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A Study on W-graphs:
Properties of Graph Models Containing Unspecified Tree Structures

by

Hua-An ZHAO

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Abstract

In this dissertation, a graph model called a W-graph is presented. The graph model Ω_w consists of an ordinary graph $G(V, E)$ and $k(> 0)$ wild-components $w_1, w_2, \cdots, w_k$, and is represented by $\Omega_w(V, E, W) = G(V, E) \cup w_1 \cup w_2 \cup \cdots \cup w_k$. Each wild-component w_i is a pair of a vertex set $V(w_i)$ having p_i vertices and a tree containing $V(w_i)$ and $p_i - 1$ edges, and is formally defined as $w_i = \{V(w_i), t^{(i)} \mid t^{(i)} \in T(w_i)\}$, where $V(w_i) = \{v_{i1},\ v_{i2},\ \cdots,\ v_{ipi}\}$, $T(w_i)$ is a set of all trees containing all vertices in $V(w_i)$ and $t^{(i)}$ is any tree in $T(w_i)$. Hence, a wild-component w_i can represent any tree containing all vertices of $V(w_i)$, where no specific tree is given. Hypergraphs and hyper-edges are related to W-graphs and wild-components. The definition of the former is more general than that of the latter, which restricting wild-components to trees leads us to more sophisticated discussion, as will be given in this dissertation.

Introduction is given in Chapter 1 and basic definitions are explained

in Chapter 2.

In Chapter 3, we introduce the concept of W-circuits and W-cutsets of a W-graph as an extension of circuits and cutsets of an ordinary graph. Also defined is an operation of W-ring sum in a W-graph. It is proved that the W-ring sum of two W-circuits is a W-circuit and that the W-ring sum of two W-cutsets is also a W-cutset. Furthermore, W-incidence, W-cutset and W-circuit matrices are introduced. In a W-incidence matrix A_w, we define a W-tree corresponding to the columns of a non-singular major submatrix of A_w. By the W-tree, a fundamental W-cutset matrix and a fundamental W-circuit matrix can be constructed where their rows corresponds to a set of linearly independent W-cutsets and a set of linearly independent W-circuits, respectively.

In Chapter 4, the relation between a W-graphs and its derived graphs is discussed. When structure of each wild-component is specified, a W-graph $\Omega_w(V, E, W)$ becomes an ordinary graph $G_d(V, E')$ which is called a derived graph. We prove (i) and (ii) as follows: (i) A W-circuit, a W-cutset and a W-tree of a W-graph can be transformed to a circuit (or edge disjoint union of circuits), a cutset (or edge disjoint union of cutsets) and a tree of any derived graph, respectively; (ii) if all elements in a set of W-circuits (W-cutsets, respectively) are linearly independent under W-ring sum, then all elements in a set of edge disjoint circuits (edge disjoint cutsets) obtained in (i) are also linearly independent under ring sum.

In Chpter 5, some applications of W-graphs are mentioned. Consider the via-minimization problem in two-layered topological routing that is often used in design of VLSI or printed wiring boards. The problem can be modeled by a W-graph $\Omega_w(V, E, W)$, where V represents a set of all terminals, E does a set of two-terminal nets and W does a set of multi-terminal nets. With this modeling, the problem is reduced to two problems of W-graphs: the one is detection of planarity of W-graphs and the other is plane drawing of planar W-graphs. At present, the two problems still remain unsolved, we are unable to evaluate our approach by W-graphs explicitly. However, if we can solve the two problems in W-graphs, the advantages of this approach will be shown. In this dissertation, some theorems are provided for testing planar W-graphs for some particular W-graphs.

Finally, unsolved problems on W-graphs left for future research are stated.

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List of Symbols

A_w	W-incidence matrix, 53
A_s	incidence matrix with star, 56
B_{ew}	W-circuit matrix, 70
$\{C_j\}$	set of W-circuits, 30
$\{C_j^*\}$	set of circuits or edge disjoint union of circuits of a derived graph, 90
E	edge set, 2
$e_{ja}^{(i)}$	edge in tree, 79
$e_{js}^{(i)}$	edge in star, 84
F_i	sequence of edge or inner path, 33
Ω_w	W-graph, 24
G_d	derived graph, 79
G_s	derived graph with stars, 56
H	boundary of routing region, 107
m_j	region, 119

n_a	net, 7
$p_{wi}(v_a, v_b)$	inner path of w_i, 23
Q_{ew}	W-cutset matrix, 60
$\{S_i\}$	set of W-cutsets, 42
$\{S_i^*\}$	set of cutsets or edge disjoint union of cutsets of a derived graph, 99
$T(w_i)$	set of spanning trees, 78
$t_a^{(i)}$	specified tree of w_i, 78
$t_s^{(i)}$	specified star of w_i, 84
V	vertex set, 2
$V(w_i)$	vertex set of a wild component w_i, 22
V_{ai}	subset of $V(w_i)$ separated by a W-cutset, 41
V_{oi}	terminal set in a W-circuit, 36
$v_j^{(i)}$	vertex in a wild component w_i, 50
$\underline{v_r^{(i)}}$	reference vertex of a wild component, 50
W	wild component set, 24
w_i	wild component, 22
Φ	closed train, 33
Γ	transformation of W-circuits, 90
Θ	transformation of W-cutsets, 99

$w_i(V_{ai} : \overline{V_{ai}})$	$V(w_i)$ separated by a W-cutset, 44
$w_i(V_{oi}/\overline{V_{oi}})$	set of inner paths in a W-circuit, 30
$\oplus$	ring sum, 4
$\hat{\oplus}$	W-ring sum of W-circuits, 37
$\tilde{\oplus}$	W-ring sum of W-cutsets, 44
$\square$	the end of example, 25
$\blacksquare$	the end or absence of a proof, 4

Contents

Abstract i

Acknowledgements v

List of Symbols vii

1 Introduction 1

1.1 Graphs 2

1.1.1 Paths and Circuits 3

1.1.2 Cutsets 4

1.1.3 Trees 5

1.2 Several Motivating Examples 7

1.3 New Graph Models – W-graphs 11

1.4 Organization of This Dissertation 15

2 Basic Concepts of W-graphs 21

2.1 Definitions of Wild-components and W-graphs 22

2.1.1 Wild-components 22

2.1.2 W-graphs 23

2.2 Classifying Properties of W-graphs 26

3 W-circuits and W-cutsets 29

3.1 W-circuits 30

3.1.1 Definition of a W-circuit 30

3.1.2 Ring Sum Operation of W-circuits 36

3.1.3 Properties of W-circuits in a W-graph 38

3.2 W-Cutsets 41

3.2.1 Definition of a W-cutset 42

3.2.2 Ring-sum Operation of W-cutsets 44

3.2.3 Properties of W-cutsets 45

3.3 Matrix Representation of W-graphs 49

3.3.1 W-incidence Matrix 51

3.3.2 W-trees 58

3.3.3 W-cutset Matrix 60

3.3.4 W-circuit Matrix 70

4 A W-graph and Its Derived Graphs 77

4.1 Derived Graphs of a W-graph 78

4.2 Relations between a W-graph and Its Derived Graphs . 83

4.2.1 Instantiation of a W-tree 83

4.2.2 Instantiation of a W-circuit 90
4.2.3 Instantiation of a W-cutset 99

5 Some Applications of W-graphs 103

5.1 An Approach to Topological Routing by W-graphs . . 105
5.1.1 Topological Routing Problems 105
5.1.2 Approach by the W-graph Model 108
5.1.3 Unsolved Problems 115
5.2 On Planarity of W-graphs 116
5.2.1 Definition of a Planar W-graph 116
5.2.2 Properties of Planar W-graphs 119
5.3 Discussion 124

6 Conclusions 131

Bibliography 135

Index of Definitions 141

Chapter 1

Introduction

Graph theory has been found useful in modeling systems arising in physical science, engineering, social science and economic problems because of their intuitive diagrammatic representation. The fact is that any system involving a binary relation can be represented by a graph.

In this introductory chapter, some basic concepts of graph theory will be reviewed and several definitions and terminologies throughout this dissertation will be introduced based on the standard texts [Mayeda1 72], [Chen 71], [Chan 69] and [Harary 69]. Through several instances, we illustrate why the concepts of wild-components are needed where each wild-component is a minimally connected subgraph with unspecified edges, then we define a W-graph which contains wild-components. Since the relation between vertices and edges in each wild-component is unspecified, a W-graph is different from an ordinary graph.

1.1 Graphs

A **graph** G or called an ordinary graph is a pair $(V(G), E(G))$, where $V(G)$ is a non-empty set of elements called **vertices**, and $E(G)$ is a family of unordered pairs of elements of $V(G)$ called **edges**. $V(G)$ and $E(G)$ are called a **vertex set** and an **edge set** of G. When there is no possibility of confusion, these can be indicated by the symbols of V and E, respectively. The graph is represented by $G(V, E)$. The number of vertices of $G(V, E)$ is usually denoted by $\mid V \mid$ and the number of edges of $G(V, E)$ is denoted by $\mid E \mid$.

It should be noticed that the relation between vertices and edges in a given graph is fixed, which makes a difference between graphs and W-graphs introduced in this dissertation.

If $e_j(v_a, v_b)$ is an edge of $G(V, E)$, the e_j is said to **join** the vertices v_a and v_b, and these vertices are then said to be **adjacent**. In this case, it is also said that e_j is **incident** at v_a and v_b, and that v_a and v_b are called **endpoints** of e_j. The number of edges incident at v_a is called a **degree** of v_a. Two edges of $G(V, E)$ incident at the same vertex will be called **adjacent edges**, and two or more edges joining the same pair of vertices will be called **parallel edges**. An edge joining a vertex to itself will be called a **self-loop**. A graph containing no self-loop or parallel edges is called **simple graph**. A simple graph in which every two vertices are adjacent is called a **complete graph**. A complete graph with n vertices and $n(n-1)/2$ edges is denoted by K_n. The **rank** of a graph is equal to $\mid V \mid -\rho$ where ρ is the number of maximally connected components. A **planar graph** is a graph which can be embedded in the plane in such a way that no two edges intersect geometrically except at a vertex. A graph drawn on a plane in this way is called a **plane drawing graph** and the areas which the plane drawing graph divides the plane are called the **regions (windows)**. The unbounded region is called the **outside region**.

1.1.1 Paths and Circuits

An **edge sequence** $\{(v_0, v_1), (v_1, v_2), \cdots, (v_{r-1}, v_r)\}$, $r \geq 2$, in a graph $G(V, E)$ is said to be closed if $v_0 = v_r$, and open otherwise. In an open edges sequence, v_0 is called the **initial vertex**, and v_r is called the **final vertex** of the edge sequence. Together they are called the **terminals** of the edge sequence. If all the edges appearing in an edge sequence are distinct, the edge sequence is called an **edge train**. If all vertices v_0, v_1, $\cdots$, v_r in an open edges train are distinct, a set of these edges is called a **path**. When the initial and the final vertices of a path are the same, it is called a **circuit**. In other words, a circuit is a closed edge train. When some circuits are edge disjoint, we call these circuits as **an edge disjoint union of circuits**. It can be seen that the degree of every vertex in a circuit or an edge disjoint union of circuits is even, that is, every vertex as an endpoint appears even times in the closed edge train.

With the aid of ring sum operation [1], we have the following important property:

Theorem 1.1.1 *The ring sum of two different circuits is a circuit or an edge-disjoint union of circuits.* ■

The proof of Theorem 1.1.1 has been given in [Chen 71] and [Mayeda1 72].

1.1.2 Cutsets

For a connected graph $G(V,E)$, let V_a and $\overline{V_a}$ be two non-empty sub-vertex sets of V such that $\overline{V_a} = V - V_a$ and $V_a \cup \overline{V_a} = V$. An edge set S consisting of all edges between a vertex in V_a and a vertex in $\overline{V_a}$ is either a **cutset** or an **edge disjoint union of cutsets**. If removal all edges of

[1] Ring-sum operation $\oplus$ is defined as $C_a \oplus C_b = (C_a \cup C_b) - (C_a \cap C_b)$.

S, the rank of $G(V, E)$ reduces by one, S is called a cutset. Otherwise S is called an edge disjoint union of cutsets. When either V_a or $\overline{V_a}$ contains one and only one vertex, the edge set S is called an **incidence set**. In other words, an incidence set is formed by the edges incident at a vertex of $G(V, E)$. The number of linearly independent cutsets or edge disjoint union of cutsets in $G(V, E)$ is $\mid V \mid -1$. [Mayeda1 72] has presented the following important relation of cutsets or edge disjoint union of cutsets under the ring sum operation.

Theorem 1.1.2 *The ring sum of two distinct cutsets or edge disjoint union of cutsets of a graph is either a cutset or an edge disjoint union of cutsets of the graph.* ■

1.1.3 Trees

A connected graph which contains no circuits is called a **tree**, and a separated graph whose maximally connected components are trees is called a **forest**. The main properties of trees are summarized in the following theorem ([Wilson 72]):

Theorem 1.1.3 *If T is a tree containing $\mid V \mid$ vertices, then*

1. *T is a connected graph with $\mid V \mid -1$ edges.*

2. *T contains no circuits.*

3. *If v_a and v_b are distinct vertices of T, then there is exactly one path between v_a and v_b.* ■

The concept of a tree is extremely important in graph theory because the number of linearly independent cutsets and circuits can be related to a tree. The discussion of the number of trees in G has been given by [Moon 67]. In particular, the number of trees in a complete graph K_n is n^{n-2} ([Harary 69]).

Some fundamental definitions and theorems in graph theory concerning this dissertation has been introduced, which establish the basic vocabulary for describing W-graphs hereafter.

1.2 Several Motivating Examples

It is well-known that any system involving a binary relation can be represented by a graph. In modern technologies, however, there are instances that the representation by graphs may not be sufficient to indicate some systems. One of instances is related to layout design of a PCB (printed circuit board) or a VLSI. Fig. 1.1(a) is a routing problem where there are two nets $n_a = \{a_1, a_2, a_3\}$ and $n_b = \{b_1, b_2\}$, all of pins (terminals) in each net must be connected by wires electrically. The net n_b is a two-terminal net, whose terminals can be connected by an edge. The net n_a is a multi-terminal net, whose terminals can be connected by any connection as long as those terminals are connected. This means that those terminals should be connected at least by a tree structure.

Another example in [Tanenbaum 81] is in modern communication technology. There exists such a computer network consisting of some terminals (hosts) and a subnet which is an unspecified structure as shown in Fig.1.1(b). The job of a subnet is to carry message from one terminal to other terminal. All terminals in a subnet must be connected but its connection is unspecified.

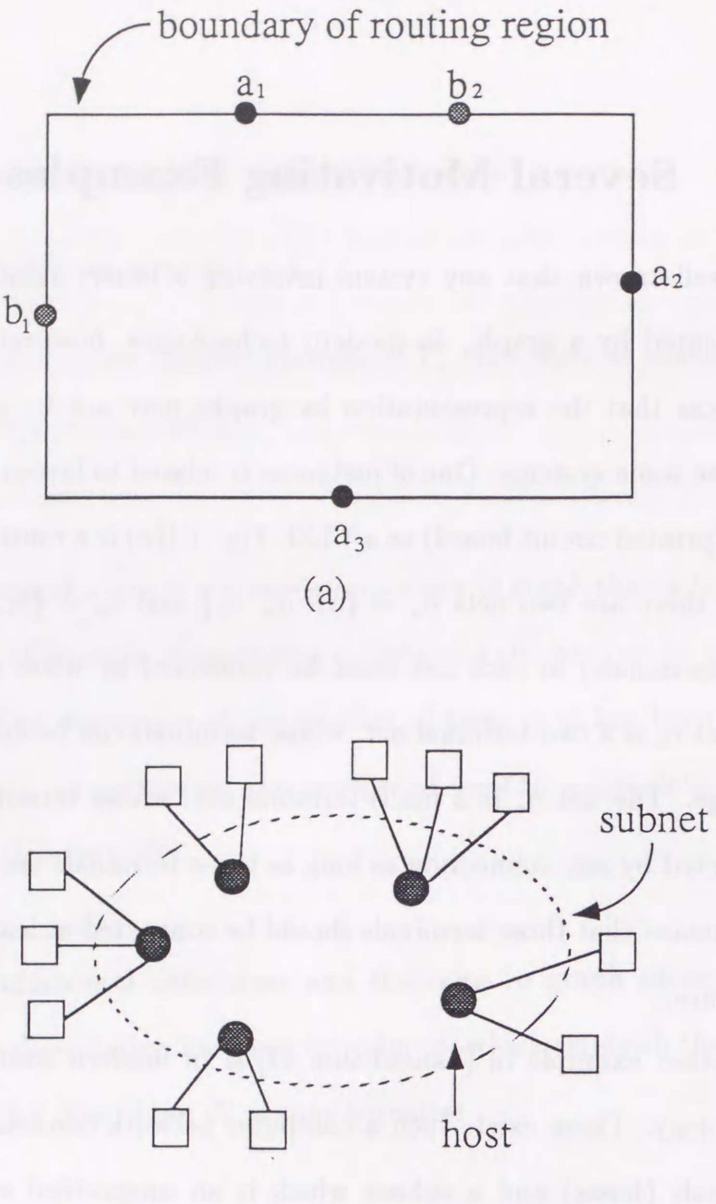


Figure 1.1: (a)A routing problem (b) a computer network

It can be seen that the connection of a multi-terminal net in a routing problem or a subnet in a computer net is unspecified though we know all terminals or all hosts must be connected. Using an ordinary graph for modeling above systems is unsuitable because the necessary requirement is that the relation between vertices and edges in the connections is unspecified, unless the connections are fixed by a particular structure such as a complete graph, a tree, a rectangle and so on [Mal. 83], [Hsu 83], [Xiong 89] and [Zhao1 89].

Should the connection be fixed by a particular structure ? Fixing it by a particular structure may produce an influence on physical design. Can we make a graph model for these systems without particular structures ? For example, we define an connected component containing vertices a_1, a_2 and a_3 as shown in Fig.1.2 to describe the connection of net n_a as shown in Fig.1.1(a). The connected component will be discussed later.

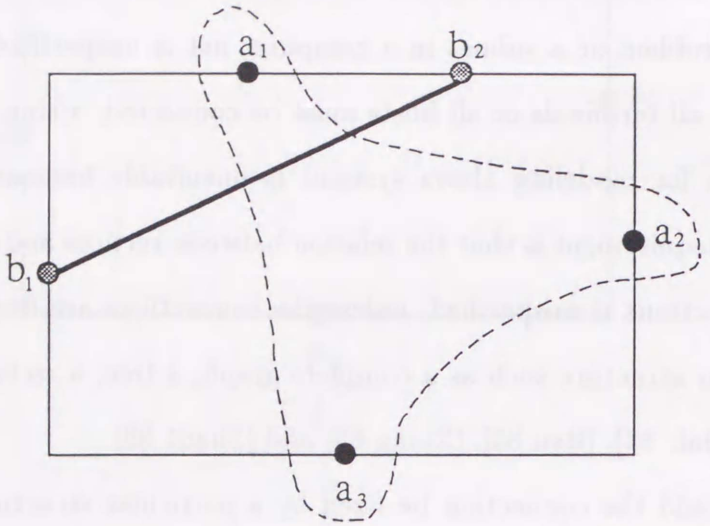


Figure 1.2: A graph containing a connected component.

1.3 New Graph Models – W-graphs

In 1988, the concept of **wild-component** in graph theory has been presented by [Mayeda2 88]. A wild-component (Definition 2.1.1) is an incompletely defined connected subgraph having p vertices and $p-1$ unspecified edges. In other words, we know there is one and only one path between any two vertices in a wild-component, but which vertices being in the path other than initial vertex and final vertex are unknown. It can be considered that a wild-component is an unspecified tree containing all vertices of the wild-component. Hence, a wild-component is a partially known graph.

The background of a wild-component is for modeling a multi-terminal net or for indicating a set of specific terminals under some requirements such as these terminals can not be separated by any wires [Zhao1 89] and [Zhao3 90]. Because a multi-terminal net is a means of minimally connecting terminals but the connecting structure is unspecified, it can be represented by a wild-component in which these terminals are represented by vertices.

When a graph $G(V, E)$ contains wild-components each of whose vertices are in V, the graph is called a **W-graph** whose formulation will be given later (Definition 2.1.3).

Because a W-graph contains some wild-components, it is a partially known graph which is different from an ordinary graph. It is very interesting and useful to discuss the properties of such a partially known graph.

Although in each wild-component the relation between vertices and edges is unspecified, some theorems related to W-graphs have been summarized in [Mayeda3 90], [Zhao5 92], [Zhao6 92] and [Zhao7 92] where knowing the structure of each wild-component being a tree is enough to study the properties of W-graphs such as circuits and cut-sets, and some properties under matrix representations. Some possible applications of W-graphs for solving the problems of layout design have been introduced in [Zhao1 89] and [Zhao3 90].

It must be pointed out that a W-graph is different from a **hypergraph** [Berge1 73]. A hypergraph is defined as follows. Let $V = \{v_1, v_2, \cdots, v_n\}$ be a finite set, and let $E = \{e_i / i \in I\}$ be a family of subsets of V. The family E is said to be a hypergraph on V if

(1) $$e_i \neq \emptyset \ (i \in I)$$

(2) $$\bigcup_{i \in I} e_i = V.$$

The couple $H = (V, E)$ is called a hypergraph. The elements v_1, v_2, $\cdots$, v_n are called the vertices and the sets e_1, e_2, $\cdots$, e_m are called the **hyper-edges**. An edge e_i with $\mid e_i \mid > 2$ is drawn as a curve encircling all the vertices of e_i. An edge e_i with $\mid e_i \mid = 2$ is drawn as a curve connecting its two vertices. An edge e_i with $| e_i |= 1$, is drawn as a self-loop.

From the definition of a hypergraph, we can see that an edge e_i with $| e_i |> 2$ is a sub-vertex set and all vertices in the edge e_i are connected but the connection is undefined. Because hypergraphs are too ambiguous to be used. However, the structure of a wild-component is defined as a minimally connected graph which is any one of p^{p-2} trees if the wild-component contains p vertices.

Although a cycle can be defined in a hypergraph which is formed by hyper-edges, however, the relation between any two cycles can not be established such as to obtain one from others and so on. Furthermore, in a hypergraph there are no concepts similar to cutset and tree of an ordinary graph [Berge2 74]. However, we will show that W-circuits, W-cutsets and W-trees which we will define in W-graphs have very similar properties as circuits, cutsets and trees of an ordinary graph.

In fact, when we fix each wild-component with a tree, a W-graph becomes an ordinary graph, called a derived graph (Definition 4.1.2). W-circuits, W-cutsets and W-trees become circuits or edge disjoint union of circuits, cutsets or edge disjoint union of cutsets and trees of the derived graph, respectively. Furthermore, without choosing a tree for each wild-component, we can show that there are linearly independent W-circuits and linearly independent W-cutsets which lead to fundamental W-circuit matrix and fundamental W-cutset which are theoretically very important.

Thus, W-graphs may be an important model in the field of circuits and systems.

1.4 Organization of This Dissertation

In this dissertation, a graph model called a W-graph will be introduced. The properties of W-graphs will be discussed.

Chapter 1 : Some basic terminologies in graph theory are reviewed and the summary of this dissertation is given. The terminologies including paths and circuits, incidence sets and cutsets, and trees are mentioned which are related to later chapters. In this introductory chapter, the concepts of *wild-components* and *W-graphs* are introduced. A wild-component w_i is defined as a pair of a vertex set and a spanning tree containing all vertices in the vertex set. In other words, wild-component can be considered as an unspecified tree-structure. A W-graph consists of an ordinary graph and $k(>0)$ wild-components so that which is partially known graph. It is pointed out that hyper-edges and hypergraphs are related to wild-components and W-graphs. The definition of the former is more general than that of the latter, which restricting wild-components to trees leads us to more sophisticated discussion, as will be given in this dissertation.

Chapter 2 : The basic concepts on W-graphs are explained. First, we give the definition of wild-components. A wild-component w_i is a pair of a vertex set $V(w_i)$ having p_i vertices and a spanning tree containing p_i and $p_i - 1$ edges, and is formally defined as $w_i = \{V(w_i), t^{(i)} \mid$

$t^{(i)} \in T(w_i)\}$, where $V(w_i) = \{v_{i1}, v_{i2}, \cdots, v_{ipi}\}$, $T(w_i)$ is a set of all trees containing all vertices in $V(w_i)$ and $t^{(i)}$ is any tree in $T(w_i)$. Hence, a wild-component w_i can represent any tree containing all vertices of $V(w_i)$, where no specific tree is given. The information available on a wild-component is only that there exists exactly one path (called an *inner path*) between any two vertices of a wild-component.

Then, we define a W-graph. A W-graph Ω_w consists of an ordinary graph $G(V, E)$ and $k(> 0)$ wild-components $w_1, w_2, \cdots, w_k$, and is represented by $\Omega_w(V, E, W) = G(V, E) \cup w_1 \cup w_2 \cup \cdots \cup w_k$. If we use colors to distinguish each wild-component in a W-graph $\Omega_w(V, E, W)$, it is clear that the total number of edges in the W-graph is equal to $\mid E \mid + \sum_{i=1}^{|W|} \mid V(w_i) \mid - \mid W \mid$. However, as we have mentioned previously, $\sum_{i=1}^{|W|} \mid V(w_i) \mid - \mid W \mid$ edges in wild-components are unspecified.

The properties of a W-graph can be classified into two types: The one is called an *arbitrary property* which holds for any tree given to each wild-component; the other is called a *restricted property* which can hold for at least one tree given to each wild-component. We will discuss some arbitrary properties of a W-graph in Chapter 3 and 4, and some restricted properties in Chapter 5 .

Chapter 3 : We introduce *W-circuits* and *W-cutsets* of a W-graph as an extension of circuits and cutsets of an ordinary graph. A W-circuits is defined as a set consisting of edges and $w_i(V_{oi}/V(w_i) - V_{oi})$ which satisfy four conditions. $w_i(V_{oi}/V(w_i) - V_{oi})$ can be considered as a set of $\mid V_{oi} \mid /2$ inner paths of w_i whose terminals are in V_{oi} and are different. No matter how we choose pairs of vertices in V_{oi} as long as each vertex is exactly in one pair, a set of $\mid V_{oi} \mid /2$ inner paths can be obtained. We replace each $w_i(V_{oi}/\overline{V_{oi}})$ in a W-circuit by the set of inner paths, the W-circuit becomes a closed train which is similar to an closed edge train in ordinary graph.

