq N \\ S \ni i}} v(S) - (n+1)2^{n-2}v(N) = \text{a constant value},$$

we have

$$\sum_{i \in N} w(i, x') = \sum_{i \in N} w(i, x^*).$$

Thus,

$$\sum_{j} \rho_j = n\varepsilon' - n\varepsilon^* > 0$$

leads to the contradiction $\varepsilon' > \varepsilon^*$. Therefore, the solution $ES(N, v)$ is equal to the solution $LS(N, v)$.

Next, we will prove the second part of the theorem. Let x^* be the solution $ES(N, v)$. From the definition of the solution $ES(N, v)$,

$$w(i, x^*) = w(j, x^*), \qquad i \neq j$$

and subsequently,

$$dw(i, j, x^*) = w(i, x^*) - w(j, x^*) = 0.$$

Let (x', ε') be the solution $LSD(N, v)$ and assume that the solution $ES(N, v)$ is not equal to the solution $LS(N, v)$, i.e., $x^* \neq x'$. From the property of the inequality constraints in the following mathematical programming problems in the algorithm for computing the solution $LSD(N, v)$,

$$\begin{array}{ll} \text{minimize} & \varepsilon \\ \text{subject to} & dw(i, j, x) = \varepsilon_1, \qquad \forall i (\neq j) \in \mathcal{T}_1 \\ & dw(i, j, x) = \varepsilon_2, \qquad \forall i (\neq j) \in \mathcal{T}_2 \\ & \vdots \\ & dw(i, j, x) = \varepsilon_{t-1}, \qquad \forall i (\neq j) \in \mathcal{T}_{t-1} \\ & dw(i, j, x) \leq \varepsilon, \qquad \forall i (\neq j) \in N, \ i \notin \mathcal{T}_1 \cup \mathcal{T}_2 \cup \cdots \cup \mathcal{T}_{t-1} \\ & \sum_{i \in N} x_i = v(N) \\ & x_i \geq 0, \qquad i = 1, 2, \ldots, n, \end{array} \tag{5.26}$$

we have $\varepsilon' > 0$. However, (x^*, ε^*) is feasible for the problem (5.26) and $\varepsilon^* = 0$. This contradicts the optimal property of (x', ε'). Therefore, the solution $ES(N, v)$ is equal to the solution $LSD(N, v)$. □

From Theorem 5.4, we can easily obtain the solutions $LS(N, v)$ and $LSD(N, v)$, which coincide, by solving the simultaneous linear equations (5.22) if the solution of (5.22) belongs to the set of imputation $X(N, v)$. Moreover, from the proof of Theorem 5.4, we can say that there exists the solution $ES(N, v)$ if and only if the optimal solution to the problem $\min_{x \in X(N,v)} \max_{i \in N} w(i, x)$ is unique.

Example 5.1

Consider the three-person cooperative game defined by the following characteristic function:

$$\begin{array}{llll} v(\phi) = 0, & v(\{1\}) = 0, & v(\{2\}) = 0, & v(\{3\}) = 0, \\ v(\{1,2\}) = 80, & v(\{1,3\}) = 20, & v(\{2,3\}) = 60, & v(\{1,2,3\}) = 200. \end{array}$$

In this example, there exists the solution $ES(N, v)$ which is equal to both the solutions $LS(N, v)$ and $LSD(N, v)$. The proposed solution and the nucleolus were computed.

Table 5.1 Imputations of the proposed solution and the nucleolus

	Player 1	Player 2	Player 3
Proposed solution	66.7	86.7	46.7
Nucleolus	70.0	70.0	60.0

It is seen from Table 5.1 that the nucleolus gives both Player 1 and Player 2 the same payoffs, but the proposed solution gives all the players different values. Even in such a simple game, the proposed solution can be considered to derive the property of the characteristic function.

5.3 n-Person Cooperative Games with Fuzzy Coalitions

In this section we consider the lexicographical solution and the related solutions in n-person cooperative games with fuzzy coalitions and present their computational methods.

5.3.1 Problem Formulation and Solution Concepts

For conventional n-person cooperative games, the coalition S can be specified by the function τ^S as follows:

$$\tau^S(i) = \begin{cases} 1 & \text{if} \quad i \in S \\ 0 & \text{if} \quad i \notin S. \end{cases} \tag{5.27}$$

A rate of participation of a player i is defined by $\tau^S(i)$, i.e., $\tau^S(i) = 1$ if a player i participates in a coalition S, and $\tau^S(i) = 0$ if a player i does not participate in S. Consequently, a coalition S is represented by $\tau^S = (\tau^S(1), \cdots, \tau^S(n))$.

A fuzzy coalition τ is defined as a coalition in which a player i can participate with a rate of participation $\tau_i \in [0,1]$ instead of $\{0,1\}$. Thus we can define a fuzzy coalition as follows:

Definition 5.14 (Fuzzy coalition) For an n-person game, let the set of all players be denoted $N = \{1, 2, \ldots, n\}$. Any nonempty fuzzy subset of N is called a fuzzy coalition.

Definition 5.15 (Characteristic function for games with fuzzy coalitions) The function f, called a characteristic function of a game with fuzzy coalitions, is a real-valued function which associates any fuzzy coalition $\tau \in [0,1]^n$ with its real value $f(\tau)$.

From the above two definitions, a game with fuzzy coalitions is denoted by (N, f).

An n-person cooperative game with fuzzy coalitions includes a conventional n-person cooperative game as a special case. Therefore, for a game (N, v), the excess of S with respect to x can be represented as follows:

$$e(S, x) = v(S) - x \cdot \tau^S, \tag{5.28}$$

where $\tau^S \in \{0,1\}^n$, i.e., for any player $i \in N$, $\tau_i^S = 1$ if $i \in S$, $\tau_i^S = 0$ if $i \notin S$ and, for the sake of simplicity, $\tau^S(i)$ is written as τ_i^S.

Similar to a game (N, v), an excess in a game with fuzzy coalitions (N, f) can be defined as follows:

Definition 5.16 (Excess of a fuzzy coalition) For a game with fuzzy coalitions (N, f), let x be a payoff vector and let τ be a fuzzy coalition. Then an excess of the fuzzy coalition $\tau \in [0,1]^n$ with respect to a payoff vector x is defined by

$$\tilde{e}(\tau, x) = f(\tau) - x \cdot \tau, \tag{5.29}$$

where $f(\tau)$ is a value of a characteristic function representing the total amount of transferable utility that the cooperation of players, based on rates of participating in the fuzzy coalition τ, can make by itself.

We now consider an excess of a player in a game with fuzzy coalitions (N, f) as well as in a game without fuzzy coalitions (N, v). For a game (N, v), an excess of a player i with respect to x can be rewritten as follows:

$$w(i,x) = \sum_{S \subseteq N} \tau_i^S \; e(S,x) = \sum_{S \subseteq N} \tau_i^S \; (v(S) - x \cdot \tau^S). \tag{5.30}$$

For a game with fuzzy coalitions (N, f), we first consider a game with a finite number of fuzzy coalitions. A fuzzy coalition τ is represented by an n-dimensional vector of which an entry satisfies $\tau_i \in [0, 1]$. Let a set of fuzzy coalitions be T, then an excess $\tilde{w}(i, x)$ of a player i with respect to a payoff vector x is

$$\tilde{w}(i,x) = \sum_{\tau \in T} \tau_i \; \tilde{e}(\tau, x). \tag{5.31}$$

Consequently, an excess $w(i, x)$ of player i in a game without fuzzy coalitions (N, v) can be regarded as a special case of an excess $\tilde{w}(i, x)$ of player i in a game with fuzzy coalitions (N, f).

Secondly, we consider a fuzzy game with an infinite number of fuzzy coalitions. The excess $\tilde{w}(i, x)$ in a game (N, f), which permits all of the fuzzy coalitions $\tau \in [0, 1]^n$, is defined by multiplying an excess $\tilde{e}(\tau, x)$ by a rate of participation τ_i of player i and integrating it from 0 to 1, i.e.,

$$\tilde{w}(i,x) = \int_0^1 \tau_i \; \tilde{e}(\tau, x) d\tau, \tag{5.32}$$

where

$$\int_0^1 d\tau \; = \int_0^1 \cdots \int_0^1 d\tau_1 d\tau_2 \cdots d\tau_n. \tag{5.33}$$

When a permissible domain of τ is limited to

$$D = \{\tau \mid \alpha_i \leq \tau_i \leq \beta_i, 0 \leq \alpha_i \leq \beta_i \leq 1, \forall i \in N\},$$

instead of $[0, 1]^n$, the excess $\tilde{w}(i, x)$ can also be considered as follows:

$$\tilde{w}(i,x) = \int_D \tau_i \; \tilde{e}(\tau, x) d\tau. \tag{5.34}$$

Therefore, in general, when both D and T are considered as permissible domains of τ, an excess $\tilde{w}(i, x)$ is defined as follows.

Definition 5.17 (Excess of a player in a game with fuzzy coalitions) For a game with fuzzy coalitions (N, f), let $\tilde{e}(\tau, x)$ be an excess of a fuzzy coalition

τ with respect to a payoff vector x. Also let D be a subset of $[0,1]^n$, consisting of a finite number of elements; and T be a subset of $[0,1]^n$, consisting of an infinite number of elements. Then an excess of a player i is defined as:

$$\tilde{w}(i,x) = \int_D \tau_i \tilde{e}(\tau,x)d\tau + \sum_{\tau \in T} \tau_i \tilde{e}(\tau,x). \tag{5.35}$$

Especially, when the permissible domain of fuzzy coalitions is $[0,1]^n$, the excess of a player i is expressed as

$$\tilde{w}(i,x) = \int_0^1 \tau_i f(\tau)dt - \frac{1}{4}(x_1 + \cdots + x_{i-1} + x_{i+1} + \cdots + x_n) - \frac{1}{3}x_i. \tag{5.36}$$

Using the concept of the lexicographical order, we consider a solution minimizing an excess of a player $\tilde{w}(i,x)$ in a game (N,f) with fuzzy coalitions.

Definition 5.18 (Lexicographical solution using an excess of a player for games with fuzzy coalitions) Let $H_n : R^n \to R^n$ be a mapping which arranges entries of an n-dimensional vector in order of decreasing magnitude. Then, for a game (N,f) with fuzzy coalitions, the lexicographical solutions using an excess of a player $\tilde{w}(i,x)$ can be defined as follows:

$$\begin{aligned} FLS(N,f) &= \{x \mid H(\tilde{w}(1,x),\cdots,\tilde{w}(n,x)) \\ &\le_L H(\tilde{w}(1,y),\cdots,\tilde{w}(n,y)), \forall y \in X(N,f)\}, \end{aligned} \tag{5.37}$$

where $X(N,f)$ is the set of all imputations, i.e.,

$$X(N,f) = \left\{x \mid x_i \ge f(\tau^{\{i\}}), \forall i \in N, \sum_{i\in N} x_i = f(\tau^N)\right\}. \tag{5.38}$$

5.3.2 Computational Method and Extension of Games

We present the computational methods for the lexicographical solution using an excess of a player in games with fuzzy coalitions, and we try to extend a characteristic function v of a game (N,v) to a characteristic function $f(v)$ of a game $(N,f(v))$ and propose the computational methods for the lexicographical solution in the extended game $(N,f(v))$.