A W-cutset separates the vertex set V of a W-graph into V_a and $\overline{V_a}$ where $V_a \cup \overline{V_a} = V$ and $V_a \cap \overline{V_a} = \emptyset$. If wild-component w_i is separated by a W-cutset such that $V(w_i)$ is divided into V_{ai} and $\overline{V_{ai}}$ where $V_{ai} \cup \overline{V_{ai}} = V(w_i)$, $V_{ai} \subseteq V_a$ and $\overline{V_{ai}} \subseteq \overline{V_a}$, w_i is represented by $w_i(V_{ai} : \overline{V_{ai}})$. Hence, a W-cutset consists of edges and $w_i(V_{ai} : \overline{V_{ai}})$.

Also defined is an operation of *W-ring sum* in a W-graph. It is proved that the W-ring sum of two W-circuits is a W-circuit and that the W-ring sum of two W-cutsets is also a W-cutset. Furthermore, W-incidence, W-cutset and W-circuit matrices are introduced. In a W-incidence matrix A_w, we define a *W-tree* corresponding to the columns of a non-singular major submatrix of A_w. By the W-tree, a fundamental W-cutset matrix and a fundamental W-circuit matrix can be constructed where their rows corresponds to a set of linearly independent W-cutsets and a set of linearly independent W-circuits, respectively.

Chapter 4 : The relation between a W-graph and its derived graphs is discussed. When structure of each wild-component is specified, a W-graph $\Omega_w(V, E, W)$ becomes an ordinary graph $G_d(V, E')$ which is called a *derived graph*. We prove (i) and (ii) as follows:

(i) A W-circuit, a W-cutset and a W-tree of a W-graph can be transformed to a circuit (or edge disjoint union of circuits), a cutset (or edge disjoint union of cutsets) and a tree of any derived graph, respectively;

(ii) if all elements in a set of W-circuits (W-cutsets, respectively) are linearly independent under W-ring sum, then all elements in a set of edge disjoint circuits (edge disjoint cutsets) obtained in (i) are also linearly independent under ring sum.

These results are theoretically very important.

Chapter 5 : Some applications of W-graphs are mentioned. Consider the via-minimization problem in *two-layered topological routing* that is often used in design of VLSI or printed wiring boards. The problem can be modeled by a W-graph $\Omega_w(V, E, W)$, where V represents a set of all terminals, E does a set of two-terminal nets and W does a set of multi-terminal nets. It is proved that a W-graph for modeling a routing problem can be embedded on either inside or outside (the inside and the outside are corresponding to two layers, respectively) of the boundary of routing region without crossing edges by *created vertices* and that the number of vias is equal to the number of created vertices. With this modeling, the routing problem can be reduced to two problems of W-graphs: The one is detection of planarity of W-graphs and the other is plane drawing of planar W-graphs.

At present, the two problems still remain unsolved, we are unable to evaluate our approach by W-graphs explicitly. However, if we can solve the two problems in W-graphs, the advantages of this approach will be shown. In this dissertation, some theorems are provided for testing planar W-graphs for some particular W-graphs. The difficulty of testing planar W-graphs are analyzed.

Chapter 6 : The properties of W-graphs introduced in this dissertation are summarized and some suggestions together with unsolved problems are stated.

Chapter 2

Basic Concepts of W-graphs

The basic concepts on a W-graph will be introduced. First, we give the definition of a wild-component. A wild-component w_i is defined by a pair of a vertex set and an unspecified tree containing all vertices in the vertex set. The information available on a wild-component is only that there exists exactly one path (called an inner path) between any two vertices of a wild-component. Then, we define a W-graph. A W-graph Ω_w consists of an ordinary graph $G(V, E)$ and $k(> 0)$ wild-components w_1, w_2, $\cdots$, w_k, and is represented by $\Omega_w(V, E, W) = G(V, E) \cup w_1 \cup w_2 \cup \cdots \cup w_k$. The properties of a W-graph can be classified into two types: The one is called an arbitrary property which holds for any tree given to each wild-component; the other is called a restricted property which can hold for at least one tree given to each wild-component. We will discuss some arbitrary properties of a W-graph in Chapter 3 and 4, and some restricted properties in Chapter 5.

2.1 Definitions of Wild-components and W-graphs

Since a W-graph is a new concept in graph theory, it is very important to notice the following definitions.

2.1.1 Wild-components

If a subsystem should be connected but there is no requirements on how the connection should be, the subsystem can be modeled by a wild-component, defined as follows:

Definition 2.1.1 (Wild-component) *A wild-component w_i containing p_i $(2 < p_i < \infty)$ vertices $v_{i1}, v_{i2}, \cdots, v_{ipi}$ is defined as:*

$$w_i = \{V(w_i), t^{(i)} \mid t^{(i)} \in T(w_i)\}$$

where $V(w_i) = \{v_{i1}, v_{i2}, \cdots, v_{ipi}\}$, $T(w_i)$ is a set of all trees containing all vertices in $V(w_i)$ and $t^{(i)}$ means any one of trees in $T(w_i)$.

It should be noticed that a wild-component can be considered as an incompletely defined tree. In other words, a wild-component is not a vertex set but is a minimally connected graph where the relation between vertices and edges is unspecified. Hence, the existence of edges in a wild-component is known but the endpoints of these edges are unspecified. The information available on a wild-component is that there exists one and only one path between any two vertices in the wild-component.

For avoiding confusions in terms of path, we give a definition of a path in wild-component as follows:

Definition 2.1.2 (Inner path of w_i) *A path in a wild-component w_i is called an inner path of w_i, denoted by $p_{wi}(v_a, v_b)$ where v_a and v_b are terminals* [1] *of the path.*

It should be noticed that the terminals of an inner path of wild-component w_i is known but the other vertices contained in the inner path are unknown though there always exists exactly one inner path between any two vertices in w_i. Of course, all vertices in the inner path are in $V(w_i)$.

2.1.2 W-graphs

If a system contains wild-components, the system can be expressed by a W-graph which is defined as follows:

[1] Terminal is either an initial vertex or a final vertex in a path whose degree is one.

Definition 2.1.3 (W-graph) *A W-graph Ω_w is represented by*

$$\Omega_w(V, E, W) = G(V, E) \cup w_1 \cup w_2 \cup \cdots \cup w_k \tag{2.1}$$

or simply denoted by $\Omega_w(V, E, W)$ where W is a set of wild-components $w_1, w_2, \cdots, w_k$, $G(V, E)$ $(V \neq \emptyset)$ is an ordinary graph and $V(w_i) \subseteq V$, for all $i = 1, 2, \cdots, k$.

It should be noticed that a wild-component w_i contains $\mid V(w_i) \mid -1$ unspecified edges which are different from unspecified edges in any other wild-components. In other words, if we use colors, one color is given to all unspecified edges in one wild-component, another color is given to all unspecified edges in another wild-component and so on. Hence, if wild-components w_i and w_j have common vertices, unspecified edges in w_i and w_j may be connected between the same vertices but those are different colors (that is, they are considered to be different).

For a given W-graph $\Omega_w(V, E, W)$, we use the symbols of $\mid V \mid$, $\mid E \mid$ and $\mid W \mid$ for indicating the number of vertices, the number of edges and the number of wild-components in the W-graph, respectively. It should be noticed that we only consider the case that $\mid V \mid$, $\mid E \mid$ and $\mid W \mid$ are finite and $V \neq \emptyset$. When the wild-component set W in a W-graph is an empty set, the W-graph is an ordinary graph. Hence, we suppose $W \neq \emptyset$ in this dissertation. Since a W-graph is a partially known graph and differs from an ordinary graph, for each wild-component, there is only one information available that each wild-component has one and only one inner path between any two vertices.

For a W-graph $\Omega_w(V, E, W)$, since each wild-component has $\mid V(w_i) \mid$ vertices and $\mid V(w_i) \mid -1$ edges, the total number of all edges in the W-graph is equal to $\mid E \mid + \sum_{i=1}^{|W|} \mid V(w_i) \mid - \mid W \mid$. However, as we have mentioned previously, $\sum_{i=1}^{|W|} \mid V(w_i) \mid - \mid W \mid$ edges in wild-components are unspecified, that is, we know these edges exist, but don't know where they exist.

Example 2.1.1 A given W-graph $\Omega_w(V, E, W)$ is shown in Fig.2.1, which contains vertices $v_1, v_2, \cdots, v_{10}$ and edges $e_1, e_2, \cdots, e_{11}$ and two wild-components $V(w_1) = \{v_2, v_5, v_6, v_7\}$ and $V(w_2) = \{v_2, v_3, v_7, v_8, v_9\}$. Hence, we can obtain that $\mid V \mid= 10$, $\mid E \mid= 11$ and $\mid W \mid= 2$. The total number of edges in the W-graph is eighteen where three edges in w_1 and four edges in w_2 are unspecified. $\square$

Definition 2.1.4 (Connected W-graph) *A W-graph is separated if there exist two vertices such that there are no paths or inner paths between them. A W-graph is said to be connected if it is not separated.*

Since there exists one and only one inner path between any two vertices in a wild-component, it should be noticed that Fig.2.1 is a

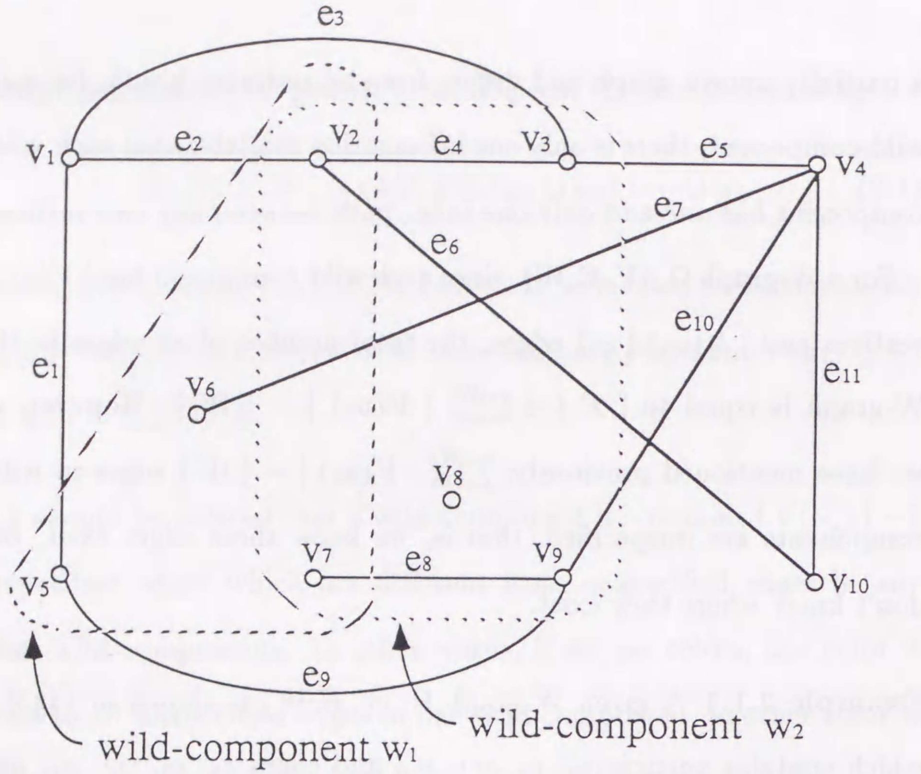


Figure 2.1: A W-graph.

connected W-graph though it contains vertex v_8.

2.2 Classifying Properties of W-graphs

A W-graph is a partially known graph where the edges in each wild-component are unspecified. When we study the properties of W-graphs, we should notice that the properties of W-graphs have two types. One is that some properties of a W-graph hold for any trees given to each wild-component, called arbitrary property and other is some properties hold only for some trees given to each wild-component, called restricted property.

1. **Arbitrary Property:** The property holds for any tree given to each wild-component in a W-graph.

2. **Restricted Property:** The property can hold for at least one tree given to each wild-component in a W-graph.

We show a simple example to explain what is the arbitrary property of W-graphs. Fig. 2.2 is a W-graph containing two edges e_1, e_2 and a wild-component w_1 where $V(w_1) = \{v_1, v_2, v_3, v_4\}$. We say that there exists one and only one path between v_5 and v_6, which is true for any tree containing v_1, v_2, v_3 and v_4 to be the structure of w_1.

On the other hand, a W-graph is said to be planar (Definition 5.2.1) if there exists at least one tree given to each wild-component in the W-graph such that it can be drawn on a plane without crossing edges. It is clear that the properties of planar W-graphs is restricted property.

We will give some arbitrary properties of W-graphs in Chapter 3 and 4 such as W-circuits and W-cutsets where those properties satisfy any tree to be the structure of each wild-component. Some restricted properties are introduced in Chapter 5.

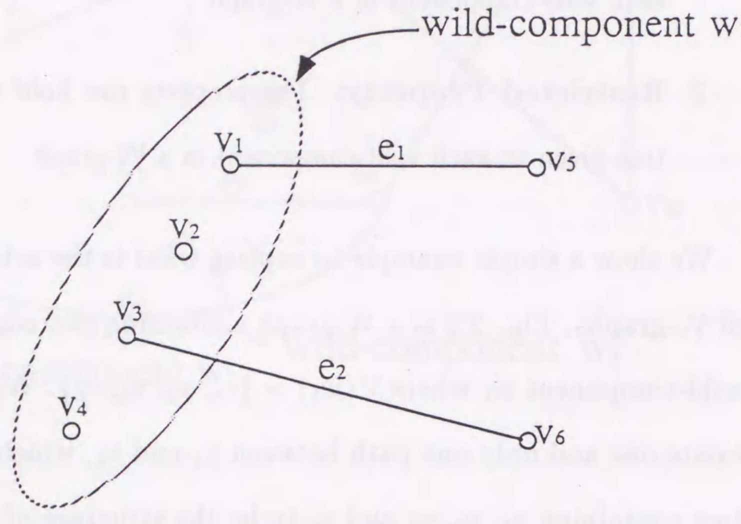


Figure 2.2: An example of arbitrary property.

Chapter 3

W-circuits and W-cutsets

Circuits and cutsets are very important subgraphs not only in terms of theories but also in applications in graph theory, [Chen 71], [Chan 69], [Wilson 72], [Mayeda1 72], [Breuer 77] and [Lauther 79]. Though W-graphs are partially specified graphs, W-circuits and W-cutsets which are similar to circuits and cutsets of ordinary graphs can be defined in W-graphs. Also defined is an operation of W-ring sum in a W-graph. It is proved that the W-ring sum of two W-circuits is a W-circuit and that the W-ring sum of two W-cutsets is also a W-cutset. Furthermore, W-incidence, W-cutset and W-circuit matrices are introduced. In a W-incidence matrix A_w, we define a W-tree corresponding to the columns of a non-singular major submatrix of A_w. By the W-tree, a fundamental W-cutset matrix and a fundamental W-circuit matrix can be constructed where their rows corresponds to a set of linearly independent W-cutsets and a set of linearly independent W-circuits, respectively.

3.1 W-circuits

A W-circuit in a W-graph corresponds to a closed train consisting of edges and inner paths (Definition 2.1.2) which is similar to a closed edge train in an ordinary graph. Under the defined W-ring sum operation, we will discuss the relation of W-circuits in a W-graph. In fact, when each wild-component is specified by a tree, a W-circuit becomes either a circuit or an edge disjoin union of circuits which will be discussed in Chapter 4.

3.1.1 Definition of a W-circuit

By Definition 2.1.1 and 2.1.2, we know that there exists one and only one inner path between any two vertices in a wild-component. It is possible to describe a closed train in a W-graph by edges and inner paths. A W-circuit is defined as follows:

Definition 3.1.1 (W-circuit) *For a W-graph $\Omega_w(V, E, W)$ where $\mid W \mid= k$, let $e_{ci}(v_{ci}, v_{di})$ be an edge in E where v_{ci} and v_{di} are endpoints of the edge, also let V_{oi} be a sub-vertex set of $V(w_i)$. A W-circuit is* *represented by:*

$$C_w = \{e_{c1}(v_{c1}, v_{d1}), e_{c2}(v_{c2}, v_{d2}), \cdots, e_{cm}(v_{cm}, v_{dm}), w_1(V_{o1}/V(w_1) - V_{o1}),$$
$$w_2(V_{o2}/V(w_2) - V_{o2}), \cdots, w_k(V_{ok}/V(w_k) - V_{ok})\} \tag{3.1}$$

which satisfies the following four conditions:

1. *Any two edges in Eq.*(3.1) *are different.*
2. *Each vertex set* V_{oi} $(i = 1, 2, \cdots, k)$ *must consists of different vertices and* $\mid V_{oi} \mid$ *is even.*
3. *If* $V_{oi} = \emptyset$, $w_i(V_{oi}/V(w_i) - V_{oi})) = \emptyset$ *by definition.*
4. *Considering vertices as endpoints of edges and vertices in* V_{oi} *of* $w_i(V_{oi}/V(w_i) - V_{oi})$ *for* $i = 1, 2, \cdots, k$*, then each vertex appears even times.*

For a W-graph as shown in Fig. 3.1, we can find a W-circuit expressed as follows:

$$\begin{aligned} C_w = \{&e_1(v_1, v_7), e_2(v_7, v_2), e_3(v_3, v_8), e_4(v_8, v_6), \\ &w_1(v_4, v_6/v_3, v_5), w_2(v_1, v_2, v_3, v_4/\emptyset)\} \end{aligned} \tag{3.2}$$

where all edges are different and $V_{o1} = \{v_4, v_6\}$ and $V_{o2} = \{v_1, v_2, v_3, v_4\}$ which satisfies Definition 3.1.1.

Consider a W-circuit in a W-graph. A W-circuit corresponds to a closed train consisting of edges and inner paths defined as follows:

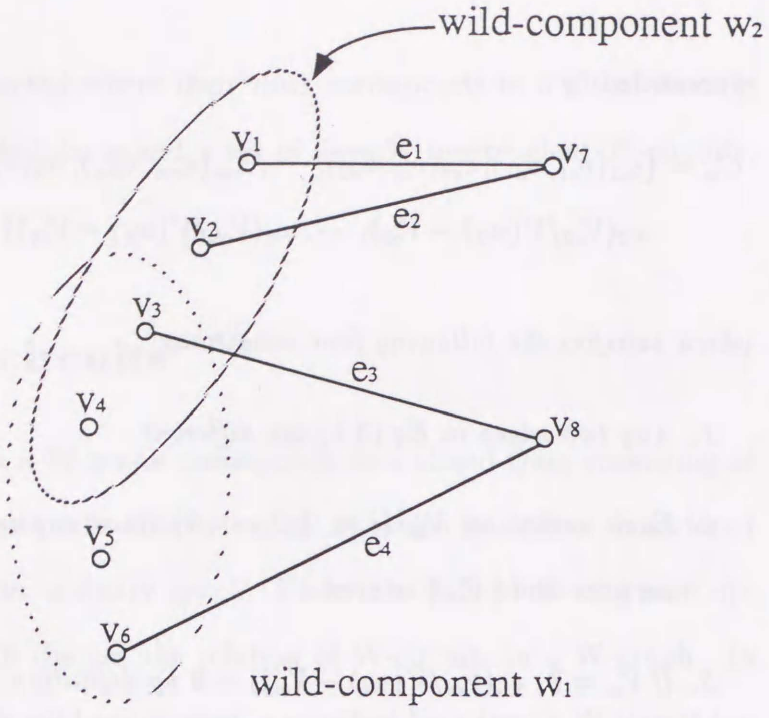


Figure 3.1: A W-graph having two wild-components.

For a W-graph $\Omega_w(V, E, W)$, we can get an sequence consisting of edges in E and inner paths of wild-components in W. Let $e(v_{rj}, v_{rj+1})$ be an edge in E whose endpoints are v_{rj}, v_{rj+1} and $p_{wi}(v_{rj}, v_{rj+1})$ be an inner path of a wild-component w_i indicated by its subscript whose terminals (Definition 2.1.2) are $v_{rj}, v_{rj+1} \in V(w_i)$.

We make the sequence composed of the edges $e(v_{rj}, v_{rj+1})$ and inner paths $p_{wi}(v_{rj}, v_{rj+1})$ as follows.

$$
\begin{aligned}
&\{F_1, F_2, \cdots, F_r, \cdots, F_m\} = \\
&\{[f(v_{11}, v_{12}), f(v_{12}, v_{13}), \cdots, f(v_{1j}, v_{1j+1}), \cdots, f(v_{1k(1)}, v_{11})], \\
&\quad [f(v_{21}, v_{22}), f(v_{22}, v_{23}), \cdots, f(v_{2j}, v_{2j+1}), \cdots, f(v_{2k(2)}, v_{21})], \\
&\qquad\qquad\qquad\qquad\vdots \\
&\quad [f(v_{r1}, v_{r2}), f(v_{r2}, v_{r3}), \cdots, f(v_{rj}, v_{rj+1}), \cdots, f(v_{rk(r)}, v_{r1})], \\
&\qquad\qquad\qquad\qquad\vdots \\
&\quad [f(v_{m1}, v_{m2}), f(v_{m2}, v_{m3}), \cdots, f(v_{mj}, v_{mj+1}), \cdots, f(v_{mk(m)}, v_{m1})]\}
\end{aligned}
\tag{3.3}
$$

where each of $\{f(v_{rj}, v_{rj+1})\}$, $r = 1, 2, \cdots, m$, $j = 1, 2, \cdots, k(r)$, is either an edge $e(v_{rj},\ v_{rj+1})$ or an inner path $p_{wi}(v_{rj},\ v_{rj+1})$. In F_r, it can be seen that each of $\{f(v_{rj}, v_{rj+1})\}$ (for r, $l < j < k(r)$), has one endpoint or terminal in common with the preceding $f(v_{rj-1},\ v_{rj})$, and the other endpoint or terminal in common with the succeeding $f(v_{rj+1},\ v_{rj+2})$ and $v_{rk(r)+1} = v_{r1}$.

Definition 3.1.2 (Closed train) *If the following two conditions are satisfied, the sequence in Eq.*(3.3) *consisting of edges and inner paths is called a closed train, denoted by* Φ.

Condition 1: *Neither each* v_{rj} *nor* v_{rj+1} *of an inner path* $p_{wi}(v_{rj},\ v_{rj+1})$ *can be a terminal of another inner path of the same wild-component* w_i *in Eq.*(3.3).

Condition 2: *Each edge and each inner path of w_i appear exactly once in Eq.(3.3).*

Consider the W-graph containing four edges and two wild-components w_1 and w_2 as shown in Fig.3.1. There is a closed train Φ:

$$
\begin{aligned}
\Phi &= \{F_1, F_2\} \\
&= \{[e_4(v_8, v_6), p_{w1}(v_6, v_4), p_{w2}(v_4, v_3), e_3(v_3, v_8)], \qquad (3.4) \\
&\quad [e_1(v_1, v_7), e_2(v_7, v_2), p_{w2}(v_2, v_1)]\}
\end{aligned}
$$

because this sequence satisfies Condition 1 and 2. It should be noticed that v_4 as a terminal appears twice but one is in an inner path of w_1 and other is in an inner path of w_2. However, another sequence $\{e_3(v_8, v_3)$, $p_{w1}(v_3, v_5)$, $p_{w1}(v_5, v_6)$, $e_4(v_6, v_8)\}$ is not a closed train since there exist $p_{w1}(v_3, v_5)$ and $p_{w1}(v_5, v_6)$ in the sequence having a common vertex v_5 as a terminal vertex not satisfying Condition 1.

Property 3.1.1 *Let a W-circuit contain $w_i(V_{oi}/\overline{V_{oi}})$ $(i \in 1, 2, \cdots, k)$ where $\mid V_{oi} \mid$ is even. No matter how we choose pairs of vertices in V_{oi} as long as each vertex is exactly in one pair, we can obtain $\mid V_{oi} \mid /2$ inner paths of w_i whose terminals are in V_{oi}, so that we can replace each $w_i(V_{oi}/\overline{V_{oi}})$ in the W-circuit by these inner paths to produce a closed train.*

Example 3.1.1 *Eq.* (3.2) is a W-circuit expressed by

$$C_w = \{e_1(v_1, v_7), e_2(v_7, v_2), e_3(v_3, v_8), e_4(v_8, v_6),$$
$$w_1(v_4, v_6/v_3, v_5), w_2(v_1, v_2, v_3, v_4/\emptyset).\}$$

For changing $w_1(v_4, v_6/v_3, v_5)$, since $V_{o1} = \{v_4, v_6\}$, there exists only one inner path $p_{w1}(v_4, v_6)$ available. However, we can replace $w_2(v_1, v_2, v_3, v_4/\emptyset)$ by any one set of $\{p_{w2}(v_1, v_2), p_{w2}(v_3, v_4)\}$, $\{p_{w2}(v_1, v_3), p_{w2}(v_2, v_4)\}$ and $\{p_{w2}(v_2, v_3), p_{w2}(v_1, v_4)\}$.

When we choose $\{p_{w2}(v_1, v_2), p_{w2}(v_3, v_4)\}$ to change $w_2(v_1, v_2, v_3, v_4/\emptyset)$, the corresponding closed train is shown in *Eq.* (3.4).

When we choose $\{p_{w2}(v_1, v_3), p_{w2}(v_2, v_4)\}$, the corresponding closed train is

$$\Phi' = \{e_1(v_1, v_7), e_2(v_7, v_2), p_{w2}(v_2, v_4), p_{w1}(v_4, v_6),$$
$$e_4(v_6, v_8), e_3(v_8, v_3), p_{w2}(v_3, v_1)\}$$

When $\{p_{w2}(v_2, v_3), p_{w2}(v_1, v_4)\}$ is chosen, the corresponding closed train is expressed as

$$\Phi'' = \{e_1(v_1, v_7), e_2(v_7, v_2), p_{w2}(v_2, v_3), e_3(v_3, v_8),$$
$$e_4(v_8, v_6), p_{w1}(v_6, v_4), p_{w2}(v_4, v_1).\}$$

□

It can be seen that $w_i(V_{oi}/\overline{V_{oi}})$ in a W-circuit is a set of $\mid V_{oi} \mid /2$ inner paths of w_i and V_{oi} is a set of terminals of those inner paths. Hence, we give a definition to describe V_{oi} as follows.

Definition 3.1.3 (Terminal set) *A vertex set* V_{oi} $(i = 1, 2, \cdots, \mid W \mid)$ *in a W-circuit in Eq.(3.1) is a terminal set of* w_i. *The total number of vertices in* V_{oi} *is always even.*

It should be noticed that for a W-circuit containing $w_i(V_{oi}\ /\ \overline{V_{oi}})$ $(i \in 1, 2, \cdots, k)$, when we change $w_i(V_{oi}\ /\ \overline{V_{oi}})$ by $\mid V_{oi} \mid /2$ inner paths of w_i whose terminals are distinct in the W-circuit, the W-circuit becomes a closed train by Property 3.1.1. Furthermore, we will show in Chapter 4 that when each wild-component is specified by a tree, there exists exactly one subgraph of the tree which consists of $\mid V_{oi} \mid /2$ edge disjoint paths suth that the W-circuit becomes either a circuit or an edge disjoin union of circuits.

We will establish a relation of W-circuits in a W-graph under an operation called W-ring sum which is defined next.

3.1.2 Ring Sum Operation of W-circuits

A theorem associated with the W-ring sum of W-circuits will be given. First, we define an operation of W-ring sum with respect to W-circuits C_α and C_β, denoted by $C_\alpha \hat{\oplus} C_\beta$, as follows.

Definition 3.1.4 (W-ring sum of W-circuit) *Let C_α and C_β be W-circuits,*

$$\begin{aligned} C_\alpha = \{&e_{\alpha 1}, e_{\alpha 2}, \cdots, e_{\alpha m}, w_1(V_{\alpha o1}/\overline{V_{\alpha o1}}), w_2(V_{\alpha o2}/\overline{V_{\alpha o2}}), \cdots, \\ &w_{|W|}(V_{\alpha o|W|}/\overline{V_{\alpha o|W|}})\} \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} C_\beta = \{&e_{\beta 1}, e_{\beta 2}, \cdots, e_{\beta n}, w_1(V_{\beta o1}/\overline{V_{\beta o1}}), w_2(V_{\beta o2}/\overline{V_{\beta o2}}), \cdots, \\ &w_{|W|}(V_{\beta o|W|}/\overline{V_{\beta o|W|}}).\} \end{aligned} \tag{3.6}$$

Then, $C_\alpha \hat{\oplus} C_\beta$ is formed by the following three parts:

Part I: *$C_\alpha \hat{\oplus} C_\beta$ contains all edges in $\{e_{\alpha 1}, e_{\alpha 2}, \cdots, e_{\alpha m}\} \oplus \{e_{\beta 1}, e_{\beta 2}, \cdots, e_{\beta n}\}$.*

Part II: *If $w_i(V_{oi}/V(w_i) - V_{oi})$ is in C_α or C_β but not in both C_α and C_β, then $w_i(V_{oi}/V(w_i) - V_{oi})$ is in $C_\alpha \hat{\oplus} C_\beta$.*

Part III: *If $V_{\alpha oi} \oplus V_{\beta oi} \neq \emptyset$ $(i = 1, 2, \cdots, \mid W \mid)$, $w_i(V_{\alpha oi} \oplus V_{\beta oi}/V(w_i) - V_{\alpha oi} \oplus V_{\beta oi})$ is in $C_\alpha \hat{\oplus} C_\beta$. Eles, w_i is not contained in $C_\alpha \hat{\oplus} C_\beta$.*

The W-ring sum of W-circuits is explained by the following example.

Example 3.1.2 In the given W-graph in Fig. 2.1, we can find a W-circuit C_α in as:

$$\begin{aligned} C_\alpha = \{&e_1(v_5, v_1), e_2(v_1, v_2), e_7(v_6, v_4), e_{10}(v_4, v_9), \\ &w_1(v_2, v_5, v_6, v_7/\emptyset), w_2(v_7, v_9/v_2, v_3, v_8)\} \end{aligned}$$

Also, we can obtain another W-circuit C_β expressed as:

$$C_\beta = \{e_5(v_3, v_4), e_8(v_9, v_7), e_{10}(v_4, v_9), w_2(v_3, v_7/v_2, v_8, v_9)\}.$$

By Definition 3.1.4, $C_\alpha \hat{\oplus} C_\beta$ can be obtained as

$$\begin{aligned} C_\gamma &= C_\alpha \hat{\oplus} C_\beta \\ &= \{e_1(v_5, v_1), e_2(v_1, v_2), e_5(v_3, v_4), e_7(v_6, v_4), e_8(v_9, v_7), \\ &\quad w_1(v_2, v_5, v_6, v_7/\emptyset), w_2(v_3, v_9/v_2, v_7, v_8)\}. \end{aligned} \tag{3.7}$$

□

It is clear that C_γ is also a W-circuit because it satisfies Definition 3.1.1. There is a question whether the W-ring sum of any two W-circuits of a W-graph is also a W-circuit of the W-graph, which will be answered in the following discussion.

3.1.3 Properties of W-circuits in a W-graph

In graph theory, we have Theorem 1.1.1 which states that the ring sum of circuits becomes either a circuit or an edge disjoint union of circuits. If we can provide a theorem corresponding to Theorem 1.1.1 in a W-graph, then any W-circuit can be obtained by the W-ring sum of linearly independent W-circuits.

From Definition 3.1.4, the following lemma is trivial.

Lemma 3.1.1 *Let C_α and C_β be two W-circuits, we have,* *1. If $C_\alpha = C_\beta$, then $C_\alpha \hat{\oplus} C_\beta = \emptyset$, and*

2. $C_\alpha \hat{\oplus} \emptyset = C_\alpha$. ■

In general, we have the following theorem.

Theorem 3.1.1 *The W-ring sum of two different W-circuits of a W-graph $\Omega_w(V, E, W)$ is also a W-circuit in the W-graph.*

Proof: Let C_α and C_β be two different W-circuits. Suppose $C_\alpha \hat{\oplus} C_\beta$ is:

$$\{e_{\xi 1}, e_{\xi 2}, \cdots, e_{\xi m}, w_1(V_{\xi o1}/\overline{V_{\xi o1}}), \cdots, w_{|W|}(V_{\xi o|W|}/\overline{V_{\xi o|W|}})\}. \quad (3.8)$$

In order to show that the set in $Eq.(3.8)$ is a W-circuit, we must show that the conditions in Definition 3.1.1 will be satisfied.

Since $\{e_{\xi 1}, e_{\xi 2}, \cdots, e_{\xi m}\} = \{e_{\alpha 1}, e_{\alpha 2}, \cdots, e_{\alpha m}\} \oplus \{e_{\beta 1}, e_{\beta 2}, \cdots, e_{\beta n}\}$, edges in $\{e_{\xi 1}, e_{\xi 2}, \cdots, e_{\xi m}\}$ are all different which satisfies Condition 1 in Definition 3.1.1.

For any terminal set $V_{\xi oi}$, since $V_{\xi oi} = V_{\alpha oi} \oplus V_{\beta oi}$ where $| V_{\alpha oi} |$ and $| V_{\beta oi} |$ are both even, $| V_{\xi oi} |$ is also even which satisfies Condition 2 in Definition 3.1.1.

When $V_{\xi oi} = \emptyset$, we will remove $w_i(V_{\xi oi}/\overline{V_{\xi oi}})$ from $Eq.(3.8)$ so it satisfies Condition 3 in Definition 3.1.1.

Now we only need to show that $C_\alpha \hat{\oplus} C_\beta$ satisfies Condition 4 in Definition 3.1.1.

Let v_ξ be any vertex in *Eq.* (3.8) as either an endpoint of an edge or a vertex in a terminal set.

Consider v_ξ is contained in both C_α and C_β. Let $d_\alpha^{(e)}(v_\xi)$ be a number of times that v_ξ appears as an endpoint of an edge in C_α and $d_\alpha^{(w)}(v_\xi)$ be a number of terminal sets $V_{\alpha oi}$ $(i \in 1, 2, \cdots, \mid W \mid)$ which contains v_ξ in C_α. Similarly, we can define those with respect to C_β as $d_\beta^{(e)}(v_\xi)$ and $d_\beta^{(w)}(v_\xi)$. By Definition 3.1.1, since C_α and C_β are W-circuits, we know $d_\alpha^{(e)}(v_\xi) + d_\alpha^{(w)}(v_\xi)$ is even number and $d_\beta^{(e)}(v_\xi) + d_\beta^{(w)}(v_\xi)$ is also even number. Hence, v_ξ in *Eq.* (3.8) appears the following times:

$$d_\alpha^{(e)}(v_\xi) + d_\alpha^{(w)}(v_\xi) + d_\beta^{(e)}(v_\xi) + d_\beta^{(w)}(v_\xi) - 2d_c^{(e)}(v_\xi) - 2d_c^{(w)}(v_\xi)$$

where $d_c^{(e)}(v_\xi)$ is a number of common edges whose one endpoint is v_ξ and in both C_α and C_β, and $d_c^{(w)}(v_\xi)$ is a number of pairs of terminal sets $V_{\alpha oi}$ and $V_{\beta oi}$ $(i \in 1, 2, \cdots, \mid W \mid)$ each of which contains v_ξ.

Since $d_\alpha^{(e)}(v_\xi) + d_\alpha^{(w)}(v_\xi) + d_\beta^{(e)}(v_\xi) + d_\beta^{(w)}(v_\xi)$ is even and $d_c^{(e)}(v_\xi)$ and $d_c^{(w)}(v_\xi)$ are multiplied by 2 so that the above result is always even which proves that every vertex in $C_\alpha \hat{\oplus} C_\beta$ appears even times. ■

Theorem 3.1.1 establishes the relation of W-circuits in a W-graph under the defined W-ring sum operation, so that we can obtain any W-circuit by W-ring sum of linearly independent W-circuits. A set of linearly independent W-circuits $\{C_i\}$, $i = 1, 2, \cdots, r$, is defined as follows.

If for some set of constants $a_i = 1$ or 0, not all of which are zero, we have

$$a_1 C_1 \hat{\oplus} a_2 C_2 \hat{\oplus} a_3 C_3 \hat{\oplus} \cdots \hat{\oplus} a_r C_r = \emptyset \tag{3.9}$$

where $1C_i = C_i$ and $0C_i = \emptyset$, then the W-circuits are said to be **linearly dependent**. If however $Eq.(3.9)$ is satisfied only when all the constants a_i are zero, the W-circuits are said to be **linearly independent**.

We will provide a method for obtaining a set of linear independent W-circuits by matrix representation later.

3.2 W-Cutsets

Consider a **cutset** or an **edge disjoint union of cutsets** separating the vertex set V of a W-graph $\Omega_w(V, E, W)$ into two vertex subsets V_a and $\overline{V_a}$ such that $V_a \cup \overline{V_a} = V$, $V_a \cap \overline{V_a} = \emptyset$. If an edge in E is connected between a vertex in V_a and a vertex in $\overline{V_a}$, we say that the cutset or the edge disjoint union of cutsets contains the edge. If v_a and v_b are two vertices in $V(w_i)$ and $v_a \in V_a$ and $v_b \in \overline{V_a}$, we will use the colon ":" to divide $V(w_i)$ into two subsets V_{ai} and $\overline{V_{ai}}$ such that $v_a \in V_{ai}$ and $v_b \in \overline{V_{ai}}$ where $\overline{V_{ai}} = V(w_i) - V_{ai}$, then we say the cutset or the edge disjoint union of cutsets contains w_i as form of $w_i(V_{ai} : \overline{V_{ai}})$.

3.2.1 Definition of a W-cutset

A W-cutset is defined as a collection of edges and wild-components which are contained by a cutset or an edge disjoint union of cutsets separating the vertex set V of $\Omega_w(V, E, W)$ into two vertex subsets V_a and $\overline{V_a}$ where $\overline{V_a} = V - V_a$.

Definition 3.2.1 (W-cutset) *For a W-graph $\Omega_w(V, E, W)$, a W-cutset corresponding to a cutset or an edge disjoint union of cutsets separating V into V_a and $\overline{V_a}$ is represented by*

$$\begin{aligned} S_w = \{&e_{s1}, e_{s2}, \cdots, e_{sm}, w_1(V_{a1} : \overline{V_{a1}}), w_2(V_{a2} : \overline{V_{a2}}), \cdots, \\ &w_{|W|}(V_{a|W|} : \overline{V_{a|W|}})\} \end{aligned} \tag{3.10}$$

where $e_{s1}, e_{s2}, \cdots, e_{sm}$ are all edges which are connected between a vertex in V_a and a vertex in $\overline{V_a} = V - V_a$. V_{ai} and $\overline{V_{ai}}$ $(i = 1, 2, \cdots, | W |)$ are two vertex subsets of $V(w_i)$ such that $V_{ai} \subseteq V_a$, $\overline{V_{ai}} \subseteq \overline{V_a}$ and $V_{ai} \cup \overline{V_{ai}} = V(w_i)$. When one of V_{ai} and $\overline{V_{ai}}$ is an empty set, then we define $w_i(V_{ai} : \overline{V_{ai}}) = \emptyset$ which is not contained in S.

Consider a W-cutset separating vertices of the W-graph in Fig.2.1 into two parts $V_a = \{v_1, v_2, v_5, v_6\}$ and $\overline{V_a} = \{v_3, v_4, v_7, v_8, v_9, v_{10}\}$ as shown in Fig.3.2. The W-cutset contains edges e_3, e_4, e_6 and e_9 which are connected between a vertex in V_a and a vertex in $\overline{V_a}$. Also, the

W-cutset separates $V(w_1)$ into $V_{a1} = \{v_2, v_5, v_6\}$ and $\overline{V_{a1}} = \{v_7\}$ and separates $V(w_2)$ into $V_{a2} = \{v_2\}$ and $\overline{V_{a2}} = \{v_3, v_7, v_8, v_9\}$. Hence, the W-cutset S_1 is as follows.

$$S_1 = \{e_3, e_4, e_6, e_7, e_9, w_1(v_2, v_5, v_6 : v_7), w_2(v_2 : v_3, v_7, v_8, v_9)\}. \quad (3.11)$$

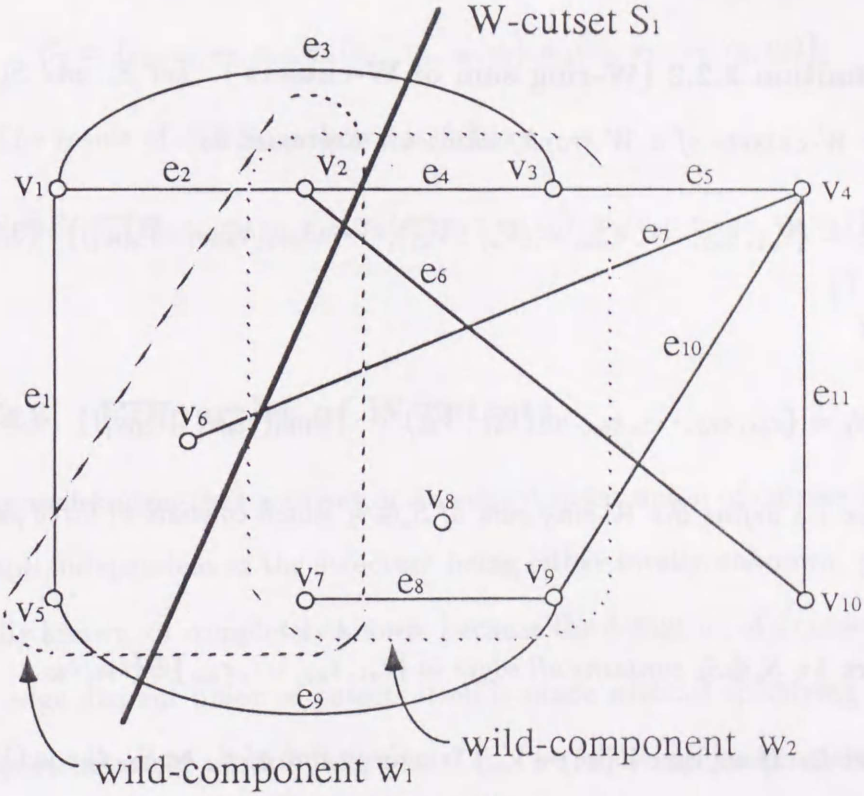


Figure 3.2: A W-cutset.

It should be noticed that the definitions of $w_i(V_{ai} : \overline{V_{ai}})$ in a W-cutset and that of $w_i(V_{oi}/\overline{V_{oi}})$ in a W-circuit. For a W-cutset, $w_i(V_{ai} : \overline{V_{ai}})$ and $w_i(\overline{V_{ai}} : V_{ai})$ have the same meanings. And we have defined that $w_i(V_{ai} : \overline{V_{ai}}) = \emptyset$ if one of V_{ai} and $\overline{V_{ai}}$ is an empty set. However, in

W-circuit, only if $V_{oi} = \emptyset$, $w_i(V_{oi}/\overline{V_{oi}}) = \emptyset$.

3.2.2 Ring-sum Operation of W-cutsets

For W-cutsets S_a and S_b, an operation of W-ring sum with respect to S_a and S_b represented by $S_a \tilde{\oplus} S_b$ is defined as :

Definition 3.2.2 (W-ring sum of W-cutsets) *Let S_a and S_b be two W-cutsets of a W-graph, which are expressed as:*

$$S_a = \{e_{a1}, e_{a2}, \cdots, e_{am}, w_1(V_{a1} : \overline{V_{a1}}), \cdots, w_{|W|}(V_{a|W|} : \overline{V_{a|W|}})\} \quad (3.12)$$

and

$$S_b = \{e_{b1}, e_{b2}, \cdots, e_{bn}, w_1(V_{b1} : \overline{V_{b1}}), \cdots, w_{|W|}(V_{b|W|} : \overline{V_{b|W|}})\} \quad (3.13)$$

Then we define the W-ring sum of $S_a \tilde{\oplus} S_b$ which consists of three parts as:

Part 1: *$S_a \tilde{\oplus} S_b$ contains all edges in $\{e_{a1}, e_{a2}, \cdots, e_{am}\} \oplus \{e_{b1}, e_{b2}, \cdots, e_{bn}\}$.*

Part 2: *If $w_i(V_{ai} : V(w_i) - V_{ai})$ is only in one of S_a or S_b, the $w_i(V_{ai} : V(w_i) - V_{ai})$ is in $S_a \tilde{\oplus} S_b$.*

Part 3: *For each wild-component w_i, a $w_i(V_{ai} \oplus V_{bi} : V(w_i) - V_{ai} \oplus V_{bi})$ is in $S_a \tilde{\oplus} S_b$ if both $V_{ai} \oplus V_{bi} \neq \emptyset$ and $V(w_i) - V_{ai} \oplus V_{bi} \neq \emptyset$ $(i = 1, 2, \cdots, \mid W \mid)$ are satisfied. Otherwise, $S_a \tilde{\oplus} S_b$ does not contain w_i.*

Consider the W-graph as shown in Fig.2.1. One W-cutset S_1 has been obtained in *Eq.* (3.11). There is another W-cutset S_2 separates vertices of the W-graph into $V_a = \{v_1, v_5, v_6, v_7, v_8, v_9\}$ and $\overline{V_a} = \{v_2, v_3, v_4, v_{10}\}$. S_2 contains edges e_2, e_3, e_7, e_{10} and separates $V(w_1)$ into $\{v_2\}$ and $\{v_5, v_6, v_7\}$ and $V(w_2)$ in to $\{v_2, v_3\}$ and $\{v_7, v_8, v_9\}$. Hence,

$$S_2 = \{e_2, e_3, e_7, e_{10}, w_1(v_2 : v_5, v_6, v_7), w_2(v_2, v_3 : v_7, v_8, v_9)\}.$$

The result of $S_1 \tilde{\oplus} S_2$ is shown as follows.

$$S_1 \tilde{\oplus} S_2 = \{e_2, e_4, e_6, e_9, e_{10}, w_1(v_2, v_7 : v_5, v_6), w_2(v_3 : v_2, v_7, v_8, v_9)\}. \tag{3.14}$$

3.2.3 Properties of W-cutsets

It is well-known that a cutset or an edge disjoint union of cutsets in a graph independent of the structure being either totally unknown, partially known, or completely known, because the definition of a cutset or an edge disjoint union of cutsets itself is made without specifying the edge structure of a graph [Mayeda1 72]. The following theorem is trivial, but the description of W-ring sum of W-cutsets and the expression related to wild-components should be verified.

Theorem 3.2.1 *If S_a and $S_b(\neq S_a)$ are two W-cutsets of a W-graph, then $S_a \tilde{\oplus} S_b$ is a W-cutset of the W-graph.*

Proof: We show that the W-ring sum of two W-cutsets becomes one shown in $Eq.(3.10)$. First, suppose V is the vertex set of a graph. Consider four subset of V as V_{11}, V_{12}, V_{21} and V_{22} which are not empty sets and satisfy

$$V_{11} \cup V_{12} \cup V_{21} \cup V_{22} = V$$

and

$$V_{pq} \cap V_{rs} = \emptyset$$

where $p, q, r, s = 1$ or 2, $(pq) \neq (rs)$ as shown in Fig.3.3.

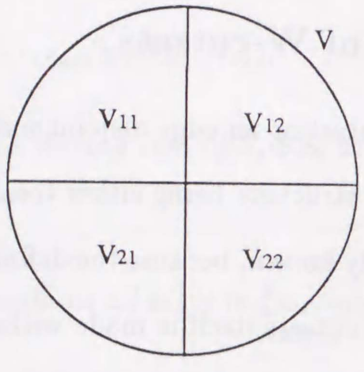


Figure 3.3: V separated into V_{11}, V_{12}, V_{21} and V_{22}.

Let a cutset or an edge disjoint union of cutsets separate V into $V_a = V_{11} \cup V_{12}$ and $\overline{V_a} = V_{21} \cup V_{22}$ and another one separate V into $V_b = V_{12} \cup V_{22}$ and $\overline{V_b} = V_{11} \cup V_{21}$. By $Eq.$(2-3-17) in [Mayeda1 72], the ring sum of the two cutsets or edge disjoint union of cutsets separates

V into $V_c = V_{11} \cup V_{22}$ and $\overline{V_c} = V_{12} \cup V_{21}$, that is,

$$\begin{aligned} S_a \oplus S_b &= \varepsilon(V_a \times \overline{V_a}) \oplus \varepsilon(V_b \times \overline{V_b}) \\ &= \varepsilon((V_{11} \cup V_{12}) \times (V_{21} \cup V_{22})) \oplus \varepsilon((V_{12} \cup V_{22}) \times (V_{11} \cup V_{21})) \\ &= \varepsilon((V_{11} \cup V_{22}) \times (V_{12} \cup \overline{V_{21}})) = \varepsilon((V_a \oplus V_b) \times (V_a \oplus \overline{V_b})) \\ &= \varepsilon(V_c \times \overline{V_c}) \\ &= S_3. \end{aligned} \tag{3.15}$$

Notice that V_c and $\overline{V_c}$ are two disjoint vertex subsets of V separated by S_3.

Suppose we separate vertex set V in a W-graph $\Omega_w(V, E, W)$ by the same way, that is, $V_{11} \cup V_{12} \cup V_{21} \cup V_{22} = V$ and $V_{pq} \cap V_{rs} = \emptyset$ where $p, q, r, s = 1$ or 2, $(pq) \neq (rs)$.

Let two W-cutsets be

$$S_a = \{\varepsilon(V_a \times \overline{V_a}), w_1(V_{a1} : \overline{V_{a1}}), \cdots, w_{|W|}(V_{a|W|} : \overline{V_{a|W|}})\}$$

and

$$S_b = \{\varepsilon(V_b \times \overline{V_b}), w_1(V_{b1} : \overline{V_{b1}}), \cdots, w_{|W|}(V_{a|W|} : \overline{V_{a|W|}})\}$$

where $V_a = V_{11} \cup V_{12}$, $\overline{V_a} = V_{21} \cup V_{22}$, $V_b = V_{12} \cup V_{22}$ and $\overline{V_b} = V_{11} \cup V_{21}$, and $\varepsilon(V_k \times \overline{V_k})$ $(k = a$ or $b)$ is an edge set connected between a vertex in V_k and a vertex in $\overline{V_k}$.

Then we will show the W-ring sum $S_a \tilde{\oplus} S_b$ becomes one in $Eq.(3.10)$.

Now, we consider only edges in a W-graph. The edges in $S_a \tilde{\oplus} S_b$ by $Eq.(3.15)$ will be

$$\varepsilon(V_a \times \overline{V_a}) \oplus \varepsilon(V_b \times \overline{V_b}) = \varepsilon((V_a \oplus V_b) \times (V_a \oplus \overline{V_b})) = \varepsilon(V_c \times \overline{V_c})$$

which satisfies edges in $Eq.(3.10)$.

For a wild-component w_i which is either in S_a or in S_b but not in both S_a and S_b, we will show that w_i will be in the W-ring sum of S_a and S_b. Suppose S_a contains $w_i(V_{ai} : \overline{V_{ai}})$, but S_b does not contain w_i. By Definition 3.2.1, $w_i(V_{ai} : \overline{V_{ai}})$ is in S_a means $V_{ai} \subseteq V_a$ and $\overline{V_{ai}} \subseteq \overline{V_a}$. Since S_b does not contain w_i, either $V(w_i) \subseteq V_b$ or $V(w_i) \subseteq \overline{V_b}$. Let $V(w_i) \subseteq \overline{V_b}$. Then, we can see that $V_{ai} \subseteq V_a \oplus V_b = V_c$ and $\overline{V_{ai}} \subseteq V_a \oplus \overline{V_b} = \overline{V_c}$. Thus, $w_i(V_{ai} : \overline{V_{ai}})$ must be in $S_a \tilde{\oplus} S_b$ which satisfies the conditions in $Eq.(3.10)$. When $V(w_i) \subseteq V_b$, exchanging V_b and $\overline{V_b}$ makes the same result. Also, we can achieve the same result when S_b contains $w_i(V_{bi} : \overline{V_{bi}})$ but S_a does not contain w_i.

When w_i is in both S_a and S_b, that is, $w_i(V_{ai} : \overline{V_{ai}})$ is in S_a and $w_i(V_{bi} : \overline{V_{bi}})$ is in S_b, let $V_{ai} \subseteq V_a$ and $V_{bi} \subseteq V_b$. Thus, $V_c = V_a \oplus V_b$ contains $V_{ai} \oplus V_{bi}$ and $\overline{V_c} = V_a \oplus \overline{V_b}$ contains $V_{ai} \oplus \overline{V_{bi}}$. Hence, $w_i(V_{ai} \oplus V_{bi} : V_{ai} \oplus \overline{V_{bi}})$ is in $S_a \tilde{\oplus} S_b$ which satisfies the conditions of w_i in $Eq.(3.10)$.