In this section we assume that the permissible domain of fuzzy coalitions is $[0,1]^n$. The algorithm for computing the solution $FLS(N,f)$ in the game (N,f) can be summarized in the following steps.

Algorithm 5.3

[**Step 1**]

Formulate the following linear programming problem:

$$\begin{array}{ll} \text{minimize} & \varepsilon \\ \text{subject to} & \tilde{w}(i,x) \leq \varepsilon, \qquad i = 1,2,\ldots,n \\ & \sum_{i \in N} x_i = f(\tau^N) \\ & x_i \geq 0, \qquad i = 1,2,\ldots,n, \end{array} \tag{5.39}$$

and solve the problem (5.39). Let the optimal solution of (5.39) be ε_1 and the set of players for the active inequality constraints be denoted $\mathcal{T}_1$.

[**Step 2**]

By fixing the active inequality constraints of Step 1 as $\varepsilon = \varepsilon_1$, the linear programming problem

$$\begin{array}{ll} \text{minimize} & \varepsilon \\ \text{subject to} & \tilde{w}(i,x) = \varepsilon_1, \qquad \forall i \in \mathcal{T}_1 \\ & \tilde{w}(i,x) \leq \varepsilon, \qquad \forall i \in N,\ i \notin \mathcal{T}_1 \\ & \sum_{i \in N} x_i = f(\tau^N) \\ & x_i \geq 0, \qquad i = 1,2,\ldots,n, \end{array} \tag{5.40}$$

can be obtained. Let the optimal solution of (5.40) be ε_2 and the set of players for the new active inequality constraints be denoted $\mathcal{T}_2$.

[**Step 3**]

In a procedure similar to [Step 2], the linear programming problem can be obtained by fixing the active inequality constraints in the previous step. Then let the optimal solution of the problem be ε_3 and the set of coalitions for the new active inequality constraints be denoted $\mathcal{T}_3$.

⋮

[**Step t**]

Let the optimal solution at Step $t-1$ be ε_{t-1}. By fixing $\varepsilon = \varepsilon_{t-1}$, the active inequality constraints containing ε are converted into equality constraints. The obtained linear programming problem

$$
\begin{array}{lll}
\text{minimize} & \varepsilon & \\
\text{subject to} & \tilde{w}(i,x) = \varepsilon_1, & \forall i \in \mathcal{T}_1 \\
& \tilde{w}(i,x) = \varepsilon_2, & \forall i \in \mathcal{T}_2 \\
& \vdots & \\
& \tilde{w}(i,x) = \varepsilon_{t-1}, & \forall i \in \mathcal{T}_{t-1} \\
& \tilde{w}(i,x) \leq \varepsilon, & \forall i \in N,\ i \notin \mathcal{T}_1 \cup \mathcal{T}_2 \cup \cdots \cup \mathcal{T}_{t-1} \\
& \sum_{i \in N} x_i = f(\tau^N) & \\
& x_i \geq 0, & i = 1, 2, \ldots, n,
\end{array}
\tag{5.41}
$$

is solved. Let the optimal solution of (5.41) be ε_t.

Theorem 5.5

For the game (N, f), Algorithm 5.3 can always determine the unique solution by at most $t = n$ steps, which is the lexicographical solution using an excess of a player $FLS(N, f)$, when the permissible domain of fuzzy coalitions is $[0, 1]^n$.

Proof Since an excess $\tilde{w}(i, x)$ of any player i with respect to x is linear when the permissible domain of fuzzy coalitions is $[0, 1]^n$, we can prove the theorem by a procedure similar to the proof of Theorem 5.2. □

Consider a solution concept related to the solution $FLS(N, f)$. Especially, when the solution $FLS(N, f)$ can be obtained by solving the linear programming problem only one time, the imputation is such that excesses of all players have the same value. From this point of view, a part of the solution $FLS(N, f)$ can be redefined as follows. The solution is more restrictive than the solution $FLS(N, f)$ and can be defined as the solution in which the excesses of all players take same values.

Definition 5.19 (Solution equating excesses of all players in a game with fuzzy coalitions) For a game (N, f), let $\tilde{w}(i, x)$ be an excess of a player i. Then the solution equating excesses of all the players is defined as follows:

$$
FES(N, f) = \{x \mid \tilde{w}(1, x) = \cdots = \tilde{w}(n, x), \forall x \in X(N, f)\}. \tag{5.42}
$$

The solution $FES(N, f)$ can be obtained by solving the following simultaneous linear equations:

$$\left.\begin{array}{l}\tilde{w}(1,x)=\varepsilon \\ \tilde{w}(2,x)=\varepsilon \\ \quad\vdots \\ \tilde{w}(n,x)=\varepsilon \\ x_1+x_2+\cdots+x_n=f(\tau^N).\end{array}\right\} \tag{5.43}$$

Let the solution be denoted (x^*, ε^*). If x^* belongs to the set of imputations $X(N, f)$, it is the solution $FES(N, f)$.

Theorem 5.6

For a game (N, f), if the permissible domain of fuzzy coalitions is $[0, 1]^n$ and there exists the solution $FES(N, f)$, it is equal to the solution $FLS(N, v)$.

Proof Since

$$\sum_{i\in N} \tilde{w}(i,x) = \sum_{i\in N} \int_0^1 \tau_i f(\tau) d\tau - \left(\frac{1}{12} + \frac{n}{4} f(\tau^N)\right) = \text{a constant value},$$

we can prove the theorem in a way similar to Theorem 5.4. □

From Theorem 5.6, we can easily obtain the solution $FLS(N, f)$ by solving the simultaneous linear equations (5.43) if the solution of the simultaneous linear equations (5.43) belongs to the set of imputations $X(N, f)$. Moreover, we can state that the solution $FES(N, f)$ exists if and only if the optimal solution to the problem $\min_{x\in X(N,f)} \max_{i\in N} \tilde{w}(i, x)$ is unique.

In general, it is difficult to identify a characteristic function of a game with fuzzy coalitions in practice. Also, when people introduce fuzzy coalitions in a conventional game, a new characteristic function of the game with fuzzy coalitions must be constructed on the basis of the characteristic function of the conventional game. The extension of a game without fuzzy coalitions to a game with fuzzy coalitions can be considered as a mapping from a characteristic function of a game without fuzzy coalitions to a characteristic function of a game with fuzzy coalitions. There exist two mappings, Owen's extension (1972) and Cornet's extension (cited in Aubin 1979).

Let o be Owen's extension operator, with the extension represented by

$$ov(\tau) = \sum_{S\subseteq N} \alpha_S(v) \left(\prod_{i\in S} \tau_i\right), \tag{5.44}$$

where $|S|$ is a number of members of a nonfuzzy coalition S, and

$$\alpha_S(v) = \sum_{T \subseteq S} (-1)^{|S|-|T|} v(T). \tag{5.45}$$

Similarly, let c be Cornet's extension operator, with the extension represented by

$$cv(\tau) = \sum_{S \subseteq N} \alpha_S(v) \left(\prod_{i \in S} \tau_i \right)^{1/|S|}. \tag{5.46}$$

Using a new extension operator a, another extension can be represented by

$$av(\tau) = \sum_{S \subseteq N} \alpha_S(v) \left(\prod_{i \in S} \tau_i \right)^{|S|}. \tag{5.47}$$

Owen's extension is expressed by summing up the influence of each nonfuzzy coalition S on a fuzzy coalition τ, according to a rate of participation τ_i, in a fuzzy coalition τ. $\prod_{i \in S} \tau_i$ is the degree of influence a coalition S has on a fuzzy coalition τ, which is represented by a real number between 0 and 1, and $\alpha_S(v)$ is a normalization for the mapping. Cornet's extension overestimates the influence of the coalition S by proportioning the number of players belonging to S when $\alpha_S(v)$ is positive, because $cv(\tau)$ is represented by taking the one over $|S|$th power of $\prod_{i \in S} \tau_i$ in ov. Notably, cv is positively homogeneous, and Aubin (1979, 1981) has shown interesting results when a characteristic function has this property in a game with fuzzy coalitions. Conversely, av underestimates the influence of a coalition S.

Moreover, we present the following extension operator m by combining these extensions,

$$mv(\tau) = \sum_{S \subseteq N} \alpha_S(v) \left(\prod_{i \in S} \tau_i \right)^{p}, \tag{5.48}$$

where p is any of $1, 1/|S|$ or $|S|$ for each $S \subseteq N$.

In general, a value $v(S)$ of a coalition S in the original game must be equal to a value $f(\tau^S)$ of a nonfuzzy coalition τ^S, which is the same as S, in the extended game, i.e., $v(S) = f(\tau^S)$. The following proposition demonstrates that the extension operators o, c, a and m satisfy this property.

Proposition 5.7

The extension operators o, c, a and f are interpolation operators; that is,

$$ov(\tau^S) = cv(\tau^S) = av(\tau^S) = mv(\tau^S) = v(S), \tag{5.49}$$

for any nonfuzzy coalition S.

Proof We first prove that $cv(\tau^S) = v(S)$. Since

$$\left(\prod_{i\in T}\tau_i^S\right)^{1/|S|} = \mu_T(S) = \begin{cases} 1 & \text{if } S \supset T \\ 0 & \text{if } S \not\supset T, \end{cases}$$

we have

$$\begin{aligned} cv(\tau^S) &= \sum_{T\subseteq N} \alpha_T(v)\Big(\prod_{i\in S}\tau_i^S\Big)^{1/|S|} \\ &= \sum_{T\subseteq N} \alpha_T(v)\mu_T(S) \\ &= v(S). \end{aligned}$$

Since

$$\prod_{i\in T}\tau_i^S = \left(\prod_{i\in T}\tau_i^S\right)^{1/|T|} = \left(\prod_{i\in T}\tau_i^S\right)^{|T|} = \left(\prod_{i\in T}\tau_i^S\right)^{p} = \mu_T(S),$$

the rest of the proof can be shown in a similar way. □

Consider an excess $\tilde{w}(i,x)$ of a player i in an n-person game extended by each extension operator. Let a permissible domain for fuzzy coalitions be $[0,1]^n$.