Suppose $V_{ai} \oplus V_{bi} = \emptyset$ or $V_{ai} \oplus \overline{V_{bi}} = \emptyset$. Since $V_{ai} \subseteq V_a$, $\overline{V_{ai}} \subseteq \overline{V_a}$, $V_{bi} \subseteq V_b$ and $\overline{V_{bi}} \subseteq \overline{V_b}$, $V_{ai} \oplus V_{bi} = \emptyset$ means $V_{ai} \oplus \overline{V_{bi}} = V(w_i) \subseteq V_a \oplus \overline{V_b} = \overline{V_c}$. Hence, W-ring sum of $S_a \tilde{\oplus} S_b$ does not contain $w_i(\emptyset : V(w_i))$. Similarly, when $V_{ai} \oplus \overline{V_{bi}} = \emptyset$, we will have the same result. Thus, w_i is not in

$S_a\tilde{\oplus}S_b$ which satisfys conditions in $Eq.(3.10)$.

These conclude that the W-ring sum of S_a and S_b gives a W-cutset separating V of $\Omega_w(V, E, W)$ into two vertex subsets V_c and $\overline{V_c}$ where $\overline{V_c} = V - V_c$ by Definition 3.2.1, so $S_a\tilde{\oplus}S_b$ is a W-cutset when $S_a \neq S_b$. ■

It is evident that $S_a\tilde{\oplus}S_b = \emptyset$ if $S_a = S_b$ and $\emptyset\tilde{\oplus}S_a = S_a$ by Definition 3.2.2. A set of W-cutsets $\{S_i\}$, $i = 1, 2, \cdots, r$, is said to be linearly independent, only when all the constants a_i are zero, the following $Eq.(3.16)$ is satisfied.

$$a_1S_1\tilde{\oplus}a_2S_2\tilde{\oplus}a_3S_3\tilde{\oplus}\cdots\tilde{\oplus}a_rS_r = \emptyset \tag{3.16}$$

By Theorem 3.2.1, we can see that $S_1\tilde{\oplus}S_2$ in $Eq.(3.14)$ is a W-cutset separating all vertices of the W-graph in Fig. 2.1 into $\{v_2, v_7, v_8, v_9\}$ and $\{v_1, v_3, v_4, v_5, v_6, v_{10}\}$.

3.3 Matrix Representation of W-graphs

Matrix representations of a graph play an important role in graph theory, each of which is a mathematical form indicating all informations such as incidence situations and characteristics of a graph. The main purpose of this section is to provide linearly independent W-cutsets and linearly independent W-circuits based on [Zhao5 92].

For a given W-graph $\Omega_w(V, E, W)$, we will prove that a W-graph can be expressed by a mixed matrix representations in spite of the existence of unspecified edges in the W-graph. We will use matrices such as W-incidence matrix, W-cutset matrix and W-circuit matrix to represent W-incidence sets, W-cutsets and W-circuits in a W-graph. Particularly, a fundamental W-cutset matrix and a fundamental W-circuit matrix are useful for obtaining linearly independent W-cutsets and W-circuits, respectively.

A matrix is here composed by either 0 or 1, whose columns consist of edges and vertices for dealing with the unspecified edges, which differs from general matrix representation of ordinary graphs. The rows of the matrix represent either W-incidence sets, or W-cutsets, or W-circuits, which is similar to general one.

In order to avoid the confusion which vertex belongs to wild component w_i in a matrix representation of W-graphs, we will indicate it by using a superscription (i) as $v_j^{(i)}$ for a vertex $v_j \in V(w_i)$. In other words, the symbols of $v_1^{(i)}, v_2^{(i)}, \cdots, v_p^{(i)}$ are employed for specifying wild component w_i containing vertices $v_1, v_2, \cdots, v_p$. Hence, $V(w_i) = \{v_1^{(i)}, v_2^{(i)}, \cdots, v_p^{(i)}\}$.

Definition 3.3.1 (Reference vertex of w_i) *For a W-graph, we choose one vertex in each wild component w_i as a reference vertex of w_i, denoted by drawing a line under the vertex such as $\underline{v_r^{(i)}}$.*

For conveniences, we suppose that W-graphs used hereafter are connected and having no self-loops. Of course, it is not difficult to extend the results of connected W-graphs to a separable one.

For a given W-graph $\Omega_w(V, E, W)$, a matrix whose columns represent all edges in E and all vertices in $\{V(w_i) - \underline{v_r^{(i)}}\}$, $i = 1, 2, \cdots, \mid W \mid$ and whose rows represent W-incidence sets (W-cutsets, W-circuits) is called a W-incidence (W-cutset, W-circuit) matrix.

3.3.1 W-incidence Matrix

Every row of a W-incidence matrix represents a W-incidence set. Hence, a W-graph without self-loops is completely characterized by its W-incidence matrix.

First, we define a W-incidence set as follows. Since a W-incidence set is also a W-cutset (Definition 3.2.1) where either V_a or $\overline{V_a}$ contains one and only one vertex.

Definition 3.3.2 (W-incidence set) *For a W-graph $\Omega_w(V, E, W)$ having k wild components, a W-incidence set $A(v_a)$ with respect to vertex $v_a \in V$ is a W-cutset where either V_a or $\overline{V_a}$ contains v_a only, represented by:*

$$A(v_a) = \{e_{a1}, e_{a2}, \cdots, e_{al}, w_1(\{v_a^{(1)}\} : \overline{\{v_a^{(1)}\}}), w_2(\{v_a^{(2)}\} : \overline{\{v_a^{(2)}\}}), \ldots,$$
$$w_k(\{v_a^{(k)}\} : \overline{\{v_a^{(k)}\}})\}$$

where $e_{a1}, e_{a2}, \cdots, e_{al}$ *are edges connected between* v_a *and a vertex in* $\overline{\{v_a\}} = V - \{v_a\}$. *When* v_a *is a vertex in* $V(w_i)$, *we denote* v_a *by* $v_a^{(i)}$ *and* $\overline{\{v_a^{(i)}\}} = V(w_i) - \{v_a^{(i)}\}$ $(i \in 1, 2, \cdots, k)$.

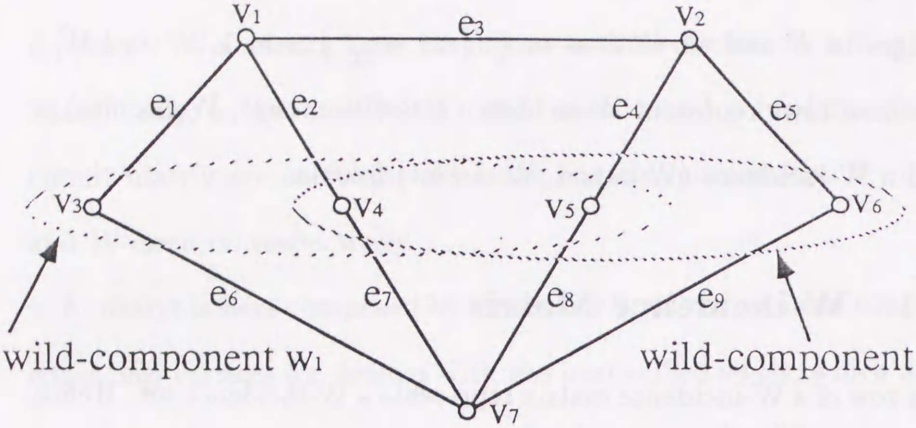


Figure 3.4: A W-graph with two wild components.

Consider the W-graph as shown in Fig.3.4, the W-incidence set $A(v_5)$ can be obtained as follows: Set $A(v_5)$ separates the vertex set of the W-graph into v_5 and $\overline{\{v_5\}} = \{v_1, v_2, v_3, v_4, v_6, v_7\}$, $V(w_1)$ is separated into $\{v_5^{(1)}\}$ and $\{v_3^{(1)}, v_4^{(1)}\}$ and $V(w_2)$ is separated into $\{v_5^{(2)}\}$ and $\{v_4^{(2)}, v_6^{(2)}\}$. Thus, $A(v_5)$ contains edges e_4, e_8 and wild component w_1 as form of $w_1(v_5^{(1)} : v_3^{(1)}, v_4^{(1)})$ and w_2 as $w_2(v_5^{(2)} : v_4^{(2)}, v_6^{(2)})$. Hence,

$$A(v_5) = \{e_4, e_8, w_1(v_5^{(1)} : v_3^{(1)}, v_4^{(1)}), w_2(v_5^{(2)} : v_4^{(2)}, v_6^{(2)})\}.$$

A W-incidence matrix is described as follows:

A W-incidence matrix $A_w = [a_{pq}]$ of a W-graph $\Omega_w(V, E, W)$ consists of rows representing W-incidence sets with respect to all vertices except one which is chosen to be the reference vertex of the W-graph Ω_w. Instead of representing edges by each column of an incidence matrix of an ordinary graph, columns of a W-incidence matrix representing all edges in E and all vertices in $\{V(w_i) - \underline{v_r^{(i)}}\}$, $i = 1, 2, \cdots, |W|$.

Definition 3.3.3 (W-incidence matrix) *A W-incidence matrix $A_w = [a_{pq}]$ of a W-graph $\Omega_w(V, E, W)$ is defined as:*

Case 1: *When column q indicates an edge e,*

$$\mathbf{a_{pq}} = \begin{cases} \mathbf{1} & \text{edge } e \text{ incidents at } v_p, \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

Case 2: *When column q corresponds to a vertex $v_j^{(i)}$, for all i, if*

(a) *v_p is not the reference vertex of w_i, then*

$$\mathbf{a_{pq}} = \begin{cases} \mathbf{1} & v_p = v_j^{(i)}, \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

(b) v_p *is the reference vertex of* w_i*, then*

$$\mathbf{a_{pq}} = \begin{cases} \mathbf{1} & v_p \neq v_j^{(i)}, \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

Example 3.3.1 Obtaining the W-incidence matrix A_w of the W-graph as shown in Fig.3.4 where $V(w_1) = \{v_3^{(1)}, v_4^{(1)}, v_5^{(1)}\}$ and $V(w_2) = \{v_4^{(2)}, v_5^{(2)}, v_6^{(2)}\}$. Let the reference vertex of wild component w_1 be $\underline{v_5^{(1)}}$ and that of wild component w_2 be $\underline{v_4^{(2)}}$. Then, the columns of the W-incidence matrix A_w consist of edges e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9 and vertices $v_3^{(1)}, v_4^{(1)}$ obtained by $V(w_1) - \{\underline{v_5^{(1)}}\}$ and $v_5^{(2)}, v_6^{(2)}$ obtained by $V(w_2) - \{\underline{v_4^{(2)}}\}$. We choose v_7 as the reference vertex of the W-graph, the rows of A_w indicate W-incidence sets with respect to every vertex other than v_7. Thus, the W-incidence matrix A_w is formed as follow:

$$A_w = \begin{array}{c} \\ A(v_1) \\ A(v_2) \\ A(v_3) \\ A(v_4) \\ A(v_5) \\ A(v_6) \end{array} \begin{array}{c} \begin{array}{ccccccccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 & v_3^{(1)} & v_4^{(1)} & v_5^{(2)} & v_6^{(2)} \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \tag{3.17}$$

In the row corresponding to $A(v_1) = \{e_1, e_2, e_3\}$, there are 1s in columns e_1, e_2 and e_3 because these edges incident at v_1 and 0s in columns representing $v_3^{(1)}$, $v_4^{(1)}$, $v_5^{(2)}$ and $v_6^{(2)}$ because v_1 is in neither $V(w_1)$ nor $V(w_2)$. In the row corresponding to $A(v_3) = \{e_1, e_6, w_1(v_3^{(1)} : v_4^{(1)}, \underline{v_5^{(1)}})\}$, there are 1s in columns e_1, e_6 and $v_3^{(1)}$ because e_1 and e_6 incident at v_3 and v_3 is in $V(w_1)$ and v_3 is not the reference vertex of w_1 so that the column of $v_3^{(1)}$ has 1. The row corresponding to $A(v_5) = \{e_4, e_8, w_1(\underline{v_5^{(1)}} : v_3^{(1)}, v_4^{(1)}), w_2(v_5^{(2)} : \underline{v_4^{(2)}}, v_6^{(2)})\}$ has 1s at columns e_4, e_8, $v_3^{(1)}$, $v_4^{(1)}$ and column $v_5^{(2)}$ because edges e_4 and e_8 incident at v_5, and v_5 is in $V(w_1)$ as the reference vertex of w_1 so the columns of $v_3^{(1)}$ and $v_4^{(1)}$ have 1s according to Case 2 (b). Also v_5 is in $V(w_2)$ and is not the reference vertex of w_2 so that the column of $v_5^{(2)}$ has 1 and column of $v_6^{(2)}$ has 0 according to Case 2 (a). The other rows also can be obtained easily by the same procedure. □

Now, we study the rank of a W-incidence matrix of a W-graph.

Consider a W-graph Ω_w consisting of only one wild component w_1 as shown in Fig.3.5(a). Suppose v_4 is the reference vertex of w_1 and also the reference vertex of Ω_w. W-incidence set with respect to v_1 is $A_w(v_1) = \{w_1(v_1^{(1)} : v_2^{(1)}, v_3^{(1)}, \underline{v_4^{(1)}}, v_5^{(1)})\}$. When we obtain a W-incidence matrix of Ω_w, the row of $A_w(v_1)$ has 1 in the column corresponding to vertex $v_1^{(1)}$ and has 0s in the other columns corresponding to $v_2^{(1)}$, $v_3^{(1)}$, $v_5^{(1)}$ according to Case 2 (a) in Definition 3.3.3. When assigning a star to the wild component w_1 where the center of the star is $\underline{v_4^{(1)}}$ (the reference vertex of w_i), Ω_w becomes a graph g_s as shown

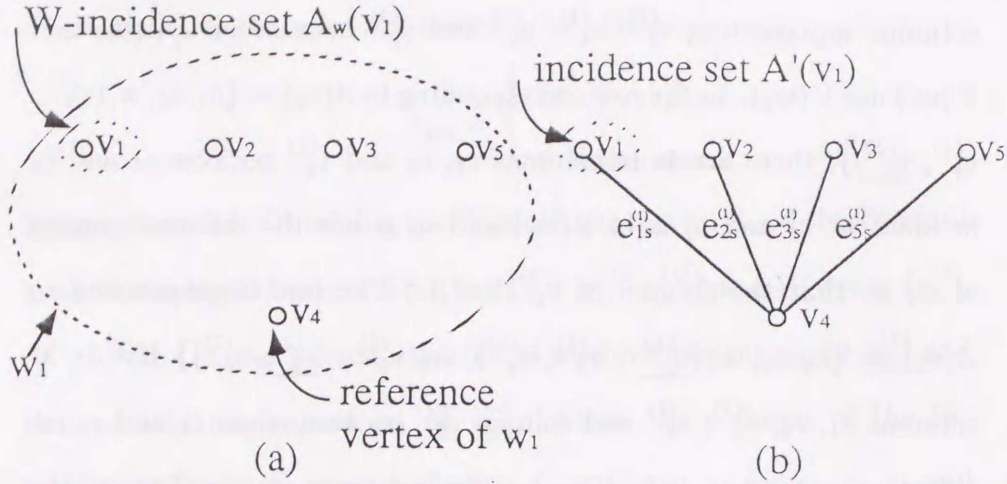


Figure 3.5: (a)A W-graph Ω_w (b) a graph g_s.

in Fig.3.5(b). In the graph g_s, the incidence set of v_1 is $A(v_1) = \{e_{1s}^{(1)}\}$ because only edge $e_{1s}^{(1)}$ incidents at vertex v_1. In the incidence matrix A_s of the graph g_s, let v_4 is the reference vertex of g_s, the row of $A(v_1)$ has 1 at the columns of $e_{1s}^{(1)}$ and has 0s in columns of $e_{2s}^{(1)}$, $e_{3s}^{(1)}$ and $e_{5s}^{(1)}$. This means that when we make correspondence between column $v_j^{(i)}$ of A_w and column $e_{js}^{(i)}$ of A_s where $e_{js}^{(i)}$ is an edge connecting between $v_j^{(i)}$ and $\underline{v_r^{(i)}}$ in the star, a W-incidence matrix A_w of a W-graph Ω_w is identical to an incidence matrix A_s of a graph g_s. In general, we have the following corollary.

Corollary 3.3.1 *A W-incidence matrix A_w of a W-graph Ω_w is identical to an incidence matrix A_s of a graph G_s obtained by assigning each wild component w_i in Ω_w by a star whose center is the reference* *vertex of w_i, if the reference vertices of Ω_w and G_s are the same one.*

Let's use an example to illustrate Corollary 3.3.1. For the W-graph Ω_w shown in Fig.3.4, when w_1 and w_2 are specified by two star-structures whose center vertices are $\underline{v_5^{(1)}}$ and $\underline{v_4^{(2)}}$ as shown in Fig.3.6(a), Ω_w becomes an ordinary graph G_s as shown in Fig.3.6(b). An incidence matrix A_s of G_s can be obtained as follows:

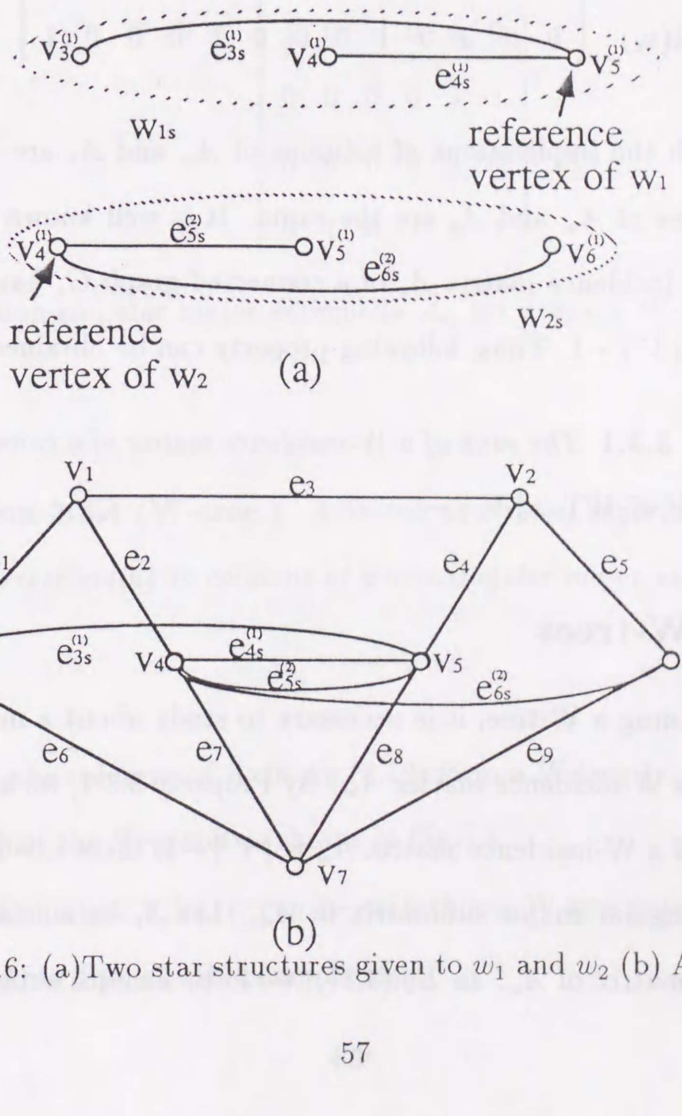


Figure 3.6: (a)Two star structures given to w_1 and w_2 (b) A graph G_s

$$A_s = \begin{array}{c} \\ A(v_1) \\ A(v_2) \\ A(v_3) \\ A(v_4) \\ A(v_5) \\ A(v_6) \end{array} \begin{array}{c} \begin{array}{ccccccccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 & e_{3s}^{(1)} & e_{4s}^{(1)} & e_{5s}^{(2)} & e_{6s}^{(2)} \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \quad (3.18)$$

Although the implications of columns of A_w and A_s are different, the matrices of A_w and A_s are the same. It is well known that the rank of an incidence matrix A_s of a connected graph G_s having $\mid V \mid$ vertices is $\mid V \mid -1$. Thus, following property can be obtained:

Property 3.3.1 *The rank of a W-incidence matrix of a connected W-graph $\Omega_w(V, E, W)$ is $\mid V \mid -1$.*

3.3.2 W-trees

Before defining a W-tree, it is necessary to study about a major submatrix of a W-incidence matrix A_w. By Property 3.3.1, we know that the rank of a W-incidence matrix A_w is $\mid V \mid -1$, there exists at least one non-singular major submatrix in A_w. Let A_t be a non-singular major submatrix of A_w. In *Eq.*(3.17), we form a major submatrix by taking $\mid V \mid -1$ columns of A_w such as columns e_1, e_5, e_6, e_9, $v_4^{(1)}$ and $v_6^{(2)}$. By Corollary 3.3.1, we know that the major submatrix is non-singular.

$$A_t = \begin{array}{c} \begin{array}{cccccc} e_1 & e_5 & e_6 & e_9 & v_4^{(1)} & v_6^{(2)} \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array} \quad (3.19)$$

For a non-singular major submatrix A_t, we define a W-tree as following:

Definition 3.3.4 (W-tree) *A W-tree is a set of edges and vertices of $v_j^{(i)}$ corresponding to columns of a non-singular major submatrix A_t of A_w.*

Hence, the columns of A_t in $Eq.$(3.19) form a W-tree $\{e_1$, e_5, e_6, e_9, $v_4^{(1)}$, $v_6^{(2)}\}$ of the W-graph as shown in Fig.3.4.

From Definition 3.3.4, it can be seen that a W-tree may consist of edges and vertices of wild-components.

3.3.3 W-cutset Matrix

By Definition 3.2.1, we know that a W-cutset can be represented by

$$S_w = \{e_{s1}, e_{s2}, \cdots, e_{sm}, w_1(V_{a1} : \overline{V_{a1}}), w_2(V_{a2} : \overline{V_{a2}}), \cdots,$$
$$w_{|W|}(V_{a|W|} : \overline{V_{a|W|}})\}.$$

Consider a W-cutset S_1 as shown in *Eq.* 3.4, here we add superscripts (i) to vertices which belong to w_i, so that S_1 becomes to

$$S_1 = \{e_3, e_4, e_6, e_7, e_9, w_1(v_2^{(1)}, v_5^{(1)}, v_6^{(1)} : v_7^{(1)}),$$
$$w_2(v_2^{(2)} : v_3^{(2)}, v_7^{(2)}, v_8^{(2)}, v_9^{(2)})\},$$

which separates vertices of the W-graph into two parts $\{v_3, v_4, v_5\}$ and $\{v_1, v_2, v_6, v_7\}$.

Now we describe a W-cutset Matrix.

For a W-graph $\Omega_w(V, E, W)$ where $\mid W \mid = k$, we define an exhaustive W-cutset matrix whose rows represent W-cutsets and columns correspond to all edges in E and all vertices in $V(w_i) - \{\underline{v_r^{(i)}}\}$ $(i = 1, 2, \cdots, k)$.

Definition 3.3.5 (Exhaustive W-cutset matrix) *An exhaustive W-cutset matrix* $Q_{ew} = [q_{pq}]$ *of a W-graph having k wild components is defined as:*

Case 1: *When column q indicates an edge* e_q,

$$\mathbf{q_{pq}} = \begin{cases} \mathbf{1} & \text{W-cutset } s_p \text{ contains the edge } e_q, \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

Case 2: *When column q corresponds to a vertex of the form* $v_q^{(i)}$,

$$\mathbf{q_{pq}} = \begin{cases} \mathbf{1} & \text{W-cutset } s_p \text{ contains } w_i(V_{ai} : \overline{V_{ai}}) \text{ and} \\ & \text{either } v_q^{(i)} \in V_{ai} \text{ and } \underline{v_r^{(i)}} \in \overline{V_{ai}} \\ & \text{or } v_q^{(i)} \in \overline{V_{ai}} \text{ and } \underline{v_r^{(i)}} \in V_{ai}, \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

By Definition 3.2.1 and Definition 3.3.2, it can be seen that a W-incidence set is a particular W-cutset such that the W-incidence matrix A_w is a submatrix of Q_{ew}. The rank of Q_{ew} is therefore equal to the rank of A_w which is $|V|-1$ by Property 3.3.1.

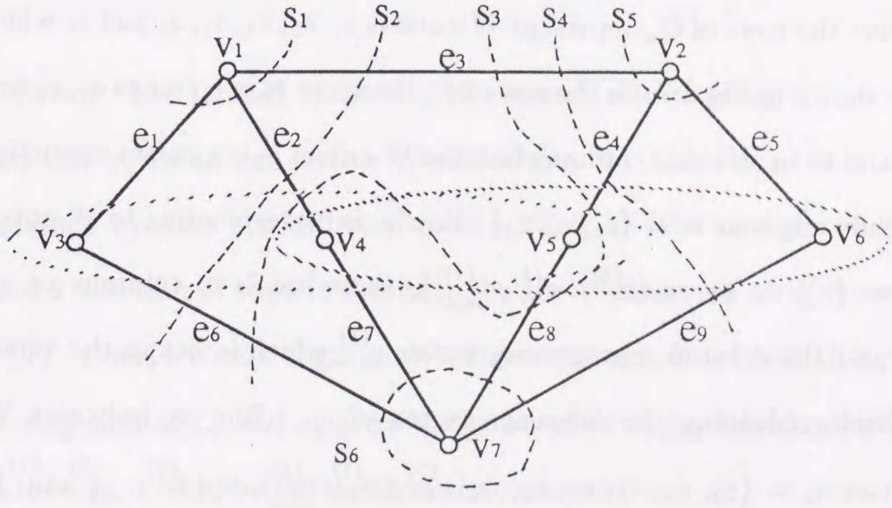


Figure 3.7: W-cutsets of a W-graph.