(i) Let $f(\tau) = ov(\tau)$. In this case,

$$\begin{aligned} \tilde{w}(i,x) &= \int_0^1 \tau_i(ov(\tau) - x\cdot\tau)d\tau \\ &= o_i - \frac{1}{3}x_i - \sum_{j\neq i}\frac{1}{4}x_j, \end{aligned} \tag{5.50}$$

$$\begin{aligned} o_i &= \int_0^1 \tau_i ov(\tau)d\tau \\ &= \int_0^1 \tau_i \sum_{S\subseteq N}\alpha_S(v)(\prod_{i\in S}\tau_i)d\tau \\ &= \int_0^1 \sum_{k=1}^{n}\Big\{\sum_{S\in a_{ki}}\alpha_S(v)\tau_i^2\prod_{\substack{j\in S\\ j\neq i}}\tau_j + \sum_{S\in a_{k\bar{i}}}\alpha_S(v)\tau_i\prod_{j\in S}\tau_j\Big\}d\tau \\ &= \sum_{k=1}^{n}\Big\{\frac{1}{3}\Big(\frac{1}{2}\Big)^{k-1}\sum_{S\in a_{ki}}\alpha_S(v) + \Big(\frac{1}{2}\Big)^{k+1}\sum_{S\in a_{k\bar{i}}}\alpha_S(v)\Big\} \\ &= \frac{1}{3\cdot 2^{n-1}}\sum_{i\in S}v(S) + \frac{1}{3\cdot 2^{n}}\sum_{i\notin S}v(S), \end{aligned} \tag{5.51}$$

where

$$\begin{aligned} a_{ki} &= \{S \mid i\in S, |S| = k\} \\ a_{k\bar{i}} &= \{S \mid i\notin S, |S| = k\}. \end{aligned} \tag{5.52}$$

(ii) Let $f(\tau) = cv(\tau)$. In this case,

$$\begin{aligned} \tilde{w}(i,x) &= \int_0^1 \tau_i(cv(\tau) - x\cdot\tau)d\tau \\ &= c_i - \frac{1}{3}x_i - \sum_{j\neq i}\frac{1}{4}x_j, \end{aligned} \tag{5.53}$$

$$
\begin{aligned}
c_i &= \int_0^1 \tau_i cv(\tau)d\tau \\
&= \sum_{k=1}^{n}\Big\{\frac{k}{2k+1}\Big(\frac{k}{k+1}\Big)^{k-1}\sum_{S\in a_{ki}}\alpha_S(v)+\frac{1}{2}\Big(\frac{k}{k+1}\Big)^{k}\sum_{S\in a_{k\bar{i}}}\alpha_S(v)\Big\}.
\end{aligned} \tag{5.54}
$$

(iii) Let $f(\tau) = av(\tau)$. In this case,

$$
\begin{aligned}
\tilde{w}(i,x) &= \int_0^1 \tau_i(av(\tau) - x\cdot\tau)d\tau \\
&= a_i - \frac{1}{3}x_i - \sum_{j\neq i}\frac{1}{4}x_j,
\end{aligned} \tag{5.55}
$$

$$
\begin{aligned}
a_i &= \int_0^1 \tau_i av(\tau)d\tau \\
&= \sum_{k=1}^{n}\Big\{\frac{1}{k+2}\Big(\frac{1}{k+1}\Big)^{k-1}\sum_{S\in a_{ki}}\alpha_S(v)+\frac{1}{2}\Big(\frac{1}{k+1}\Big)^{k}\sum_{S\in a_{k\bar{i}}}\alpha_S(v)\Big\}.
\end{aligned} \tag{5.56}
$$

(iv) Let $f(\tau) = fv(\tau)$. In this case,

$$
\begin{aligned}
\tilde{w}(i,x) &= \int_0^1 \tau_i(fv(\tau) - x\cdot\tau)d\tau \\
&= f_i - \frac{1}{3}x_i - \sum_{j\neq i}\frac{1}{4}x_j,
\end{aligned} \tag{5.57}
$$

$$
\begin{aligned}
f_i &= \int_0^1 \tau_i fv(\tau)d\tau \\
&= \sum_{k=1}^{n}\Big\{\sum_{S\in a_{ki}}G_k^S\alpha_S(v)+\sum_{S\in a_{k\bar{i}}}H_k^S\alpha_S(v)\Big\},
\end{aligned} \tag{5.58}
$$

where

$$
G_k^S = \begin{cases}
\Big(\frac{1}{3}\Big)\Big(\frac{1}{2}\Big)^{k-1} & \text{if } \ S:p=1 \\
\frac{k}{2k+1}\Big(\frac{k}{k+1}\Big)^{k-1} & \text{if } \ S:p=1/|S| \\
\frac{1}{k+2}\Big(\frac{1}{k+1}\Big)^{k-1} & \text{if } \ S:p=|S|,
\end{cases} \tag{5.59}
$$

$$
H_k^S = \begin{cases}
\Big(\frac{1}{2}\Big)^{k+1} & \text{if } \ S:p=1 \\
\frac{1}{2}\Big(\frac{k}{k+1}\Big)^{k} & \text{if } \ S:p=1/|S| \\
\frac{1}{2}\Big(\frac{1}{k+1}\Big)^{k} & \text{if } \ S:p=|S|.
\end{cases} \tag{5.60}
$$

We are ready to apply the proposed solutions to the extended game. Using Owen's extension, the excess of a player is represented by (5.50) and (5.51). Therefore, the solution $FES(N, ov)$ can be obtained by solving the following simultaneous equations:

$$\left.\begin{array}{l}\frac{1}{3}x_1+\frac{1}{4}x_2+\cdots+\frac{1}{4}x_n+\varepsilon=o_1\\ \frac{1}{4}x_1+\frac{1}{3}x_2+\cdots+\frac{1}{4}x_n+\varepsilon=o_2\\ \cdots\\ \frac{1}{4}x_1+\frac{1}{4}x_2+\cdots+\frac{1}{3}x_n+\varepsilon=o_n\\ x_1+x_2+\cdots+x_n=v(N).\end{array}\right\} \tag{5.61}$$

Then the solution of (5.61) is

$$\begin{aligned} x_i^* &= \frac{1}{2^{n-2}}\left\{\frac{2^{n-2}}{n}v(N)+\sum_{i\in S}v(S)-\frac{1}{n}\sum_{\substack{j\in N\\ j\neq i}}\Big(\sum_{\substack{j\in S\\ j\neq i}}v(S)\Big)\right\},\ i=1,2,\ldots,n\\ \varepsilon^* &= \frac{1}{n}\left\{\frac{1}{3\cdot 2^{n-1}}\Big(\sum_{i\in N}\sum_{i\in S}v(S)+2\sum_{i\in N}\sum_{i\notin S}v(S)\Big)-\frac{3n+1}{12}v(N)\right\}.\end{aligned} \tag{5.62}$$

The following proposition can be directly obtained from the equations (5.24) and (5.62).

Proposition 5.8

If there exists a solution $ES(N, v)$ in a game (N, v), the solution $ES(N, v)$ is equal to the solution $FES(N, ov)$ in the game (N, ov) with fuzzy coalitions extended by Owen's extension operator o.

Similarly, for the other extensions, the solution FES can be obtained as:

$$\begin{aligned} x_i^* &= \frac{12}{n}\left\{\frac{1}{12}v(N)+nB_i-\sum_{\substack{j\in N\\ j\neq i}}B_j,\right\} \qquad i=1,2,\ldots,n\\ \varepsilon^* &= \frac{1}{n}\left\{\sum_{i\in N}B_i-\frac{3n+1}{12}v(N)\right\},\end{aligned} \tag{5.63}$$

where, for the extension operator c,

$$B_i=\int_0^1 \tau_i\, cv(\tau)d\tau, \qquad i=1,2,\ldots,n, \tag{5.64}$$

for the extension operator a,

$$B_i=\int_0^1 \tau_i\, av(\tau)d\tau, \qquad i=1,2,\ldots,n \tag{5.65}$$

and, for the extension operator m,

$$B_i=\int_0^1 \tau_i\, mv(\tau)d\tau, \qquad i=1,2,\ldots,n. \tag{5.66}$$

Therefore, for the games extended by the extension operators o, c, a and m, if the conditions

$$\frac{1}{12}v(N) + nB_i - \sum_{\substack{j \in N \\ j \neq i}} B_j \geq 0, \qquad i = 1, 2, \ldots, n \tag{5.67}$$

are satisfied, there exist the solutions $FES(N, ov)$, $FES(N, cv)$, $FES(N, av)$ and $FES(N, mv)$ for the extended games, respectively. In the original game (N, v), if the value $v(S)$ of coalitions $S \ni i$ to which any player i belongs is not so small, compared with others, these conditions can be satisfied.

Even if the solutions $FES(N, ov)$, $FES(N, cv)$, $FES(N, av)$ and $FES(N, mv)$ do not exist, we can always obtain the solutions $FLS(N, ov)$, $FLS(N, cv)$, $FLS(N, av)$ and $FLS(N, mv)$ uniquely by Algorithm 5.3.

Example 5.2

Consider a four-person game defined by the following characteristic function v:

$$\begin{array}{llll} v(\phi) = 0, & v(\{1\}) = 1, & v(\{2\}) = 2, & v(\{3\}) = 2, \\ v(\{4\}) = 3, & v(\{1,2\}) = 50, & v(\{1,3\}) = 55, & v(\{1,4\}) = 70, \\ v(\{2,3\}) = 60, & v(\{2,4\}) = 80, & v(\{3,4\}) = 90, & v(\{1,2,3\}) = 120, \\ v(\{1,2,4\}) = 130, & v(\{1,3,4\}) = 150, & v(\{2,3,4\}) = 200, & v(\{1,2,3,4\}) = 300. \end{array}$$

Let the permissible domain of a coalition be $[0,1]^n$ and let us extend the four-person game by the extension operators o, c and a. The solutions $ES(N, v)$, $FES(N, ov)$, $FES(N, cv)$ and $FES(N, av)$ exist for the four-person games (N, v), (N, ov), (N, cv) and (N, av), and the solutions can be computed easily.

Table 5.2 Solutions ES and FES

	Player 1	Player 2	Player 3	Player 4
$ES(N, v)$	55.37501	71.87498	80.62500	92.12501
$FES(N, ov)$	55.37501	71.87498	80.62500	92.12501
$FES(N, cv)$	51.90225	72.85343	82.66145	92.58291
$FES(N, av)$	64.17812	71.53647	76.81771	87.46770

It can be seen from Table 5.2 that the imputations differ from one another according to extension operators. In the solution by Cornet's extension, the differences in the payoffs are relatively large. In contrast, they are relatively small in the solution by the extension operator a. Owen's extension is between the other two extensions. To be more specific, the discrepancy between the payoff of Player 1 and that of Player 4, which is largest among all of the players, is 40.22276 for Cornet's extension, 36.76000 for Owen's extension and 23.28958 for the extension operator a.