Example 3.3.2 Obtaining a submatrix Q_w of an exhaustive W-cutset matrix Q_{ew} in the W-graph as shown in Fig.3.4 where $V(w_1) = \{v_3^{(1)},$

$v_4^{(1)}$, $v_5^{(1)}\}$ and $V(w_2) = \{v_4^{(2)}, v_5^{(2)}, v_6^{(2)}\}$. Let $\underline{v_r^{(1)}} = \underline{v_5^{(1)}}$ and $\underline{v_r^{(2)}} = \underline{v_4^{(2)}}$, then we have Q_w as follow:

$$Q_w = \begin{array}{c} \\ s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \\ s_6 \end{array} \begin{array}{c} \begin{array}{ccccccccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 & v_3^{(1)} & v_4^{(1)} & v_5^{(2)} & v_6^{(2)} \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \tag{3.20}$$

where the rows of Q_w represent W-cutsets s_1, s_2, s_3, s_4, s_5 and s_6 which are shown in Fig.3.7. In the row of s_1, there are 1s in columns e_1, e_2 and e_3 and 0s in all other columns because W-cutset s_1 is a cutset consisting of only edges as $s_1 = \{e_1, e_2, e_3\}$. Row s_2 is corresponding to W-cutset $s_2 = \{e_2, e_3, e_6, w_1(v_3^{(1)} : v_4^{(1)}, \underline{v_5^{(1)}})\}$, there are 1s in columns e_2, e_3, e_6 and the column representing vertex $v_3^{(1)}$ which is not in the vertex subset containing the reference vertex of w_1. Row s_3 indicates W-cutset $s_3 = \{e_2, e_3, e_4, e_6, e_8, w_1(v_4^{(1)} : v_3^{(1)}, \underline{v_5^{(1)}}), w_2(v_5^{(2)} : \underline{v_4^{(2)}}, v_6^{(2)})\}$, there are 1s in columns e_2, e_3, e_4, e_6, e_8 and columns $v_4^{(1)}$ and $v_5^{(2)}$. The reason is that $v_4^{(1)}$ is in the vertex subset of w_1 which does not contain the reference vertex $\underline{v_5^{(1)}}$. Also $v_5^{(2)}$ is in the vertex subset of w_2 not containing $\underline{v_4^{(2)}}$. Row s_4 corresponds to W-cutset $s_4 = \{e_4, e_9,$

$w_2(v_6^{(2)} : v_5^{(2)}, \underline{v_4^{(2)}})\}$ where there are 1s in columns e_4, e_9 and column corresponding to vertex $v_6^{(2)}$. Rows s_5 and s_6 indicate W-cutsets s_5 and s_6 where $s_5 = \{e_3, e_4, e_5\}$ and $s_6 = \{e_6, e_7, e_8, e_9\}$ which contain only edges. $\square$

Consider the relations of two rows of an exhaustive W-cutset matrix under mod 2 operation.

Theorem 3.3.1 *Adding (mod 2) two different rows in an exhaustive W-cutset matrix produces a row indicating a W-cutset.*

Proof: Let s_m and s_n be two W-cutsets corresponding to row s_m and row s_n in an exhaustive W-cutset matrix. Also let s_r be a row obtained by mod 2 addition of rows s_m and s_n. We need prove that mod 2 addition of rows is equivalent to the W-ring sum of W-cutsets.

(1). Consider only edges in W-cutsets s_m, s_n. It is clear that the W-ring sum of s_m and s_n gives the same result as the mod 2 addition of rows s_m and s_n as far as edges are concerned.

(2). Consider the case when w_i is either in s_m or in s_n but not in both s_m and s_n. By Definition 3.2.2, $w_i(V_{ai} : \overline{V_{ai}})$ will be in $s_m \tilde{\oplus} s_n$. Let $w_i(V_{ai} : \overline{V_{ai}})$ be in s_m, also let the reference vertex of w_i be in $\overline{V_{ai}}$. Then by Definition 3.3.5, the columns of an exhaustive W-cutset matrix corresponding to every vertex in V_{ai} have 1s in row s_m but have 0s in row s_n. On the other hand, all other columns corresponding to a vertex in $\overline{V_{ai}}$ except the reference vertex will have 0s in both row s_m and row s_n. Notice that there is no column corresponding to the reference vertex of w_i. Hence, the mod 2 addition of the rows s_m and s_n will have 1 only at columns corresponding to vertices in V_{ai} which will be the same when row s_r is employed for indicating $s_m \tilde{\oplus} s_n$ for w_i being in either s_m or s_n but not in both s_m and s_n.

The same result can be obtained when the reference vertex is in V_{ai}. Also when $w_i(V_{ai} : \overline{V_{ai}})$ is in s_n rather than s_m gives the same result. Thus, in the columns corresponding to w_i, the row s_r obtained by mod 2 addition of rows s_m and s_n is identical with $s_m \tilde{\oplus} s_n$.

(3). Consider the case when w_i is in both s_m and s_n. Suppose $w_i(V_{ai} : \overline{V_{ai}})$ is in s_m and $w_i(V_{bi} : \overline{V_{bi}})$ is in s_n.

Case 1: The reference vertex of w_i is in either V_{ai} or V_{bi}. If the reference vertex of w_i is in V_{ai}, by Definition 3.3.5, in the row s_m there are 1s in the columns corresponding to the vertices in $\overline{V_{ai}}$. In the row s_n, there are 1s in the columns corresponding to the vertices in V_{bi}. When we add (mod 2) rows s_m and s_n, in the resultant row s_r there are 1s in the columns which are corresponding to the vertices either in $\overline{V_{ai}}$ or in V_{bi} but not in both $\overline{V_{ai}}$ and V_{bi}. This means that the resultant row s_r has 1 at the columns corresponding to vertices in $\overline{V_{ai}} \oplus V_{bi}$ and has 0 at the columns corresponding to vertices in $V(w_i) - (\overline{V_{ai}} \oplus V_{bi})$. Thus, the columns for w_i in row s_r indicate $w_i(\overline{V_{ai}} \oplus V_{bi} : V(w_i) - (\overline{V_{ai}} \oplus V_{bi}))$ $= w_i(V_{ai} \oplus V_{bi} : V(w_i) - (V_{ai} \oplus V_{bi}))$ which is the form of w_i in $s_m \tilde{\oplus} s_n$. We can obtain the same result when the reference vertex of w_i is in V_{bi} but not in V_{ai}.

Case 2: Both of V_{ai} and V_{bi} contain the reference vertex of w_i. By Definition 3.3.5, the row s_m has 1s in the columns corresponding to the vertices in $\overline{V_{ai}}$. Also row s_n has 1s in the columns corresponding to the vertices in $\overline{V_{bi}}$. When we add (mod 2) rows s_m and s_n, the resultant row s_r has 1s in the columns which are corresponding to the vertices either in $\overline{V_{ai}}$ or in $\overline{V_{bi}}$ but not in both $\overline{V_{ai}}$ and $\overline{V_{bi}}$. This means the resultant s_r has 1 at the columns corresponding to vertices in $\overline{V_{ai}} \oplus \overline{V_{bi}}$ and has 0 at the columns corresponding to vertices in $V(w_i) - (\overline{V_{bi}} \oplus \overline{V_{bi}})$. Hence, columns corresponding to w_i in the resultant s_r indicate $w_i(\overline{V_{ai}} \oplus \overline{V_{bi}} : V(w_i) - (\overline{V_{ai}} \oplus \overline{V_{bi}})) = w_i(V(w_i) - (V_{ai} \oplus V_{bi}) : V_{ai} \oplus V_{bi})$ which is the form of w_i in $s_m \tilde{\oplus} s_n$.

Case 3: Neither V_{ai} nor V_{bi} contain the reference vertex of w_i. By Definition 3.3.5, the row s_m has 1s in the columns corresponding to the vertices in V_{ai}. Also row s_n has 1s in the columns corresponding to the vertices in V_{bi}. When we add (mos 2) rows of s_m and s_n, the resultant row s_r has 1s in the columns which are corresponding to the vertices either in V_{ai} or in V_{bi} but not in both V_{ai} and V_{bi}. This means that the resultant row s_r has 1 at columns corresponding to vertices in $V_{ai} \oplus V_{bi}$ and has 0 at the columns corresponding to vertices in $V(w_i) - (V_{ai} \oplus V_{bi})$. Hence, the columns corresponding to w_i in the resultant s_r indicate $w_i(V_{ai} \oplus V_{bi} : V(w_i) - (V_{ai} \oplus V_{bi}))$ which is the form w_i in $s_m \tilde{\oplus} s_n$.

(4). If w_i is neither in s_m nor s_n, the resultant row s_r obtained by mod 2 addition of rows s_m and s_n contains no 1 in the columns corresponding to vertices of w_i. This means that columns corresponding to w_i in resultant s_r are all 0 which indicates that w_i is not in $s_m \tilde{\oplus} s_n$.

These conclude that mod 2 addition of two rows in an exhaustive W-cutset matrix is equivalent to operating W-ring sum of two W-cutsets corresponding to the two rows. By Theorem 3.2.1, this theorem is true. ■

According to an ordinary graph, we define a **fundamental W-cutset matrix** which is a submatrix of exhaustive W-cutset matrix having the form as:

$$Q_{wf} = [Q_{f11} \mid U] \tag{3.21}$$

where U is an unit matrix and the columns of U are corresponding to a chosen W-tree. A set of W-cutsets corresponding to rows of a fundamental W-cutset matrix is called a **fundamental W-cutsets.**

For a W-graph, when a W-tree is chosen, we provide the following method to obtain a fundamental matrix from the W-graph directly.

By Corollary 3.3.1, from a W-graph we obtain a graph G_s by assigning each wild component w_i in the W-graph by a star whose center is the reference vertex of w_i, then find a tree corresponding to the W-tree where each vertex of $v_j^{(i)}$ is changed by an edge $e_{js}^{(i)}$ connected between v_j and the reference vertex of w_i. We can obtained a fundamental cutset matrix of the graph G_s. Then, we change each edge of the form $e_{js}^{(i)}$ on column of the fundamental cutset matrix by vertex of the form $v_j^{(i)}$, the result becomes a fundamental W-cutset matrix of the W-graph.

Example 3.3.3 Finding a fundamental W-cutset matrix of the W-graph in Fig.3.4 under the W-tree $\{e_1,\ e_5,\ e_6,\ e_9,\ v_4^{(1)},\ v_6^{(2)}\}$. First we assign stars to every wild-components w_1 and w_1 to make a graph as shown in Fig.3.8(a) where $\{e_1, e_5, e_6, e_9, e_{4s}^{(1)}, e_{6s}^{(2)}\}$ is a tree. Then obtain a set of fundamental cutsets of the graph as follows:

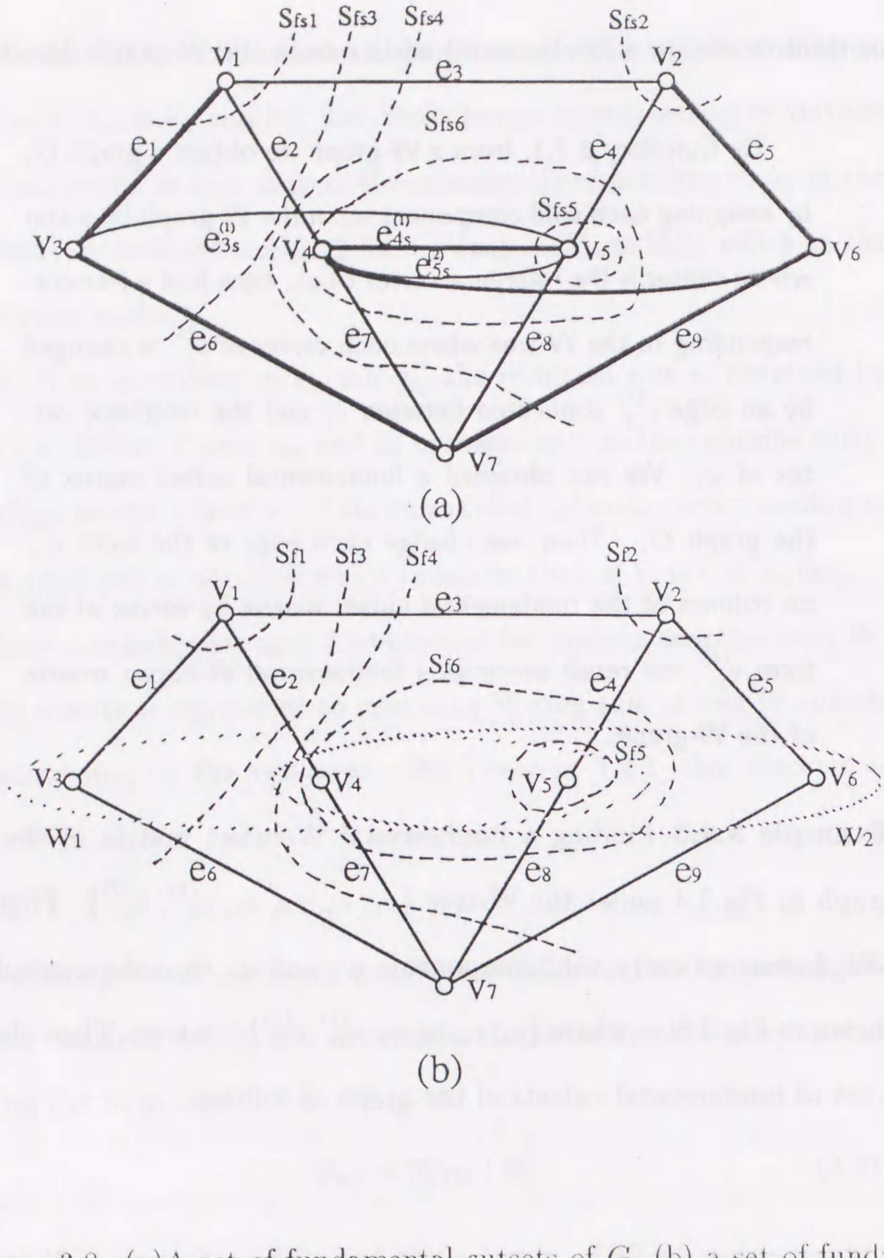


Figure 3.8: (a)A set of fundamental cutsets of G_s (b) a set of fundamental W-cutsets of a W-graph.

$$Q_f = \begin{array}{c} \\ s_{fs1} \\ s_{fs2} \\ s_{fs3} \\ s_{fs4} \\ s_{fs5} \\ s_{fs6} \end{array} \begin{array}{c} \begin{array}{ccccccccccccc} e_2 & e_3 & e_4 & e_7 & e_8 & e_{3s}^{(1)} & e_{5s}^{(2)} & e_1 & e_5 & e_6 & e_9 & e_{4s}^{(1)} & e_{6s}^{(2)} \end{array} \\ \left[\begin{array}{ccccccc|cccccc} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \end{array} \quad (3.22)$$

When we change $e_{3s}^{(1)}$, $e_{4s}^{(1)}$, $e_{5s}^{(2)}$ and $e_{6s}^{(2)}$ by vertices $v_3^{(1)}$, $v_4^{(1)}$,$v_5^{(2)}$ and $v_6^{(2)}$, respectively, a fundamental W-cutset matrix can be obtain. A set of fundamental W-cutsets are shown in Fig. 3.8(b).

$$Q_f = \begin{array}{c} \\ s_{f1} \\ s_{f2} \\ s_{f3} \\ s_{f4} \\ s_{f5} \\ s_{f6} \end{array} \begin{array}{c} \begin{array}{ccccccccccccc} e_2 & e_3 & e_4 & e_7 & e_8 & v_3^{(1)} & v_5^{(2)} & e_1 & e_5 & e_6 & e_9 & v_4^{(1)} & v_6^{(2)} \end{array} \\ \left[\begin{array}{ccccccc|cccccc} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \end{array} \quad (3.23)$$

□

Since every fundamental W-cutset has one element which is not in the others, fundamental W-cutset matrix provides $\mid V \mid -1$ linearly independent W-cutsets in a W-graph. Also the rank of an exhaustive W-cutset matrix is $| V | -1$, so we have the following property.

Property 3.3.2 *Any row of an exhaustive W-cutset matrix can be obtained by adding (mod 2) some rows of a fundamental W-cutsets matrix.*

3.3.4 W-circuit Matrix

By Definition 3.1.1, we can obtain a W-circuit of the W-graph in Fig.3.4 as follows:

$$C_w = \{e_1, e_2, e_6, e_9, w_1(v_4^{(1)}, v_5^{(1)}/v_3^{(1)}), w_2(v_5^{(2)}, v_6^{(2)}/v_4^{(2)})\}.$$

For a W-graph $\Omega_w(V, E, W)$, we use a matrix whose rows represent W-circuits indicated by $Eq.(3.1)$ and whose columns represent all edges in E and all vertices of each wild component except $\underline{v_r^{(i)}}$ $(i = 1, 2, \cdots, | W |)$ to show all W-circuits in a W-graph.

Definition 3.3.6 (Exhaustive W-circuit matrix) *An exhaustive W-circuit matrix $B_{ew} = [b_{pq}]$ of a W-graph is defined as follows:*

1. When column q indicates an edge e_q,

$$\mathbf{b_{pq}} = \begin{cases} \mathbf{1} & \text{W-circuit } c_p \text{ contains the edge } e_q, \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

2. When column q corresponds to a vertex of form $v_q^{(i)}$,

$$\mathbf{b_{pq}} = \begin{cases} \mathbf{1} & \text{W-circuit } c_p \text{ contains } w_i(V_{oi}/V_{ei}),\ v_q^{(i)} \in V_{oi}, \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

For a given W-tree, consider a submatrix of B_{ew} whose rows represent fundamental W-circuits each of which has only one element that is not in the W-tree. The submatrix can become a form of

$$B_{wf} = [U \mid B_{f12}] \tag{3.24}$$

where U is an unit matrix and the columns of B_{f12} are corresponding to the given W-tree. B_{wf} is called a **fundamental W-circuit matrix**.

Since a fundamental W-circuit matrix is a submatrix of an exhaustive W-circuit matrix B_{ew}, the rank of an exhaustive W-circuit matrix is equal to the rank of the fundamental W-circuit matrix which is

$$r = \mid E \mid + \sum_{i=1}^{|W|} \mid V(w_i) \mid - \mid W \mid - \mid V \mid + 1.$$

We can obtain a set of fundamental W-circuits by the same method as a fundamental W-cutset matrix.

Example 3.3.4 Under the chosen W-tree $\{e_1, e_5, e_6, e_9, v_4^{(1)}, v_6^{(2)}\}$, we can obtain a set of fundamental W-circuits of the W-graph as shown in Fig.3.4 by using the graph in Fig.3.8(a).

We use a symbol $c_x(e_y)$ to indicate a fundamental W-circuit c_x containing edge e_y which is not in the W-tree. Also, we employ symbol $c_x(v_y^{(i)})$ for indicating a fundamental W-circuit c_x which contains $w_i(V_{oi}/V_{ei})$ where $v_y^{(i)} \in V_{oi}$, $v_y^{(i)} \neq \underline{v_r^{(i)}}$ and $v_y^{(i)}$ is not in the W-tree.

$$
\begin{aligned}
c_1(e_2) &= \{e_1, e_2, e_6, e_9, w_2(\underline{v_4^{(2)}}, v_6^{(2)}/v_5^{(2)})\} \\
c_2(e_3) &= \{e_1, e_3, e_5, e_6, e_9\} \\
c_3(e_4) &= \{e_4, e_5, w_1(v_4^{(1)}, \underline{v_5^{(1)}}/v_3^{(1)}), w_2(\underline{v_4^{(2)}}, v_6^{(2)}/v_5^{(2)})\} \\
c_4(e_7) &= \{e_7, e_9, w_2(\underline{v_4^{(2)}}, v_6^{(2)}/v_5^{(2)})\} \\
c_5(e_8) &= \{e_8, e_9, w_1(v_4^{(1)}, \underline{v_5^{(1)}}/v_3^{(1)}), w_2(\underline{v_4^{(2)}}, v_6^{(2)}/v_5^{(2)})\} \\
c_6(v_3^{(1)}) &= \{e_6, e_9, w_1(v_3^{(1)}, v_4^{(1)}/\underline{v_5^{(1)}}), w_2(\underline{v_4^{(2)}}, v_6^{(2)}/v_5^{(2)})\} \\
c_7(v_5^{(2)}) &= \{w_1(v_4^{(1)}, \underline{v_5^{(1)}}/v_3^{(1)}), w_2(\underline{v_4^{(2)}}, v_5^{(2)}/v_6^{(2)})\}
\end{aligned}
$$

When each row represents one of above fundamental W-circuits, a fundamental W-circuit matrix can be obtained as form of $Eq.(3.24)$ as:

$$B_{wf} = \begin{array}{cc} & \begin{array}{ccccccccccccc} e_2 & e_3 & e_4 & e_7 & e_8 & v_3^{(1)} & v_5^{(2)} & e_1 & e_5 & e_6 & e_9 & v_4^{(1)} & v_6^{(2)} \end{array} \\ \begin{array}{c} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \end{array} & \left[\begin{array}{ccccccc|cccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array} \qquad (3.25)$$

□

The following theorem shows that we can obtain any W-circuit by mod 2 addition of some rows of a fundamental W-circuit matrix.

Theorem 3.3.2 *Mod 2 addition of two different rows of a W-circuit matrix produces a row representing a W-circuit.*

Proof: Let c_m and c_n be two W-circuits which are

$$c_m = \{e_{m1}, e_{m2}, \cdots, e_{mp}, w_1(V_{mo1}/\overline{V_{mo1}}), \cdots, w_k(V_{mok}/\overline{V_{mok}})\}$$

and

$$c_n = \{e_{n1}, e_{n2}, \cdots, e_{nq}, w_1(V_{no1}/\overline{V_{no1}}), \cdots, w_k(V_{nok}/\overline{V_{nok}})\}.$$

(1). When we add (mod 2) rows c_m and c_n, it is evident that the resultant row c_r contains all edges which are in $\{e_{m1}, e_{m2}, \cdots, e_{mp}\} \oplus$

$\{e_{n1}, e_{n2}, \cdots, e_{nq}\}$. Hence, when only edges are concerned, mod 2 addition of two rows of an exhaustive W-circuit matrix is equivalent to the W-ring sum of two W-circuits represented by these two rows.

(2). Consider the case when w_i is in either c_m or c_n but not in both c_m and c_n. If c_n contains $w_i(V_{noi}/\overline{V_{noi}})$, row c_m has no 1's in the columns corresponding to the vertices of V_{oi}. Hence, by adding (mod 2) rows c_m and c_n, the columns corresponding to vertices of w_i in the resultant row c_r are the same as those of c_n. This means that the resultant c_r contains $w_i(V_{noi}/\overline{V_{noi}})$. If c_m contains $w_i(V_{moi}/\overline{V_{moi}})$, we can show that the resultant c_r contains $w_i(V_{moi}/\overline{V_{moi}})$. Hence, the columns corresponding to vertices of w_i in the resultant row c_r is identical with those indicated in $c_m \hat{\oplus} c_n$.

(3). Suppose $w_i(V_{moi}/\overline{V_{moi}})$ is in c_m and $w_i(V_{noi}/\overline{V_{noi}})$ is in c_n. Then row c_m has 1's in the columns corresponding to the vertices in V_{moi}. Also row c_n has 1's in the columns corresponding to the vertices in V_{noi}. When we add (mod 2) rows c_m and c_n, the resultant row c_r has 1's in the columns corresponding to the vertices in either V_{moi} or V_{noi} but not in both V_{moi} and V_{noi}. This means that the resultant row c_r has 1 at the columns corresponding to vertices in $V_{moi} \oplus V_{noi}$ and has 0 at the columns corresponding to vertices in $V(w_i) - (V_{moi} \oplus V_{noi})$. By Definition 3.3.6, the resultant c_r represent $w_i(V_{moi} \oplus V_{noi}/V(w_i) - (V_{moi} \oplus V_{noi}))$ which is in $c_m \hat{\oplus} c_n$.

(4). Suppose w_i is neither in c_m nor in c_n. Then adding (mod 2) of rows c_m and c_n will not produce 1's in columns corresponding to vertices of w_i. Since $c_m \hat{\oplus} c_n$ will not contain w_i, columns for w_i in a row representing $c_m \hat{\oplus} c_n$ will be the same as the result obtained by adding (mod 2) rows c_m and c_n.

Hence, the above results and Theorem 3.1.1 lead to the conclusion that this theorem is true. ■

Chapter 4

A W-graph and Its Derived Graphs

For a W-graph $\Omega_w(V, E, W)$, we know the structure of each wild-component is unspecified. Some properties and some theorems of a W-graph have been obtained in Chapter 3 without considering the structure of each wild-component as long as we know that there exists exactly one inner path between any two vertices in each wild-component. In fact, when the structure of each wild-component in a W-graph is given, the W-graph becomes an ordinary graph, called a derived graph. When we use different structure to specify each wild-component, the W-graph produces a family of derived graphs. In other words, a W-graph corresponds to many derived graphs. In this chapter, we will consider the relation between a W-graph and its derived graphs. We will show that W-circuits, W-cutsets and W-tree of a W-graph can become circuits or edge disjoint union of circuits, cutsets or edge disjoint union of cutsets and trees of a derived graph, respectively. Furthermore, we can obtain linearly independent circuits or edge disjoint union of circuits of any derived graph from linearly independent W-circuits and also a set of linearly independent cutsets or edge disjoint union of cutsets can be obtained from a set of linearly independent W-cutsets. These results are theoretically very important.

4.1 Derived Graphs of a W-graph

In a W-graph, the structure of each wild-component is unspecified, and also we need not specify it when studying some properties of the W-graph. However, in applications of W-graphs ([Zhao4 91]) we need to choose a proper tree for the structure of a wild-component under given requirements. In this case, it is important to know the follows:

Let $T(w_i)$ be a set of all tree with respect to all vertices of $V(w_i)$. When the structure of wild-component w_i is specified by a tree which is one of $T(w_i)$, the specified w_i is described by following definition.

Definition 4.1.1 (Specified tree $t_a^{(i)}$) *When the structure of wild-component w_i in a W-graph $\Omega_w(V, E, W)$ is given by a tree $t_a^{(i)}$*

$\in T(w_i)$, *the wild-component* w_i *becomes the chosen tree* $t_a^{(i)}$. *Each edge in* $t_a^{(i)}$ *is denoted by* $e_{ja}^{(i)}$ *where* $j = 1, 2, \cdots, \mid V(w_i) \mid -1$.

When the structure of each wild-component w_i ($i = 1, 2, \cdots, \mid W \mid$) is changed by $t_a^{(i)}$, respectively, the W-graph in $Eq.(2.1)$ becomes an ordinary graph G_d as follows.

Definition 4.1.2 (Derived graph) *A graph obtained from a W-graph* $\Omega_w(V, E, W)$ *by changing each wild-component* w_i *by a tree* $t_a^{(i)}$ *is called a derived graph* G_d *represented by*

$$G_d(V, E') = G(V, E) \cup t_a^{(1)} \cup t_a^{(2)} \cup \cdots \cup t_a^{(|W|)} \qquad (4.1)$$

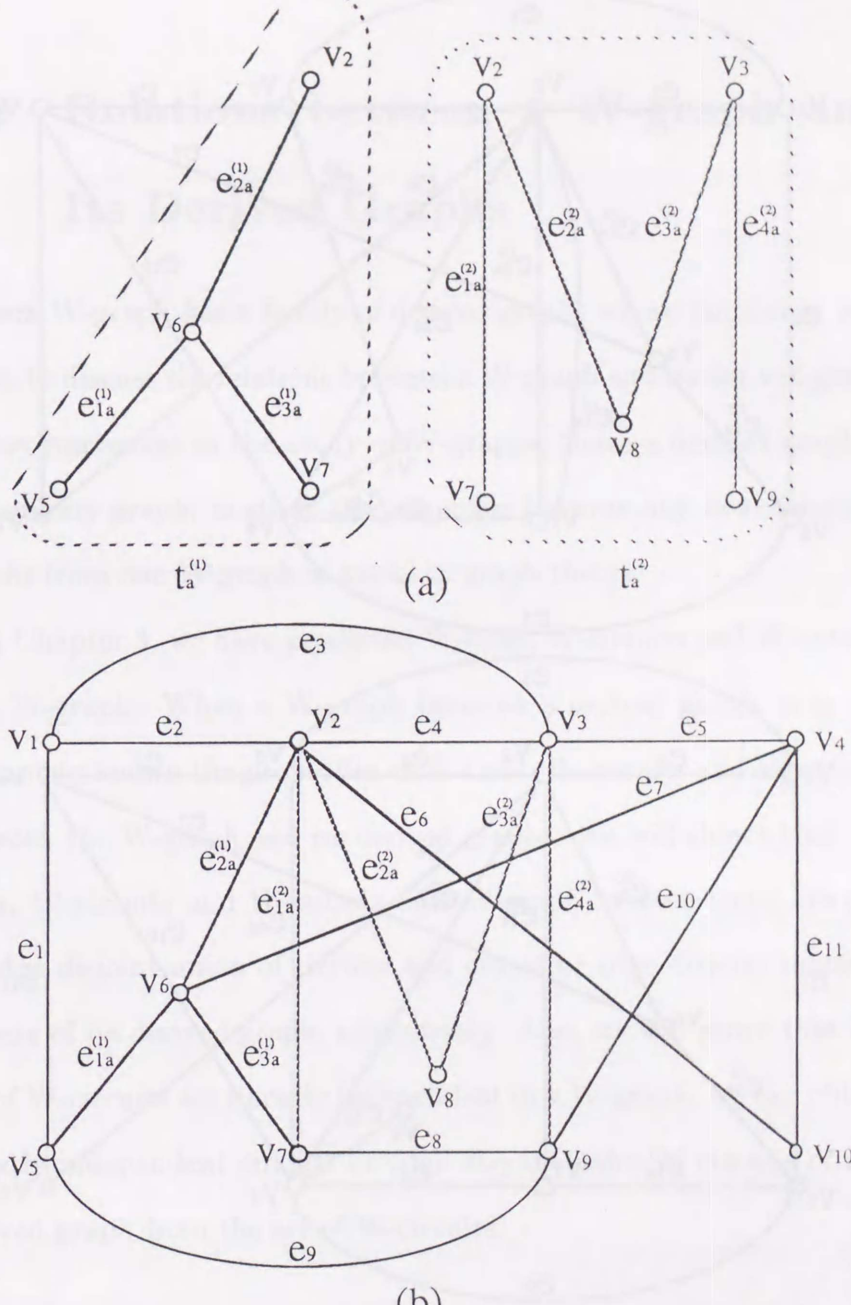
where $\mid E' \mid$ *is equal to* $\mid E \mid + \sum_{i=1}^{|W|} \mid V(w_i) \mid - \mid W \mid$.

When we choose different tree in $T(w_i)$ as $t_a^{(i)}$, the W-graph becomes different derived graph. In other words, a W-graph Ω_w can produce a family of derived graphs when we specify each wild-component in Ω_w by different trees. The number of different derived graphs from a W-graph $\Omega_w(V, E, W)$ is

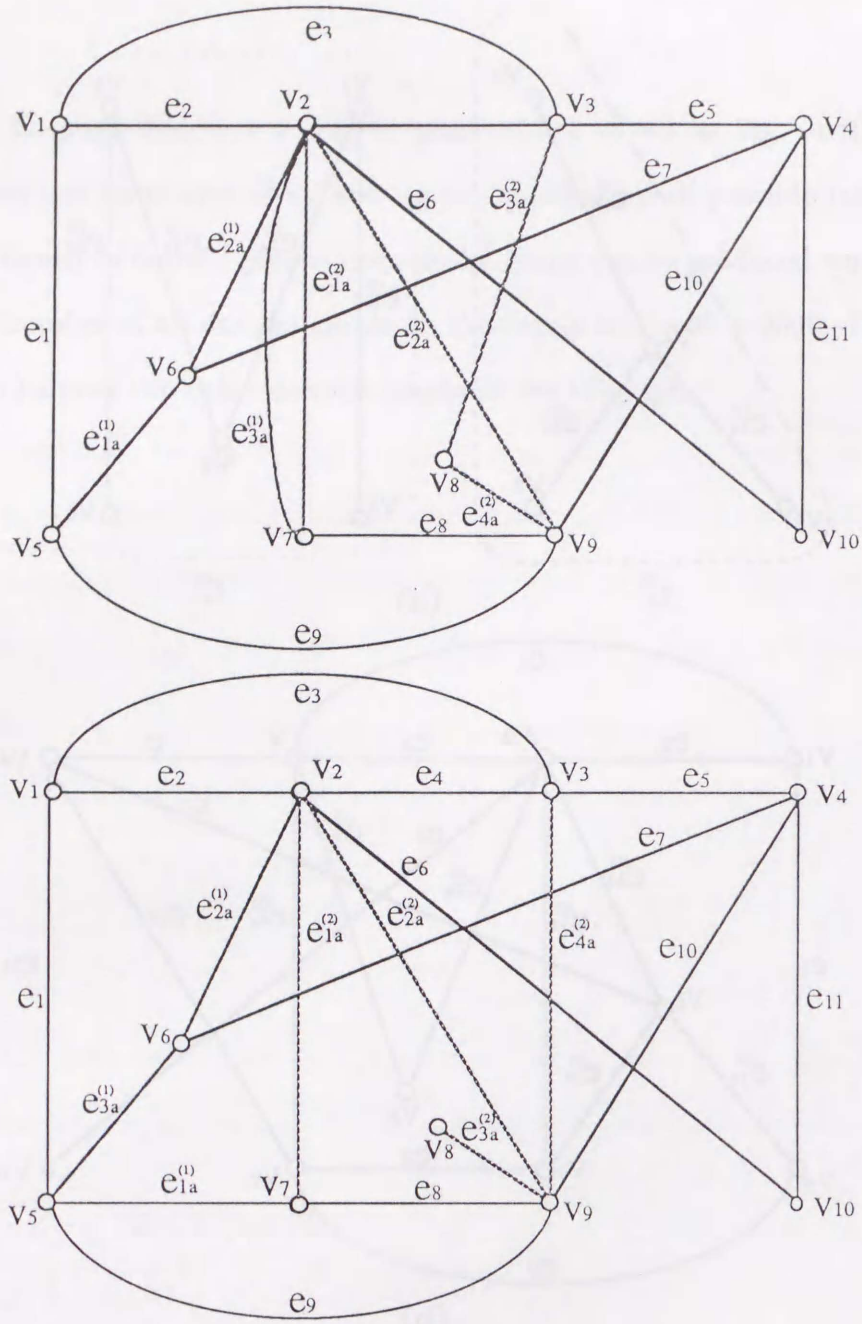
$$\prod_{i=1}^{|W|} \mid V(w_i) \mid^{|V(w_i)|-2}$$

because a wild-component w_i has $\mid V(w_i) \mid^{|V(w_i)|-2}$ trees.

Example 4.1.1 For the W-graph as shown in Fig.2.1, when the structures of w_1 and w_2 are specified by $t_a^{(1)}$ and $t_a^{(2)}$ as shown in Fig. 4.1(a), the W-graph becomes a derived graph G_d as shown as Fig. 4.1(b). When the structures of w_1 and w_2 are specified by all possible trees, the family of derived graphs from the W-graph can be produced where the number of all derived graphs in the family is $4^2 * 5^3 = 2000$. Fig. 4.1(c) shows two other derived graphs of the W-graph. $\square$



v2
v6
v5
v7
e(1)2a
e(1)1a
e(1)3a
t(1)a
v2
v3
v7
v8
v9
e(2)1a
e(2)2a
e(2)3a
e(2)4a
t(2)a
(a)
e3
v1
e2
v2
e4
v3
e5
v4
e7
e6
e(2)3a
e(1)2a
e(2)2a
e(2)1a
e(2)4a
e10
e1
e11
v6
e(1)1a
e(1)3a
v8
v5
v7
e8
v9
v10
e9
(b)



(c)

Figure 4.1: (a) Two tree-structures $t_a^{(1)}$ and $t_a^{(2)}$ corresponding to w_1 and w_2, (b) a derived graph, (c) two other derived graphs

4.2 Relations between a W-graph and Its Derived Graphs

Since a W-graph has a family of derived graphs whose number is very large, to discuss the relations between a W-graph and its derived graph is very important in the study of W-graphs. Since a derived graph is an ordinary graph, to study the properties between any two of derived graphs from one W-graph is useful in graph theory.

In Chapter 3, we have presented W-trees, W-circuits and W-cutsets in a W-graph. When a W-graph becomes a derived graph, it is important to known the properties of W-trees, W-circuits and W-cutsets between the W-graph and its derived graphs. We will shown that W-trees, W-circuits and W-cutsets of a W-graph become trees, circuits or edge disjoint union of circuits and cutset or edge disjoint union of cutsets of its derived graph, respectively. Also, we will prove that if a set of W-circuits are linearly independent in a W-graph, we can obtain linearly independent circuits or edge disjoint union of circuits of any derived graph from the set of W-circuits.

4.2.1 Instantiation of a W-tree

In graph theory, the concept of a tree is very important because the number of linearly independent cutsets and circuits can related to a tree. Also, trees widely be used for analysis and synthesis of systems [Mayedal 72], [Chen 71], [Chan 69], [Mal. 83], [Breuer 77] and [Lauther 79].

A W-tree (Definition 3.3.4) is useful for a fundamental W-cutset matrix and a fundamental W-circuit matrix whose rows correspond to a set of linearly independent W-cutsets and W-circuits, respectively. Here, we will present an important and interesting property on W-trees, that is, when a W-graph becomes a derived graph G_d, the chosen W-tree can become a tree of the derived graph G_d.

Before giving the property of W-trees, we replenish Definition 4.1.1 as follows:

Let $t_a^{(i)}$ be any specified tree given to wild-component w_i and G_d be a derived obtained as *Eq.* (4.1) by changing each wild-component w_i by $t_a^{(i)}$ $(i = 1, 2, \cdots, \mid W \mid)$. Let $t_s^{(i)}$ be a star in $T(w_i)$ given to w_i and G_s be a derived graph formed by replacing every wild-components w_i by $t_s^{(i)}$. The symbol of $e_{js}^{(i)}$ is an edge in $t_s^{(i)}$ connecting between vertex $v_j^{(i)}$ and the center of the star.

By Corollary 3.3.1, when we replace each vertex $v_j^{(i)}$ in a W-tree by an edge $e_{js}^{(i)}$, the W-tree becomes a tree of G_s because a W-tree is defined by a non-singular major submatrix A_t (Definition 3.19) .

The following theorem shows that a W-tree of a W-graph Ω_w can become a tree of any derived graph G_d.

Theorem 4.2.1 *When a W-graph $\Omega_w(V,E,W)$ becomes a derived graph G_d where each wild-component w_i is changed by $t_a^{(i)}$ $(i = 1, 2, \cdots, |W|)$, there exists an edge in $t_a^{(i)}$ to replace each $v_j^{(i)}$ in a W-tree so that the W-tree becomes a tree of the derived graph G_d.*

Proof: Suppose τ_0 is a tree of G_s obtained from a W-tree by replacing each vertex of form $v_j^{(i)}$ in the W-tree by an edge $e_{js}^{(i)}$ in $t_s^{(i)}$. To prove that a W-tree can become a tree of G_d is equivalent to prove that τ_0 becomes a tree of G_d by replacing each $e_{js}^{(i)}$ in τ_0 to an appropriate edge of $t_a^{(i)}$ in G_d.

We will show the following process to replace each edge of form $e_{js}^{(i)}$ in τ_0 to an appropriate edge in $t_a^{(i)}$ one by one such that the resultant tree is a tree of G_d.

For any one edge $e_{js}^{(i)}$ in τ_0, we remove $e_{js}^{(i)}$ from τ_0, τ_0 becomes two subtrees $\tau_a(e_{js}^{(i)})$ and $\tau_b(e_{js}^{(i)})$ because τ_0 is a tree. It must be noticed that one endpoint of $e_{js}^{(i)}$ is in $\tau_a(e_{js}^{(i)})$ and other endpoint is in $\tau_b(e_{js}^{(i)})$. Hence, $\tau_a(e_{js}^{(i)})$ contains at least one vertex of w_i and $\tau_b(e_{js}^{(i)})$ also contains at least one vertex of w_i because the edge $e_{js}^{(i)}$ is in $t_s^{(i)}$ as shown in Fig.4.2(a). On the other hand, since there is exactly one path between any two vertices in $t_a^{(i)}$, there must exist an edge $e_{ja}^{(i)}$ in the path connecting between a vertex in $\tau_a(e_{js}^{(i)})$ and a vertex in $\tau_b(e_{js}^{(i)})$ as shown in Fig.4.2(b). We use the edge $e_{ja}^{(i)}$ to connect $\tau_a(e_{js}^{(i)})$ and $\tau_b(e_{js}^{(i)})$. The resultant tree $\tau_a(e_{js}^{(i)}) \cup \tau_b(e_{js}^{(i)}) \cup e_{ja}^{(i)}$ is clearly a tree, too. According to the same reason, we repeat above process until all edges of form $e_{js}^{(i)}$ in τ_0 are replaced by edges in $t_a^{(i)}$.

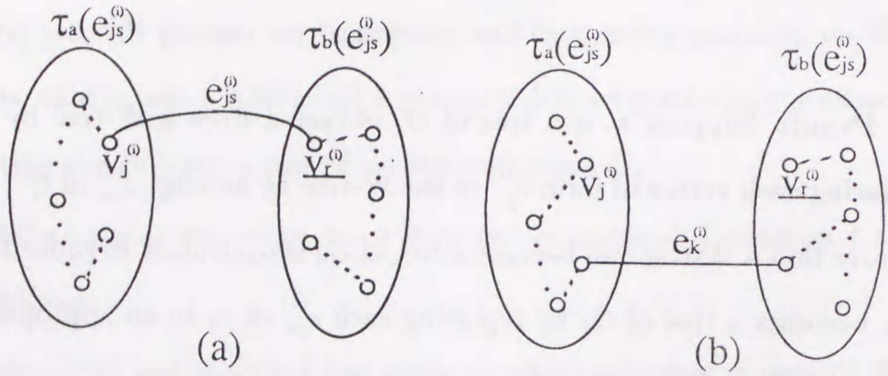


Figure 4.2: (a)$\tau_a(e_{js}^{(i)})$ and $\tau_b(e_{js}^{(i)})$ (b) there exists an edge $e_{ja}^{(i)}$ to connect $\tau_a(e_{js}^{(i)})$ and $\tau_b(e_{js}^{(i)})$.

Furthermore, if the edge $e_{ja}^{(i)}$ has been used for replacing edge $e_{js}^{(i)}$ in successive process, $e_{ja}^{(i)}$ can not be chosen more than once because used $e_{ja}^{(i)}$ is either in $\tau_a(e_{qs}^{(p)})$ or in $\tau_b(e_{qs}^{(p)})$ where $e_{qs}^{(p)}$ is another edge in τ_0. Hence, to do the process successively, we will obtain a tree of G_a from τ_0. ■

For example, suppose a W-tree of a W-graph Ω_w in Fig.3.4 is chosen as e_1, e_5, e_6, e_9 and $v_4^{(1)}$, $v_6^{(2)}$. When $t_a^{(1)}$ and $t_a^{(2)}$ are specified as shown in Fig.4.3(a), the W-graph Ω_w becomes a derived graph G_d in Fig.4.3(b). We show that the W-tree becomes a tree of G_d as follows:

When we give a star $t_s^{(1)}$ whose center is $v_5^{(1)}$ to w_1 and a star $t_s^{(2)}$ whose center is $v_4^{(2)}$ to w_2 in the W-graph as shown in Fig.3.4, the W-graph becomes a derived graph G_s as shown in Fig.4.3(c). Replacing $v_4^{(1)}$ and $v_6^{(2)}$ in the W-tree to edge $e_{4s}^{(1)}$ and $e_{6s}^{(2)}$ in G_s, the W-tree becomes a tree τ_0 of G_s consisting of e_1, e_5, e_6, e_9 and $e_{4s}^{(1)}$, $e_{6s}^{(2)}$ as shown in Fig.4.3(d). We must replace $e_{4s}^{(1)}$ and $e_{6s}^{(2)}$ in τ_0 by one edge in $t_s^{(1)}$ and one edge in $t_s^{(2)}$ so that τ_0 becomes a tree G_d in Fig.4.3(b).

First, we delete $e_{4s}^{(1)}$ in τ_0, we obtain two subtrees $\tau_a(e_{4s}^{(1)})$ and $\tau_b(e_{4s}^{(1)})$ as shown in Fig.4.3(e), $\tau_a(e_{4s}^{(1)})$ contains vertices v_1, v_2, v_3, v_4, v_6 and v_7, and $\tau_b(e_{4s}^{(1)})$ contains vertex v_5. Between vertex $v_4^{(1)}$ in $\tau_a(e_{4s}^{(1)})$ and vertex $v_5^{(1)}$ in $\tau_b(e_{4s}^{(1)})$, there is a path $\{e_{1a}^{(1)}, e_{2a}^{(1)}\}$ in $t_a^{(1)}$. We use the edge $e_{2a}^{(1)}$ which is in the path to connect $\tau_a(e_{4s}^{(1)})$ and $\tau_b(e_{4s}^{(1)})$ such that the result of $\tau_a(e_{4s}^{(1)}) \cup \tau_b(e_{4s}^{(1)}) \cup e_{2a}^{(1)}$ is also a tree as shown in Fig.4.3(f). From the resultant tree, we delete $e_{6s}^{(2)}$ in Fig.4.3(f), we obtain $\tau_a(e_{6s}^{(2)})$ and $\tau_b(e_{6s}^{(2)})$ as shown in Fig.4.3(g) where $\tau_a(e_{6s}^{(2)})$ contains vertices v_4 and $\tau_b(e_{4s}^{(1)})$ contains vertices v_1, v_2, v_3, v_5, v_6 and v_7. Between vertex $v_4^{(2)}$ in $\tau_a(e_{6s}^{(2)})$ and vertex $v_6^{(2)}$ in $\tau_b(e_{6s}^{(2)})$, there is a path $\{e_{1a}^{(2)}, e_{2a}^{(2)}\}$ in $t_a^{(2)}$ where the edge $e_{1a}^{(2)}$ is connected between $\tau_a(e_{6s}^{(2)})$ and $\tau_b(e_{6s}^{(2)})$. We use the edge $e_{1a}^{(2)}$ to make a new tree $\tau_a(e_{6s}^{(2)}) \cup \tau_b(e_{6s}^{(2)}) \cup e_{1a}^{(2)}$ as shown in Fig.4.3(h). Since there no edge of the form $e_{js}^{(i)}$ in Fig.4.3(h), Fig.4.3(h) is a tree of G_d where $e_{4s}^{(1)}$ and $e_{6s}^{(2)}$ in τ_0 are replaced by edges $e_{2a}^{(1)}$ and $e_{1a}^{(2)}$ which are in $t_a^{(1)}$ and $t_a^{(2)}$, respectively.

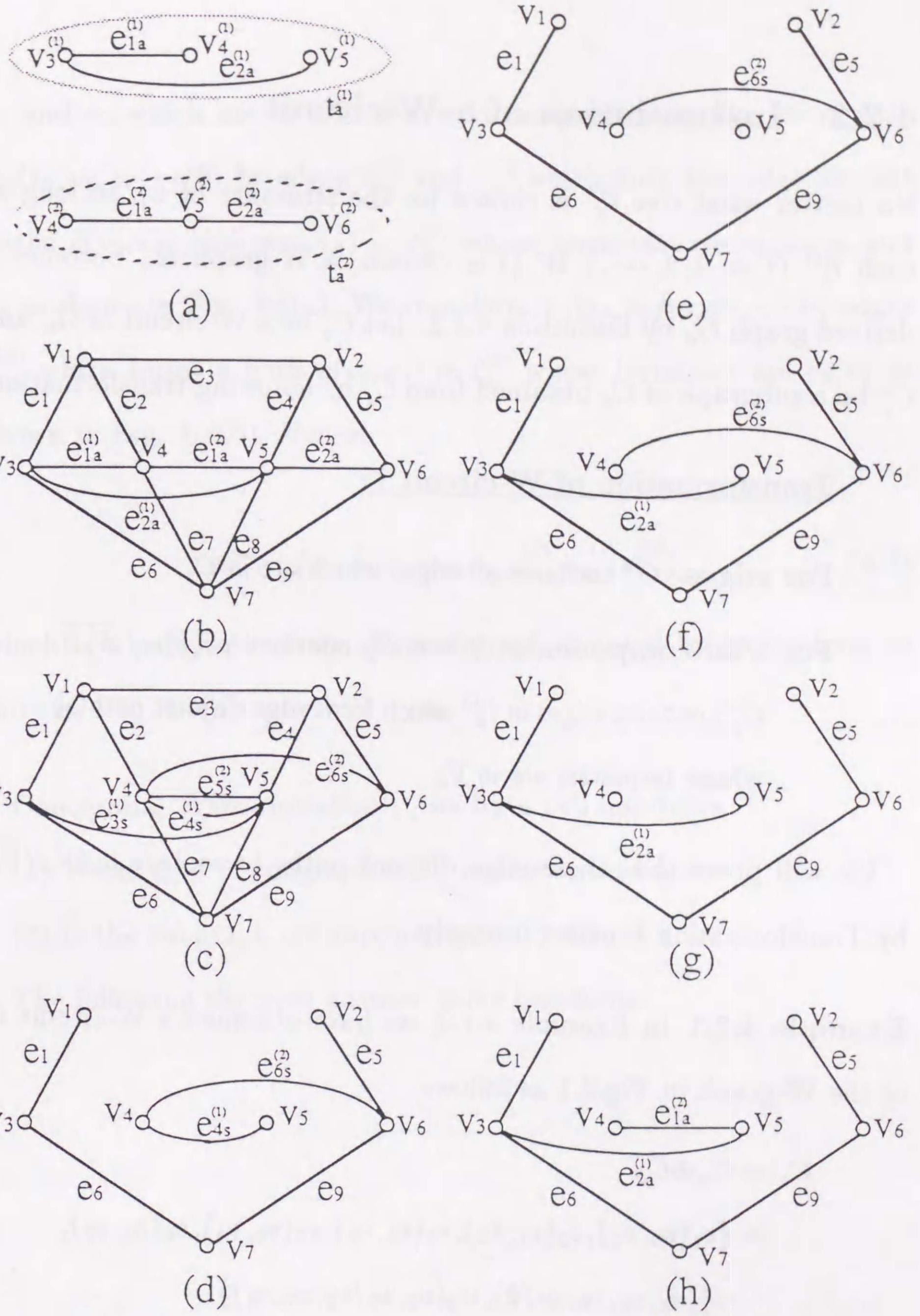


Figure 4.3: (a)$t_a^{(1)}$ and $t_a^{(2)}$ (b)a graph G_d (c)a graph G_s (d)a tree of τ_0 (e)$\tau_a(e_{4s}^{(1)})$ and $\tau_b(e_{4s}^{(1)})$ (f)a tree (g) $\tau_a(e_{6s}^{(2)})$ and $\tau_b(e_{6s}^{(2)})$ (h) the resultant tree.

4.2.2 Instantiation of a W-circuit

No matter what tree $t_a^{(i)}$ is chosen for the structure of w_i, so long as each $t_a^{(i)}$ ($i = 1, 2, \cdots, \mid W \mid$) is chosen, a W-graph Ω_w becomes a derived graph G_d by Definition 4.1.2. Let C_j be a W-circuit of Ω_w and C_j^* be a subgraph of G_d obtained from C_j by following transformation.

<u>**Transformation of W-circuit Γ:**</u>

For edges: C_j^* contains all edges which are in C_j.

For wild-components: When C_j contains $w_i(V_{oi} \ / \ \overline{V_{oi}})$, C_j^* contains edges in $t_a^{(i)}$ which form edge disjoint path(s) whose terminals are in V_{oi}.

We will prove that these edge disjoint paths to replace $w_i(V_{oi}/\overline{V_{oi}})$ by Transformation Γ exists uniquely.

Example 4.2.1 In Example 3.1.2, we have obtained a W-circuit C_γ of the W-graph in Fig.2.1 as follows:

$$\begin{aligned} C_\gamma &= C_\alpha \hat{\oplus} C_\beta \\ &= \{e_1(v_5, v_1), e_2(v_1, v_2), e_5(v_3, v_4), e_7(v_6, v_4), e_8(v_9, v_7), \\ &\quad w_1(v_2, v_5, v_6, v_7/\emptyset), w_2(v_3, v_9/v_2, v_7, v_8)\}. \end{aligned}$$

When the W-graph becomes a derived graph as shown in Fig. 4.1(c), C_γ can be transformed by Transformation Γ to be a subgraph C_γ^* of the derived graph. For edges, C_γ^* contains all edges e_1, e_2, e_5,

e_7 and e_8 which are in C_γ. For wild-components in C_γ, we replace $w_1(v_2, v_5, v_6, v_7/\emptyset)$ by edges $e_{1a}^{(1)}$ and $e_{2a}^{(1)}$ which form two edge disjoint paths $p(v_5, v_7)$ and $p(v_2, v_6)$ in $t_a^{(1)}$ whose terminals are v_2, v_5, v_6 and v_7 as shown in Fig. 4.4(a). We transform $w_2(v_3, v_9/v_2, v_7, v_8)$ by edges $e_{4a}^{(2)}$ which forms a path $p(v_3, v_9)$ in $t_a^{(2)}$ whose terminals are v_3, v_9 as shown in Fig. 4.4(b). Hence,

$$C_\gamma^* = \{e_1, e_2, e_5, e_7, e_8, e_{1a}^{(1)}, e_{2a}^{(1)}, e_{4a}^{(2)}\} \tag{4.2}$$

which is a subgraph of the derived graph denoted by heavy lines as show in Fig. 4.4(c). □

Concerning Transformation Γ, we have two questions:

(1) What kind of subgraph is C_j^* in the derived graph ?

(2) Is the subgraph corresponding to C_j^* unique ?

The following theorem answers these questions.