5.4 Conclusion

We have introduced the concept of the excess of a player in a game with fuzzy coalitions as well as in a conventional game and examined the lexicographical solution concepts in both of the games.

In a game with fuzzy coalitions, we have mainly considered the proposed solutions but not the nucleolus. Since fuzzy coalitions exist infinitely, excesses of fuzzy coalitions also exist infinitely. Therefore, to consider the nucleolus of games with fuzzy coalitions, we have to extend the lexicographical order defined in this chapter to the lexicographical order for infinite sequences. Moreover, it would seem that computing the nucleolus in the game with fuzzy coalitions is difficult, and even if we could present the method, many assumptions would be required. As we mentioned, the main aim of this thesis is to present resolution methods for competitive systems. Therefore, although the studies on the nucleolus in games with fuzzy coalitions are important, such analyses are outside the scope of this thesis and will be addressed in another paper.

To conclude, the results of this chapter are summarized as follows:

1) To evaluate payoff vectors in terms of players, we have introduced the concept of the excess of a player for games with fuzzy coalitions as well for conventional games.

2) The lexicographical framework has given rise to an important class of solution concepts, but for games with fuzzy coalitions, solution concepts incorporating such a framework have not been examined. The lexicographical solution using the excess of a player has been considered and the computational methods were presented.

3) We also proposed the solution concepts related to the lexicographical solution using an excess of a player, which include the solution equating excesses of all the players as a special case; and the relationship between the two solution concepts was analyzed.

4) When fuzzy coalitions are introduced in a conventional game, the characteristic

function of the game with fuzzy coalitions must be formulated on the basis of the original characteristic function. We have provided some extension methods and presented the computational methods to facilitate the proposed solutions in the extended games.

In this chapter, games with fuzzy coalitions were examined to consider ambiguity in the forming of an organization. In the following chapter, we will deal with other n-person cooperative games, which are described by a fuzzy goal expressing a degree of the coalition's satisfaction for the payoffs instead of the value $v(S)$.

CHAPTER 6

N-PERSON COOPERATIVE GAMES WITH FUZZY GOALS

6.1 Introduction

In this chapter, we consider n-person cooperative games in a fuzzy environment different from a fuzzy coalition. A characteristic function v describes an n-person cooperative game and associates a coalition S with the worth $v(S)$ of the coalition, represented by a real number which is considered to be the max-min value of the two-person game played between the coalition S and the other coalition $N - S$ which consists of the rest of the players, as von Neumann and Morgenstern (1944) have suggested. As we mentioned in Chapters 2, 3 and 4, there is a two-person game in which payoffs cannot be accurately determined because of some imprecision in the information, so the value $v(S)$ of a coalition S, derived by the two-person game, becomes ambiguous. Therefore, it is meaningful to consider games with fuzzy values of coalitions where the characteristic function of such games, which should be a characteristic mapping, associates a value of a coalition with a fuzzy set (Sakawa and Nishizaki 1992a; Nishizaki and Sakawa 1992a; Seo and Nishizaki 1991, 1992, 1993; and Seo, Sakawa and Nishizaki 1992, 1993).

A value $v(S)$ of a coalition S refers to the gain which the coalition S can acquire only through the action of S; hence, the game (N, v) can be interpreted as a game described by the value $v(S)$ with which the coalition S is minimally satisfied. By utilizing a coalition's satisfaction with a payoff vector, we can present another representation of a game to accommodate the imprecision of information. Namely, the game is described by the fuzzy goal $\tilde{G}_S$, which expresses the degree of the coalition's satisfaction with a payoff vector, instead of the value $v(S)$. Membership functions of fuzzy goals assign degrees of satisfaction continuously from the coalition's minimal satisfying value to its maximal satisfying value. This game is defined by a 3-tuple $(N, \mu_{\tilde{G}_S}, P)$, where N is the set of all players, $\mu_{\tilde{G}_S}$ is a membership function of a fuzzy goal of a coalition S and P is the payoff amount, which is divided among all players on the assumption that the players will form the grand coalition N.

In Section 6.2, we review some definitions for the game $(N, \mu_{\tilde{G}_S}, P)$ and propose a new solution concept based on the fuzzy decision rule by Bellman and Zadeh (1970). Several methods for identifying a membership function are presented. We can also construct a membership function $\mu_{\tilde{G}_S}$ using the characteristic function v in a conventional game (N, v), which facilitates the transformation of the game (N, v) to the fuzzy game $(N, \mu_{\tilde{G}_{S(v)}}, P)$.

In Section 6.3, when all of the membership functions are linear functions or hyperbolic functions, the proposed solutions can be obtained by solving linear programming problems (Zimmermann 1976 and Leberling 1981). We also adopt for each coalition one of five types of membership functions, which include linear, hyperbolic, exponential, hyperbolic inverse and piecewise linear functions. In this case, the proposed solution corresponds to the optimum solution of a nonlinear programming problem. However, it can be obtained by a combined use of the bisection method and phase one of the simplex method (Sakawa 1983). Finally, an illustrative numerical example, where all of the membership functions are linear, is presented.

6.2 Problem Formulation and Solution Concepts

For an n-person cooperative game (N, v) in characteristic function form, $v(S)$ denotes the joint payoff which the members of any given coalition $S \subseteq N$ achieve if they cooperate among themselves but not with the remaining players. In this chapter, we employ a representation of an n-person cooperative game instead of the usual characteristic function v. We introduce the concept of a fuzzy goal and its corresponding membership function to represent a degree of satisfaction of a coalition S with respect to a payoff vector x. Given the payoff vector, the membership function of the fuzzy goal provides a value in [0,1] which represents the coalition S's degree of satisfaction.

Definition 6.1 (A fuzzy goal) Let N be the set of all players and let S, which is a subset of N, be a coalition. Let X denote a set of payoff vector x and let x_S denote coalition S's payoff where $x_S = \sum_{i \in S} x_i$. Then a membership function of a fuzzy goal for coalition S is represented by $\mu_{\tilde{G}_S}(x_S)$, where the fuzzy goal $\tilde{G}_S$ is a fuzzy set which represents the degree of satisfaction of the coalition S. The fuzzy goal $\tilde{G}_S$ is expressed by a pair x_S and $\mu_{\tilde{S}}(x_S)$, i.e.,

$$\tilde{G}_S = \{(x_S, \mu_{\tilde{G}_S}(x_S)) \mid x \in X\}. \tag{6.1}$$

Then the fuzzy goal $\tilde{G}_S$, which represents the degree of satisfaction, is characterized by a membership function $\mu_{\tilde{G}_S}(x_S)$. By assuming collective rationality, the following is considered as a set of payoff vectors:

$$X = \{x \mid x_1 + x_2 + \cdots + x_n = P,\ x_i \geq 0,\ i \in N\}, \tag{6.2}$$

where P is the payoff amount, which is divided among all players on the assumption that the players will form the grand coalition N.

We present the five types of fuzzy goals which are comprised of linear, hyperbolic, exponential, hyperbolic inverse and piecewise linear functions.

1) Linear membership function

Consider a membership function $\mu_{\tilde{G}_S}(x_S)$ which increases linearly from 0 to 1. $\mu_{\tilde{G}_S}(x_S) = 0$ is interpreted as the minimum degree of satisfaction for a coalition S and $\mu_{\tilde{G}_S}(x_S) = 1$ is interpreted as the maximum degree of satisfaction. Let $\underline{a}_S$ be the maximal value x_S satisfying $\mu_{\tilde{G}_S}(x_S) = 0$ and let $\overline{a}_S$ be the minimal value x_S satisfying $\mu_{\tilde{G}_S}(x_S) = 1$. Then the linear membership function can be expressed as

$$\mu_{\tilde{G}_S}(x_S) = \begin{cases} 0 & \text{if } x_S \leq \underline{a}_S \\ 1 - \dfrac{\overline{a}_S - x_S}{\overline{a}_S - \underline{a}_S} & \text{if } \underline{a}_S < x_S \leq \overline{a}_S \\ 1 & \text{if } \overline{a}_S < x_S. \end{cases} \tag{6.3}$$

2) Hyperbolic membership function

When smooth changes in the degree of satisfaction are required everywhere, nonlinear membership functions should be considered. A typical nonlinear function is a hyperbolic function. This function can be expressed as

$$\mu_{\tilde{G}_S}(x_S) = \frac{1}{2}\tanh\Big(\big(x_S - \frac{\overline{a}_S - \underline{a}_S}{2}\big)\alpha_S\Big) + \frac{1}{2}, \tag{6.4}$$

where α_S is the parameter of distortion, and $\overline{a}_S$ and $\underline{a}_S$ are assessed values. α_S can be determined by assessing the values of $\overline{a}_S$ such that $\mu_{\tilde{G}_S}(\overline{a}_S) = 0.9$; and

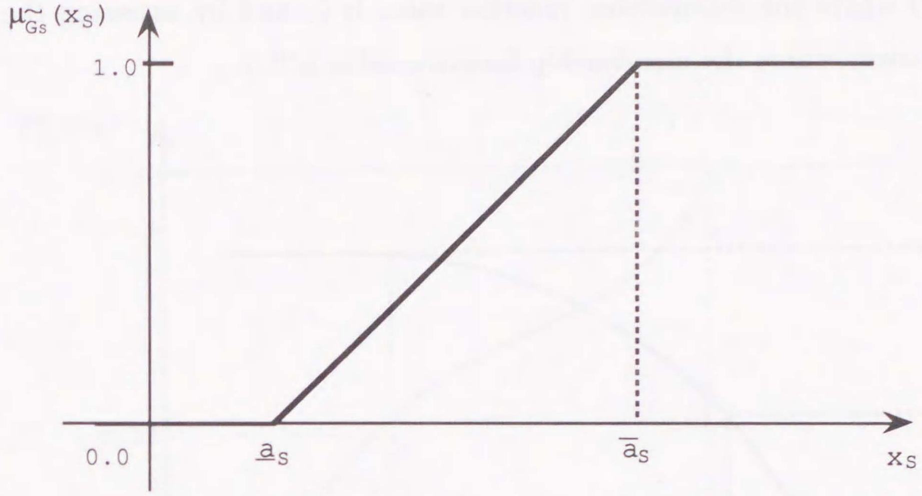

Figure 6.1 Linear membership function

$\underline{a}_S$ such that $\mu_{\tilde{G}_S}(\underline{a}_S) = 0.1$. Then $\overline{a}_S$ can be interpreted as the value having the degree of satisfaction 0.9 and $\underline{a}_S$ as the value having the degree of satisfaction 0.1.