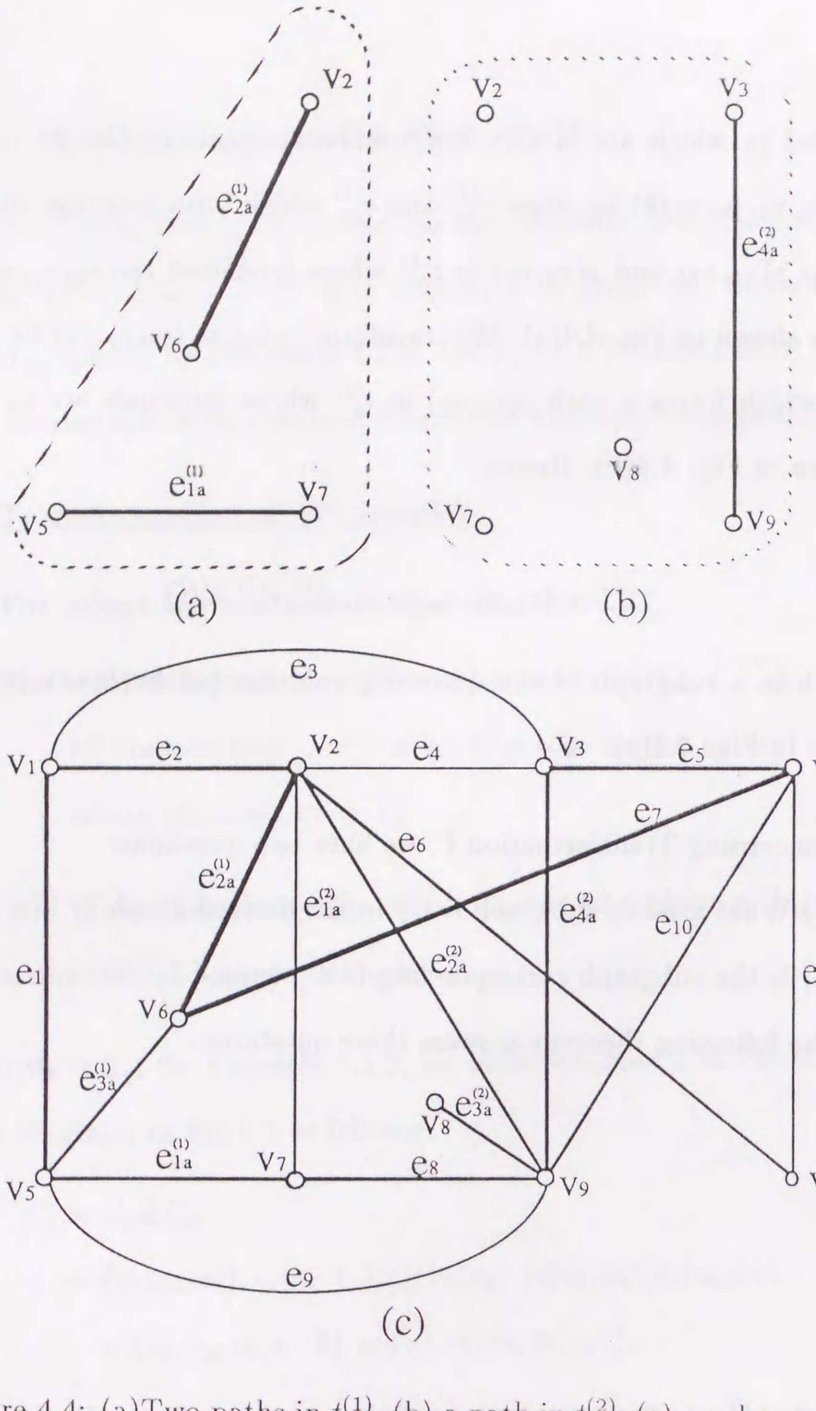


Figure 4.4: (a)Two paths in $t_a^{(1)}$, (b) a path in $t_a^{(2)}$, (c) a subgraph C_γ^*.

Theorem 4.2.2 *If C_j^* is obtained from a W-circuit C_j of a W-graph by Transformation Γ, then C_j^* is one and only one subgraph of a derived graph corresponding to the W-graph and the subgraph is either a circuit or an edge disjoint union of circuits.*

Proof : Let C_j be a W-circuit of a W-graph $\Omega_w(V, E, W)$ containing $w_i(V_{oi}/\overline{V_{oi}})$ $(i \in 1, 2, \cdots, \mid W \mid)$ where $V_{oi} = \{v_{ia1}, v_{ib1}, v_{ia2}, v_{ib2}, \cdots, v_{ian}, v_{ibn}\}$ and $\mid V_{oi} \mid$ is even. By Property 3.1.1, we replace each $w_i(V_{oi}/\overline{V_{oi}})$ by $\mid V_{oi} \mid /2$ inner paths whose terminals are in V_{oi} so that the W-circuit becomes a closed train. Suppose these inner paths are $p_{wi}(v_{ia1}, v_{ib1}), p_{wi}(v_{ia2}, v_{ib2}), \cdots, p_{wi}(v_{ian}, v_{ibn})$. When wild-component w_i is specified by $t_a^{(i)}$, each of these inner paths becomes one and only one path in $t_a^{(i)}$, that is, $p(v_{ia1}, v_{ib1}), p(v_{ia2}, v_{ib2}), \cdots, p(v_{ian}, v_{ibn})$. When we make the ring sum of these paths $p(v_{ia1}, v_{ib1}) \oplus p(v_{ia2} \oplus v_{ib2}) \oplus \cdots \oplus p(v_{ian}, v_{ibn})$, the result of $p(v_{ia1}, v_{ib1}) \oplus p(v_{ia2} \oplus v_{ib2}) \oplus \cdots \oplus p(v_{ian}, v_{ibn})$ is a subgraph of $t_a^{(i)}$ consisting of edge disjoint paths whose terminals are also in $\{v_{ia1}, v_{ib1}, v_{ia2}, v_{ib2}, \cdots, v_{ian}, v_{ibn}\}$ ([Mayeda1 72]). Hence, when we change each $w_i(V_{oi}/\overline{V_{oi}})$ by $p(v_{ia1}, v_{ib1}) \oplus p(v_{ia2} \oplus v_{ib2}) \oplus \cdots \oplus p(v_{ian}, v_{ibn})$, the W-circuit becomes an closed edge train of a derived graph because all inner paths are replaced by edges of $t_a^{(i)}$. We can see that the closed edge train is C_j^* which is either a circuit or an edge disjoint union of circuits.

Furthermore, since each inner path in a wild-component w_i corresponds to exactly one path in $t_a^{(i)}$, it is clear that $p(v_{ia1}, v_{ib1}) \oplus p(v_{ia2} \oplus v_{ib2}) \oplus \cdots \oplus p(v_{ian}, v_{ibn})$ corresponds to one and only one subgraph of $t_a^{(i)}$. Thus, C_j^* is unique. ■

As an example of Theorem 4.2.2, it can be verified that C_γ^* in $Eq.(4.2)$ transformed from C_γ in $Eq.(3.7)$ is a circuit in the derived graph as shown in Fig.4.4(c).

Let Ω_w be a W-graph and G_d be a derived graph of Ω_w, and let $\{C_j\}$, $j = 1, 2, \cdots, r$, be a set of W-circuits of Ω_w and $\{C_j^*\}$ be a set of circuits or edge disjoint unions of circuits of G_d obtained from $\{C_j\}$ by Transformation Γ. We will prove that the members in $\{C_j^*\}$ are linearly independent if and only if the members in $\{C_j\}$ are linearly independent.

Theorem 4.2.3 *Let $\{C_j^*\}$, $j = 1, 2, \cdots, r$, be obtained from $\{C_j\}$ by Transformation Γ. All circuits or edge disjoint unions of circuits in $\{C_j^*\}$ are linearly independent if and only if all W-circuits in $\{C_j\}$ are linearly independent.*

Proof: We have proved that transforming Γ on C_j gives one and only one C_j^* by Theorem 4.2.2, we need show any circuit or edge disjoint union of circuits C_a^* in a derived graph can form exactly one W-circuit C_a of its W-graph. Let C_a^* be

$$C_a^* = \{e_1, e_2, \cdots, e_m, e_{1a}^{(1)}, e_{2a}^{(1)}, \cdots, e_{\alpha 1a}^{(1)}, \\ e_{1a}^{(2)}, e_{2a}^{(2)}, \cdots, e_{\alpha 2a}^{(2)}, \cdots, e_{1a}^{(|W|)}, e_{2a}^{(|W|)}, \cdots, e_{\alpha|W|a}^{(|W|)}\}$$

where each $e_{ja}^{(i)} (j = 1, 2, \cdots, \alpha i)$ is an edge in $t_a^{(i)}$ of a derived graph.

By Transformation Γ, we know that C_a also contains the edges e_1, e_2, $\cdots$, e_m where these edges are different.

Since C_a^* is either a circuit or an edge disjoint union of circuits, we can consider C_a^* to be a subgraph of the derived graph. Consider all the edges $e_{1a}^{(i)}$, $e_{2a}^{(i)}$, $\cdots$ and $e_{\alpha 1a}^{(i)}$, these edges compose a subgraph of $t_a^{(i)}$ which is a set of edge disjoint paths because $t_a^{(i)}$ is a tree. Make a vertex set V_{oi} by collecting all terminals of these edge disjoint paths, it is clear that all vertex in V_{oi} are different and $\mid V_{oi} \mid$ is even. Then replace each set of these edge disjoint paths in $t_a^{(i)}$ $(i = 1, 2, \cdots, \mid W \mid)$ by $w_i(V_{oi}/\overline{V_{oi}})$ so that C_a^* becomes exactly one W-circuit C_a because C_a satisfies the conditions in Definition 3.1.1. ■

Since Theorem 4.2.3 is a necessary and sufficient condition, we can obtain a set of linearly independent circuits or edge disjoint unions of circuits of any derived graph from a set of linearly independent W-circuits of its W-graph, also we can form a set of linearly independent W-circuits of a W-graph by a set of linearly independent circuits or edge disjoint unions of circuits of a derived graph of the W-graph. It implies that we can obtain a set of linearly independent circuits or edge disjoint unions of circuits of a derived graph from those of another derived graph by means of W-circuits. We establish a relation between any two derived graphs by the following property.

Property 4.2.1 *A set of linearly independent circuits or edge disjoint unions of circuits of a derived graph can be obtained from those of another derived graph where the two derived graphs are from the same W-graph.*

For example, there are two graphs G_a and G_b as shown in Fig. 4.5(a) and Fig. 4.5(b). When we have linearly independent circuits of G_a as $\{e_1, e_{2a}^{(2)}, e_{3a}^{(1)}\}$, $\{e_1, e_2, e_{3a}^{(1)}\}$ and $\{e_2, e_3, e_{1a}^{(1)}\}$, we can obtain a set of linearly independent circuits or edge disjoint unions of circuits of G_b by the following method. Since G_a and G_b are two derived graphs of a W-graph as shown in Fig. 4.5(c) where there is a wild-component w_1 and $V(w_1) = \{v_1, v_2, v_3, v_4\}$. We can transform the linearly independent circuits of G_a to W-circuits of the W-graphs as $\{e_1, w_1(v_1, v_2/v_3, v_4)\}$,

$\{e_1, e_2, w_1(v_2, v_4/v_1, v_3)\}$ and $\{e_2, e_3, w_1(v_1, v_3 /v_2, v_4)\}$ which are linearly independent by Theorem 4.2.3. Then, By Transformation Γ, we can get a set of linearly independent circuits or edge disjoint unions of circuits of G_b as $\{e_1, e_{b2}^{(1)}\}$, $\{e_1, e_2, e_{b2}^{(1)}, e_{b3}^{(1)}\}$ and $\{e_2, e_3, e_{b1}^{(1)}, e_{b2}^{(1)}\}$.

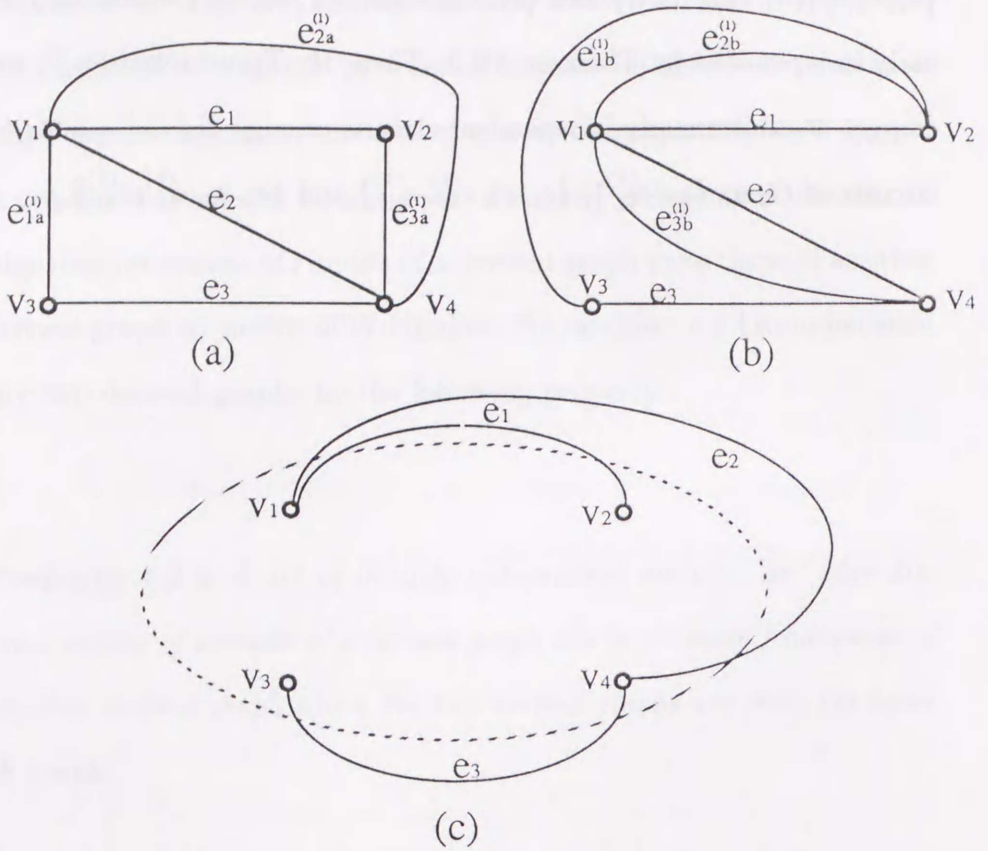


Figure 4.5: (a) A graph G_a (b) a graph G_b (c) a W-graph.

In graph theory, there are some relations between graphs such as dual graphs, isomorphic graphs and 2-isomorphic graphs, we know that a cutset in a graph is a circuit of its dual graph and two isomorphic graphs have the same incidence sets. Here, Property 4.2.1 shows a new relation of two graphs with respect to circuits.

4.2.3 Instantiation of a W-cutset

Let S_j be a W-cutset of a W-graph Ω_w and S_j^* be a subgraph of a derived graph of Ω_w. We can obtain S_j^* from S_j by following transformation.

<u>**Transformation of W-cutset Θ:**</u>

For edges: S^* contains all edges which are in S.

For wild-components: When S contains $w_i(V_{ai} : \overline{V_{ai}})$, S^* contains a set of edges in $t_a^{(i)}$ where endpoints of the edge are in V_{ai} and $\overline{V_{ai}}$, respectively.

Example 4.2.2 For the W-cutset S_1 in $Eq.(3.11)$ of Ω_w as shown in Fig.2.1, it can be transformed to a cutset S_1^* of G_a as shown in Fig.4.1(b) by Transformation Θ. S_1^* has all edges which are in S_1 and change $w_1(v_2, v_5, v_6 : v_7)$ in S_1 by edge $e_{3a}^{(1)}$ which is connected between $\{v_7\}$ and $\{v_2, v_5, v_6\}$ in $t_a^{(1)}$ and changing $w_2(v_2 : v_3, v_7, v_8, v_9)$ by edges $e_{1a}^{(2)}, e_{2a}^{(2)}$ which are connected between $\{v_3, v_7, v_8, v_9\}$ and $\{v_2\}$ in $t_a^{(i)}$ as shown as Fig.4.6. Hence,

$$S_1^* = \{e_3, e_4, e_6, e_7, e_9, e_{3a}^{(1)}, e_{1a}^{(2)}, e_{2a}^{(2)}\}.$$

□

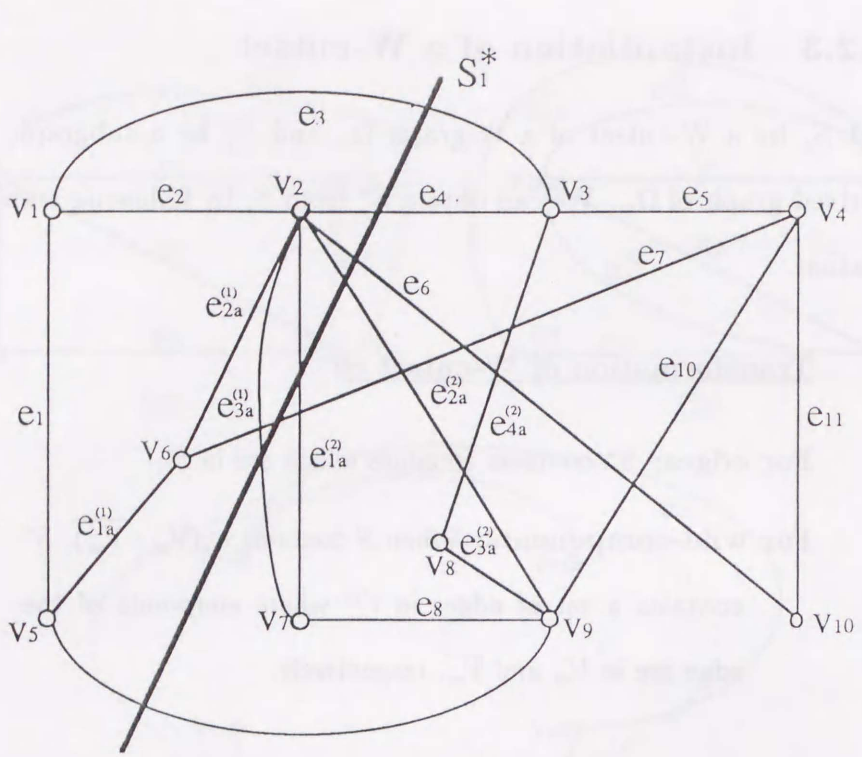


Figure 4.6: A cutset transformed from a W-cutset.

Since a W-cutset separates the vertex set V in a W-graph $\Omega_w(V, E, W)$ into two sub-vertex sets, it is evident that a W-cutset is a cutset or as edge disjoint union of cutsets of a derived graph of the W-graph.

Property 4.2.2 *Let $\{S_j\}$ be a set of linearly independent W-cutsets of a W-graph, and $\{S_j^*\}$ be a set of cutsets or edge disjoint union of cutsets in a derived graph of the W-graph obtained from $\{S_j\}$ by Transformation Θ. The member of $\{S_j^*\}$ are linearly independent.*

Although Property 4.2.2 is evident, it is an important property giving a relation between W-cutsets of a W-graph and cutsets or edge disjoint union of cutsets of a derived graph of the W-graph.

Chapter 5

Some Applications of W-graphs

In this chapter, some possible applications of W-graphs for layout design are introduced. A wild-component can be employed for modeling a multi-terminal net and a specific terminal set related to routing problems. A multi-terminal net is a means of minimally connecting a terminal to another by wires electrically whose structure is unspecified. Hence, the structure of a multi-terminal net can be represented by a wild-component where these terminals are represented by vertices of the wild-component. The specific terminal set means that any wires are forbidden to separate these terminals. In this chapter, an approach for topological routing is provided for minimizing vias [Zhao3 90] by

W-graphs. The via-minimization problem in two-layered topological routing that is often used in design of VLSI or printed wiring boards can be modeled by a W-graph $\Omega_w(V, E, W)$, where V represents a set of all terminals, E does a set of two-terminal nets and W does a set of multi-terminal nets. It is proved that a W-graph for modeling a routing problem can be embedded on either inside or outside (the inside and the outside are corresponding to two layers, respectively) of the boundary of routing region without crossing edges by created vertices and that the number of vias is equal to the number of created vertices. With this modeling, the routing problem can be reduced to two problems of W-graphs: The one is detection of planarity of W-graphs and the other is plane drawing of planar W-graphs. At present, the two problems still remain unsolved, we are unable to evaluate our approach by W-graphs explicitly. However, if we can solve the two problems in W-graphs, the advantages of this approach will be shown. In this dissertation, some theorems are provided for testing planar W-graphs for some particular W-graphs. The difficulty of testing planar W-graphs are analyzed.

5.1 An Approach to Topological Routing by W-graphs

A new approach for topological routing with via minimization is proposed by W-graphs. We employ a W-graph $\Omega_w(V, E, W)$ for indicating all nets which will be assigned to two layers, where V is a set of all terminals, E is a set of edges corresponding to two-terminals nets and W is a set of wild-components corresponding to multi-terminal nets. In other words, the topological routing problem can be considered as follows: Let H be a circle containing all vertices in the sequence corresponding to terminals on the boundary of routing region. Then we specify the structure of all wild-components in Ω_w so that $H \cup \Omega_w$ can be drawn on a plane with minimum number of created vertices (Definition 5.1.3).

5.1.1 Topological Routing Problems

For two-layer routing problem, a via minimization is desirable because minimizing vias increases the chip space usage and decreases the manufacturing cost. The problem of via minimization can be divided into two types (1) a constrained via minimization (CVM) and (2) an unconstrained via minimization (UVM). The former is that the routing geometrical assignment is given, the wires are to be assigned to one of both layers such that the number of vias needed is minimum [Chen 83]. The latter is where both routing geometrical assignment and layer assignment of wires are needed to be decided for satisfying via minimization.

The CVM problem originated in the pioneer PCB design work of Hashimato and Stevens [Hashimoto 77] in 1971. For a long time, it has been believed that the CVM belongs to the class of NP-complete problem. A number of algorithms besed on different heuristics were proposed for the problem [Sakamoto 75], [Servit 77] and [Stevens 79]. In 1980, Kajitani [Kajitani 80] proposed a polynomial-time algorithm for a special case of the problem. Ciesielski and Kinnen [Ciesielski 81] introduced an integer programming formulation to the problem with a solution which is exponential in time complexity. Chen, Kajitani and Chan extended Kajitani's earlier work to a more general, but still restricted situation and proposed an optimal solution. Independently, Pinter [Pinter 82] found a polynomial-time algorithm for the same situation where each via is to connect at most three wires.

For the UVM problem, the first work on topological via minimization was proposed by [Hsu 83] in 1983 based on a net intersection graph. In a topological routing problem, the routing region is a simple connected region whose boundary contains all terminals. We don't consider the geometrical constraints and the only information we need is the sequence of terminals along the boundary, so we will use a circle to represent the boundary and mark all terminals on the boundary to the circle by using the same counterclockwise sequence. It should be noticed that the primary aim of a topological routing is via minimization.

Hsu restricted the UVM problem to two-terminals nets and presented that UVM is a problem of "minimum node deletion bipartite subgraph" in a intersection graph. In 1984, Marek-Sadowska [Marek 84] showed that the problem is NP-complete (as far as we know, the proof were not perfect). In 1987, Du and Chang [Chang 87] proposed another heuristic algorithm for this problem based on bipartitioning of a graph. In 1989, Xiong and Kuh [Xiong 89] treated the UVM problem as a unified $\{0, 1\}$ linear programming formulation and considered this problem as finding "maximal cut" in a weighted cluster graph.

Here, we will employ a W-graph $\Omega_w(V, E, W)$ for indicating a set of terminals by V, a set of two-terminal nets by E and a set of multi-terminal nets by W, and employ a circle H containing all vertices of V for indicating the boundary of routing region. Then it will be shown that topological routing problem can be transformed to problems of a W-graph.

Definition 5.1.1 (Via) *A via is either a hole or a contact, other* *then a terminal (pin), where wire on different layers is connected.*

Definition 5.1.2 (Net) *A net* $n_j = \{v_{j1},\ v_{j2},\ \cdots,\ v_{jp}\}$ $(p \geq 2)$ *is a set of all equipotential terminals (pins) which must be connected by wires electrically. When* $p > 2$*, the net is called a multi-terminal net. When* $p = 2$*, the net is called a two-terminal net.*

We assume that every terminal contacts with both layers. The assumption means that we can connect a net by wires assigned to every layer. A way of connecting terminals in a multi-terminal net need not be specified, but those are usually connected by minimum wires. This means that a multi-terminal net is a connected subgraph having minimum number of edges. Thus, a multi-terminal net n_j can be indicated by a wild-component w_j where terminals are represented by vertices.

5.1.2 Approach by the W-graph Model

The approach for topological routing is described as follows: Let a W-graph $\Omega_w(V, E, W)$ correspond to all nets which will be assigned to two-layer, V be a set of all terminals, E be a set of edges corresponding to two-terminal nets and W be a set of wild-components corresponding to multi-terminal nets.

A vertical-horizontal routing is shown in Fig.5.1(a) where there are three nets called net n_a, net n_b and net n_c, and three vias indicated by triangles. Net n_a has three terminals, net n_b and net n_c have two. We make a circle H containing all terminals in the sequence as like as those are on the boundary of routing region. The routing problem can be modeled by a W-graph $H \cup \Omega_w$ as shown is Fig.5.1(b). Since the W-graph $H \cup \Omega_w$ is planar, it can be drawn on a plane without crossing edges as Fig.5.1(b), which is called a topological solution. We map the solution onto a rectilinear plane, when edges being on inside of H should be assigned to one of layers and those on outside of H should be assigned to the other layer, the resultant routing is shown in Fig.5.1(c) where there are no vias. By the assumption that the terminals contact with both layers, it can be seen that net n_a is connected by wires in two layers and the terminal a_2 is not regarded as a via. Note that there are wires of net n_a on both layers indicated by a rectangle in Fig.5.1(c).

It is clear that the problem of topological routing can be changed to a problem of W-graph, that is, how to draw edges in E and find a suitable tree for each wild-component in W of a W-graph on either inside or outside of the circle H to connect every net without crossing edge or with minimum number of crossing points of edges possibly.

It is evidently that the following argument is true.