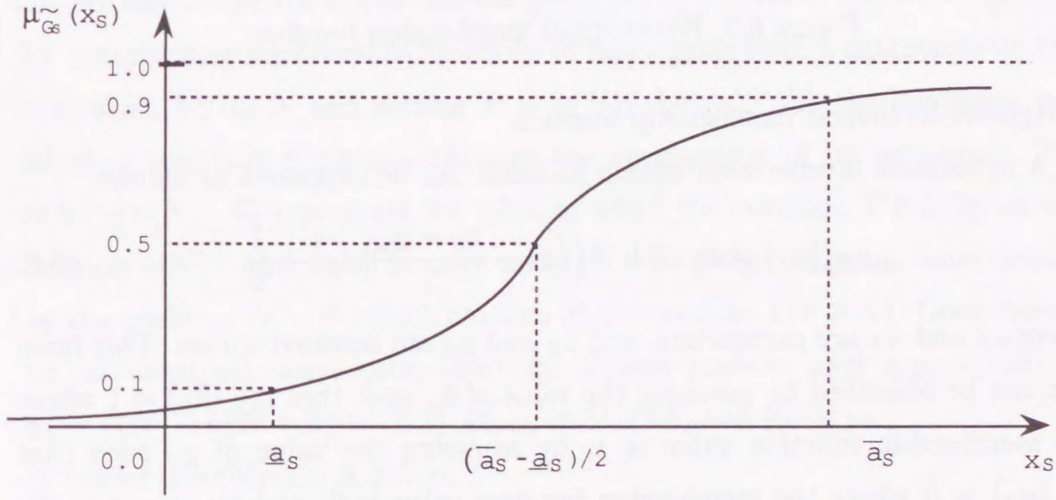

Figure 6.2 Hyperbolic membership function

3) Exponential membership function

An exponential membership function is expressed as follows:

$$\mu_{\tilde{G}_S}(x_S) = a_S\left(\exp\left(\frac{b_S(x_S - \underline{a}_S)}{\overline{a}_S - \underline{a}_S}\right) - 1\right), \tag{6.5}$$

where a_S and b_S are parameters, and $\overline{a}_S$ and $\underline{a}_S$ are assessed values. This function can be identified by assessing the value of $\overline{a}_S$ such that $\mu_{\tilde{G}_S}(\overline{a}_S) = 1$ where the membership function value is 1; by assessing the value of $\underline{a}_S$ such that

$\mu_{\tilde{G}_S}(\underline{a}_S) = 0$ where the membership function value is 0; and by assessing the value subjectively where the membership function value is 0.5.

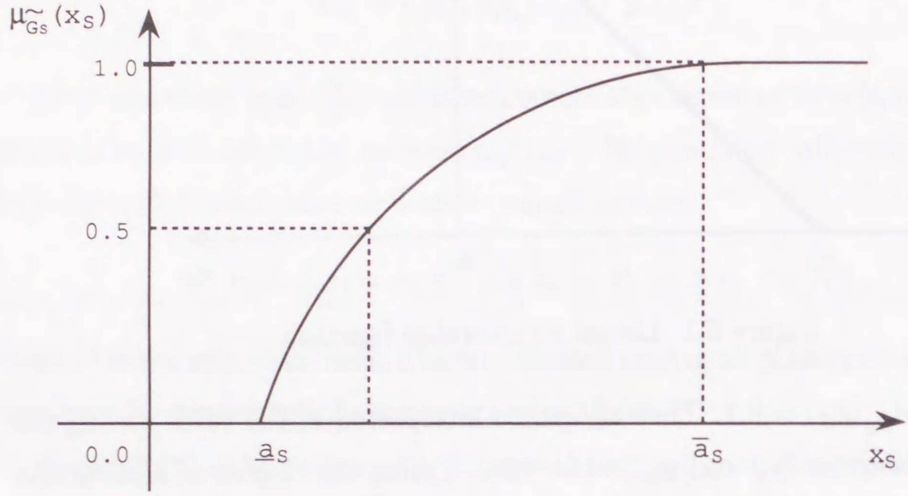

Figure 6.3 Exponential membership function

4) Hyperbolic inverse membership function

A hyperbolic inverse membership function can be expressed as follows:

$$\mu_{\tilde{G}_S}(x_S) = a_S \tanh^{-1}\left(\left(x_S - \frac{\overline{a}_S - \underline{a}_S}{2}\right)\alpha_S\right) + \frac{1}{2}, \tag{6.6}$$

where a_S and α_S are parameters, and $\overline{a}_S$ and $\underline{a}_S$ are assessed values. This function can be identified by assessing the value of $\overline{a}_S$ such that $\mu_{\tilde{G}_S}(\overline{a}_S) = 1$ where the membership function value is 1; by assessing the value of $\underline{a}_S$ such that $\mu_{\tilde{G}_S}(\underline{a}_S) = 0$ where the membership function value is 0; and by assessing the value subjectively where the membership function value is 0.25.

5) Piecewise linear membership function

A piecewise linear membership function can be expressed as follows:

$$\mu_{\tilde{G}_S}(x_S) = t_{ir} x_S + s_{ir}, \tag{6.7}$$

where t_{ir} and s_{ir} are parameters corresponding to the rth straight line, and $\mu_{\tilde{G}_S}(\overline{a}_S) = 1$ and $\mu_{\tilde{G}_S}(\underline{a}_S) = 0$. This function can also be identified by assessing each break-point.

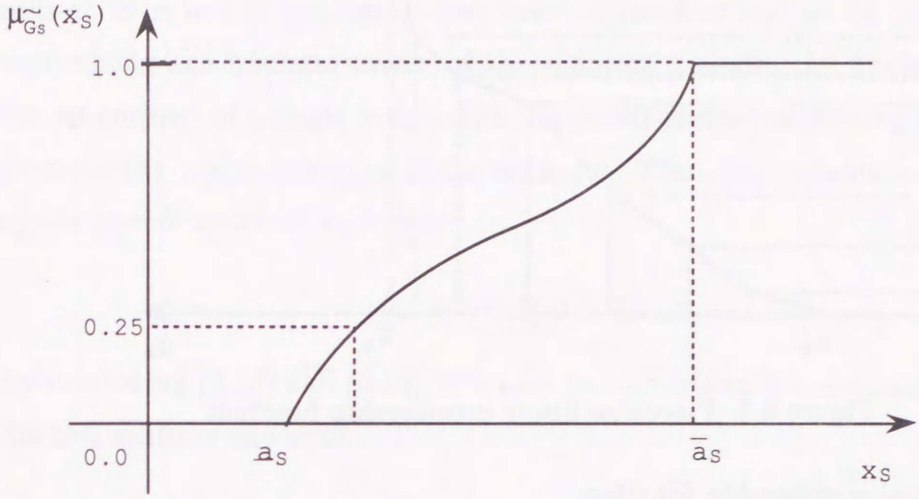

Figure 6.4 Hyperbolic inverse membership function

We can transform a conventional game (N, v) to a fuzzy game $(N, \mu_{\tilde{G}_{S(v)}}, P)$ by constructing membership functions of fuzzy goals from a characteristic function $v(S)$, $\forall S \subseteq N$ and setting $P = v(N)$. Let $v(S)$ be the minimum value which a coalition S obtains through the cooperation of its members. Then $v(N) - v(N - S)$ represents the value at which the coalition S is fully satisfied because $v(N)$ is the payoff amount and $v(N - S)$ is the minimum value obtained by the coalition $N - S$ (which consists of all members not in S). Consequently, we can construct membership functions of fuzzy goals by setting $\underline{a}_S = v(S)$ and $\bar{a}_S = v(N) - v(N - S)$ in (6.3), (6.4), (6.5), (6.6) and (6.7), i.e.,

1) Linear membership function:

$$\mu_{\tilde{G}_S}(x_S) = \begin{cases} 0 & \text{if } x_S \leq v(S) \\ 1 - \dfrac{v(N) - v(N-S) - x_S}{v(N) - v(N-S) - v(S)} & \text{if } v(S) < x_S \leq v(N) - v(N-S) \\ 1 & \text{if } v(N) - v(N-S) < x_S. \end{cases} \tag{6.8}$$

2) Hyperbolic membership function:

$$\mu_{\tilde{G}_S}(x_S) = \frac{1}{2} \tanh\left(\left(x_S - \frac{v(N) - v(N-S) - v(S)}{2}\right)\alpha_S\right) + \frac{1}{2}. \tag{6.9}$$

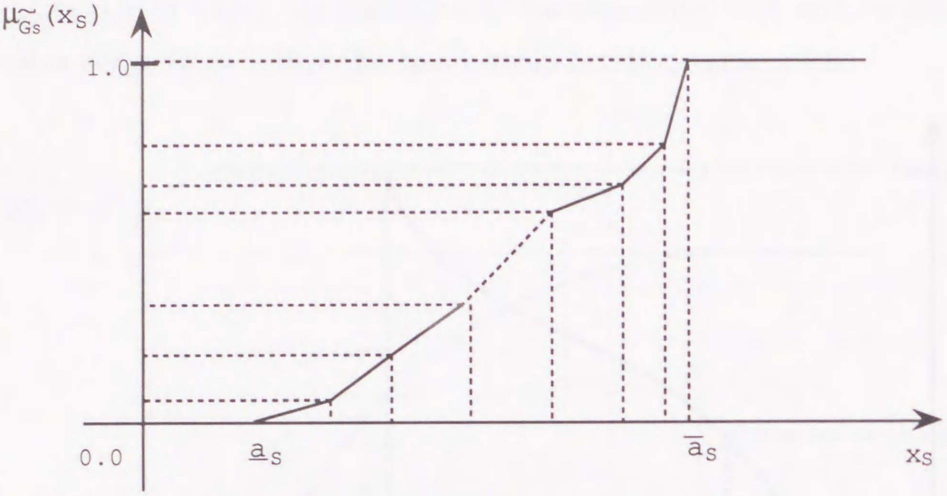

Figure 6.5 Piecewise linear membership function

3) Exponential membership function:

$$\mu_{\tilde{G}_S}(x_S) = a_S\left(\exp\left(\frac{b_S(x_S - v(S))}{v(N) - v(N-S) - v(S)}\right) - 1\right). \tag{6.10}$$

4) Hyperbolic inverse membership function:

$$\mu_{\tilde{G}_S}(x_S) = a_S \tanh^{-1}\left(\left(x_S - \frac{v(N) - v(N-S) - v(S)}{2}\right)\alpha_S\right) + \frac{1}{2}. \tag{6.11}$$

5) Piecewise linear inverse membership function:

$$\mu_{\tilde{G}_S}(x_S) = t_{ir}x_S + s_{ir}, \tag{6.12}$$

where $\mu_{\tilde{G}_S}(v(N) - v(N-S)) = 1$ and $\mu_{\tilde{S}}(v(S)) = 0$.

We now consider a new solution concept for an n-person cooperative fuzzy game $(N, \mu_{\tilde{G}_S}, P)$, defined by maximizing the minimum value of fuzzy goals $\mu_{\tilde{G}_S}(x_S)$. This problem is formulated using the fuzzy decision rule by Bellman and Zadeh (1970).

Definition 6.2 (Solution maximizing the minimal fuzzy goal) Let N be the set of all players, let $\mu_{\tilde{G}_S}$ be a membership function of a fuzzy goal of a coalition S and let P be the payoff amount. In an n-person cooperative fuzzy game $(N, \mu_{\tilde{G}_S}, P)$, the solution maximizing the minimal fuzzy goal is defined by a payoff vector x^*, where

$$\mu_D(x^*) = \max_{x \in X} \min_{S \subset N} \mu_{\tilde{G}_S}(x_S). \tag{6.13}$$

Consider the relation between the solution and the nucleolus. The nucleolus is defined by minimizing the excess $e(S, x) = v(S) - x_S$ of a coalition S with respect to x in the lexicographical order. Therefore the set of payoff vectors maximizing the minimal excess of the coalition includes the nucleolus, and if the set consists of a single component, the payoff vector maximizing the minimal excess of the coalition is equal to the nucleolus. Then the nucleolus is represented by the payoff vector x^* such that

$$\min_{x \in X} \max_{S \subseteq N} e(S, x). \tag{6.14}$$

By comparing (6.13) and (6.14), it is easy to understand the similarities between the two solution concepts.

6.3 Computational Method

We present the computational methods for the proposed solution 1) when all of the membership functions of fuzzy goals consist only of linear functions, 2) when they consist only of hyperbolic functions and 3) when, as a general case, they consist of five kinds of membership functions, which include linear, hyperbolic, exponential, hyperbolic inverse and piecewise linear functions.

1) Linear membership function

When all of the membership functions of fuzzy goals are linear functions, the proposed solution is x^*, which is obtained from

$$\mu_D(x^*) = \max_{x \in X} \min_{S \subset N} \left(1 - \frac{\overline{a}_S - \sum_{i \in S} x_i}{\overline{a}_S - \underline{a}_S} \right). \tag{6.15}$$

By introducing an auxiliary variable λ, x^* can be determined by solving the following linear programming problem (Zimmermann 1976):

$$\begin{array}{ll} \text{maximize} & \lambda \\ \text{subject to} & \lambda \leq 1 - \dfrac{\overline{a}_S - \sum_{i \in S} x_i}{\overline{a}_S - \underline{a}_S}, \quad \forall S \subset N \\ & x_1 + x_2 + \cdots + x_n = P \\ & x_i \geq 0, \qquad \forall i \in N. \end{array} \tag{6.16}$$

A remarkable relation can be found between the core, which is defined as the set of all undominated imputations, in a game (N, v) and the proposed solution in the transformed game $(N, \mu_{\tilde{G}_S}, P)$.

Proposition 6.1

Let $(N, \mu_{\tilde{G}_S}, P)$ be a game transformed from a game (N, v) in which the core is empty, and let $\mu_{\tilde{G}_S}$ be defined by the linear membership function (6.8). Then the max-min value of the membership function of the proposed solution based on the fuzzy decision rule is 0, i.e., $\mu_D(x^*) = 0$.

Proof The core is defined by the set of payoff vectors for which the following conditions hold:

$$\begin{aligned} v(S) - \sum_{i \in S} x_i \leq 0, \quad \forall S \subset N \\ v(N) = \textstyle\sum_{i \in N} x_i. \end{aligned} \tag{6.17}$$

If, in the game (N, v), the set of the core is empty, then there exists a coalition S holding the following condition:

$$v(S) - \sum_{i \in S} x_i > 0. \tag{6.18}$$

From (6.8) and (6.18), there always exists at least one coalition such that, for any payoff vector x, $\mu_{\tilde{G}_S}(x_S) = 0$ holds. Then, $\mu_D(x^*) = 0$. □

Proposition 6.1 states that if the set of the core is empty in the game (N, v) and the game (N, v) is transformed to the game $(N, \mu_{\tilde{G}_{S(v)}}, P)$ of which the membership functions $\mu_{\tilde{G}_{S(v)}}$ are defined by (6.8), there exists at least one coalition which cannot be satisfied with any payoff vector in the sense of a degree of attainment of a fuzzy goal in the transformed game $(N, \mu_{\tilde{G}_{S(v)}}, P)$.

2) Hyperbolic membership function

When all of the membership functions of fuzzy goals are hyperbolic functions, the proposed solution is x^*, which is obtained from

$$\mu_D(x^*) = \max_{x \in X} \min_{S \subset N} \left\{ \frac{1}{2} \tanh\left(\Big(\sum_{i \in S} x_i - \frac{\overline{a}_S - \underline{a}_S}{2} \Big) \alpha_S \right) + \frac{1}{2} \right\}. \tag{6.19}$$

By introducing an auxiliary variable λ, x^* can be determined by solving the following nonlinear programming problem (Leberling 1981):

$$\begin{aligned} &\text{maximize} \quad \lambda \\ &\text{subject to} \quad \lambda \leq \frac{1}{2} \tanh\left(\Big(\sum_{i \in S} x_i - \frac{\overline{a}_S - \underline{a}_S}{2} \Big) \alpha_S \right) + \frac{1}{2}, \quad \forall S \subset N \\ &\qquad\qquad x_1 + x_2 + \cdots + x_n = P \\ &\qquad\qquad x_i \geq 0, \qquad \forall i \in N. \end{aligned} \tag{6.20}$$

By the strictly monotonicity of $\tanh^{-1}(\cdot)$, the problem (6.20) can be transformed to the following equivalent linear programming problem:

$$
\begin{aligned}
&\text{maximize} \quad \theta \\
&\text{subject to} \quad \alpha_S \sum_{i \in S} x_i - \theta \geq \alpha_S b_S, \qquad \forall S \subset N \\
&\qquad\qquad x_1 + x_2 + \cdots + x_n = P \\
&\qquad\qquad x_i \geq 0, \qquad \forall i \in N,
\end{aligned}
\tag{6.21}
$$

where

$$
\theta = \tanh^{-1}(2\lambda - 1). \tag{6.22}
$$

3) Five kinds of membership functions

We have shown the methods for computing the solutions when all of membership functions are only linear or only hyperbolic functions. We now adopt five different types of membership functions: linear, hyperbolic, exponential, hyperbolic inverse and piecewise linear. First, we should explain the necessity of the selection of membership functions. In a problem where the joint development of water resources is considered, the set of players consists of agricultural associations and city services (Suzuki and Nakayama 1976). Three types of coalitions are given: coalitions consisting of only agricultural associations, only city services and a mixture of the two. It is important to adopt a different type of membership function for each coalition type. Therefore, we should consider several types of membership functions when a set of players, as will often be the case, consists of different types of players. Conversely, when a set of players consists of only one type of player, we should adopt a single type of membership function.

The proposed solution can be obtained by solving the following nonlinear programming problem:

$$
\begin{aligned}
&\text{maximize} \quad \lambda \\
&\text{subject to} \quad \lambda \leq \mu_{\tilde{G}_S}\left(\sum_{i \in S} x_i\right), \quad \forall S \subset N \\
&\qquad\qquad x_1 + x_2 + \cdots + x_n = P \\
&\qquad\qquad x_i \geq 0, \qquad \forall i \in N,
\end{aligned}
\tag{6.23}
$$

where if a coalition S has a fuzzy goal with a linear membership function, the membership function $\mu_{\tilde{G}_S}$ is expressed as (6.3); if a coalition S has a fuzzy goal with a hyperbolic membership function, the membership function $\mu_{\tilde{G}_S}$ is expressed as (6.4); if a coalition S has a fuzzy goal with an exponential membership function, the membership function $\mu_{\tilde{G}_S}$ is expressed as (6.5); if a coalition S has a fuzzy goal with a hyperbolic inverse membership function, the membership function $\mu_{\tilde{G}_S}$ is expressed as (6.6); and if a coalition S has a fuzzy goal with a piecewise linear membership function, the membership function $\mu_{\tilde{G}_S}$ is expressed as (6.7).

The solution to problem (6.23) cannot be calculated as easily as the linear function or the hyperbolic function. However, an approximate solution can be derived using the method of Sakawa (1983). A method based on linear programming problems is selected, because this problem would be difficult to solve as a nonlinear programming problem. First, we transform the constraints by using the strictly monotone increasing properties of logarithmic, hyperbolic inverse and hyperbolic functions, i.e.,
when membership functions are linear functions, the inequality constraints in (6.23) are transformed to

$$\sum_{i \in S} x_i \leq \overline{a} - (1-\lambda)(\overline{a} - \underline{a}); \tag{6.24}$$

when membership functions are exponential functions, the inequality constraints in (6.23) are transformed to

$$\sum_{i \in S} x_i \leq \frac{\overline{a} - \underline{a}}{b_S} \log\left(\frac{\lambda}{a_S} + 1\right) - \underline{a}; \tag{6.25}$$

when membership functions are hyperbolic functions, the inequality constraints in (6.23) are transformed to

$$\sum_{i \in S} x_i \leq \frac{1}{\alpha_S} \tanh^{-1}(2\lambda - 1) + b_S; \tag{6.26}$$

when membership functions are hyperbolic inverse functions, the inequality constraints in (6.23) are transformed to

$$\sum_{i \in S} x_i \leq \frac{1}{\alpha_S} \tanh\left(\frac{2\lambda - 1}{2a_S}\right) + b_S; \tag{6.27}$$

and when membership functions are piecewise linear functions, the inequality constraints in (6.23) are transformed to

$$\sum_{i \in S} x_i \leq \frac{\lambda - s_{ir}}{t_{ir}}. \tag{6.28}$$

Using these constraints, the following problem is equivalent to the original nonlinear programming problem (6.23).

$$
\begin{array}{ll}
\text{maximize} & \lambda \\
\text{subject to} & \sum_{i \in S} x_i \leq \overline{a} - (1-\lambda)(\overline{a} - \underline{a}), \quad \forall S \in T_1 \\
& \sum_{i \in S} x_i \leq \dfrac{\overline{a} - \underline{a}}{b_S} \log\left(\dfrac{\lambda}{a_S} + 1\right) - \underline{a}, \quad \forall S \in T_2 \\
& \sum_{i \in S} x_i \leq \dfrac{1}{\alpha_S} \tanh^{-1}(2\lambda - 1) + b_S, \quad \forall S \in T_3 \\
& \sum_{i \in S} x_i \leq \dfrac{1}{\alpha_S} \tanh\left(\dfrac{2\lambda - 1}{2a_S}\right) + b_S, \quad \forall S \in T_4 \\
& \sum_{i \in S} x_i \leq \dfrac{\lambda - s_{ir}}{t_{ir}}, \quad \forall S \in T_5 \\
& x_1 + x_2 + \cdots + x_n = P \\
& x_i \geq 0, \quad \forall i \in N,
\end{array}
\tag{6.29}
$$

whereby T_1 denotes a set of coalitions with linear membership functions; T_2 denotes a set of coalitions with exponential functions; T_3 denotes a set of coalitions with hyperbolic functions; T_4 denotes a set of coalitions with hyperbolic inverse functions; and T_5 denotes a set of coalitions with piecewise linear functions.

The above problem could be reduced to a linear programming problem if the values of λ in the constraints were fixed. Since the value of λ satisfies $0 \leq \lambda \leq 1$, we can solve this problem by combining the bisection method and phase one of the simplex method.

When λ in the constraints is fixed, the test for feasibility (i.e., whether the problem of which λ is fixed is feasible or not) can be accomplished by using phase one of the simplex method. If it is feasible, renew the constant value λ as follows:

$$\lambda \leftarrow \lambda + \frac{1}{2}\lambda. \tag{6.30}$$

If it is not feasible, renew the constant value λ as follows:

$$\lambda \leftarrow \lambda - \frac{1}{2}\lambda. \tag{6.31}$$

Then the test for feasibility is executed again after renewing the constant value λ. We can get the feasible problem with the maximal value of λ by repeating this procedure in a finite number of iterations and then the feasible solution x^* and the maximal constant value λ^* must be the optimal solution (x^*, λ^*) of the problem (6.29).

Example 6.1

We can transform a game (N, v) to the fuzzy game $(N, \mu_{\tilde{G}_S(v)}, P)$ by constructing the membership functions expressed by the linear membership function (6.8). Then the proposed solution is calculated and compared with the nucleolus, which is a related solution. Let $N = \{1, 2, 3, 4\}$ and let the coalition values be

$$
\begin{array}{cccc}
v(\phi) = 0, & v(\{1\}) = 1, & v(\{2\}) = 2, & v(\{3\}) = 2, \\
v(\{4\}) = 3, & v(\{1,2\}) = 50, & v(\{1,3\}) = 55, & v(\{1,4\}) = 70, \\
v(\{2,3\}) = 60, & v(\{2,4\}) = 80, & v(\{3,4\}) = 90, & v(\{1,2,3\}) = 120, \\
v(\{1,2,4\}) = 130, & v(\{1,3,4\}) = 150, & v(\{2,3,4\}) = 200, & v(\{1,2,3,4\}) = 300.
\end{array}
$$

The payoff vectors were calculated both for the proposed solution and for the nucleolus. The proposed solution can be obtained by solving the following linear programming problem:

$$
\begin{array}{llllllll}
\text{maximize} & \lambda & & & & & & \\
\text{subject to} & x_1 & & & & -99\lambda & \geq & 1 \\
 & & x_2 & & & -148\lambda & \geq & 2 \\
 & & & x_3 & & -168\lambda & \geq & 2 \\
 & & & & x_4 & -177\lambda & \geq & 3 \\
 & x_1 & +x_2 & & & -160\lambda & \geq & 50 \\
 & x_1 & & +x_3 & & -165\lambda & \geq & 55 \\
 & x_1 & & & +x_4 & -170\lambda & \geq & 70 \\
 & & x_2 & +x_3 & & -170\lambda & \geq & 60 \\
 & & x_2 & & +x_4 & -165\lambda & \geq & 80 \\
 & & & x_3 & +x_4 & -160\lambda & \geq & 90 \\
 & x_1 & +x_2 & +x_3 & & -177\lambda & \geq & 120 \\
 & x_1 & +x_2 & & +x_4 & -168\lambda & \geq & 130 \\
 & x_1 & & +x_3 & +x_4 & -148\lambda & \geq & 150 \\
 & & x_2 & +x_3 & +x_4 & -99\lambda & \geq & 200 \\
 & x_1 & +x_2 & +x_3 & +x_4 & & = & 300 \\
 & \multicolumn{7}{l}{x_i \geq 0,\ i = 1, 2, 3, 4.}
\end{array}
$$

The results are shown in Table 6.1. The degree of attainment of the fuzzy goal for the proposed solution was 0.465, with all of the coalitions being satisfied more than 0.465. In contrast, the degree of attainment of the fuzzy goal for the nucleolus was 0.451. The proposed solution shows a better outcome than the nucleolus in terms of the degree of attainment of the fuzzy goal.

Table 6.1 Payoff vectors

	Player 1	Player 2	Player 3	Player 4
Proposed solution	53.580	70.823	80.123	95.473
Nucleolus	55.375	71.875	80.625	92.125

It is seen from Table 6.1 that the payoffs in the proposed solution are similar, but slightly larger than the payoffs in the nucleolus except for Player 4.

6.4 Conclusion

This chapter has considered n-person cooperative games for coalitions with fuzzy goals, and we have proposed a new solution concept using the fuzzy decision rule.

To conclude, the results of this chapter are summarized as follows:

1) To consider the imprecise nature of human judgment, we have employed fuzzy goals for coalitions in n-person cooperative games and have described the fuzzy games $(N, \mu_{\tilde{G}_S}, P)$ by the fuzzy goals which express degrees of the coalitions' satisfaction with payoffs, instead of the values $v(S)$.

2) Several methods for identifying membership functions have been presented, and we have shown how to construct a membership function $\mu_{\tilde{G}_S}$ using the characteristic function v in the conventional game (N, v). Consequently, the game (N, v) can be transformed to the fuzzy game $(N, \mu_{\tilde{G}_{S(v)}}, P)$.

3) When all of the membership functions are either only linear or hyperbolic functions, the methods for computing the proposed solutions have been developed, using Zimmermann's and Leberling's methods. We have also presented the computational methods for the proposed solution to games with coalitions which have five types of membership functions of fuzzy goals, including linear, hyperbolic, exponential, hyperbolic inverse and piecewise linear functions.

CHAPTER 7

CONCLUSION

In this thesis, game theory has been applied for resolution of conflict in competitive systems. We must recognize the existence of ambiguity in decision makers' judgements as well as imprecision that occurs in information in such systems; moreover, decision makers need to be able to accommodate multiple objectives in the solution of the conflict problems. Consequently, new solution concepts which take the ambiguity and the multiplicity of objectives into consideration should be introduced for the application of game theoretic approach. We have shown several game representations for the resolution of conflict in competitive systems and have developed computational methods for the proposed solutions.

Chapters 2, 3 and 4 dealt with noncooperative games and Chapters 5 and 6 were devoted to investigating cooperative n-person games. These chapters are summarized as follows.

1) Chapter 2 was concerned with two-person zero-sum matrix games with fuzzy goals. The max-min solution with respect to the degree of attainment of the fuzzy goal was defined, and the computational methods were presented when membership functions of fuzzy goals were linear functions or piecewise linear functions. Particularly when the membership function was linear, the equilibrium property of the max-min solution was found with respect to the degree of attainment of the fuzzy goal in single-objective games.

2) In Chapter 3, to incorporate not only ambiguity of decision makers' judgements but also imprecision of information in the competitive system, two-person zero-sum games with fuzzy payoff matrices having entries represented as fuzzy numbers were examined. The max-min solution with respect to a degree of attainment of a fuzzy goal was defined, and it was shown that the solution was equal to an optimal solution for the nonlinear programming problem. In cases where the membership functions of the fuzzy goals were linear functions, we have developed computational methods based on the simplex method using Sakawa's

method, Shimizu and Aiyoshi's relaxation procedure, and Charnes and Cooper's variable transformations.

3) Chapter 4 dealt with two-person non-zero-sum bimatrix games with single and multiple payoffs, which were more general than the games discussed in Chapters 2 and 3. We introduced the fuzzy goal for a payoff in a procedure similar to Chapter 2 and 3 and defined equilibrium solutions with respect to the degree of attainment of the fuzzy goal. Methods by weighting coefficients and by a minimum component were employed to aggregate multiple goals, and computational methods for the equilibrium solutions were also proposed.

4) In Chapter 5, n-person cooperative games with fuzzy coalitions were examined. Fuzzy coalitions were introduced to ease the strict regulation in which any player participating in a coalition must completely accept the decisions of the coalition. We newly defined an excess of a player and proposed lexicographical solutions based on this excess. The relationship between the lexicographical solutions and other solutions was considered, and the methods for computing the proposed solutions were developed.

5) In Chapter 6, n-person cooperative games in fuzzy environments other than for a fuzzy coalition were considered. A value of a coalition referred to the gain acquired only through the actions of the coalition, and n-person cooperative games were described by the values of coalitions. However, since ambiguity of information in competitive systems could not be fully expressed by such a game representation, we proposed a new game representation based on fuzzy goals of coalitions—instead of the value of the coalition—which represent the coalition's degree of satisfaction for a payoff. For the new game representation, we defined a solution concept based on the fuzzy decision rule and presented its computational methods.

As we mentioned, we have dealt with five kinds of games to resolve conflicts in the competitive systems and presented the solution concepts and their computational methods for the games. It does not necessarily mean that all conflict situations in competitive systems can be expressed by these five models of games. We are now considering the possibility of other useful game representations in several future research directions.

In this thesis, the multiplicity of objectives was considered in two-person games but not in n-person cooperative games. As we mentioned in the Introduction, only a few attempts have been made at n-person cooperative multiobjective games and no studies have ever tried to examine n-person cooperative multiobjective games with fuzzy environments. However, such attempts are interesting research topics, and as they relate to our research, the multiobjective version of n-person cooperative games with fuzzy coalitions discussed in Chapter 5 will be considered in the future; that is, solution concepts and their computational methods will be examined in n-person cooperative multiobjective games with fuzzy coalitions.

In multiobjective mathematical programming, the idea for regarding each objective as a player in a game can be found in Belenson and Kaupur (1973), but we think it more effective that game theory be incorporated in multiobjective mathematical programming with multiple decision makers. In a case where decision makers construct multiobjective mathematical programming problems jointly, it is rare for interests of decision makers to actually be in complete conflict. Therefore, in a single-objective mathematical programming problem, all of the decision makers maximize or minimize an objective function, and the result will be the same as the problem with a single decision maker. However, when there is a multiplicity of objectives in the problem, differences between decision makers' preferences must be considered and the problem cannot be managed as an ordinary multiobjective decision problem with a single decision maker. Game theoretical approach is expected to be effective in such multiobjective mathematical programming problems.

So far we have discussed the theoretical aspect of the conflict analysis, but for public decision makers, practical applications for the conflict resolution methods, including our own methods, are still open to question; we are concerned as to how well they can manage to utilize these methods. To use these methods, they must analyze the conflict problem and learn methodologies for optimization theories, and further develop the software for computing the solutions. This suggests the necessity of decision support systems using computers. We feel that the development of decision support systems implementing methodologies for optimization theories is one of the most important future research directions. Several attempts have been made for implementing restricted fields of the methodology in decision support systems (Ruszczyński, Rogowski and Wierzbicki (Eds.) 1990; Korhonen, Lewandowski and Wallenius (Eds.) 1991; Nishizaki and Seo 1992; and Seo and Nishizaki 1993a), and more comprehensive decision support systems based on game theoretic approaches are also expected to be developed.

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Appendix

In this Appendix, three theorems are presented. The first theorem, which concerns a convergence of the relaxation procedure for a min-max problem, is proved by Shimuzu and Aiyoshi (1980). The second theorem by Parthasarathy and Raghavan (1971) shows how the problem of finding an equilibrium solution is closely related to nonlinear programming problems. The third theorem by Wierzbicki (1990) concerns the relationship between equilibrium solutions and scalarizing functions in n-person noncooperative multiobjective games.

Algorithm (Shimuzu and Aiyoshi)

[Step 1]

Choose any initial point $\boldsymbol{y}^1 \in Y$. Set $k = 1$.

[Step 2]

Solve the current relaxed problem

$$\begin{array}{ll} \underset{(\boldsymbol{x},\sigma)}{\text{minimize}} & \sigma \\ \text{subject to} & \boldsymbol{x} \in X \\ & f(\boldsymbol{x}, \boldsymbol{y}^i) \leq \sigma, \quad i = 1, 2, \ldots, k, \end{array} \tag{A1}$$

and obtain an optimal solution $(\boldsymbol{x}^k, \sigma^k)$ for the relaxed problem.

[Step 3]

Solve a maximization problem

$$\max_{\boldsymbol{y} \in Y} f(\boldsymbol{x}^k, \boldsymbol{y}) \tag{A2}$$

and obtain an optimal solution $\boldsymbol{y}^{k+1} = \hat{\boldsymbol{y}}(\boldsymbol{x}^k)$ and the maximal value $\phi(\boldsymbol{x}^k) = f(\boldsymbol{x}^k, \hat{\boldsymbol{y}}(\boldsymbol{x}^k))$.

[Step 4]

If $\phi(\boldsymbol{x}^k) \leq \sigma^k + \varepsilon$, terminate, where ε is a predetermined constant. Then, $\boldsymbol{x}^k$ is a min-max solution for the problem

$$\begin{array}{l} \underset{\boldsymbol{x}}{\text{minimize}} f(\boldsymbol{x}, \hat{\boldsymbol{y}}(\boldsymbol{x})) \\ \text{subject to } \boldsymbol{x} \in X = \{\boldsymbol{x} | \boldsymbol{g}(\boldsymbol{x}) \leq \boldsymbol{0}\} \\ \qquad f(\boldsymbol{x}, \hat{\boldsymbol{y}}(\boldsymbol{x})) = \underset{\boldsymbol{y}}{\text{maximize}} f(\boldsymbol{x}, \boldsymbol{y}) \\ \qquad\qquad \text{subject to } \boldsymbol{y} \in Y. \end{array} \tag{A3}$$

Otherwise, i.e., if $\phi(\boldsymbol{x}^k) > \sigma^k + \varepsilon$, set $k \leftarrow k+1$ and go back to Step 2.

Theorem A.1 (Shimuzu and Aiyoshi)

Let the set Y be nonempty and compact. Let f and $\boldsymbol{g}$ be differentiable with respect to x and their partial derivatives $\partial f(\boldsymbol{x}, \boldsymbol{y})/\partial x_i$, $\partial \boldsymbol{g}(\boldsymbol{x})/\partial x_i$, $i = 1, 2, \ldots, n$ be continuous in $\boldsymbol{x}$. Let f be continuous in $\boldsymbol{y}$ and X be a compact set. Then, for any given $\varepsilon > 0$, the relaxation procedure for the min-max problem (A3) terminates in a finite number of iterations.

Proof Let ($\boldsymbol{x}^k$, σ^k) be an optimal solution to the relaxed problem (A1). By taking a subsequence, if necessary, the sequence $\{(\boldsymbol{x}^k, \sigma^k)\}$ converges to a point $(\bar{\boldsymbol{x}}, \bar{\sigma})$, since $\{\boldsymbol{x}^k\}$ is in the compact set X and $\{\sigma^k\}$ is a nondecreasing sequence bounded above. Similarly, by the compactness of Y, the sequence $\{\boldsymbol{y}^{k^i+1}\}$, generated from the problem (A2) corresponding to some subsequence $\{(\boldsymbol{x}^{k^i}, \sigma^{k^i})\}$, converges to a point $\bar{\boldsymbol{y}} \in Y$. Consider the relaxed problem in the k^{i+1}th iteration, then the constant

$$f(\boldsymbol{x}, \boldsymbol{y}^{k^i+1}) \leq \sigma$$

exists because $k^{i+1} \leq k^i + 1$. Therefore, for the solution $(\boldsymbol{x}^{k^{i+1}}, \sigma^{k^{i+1}})$, it holds that

$$f(\boldsymbol{x}^{k^{i+1}}, \boldsymbol{y}^{k^i+1}) \leq \sigma^{k^{i+1}}.$$

Thus, taking limits of $k \rightarrow \infty$, by $\boldsymbol{x}^{k^{i+1}} \rightarrow \bar{\boldsymbol{x}}$, $\sigma^{k^{i+1}} \rightarrow \bar{\sigma}$, $\boldsymbol{y}^{k^i+1} \rightarrow \bar{\boldsymbol{y}}$ and the continuity of f, we have

$$f(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) \leq \bar{\sigma}. \tag{A4}$$

Incidentally, since the point-to-set mapping $\hat{Y}(\boldsymbol{x})$ is upper semicontinuous at $\bar{\boldsymbol{x}}$, which was proved by Meyer (1970), we have $\bar{\boldsymbol{y}} \in \hat{Y}(\bar{\boldsymbol{x}})$. Thus,

$$\phi(\bar{x}) = \max_{\boldsymbol{y} \in Y} f(\bar{\boldsymbol{x}}, \boldsymbol{y}) = f(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}). \tag{A5}$$

By (A4) and (A5), $\phi(\bar{\boldsymbol{x}}) \leq \bar{\sigma}$. Furthermore, from upper semicontinuity of $\phi(\boldsymbol{x})$ at $\bar{\boldsymbol{x}}$ by Meyer (1970),

$$\phi(\boldsymbol{x}^k) \leq \sigma^k + \varepsilon, \quad \text{for some } k \text{ sufficiently large,}$$

which implies that the termination criterion at Step 4 is satisfied in a finite number of iterations. □

Theorem A.2 (Parthasarathy and Raghavan)

(x^0, y^0) is an equilibrium solution to a bimatrix game with payoff matrices A, B, if and only if (x^0, y^0, p, q) is an optimal solution to the problem

$$\begin{aligned} \underset{(x,y,p,q)}{\text{maximize}} \quad & xAy + xBy - p - q \\ \text{subject to} \quad & Ay \leq pe^m \\ & Bx \leq qe^n \\ & \sum_{i=1}^{m} x_i = 1 \\ & \sum_{j=1}^{n} y_j = 1, \end{aligned} \tag{A6}$$

where e^m and e^n are m- and n-dimensional column vectors for which each of the entries is 1, respectively, i.e.,

$$e^m = (\underbrace{1, 1, \ldots, 1}_{m})^T \quad \text{and} \quad e^n = (\underbrace{1, 1, \ldots, 1}_{n})^T.$$

Proof The constraints obviously imply that

$$xAy + xBy - p - q \geq 0.$$

Thus the optimal value of the objective function is nonpositive.

Let (x^0, y^0) be an equilibrium solution. Clearly

$$(x^0, y^0, p = x^0Ay^0, q = x^0By^0)$$

is feasible, and at (x^0, y^0, p, q), the value of the objective function is zero. Thus it is an optimal solution to the nonlinear problem. Conversely, let (x^0, y^0, p, q) be an optimal solution to the problem. Since by Nash's theorem we always have an equilibrium solution, and since at such a solution the optimal value is zero, so it is at (x^0, y^0, p, q). Thus

$$x^0Ay^0 + x^0By^0 - p - q = 0.$$

An elementary argument shows that (x^0, y^0) is actually an equilibrium solution. □

Theorem A.3 (Wierzbicki)

Suppose that scalarizing functions $s_i(p_i(x,y), w_i), i = 1, 2$ such that

$$\operatorname{Arg} \max_{p_1(x,y) \in Z_1(y)} s_1(p_1(x,y), w_1) \subset P_1(y), \forall w_1 \in W_1, \forall y \in Y \qquad (A7)$$

and

$$\operatorname{Arg} \max_{p_2(x,y) \in Z_2(x)} s_2(p_2(x,y), w_2) \subset P_2(x), \forall w_2 \in W_2, \forall x \in X \qquad (A8)$$

are used for an aggregation of all objectives in a noncooperative two-person multiobjective game in normal form. Then an equilibrium solution of the single-objective game with payoffs $s_i(p_i(x,y), w_i), i = 1, 2$ for any w_i is a Pareto optimal equilibrium solution of the multiobjective game, where w_i is a parameter of the scalarizing function and W_i is a set of the parameters.

Proof If (x^*, y^*) is an equilibrium solution of the single-objective game with payoffs $s_i(p_i(x,y), w_i), i = 1, 2$ for any w_i, then $s_1(p_1(x, y^*), w_1) \le s_1(p_1(x^*, y^*), w_1)$ for all $x \in X$ and $s_2(p_2(x^*, y), w_2) \le s_2(p_2(x^*, y^*), w_2)$ for all $y \in Y$. Since $Z_1(x^*)$ and $Z_2(y^*)$ are sets of attainable payoffs, taking into account (A7) and (A8) we obtain

$$p_1(x^*, y^*) \in \operatorname{Arg} \max_{p_1 \in Z_1(y)} s_1(p_1(x,y), w_1) \subset P_1(y^*),$$

$$p_2(x^*, y^*) \in \operatorname{Arg} \max_{p_2 \in Z_2(x)} s_2(p_2(x,y), w_2) \subset P_2(x^*).$$

We observe that the definition of a Pareto optimal equilibrium solution is satisfied. □

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