Fact 5.1.1 *A W-graph Ω_w indicating all nets can be assigned to two layers without via if and only if $H \cup \Omega_w$ is planar.*

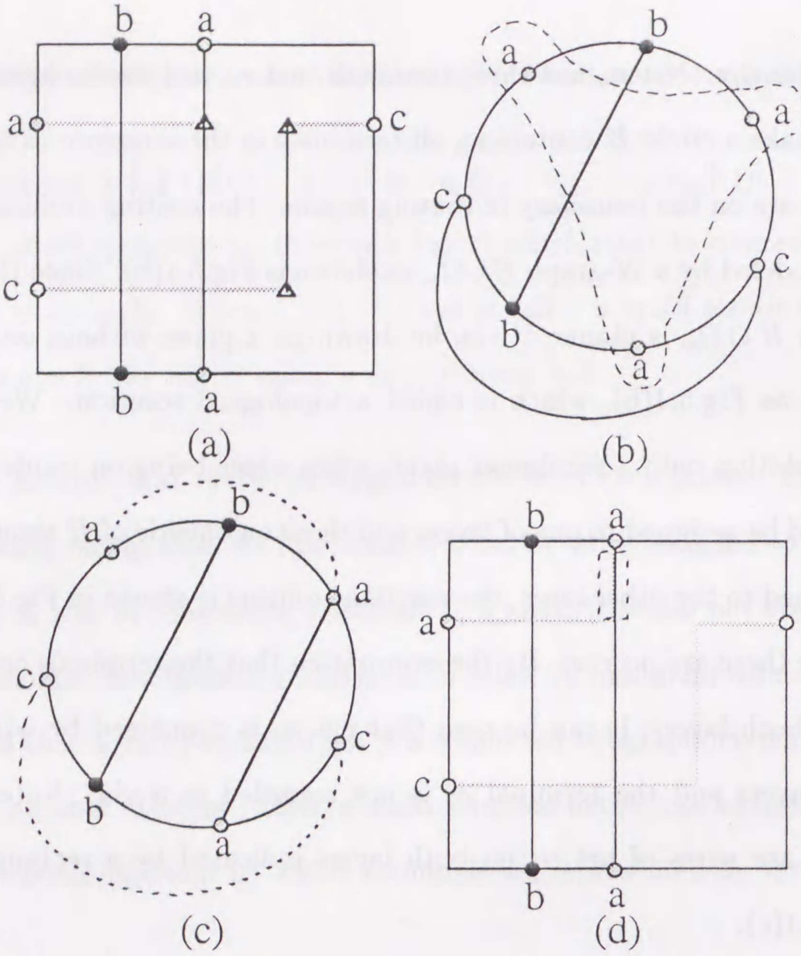


Figure 5.1: (a) A V-H routing, (b)a W-graph $H \cup \Omega_w$, (c)topological solution,(d)resultant routing.

However, when $H \cup \Omega_w$ is nonplanar, for any drawing of $H \cup \Omega_w$ on a plane, there exist some crossing points of edges surely. It means that via is necessary.

Consider a non-planar graph as shown in Fig.5.2(a) where p is a crossing point, if we create a vertex at the point p, the graph can be embedded on a plane as shown in Fig.5.2(b). We give a definition of created vertex.

Definition 5.1.3 (Created vertex) *When an edge crosses H, we create a vertex at the crossing point such that the edge and H can be embedded on a plane. The vertex is called a created vertex.*

It should be noticed that the created vertices only appear on H, so the vertex p in Fig.5.2(b) is not a created vertex. However, we can draw the non-planar graph of Fig.5.2(a) on a plane by a created vertex as shown in Fig.5.2(c).

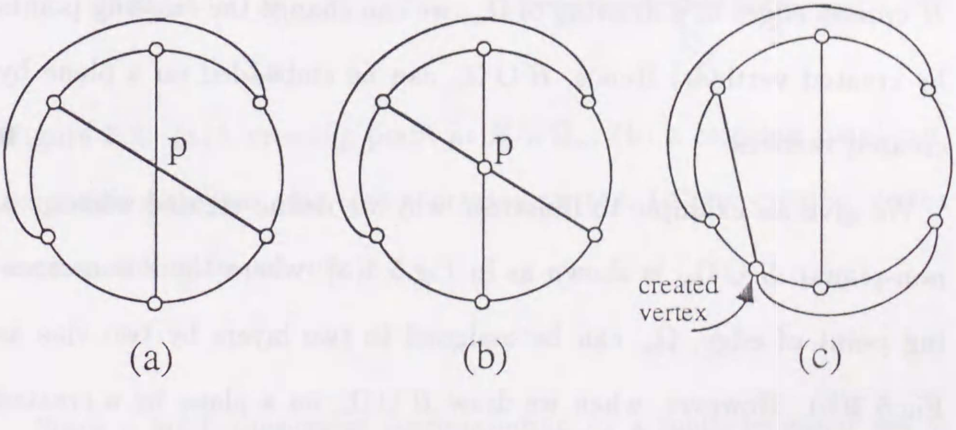


Figure 5.2: (a) A Non-planar graph, (b)a plane drawing (c)another plane drawing.

Suppose that there are no common terminals in any two nets. The following theorem shows that a non-planar $H \cup \Omega_w$ can be drawn on a plane by crossing edges to H such that all crossing points are created vertices.

Theorem 5.1.1 *Any $H \cup \Omega_w$ can be drawn on a plane by created vertices if necessary.*

Proof: Since Ω_w is a collection of nets and every net can be connected by a tree, it is clear that Ω_w can be drawn on a plane without crossing edges.

We make a Hamilton circuit H to connected all vertices of Ω_w by the sequence as like as those on the boundary of routing region. When H crosses edges in a drawing of Ω_w, we can change the crossing points by created vertices. Hence, $H \cup \Omega_w$ can be embedded on a plane by created vertices. ■

We give an example to illustrate why we define created vertex. A non-planar $H \cup \Omega_w$ is shown as in Fig.5.3(a), where there is a crossing point of edge, Ω_w can be assigned to two layers by two vias as Fig.5.3(b). However, when we draw $H \cup \Omega_w$ on a plane by a created vertex as shown in Fig.5.3(c), Ω_w can be assigned to two layers by only one via as Fig.5.3(d). Hence, the number of created vertices corresponds to the number of vias uniquely, also a created vertex implies where a via must be generated.

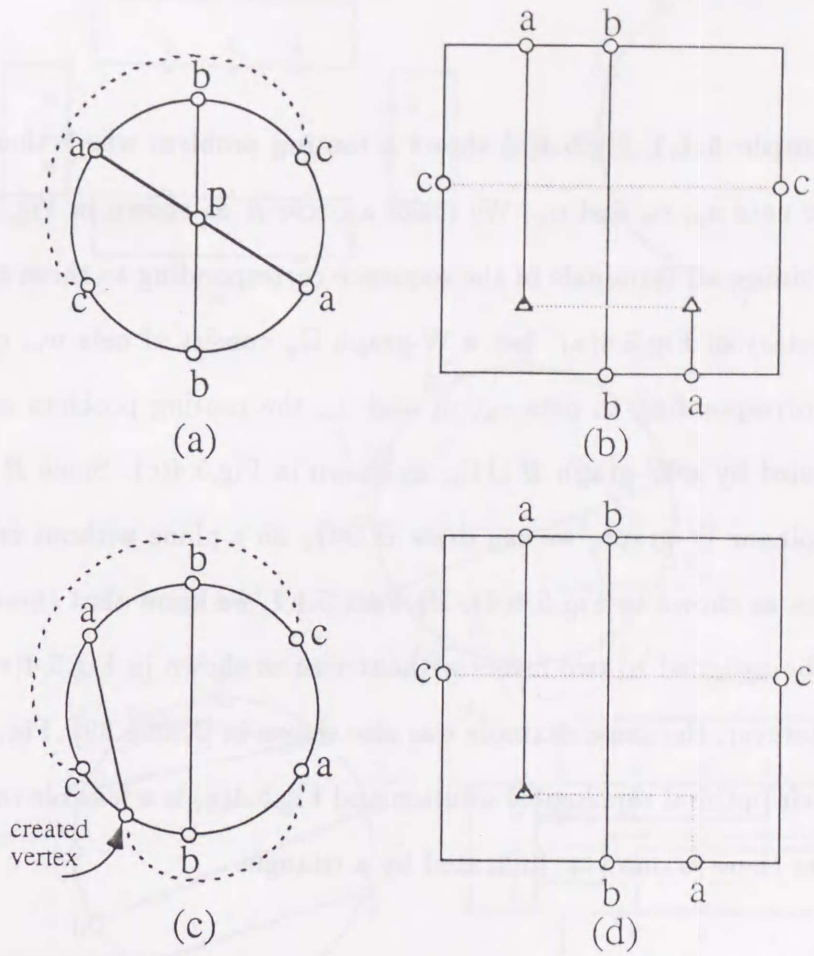


Figure 5.3: (a)A crossing point in $H \cup \Omega_w$, (b) a crossing point corresponding to two vias, (c) a created vertex, (d)one created vertex corresponding to one via.

Since a wild-component corresponding to a multi-terminal net is usually connected by a tree, the number of created vertices in a planar drawing of $H \cup \Omega_w$ will be changed by given different tree. It has not been solved how to obtain an optimal planar drawing of $H \cup \Omega_w$ which contains minimum number of created vertices.

Example 5.1.1 Fig.5.4(a) shows a routing problem where there are three nets n_a, n_b and n_c. We make a circle H as shown in Fig.5.4(b) containing all terminals in the sequence corresponding to those on the boundary of Fig.5.4(a). Let a W-graph Ω_w consist of nets w_a, e_b and w_c corresponding to nets n_a, n_b and n_c, the routing problem can be modeled by a W-graph $H \cup \Omega_w$ as shown in Fig.5.4(c). Since $H \cup \Omega_w$ is a planar W-graph, we can draw $H \cup \Omega_w$ on a plane without created vertex as shown in Fig.5.4(d). By Fact 5.1.1, we know that these nets can be assigned to two layers without vias as shown in Fig.5.4(e).

However, the same example was also shown in [Xiong 89]. Fig.5.4(f) is their optimal topological solution and Fig.5.4(g) is a feasible routing where there is one via. indicated by a triangle. □

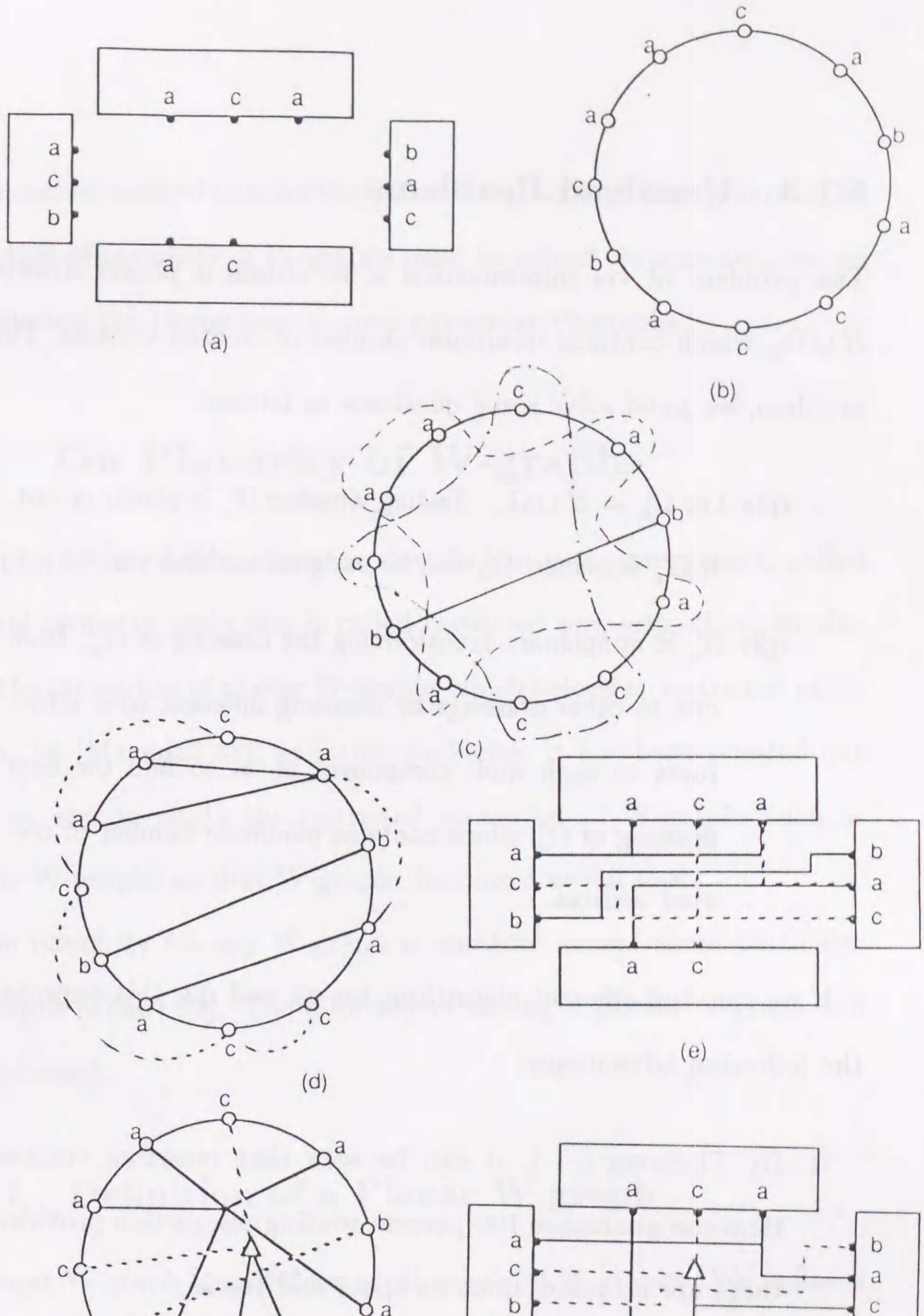


Figure 5.4: (a) A routing problem, (b) a circle H, (c) a W-graph $H \cup \Omega_w$, (d) a topological solution, (e) a mapping, (f) an optimal topological solution, (g) a feasible routing.

5.1.3 Unsolved Problems

The problem of via minimization is to obtain a planar drawing of $H \cup \Omega_w$ which contains minimum number of created vertices. For this problem, we must solve some questions as follows:

q1: Let $\Omega'_w = H \cup \Omega_w$. Testing whether Ω'_w is planar or not. If Ω'_w is planar, Ω_w can be assigned without via.

q2: Ω'_w is nonplanar. Transforming the drawing of G'_w, from one to other drawings by choosing different tree- structures to each wild- component so as to find the best drawing of Ω'_w which contains minimum number of created vertices.

If we can find efficient algorithms for **q1** and **q2**, this approach has the following advantages:

1. By Theorem 5.1.1, it can be seen that inserting created vertices can guarantee 100-percent routing completion provided that there are no restrictions on space and tracks.

2. Minimizing vias is equivalent to find a proper tree for each wild-component in Ω_w so that $H \cup \Omega_w$ can be drawn on a plane with minimal created vertices. Particularly, when $H \cup \Omega_w$ is a planar W-graph, there exists at least one routing scheme without vias.

It must be noticed to solve the questions **q1** and **q2** is very hard. The detection of planarity of W-graphs must be solved. In next section, we will discuss the planarities of some particular W-graphs.

5.2 On Planarity of W-graphs

The properties of W-graphs are classified to two types: one is called general property and other is called restricted property. Here, we discuss the properties of planar W-graphs which belong to restricted properties. In [Mayeda2 88], as future problems, it has been pointed out that we should study the restricted properties of W-graphs such as planar W-graphs so that W-graphs become a useful tool.

The planarity for any W-graph is unsolved except some particular W-graphs [Zhao4 91]. The difficulty of testing a planar W-graph will be discussed.

5.2.1 Definition of a Planar W-graph

A planar W-graph is useful for applications of W-graphs. We define a planar W-graph as follows:

Definition 5.2.1 (Planar W-graph) *A W-graph Ω_w is said to be a planar W-graph if and only if there exist at least one planar derived graph of the W-graph.*

The W-graph, as an example, as shown in Fig.5.5(a) is planar because there exists a derived graph as shown in Fig.5.5(b) which can be drawn on a plane without crossing edges in spite of the existence of a non-planar derived graphs as shown in Fig.5.5(c).

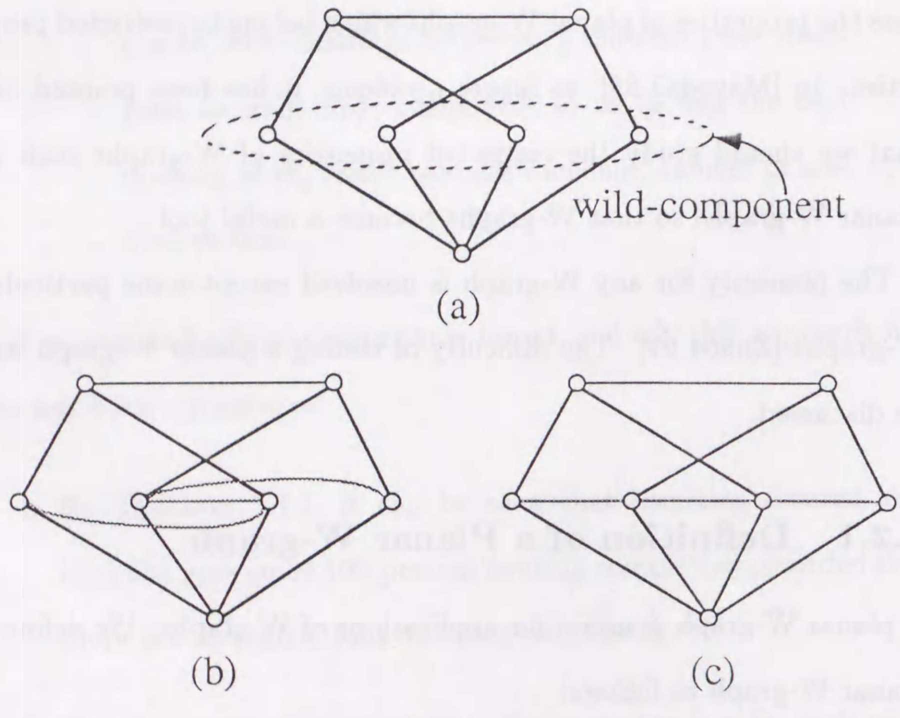


Figure 5.5: (a)A W-graph, (b) a planar derived graph, (c) a non-planar derived graph.

By Definition 4.1.2 and 5.2.1, for a W-graph $\Omega_w(V, E, W)$, if $G(V, E)$ is not planar, it is impossible that there exist planar derived graphs of Ω_w. In other words, when we discuss the planarity of a W-graph, it is necessary for $G(V, E)$ being planar. We suppose that $G(V, E)$ corresponding to a W-graph $\Omega_w(V, E, W)$ is planar hereafter. Since $G(V, E)$ is an ordinary graph, a number of algorithms are available to test whether a graph is planar or not where some of these are constructive algorithms, if the graph is planar, a plane drawing can be produced. The plane drawing of planar graphs is also introduced in [Wing 66] and [Mayeda4 85].

Definition 5.2.2 (Plane drawing) *The symbol of $D[G(V, E)]$ indicates a plane drawing of $G(V, E)$. Hence, it is also employed for expressing that $G(V, E)$ is planar.*

It should be noticed that a W-graph $\Omega_w(V, E, W)$ is planar as long as there exist at least one tree $t_a^{(i)}$ with respect to each wild-component w_i so that $D[G(V, E)]$ with all tree $t_a^{(i)}$ $(i = 1, 2, \cdots, \mid W \mid)$ becomes a planar graph, that is, $D[G(V, E)] \cup t_a^{(1)} \cup t_a^{(2)} \cdots \cup t_a^{(|W|)}$ can be drawn on a plane without crossing edges.

5.2.2 Properties of Planar W-graphs

For a W-graph $\Omega_w(V, E, W)$, if $G(V, E)$ is corresponding to $\Omega_w(V, E, W)$ is planar, a plane drawing $D[G(V, E)]$ divides the plane into some regions.

Definition 5.2.3 (Boundary of a region) *The symbol of $V(m)$ is a vertex set containing all vertices in a boundary of a region m.*

When a region m in a plane drawing $D[G(V, E)]$ contains all vertices of a wild-component w_i, we have,

Lemma 5.2.1 *If $V(w_i) \subseteq V(m)$, $D[G(V, E)] \cup t_a^{(i)}$ is planar where m is a region of $D[G(V, E)]$.*

Proof: Since the structure of a wild-component is a tree, we can draw a wild-component on a plane without crossing edges. Let m be region in $D[G(V, E)]$, we draw a tree $t_a^{(i)}$ on m such that $D[G(V, E)] \cup t_a^{(i)}$ can be drawn on a plane without crossing edges. ■

Expanding Lemma 5.2.1, the following theorem is trivial.

Theorem 5.2.1 *A W-graph $\Omega_w(V, E, W)$ is planar if there exist $\mid W \mid$ regions $m_1,\ m_2,\ \cdots,\ m_{|W|}$ in $D[G(V,\ E)]$ such that $V(w_i) \in V(m_i)$, $i = 1,\ 2,\ \cdots,\ \mid W \mid$.* ■

Example 5.2.1 Fig.5.6(a) shows a W-graph $\Omega_w(V, E, W)$ containing two wild-components w_1 and w_2 where $V(w_1) = \{v_2, v_4, v_7\}$ and $V(w_2) = \{v_6, v_9, v_{11}, v_{13}\}$. Fig.5.6(b) shows a graph $G(V, E)$ corresponding to Ω_w, where there are two regions m_1 and m_2 and $V(m_1) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9\}$ and $V(m_2) = \{v_1, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}\}$. It is clear that $V(w_1) \in V(m_1)$ and $V(w_2) \in V(m_2)$ so that the W-graph is planar, one of plane drawings of $G(V, E) \cup t_a^{(1)} \cup t_a^{(2)}$ is shown in Fig.5.6(c). □

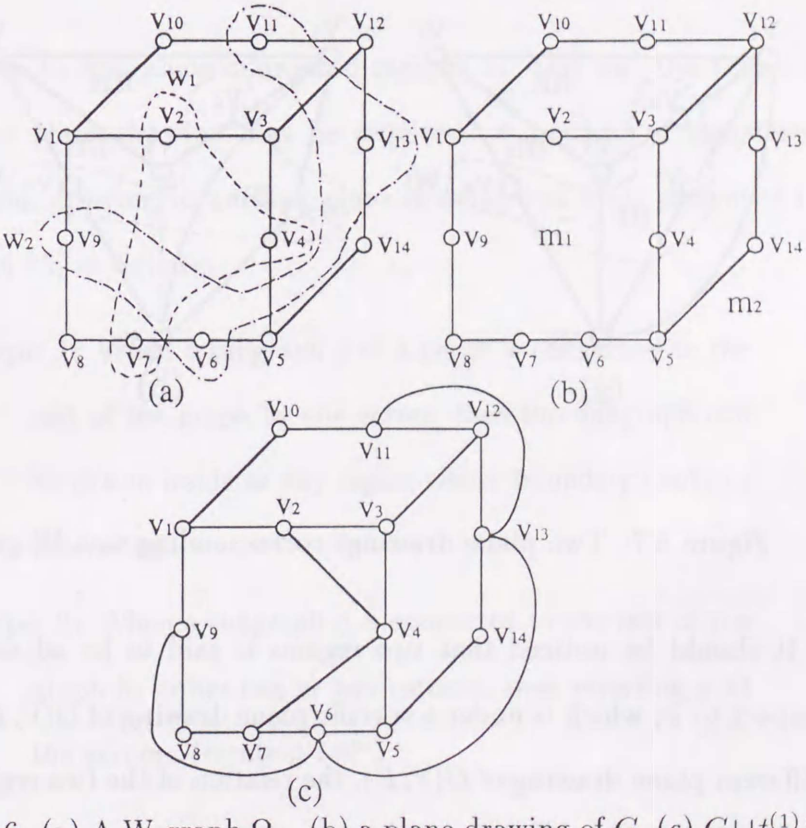


Figure 5.6: (a) A W-graph Ω_w (b) a plane drawing of G (c) $G \cup t_a^{(1)} \cup t_a^{(2)}$.

When all vertices in a wild-component are not in one boundary, we define adjacent regions and chain-connected regions as followings:

Definition 5.2.4 (Adjacent region of w_i) *Let m_a and m_b be two regions whose boundaries containing vertices of w_i in $D[G(V, E)]$, m_a and m_b are said to be adjacent regions with respect to w_i, if $V(m_a) \cap V(m_b) \cap V(w_i) \neq \emptyset$, denoted by $m_a @_i m_b$.*

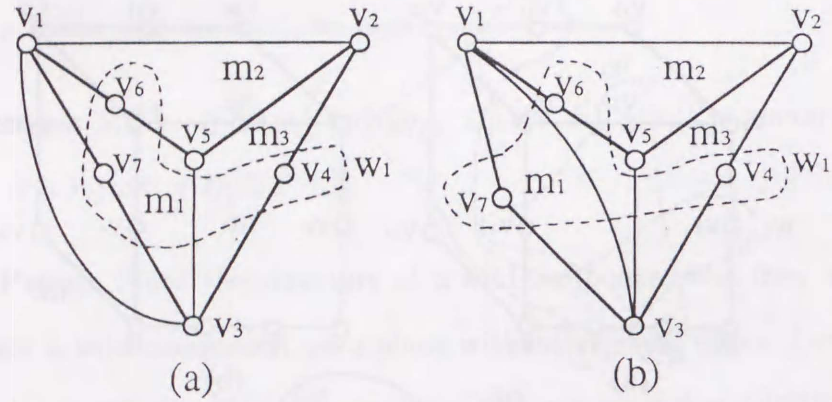


Figure 5.7: Two plane drawings corresponding to a W-graph.

It should be noticed that two regions is said to be adjacent with respect to w_i which is under a specific plane drawing of $G(V, E)$. With different plane drawing of $G(V, E)$, the relation of the two regions may be changed. There is a plane drawing $D[G(V, E)]$ corresponding to a W-graph as shown in Fig.5.7(a), m_1 and m_2 are adjacent regions of

w_1 because $V(m_1) \cap V(m_2) \cap V(w_1) = \{v_1, v_3, v_5, v_6, v_7\} \cap \{v_1, v_2, v_5, v_6\} \cap \{v_4, v_6, v_7\} = \{v_6\} \neq \emptyset$. But in Fig.5.7(b) which is another plane drawing, m_1 and m_2 are not adjacent regions with respect to w_1.

Definition 5.2.5 (Chain-connected regions) *Two regions m_a and m_b are said to be chain-connected with respect to w_i, if there exists a sequence of adjacent regions $m_1, m_2, \cdots, m_r$ which satisfies following relation,*

$$m_a @_i m_1 @_i m_2 @_i \cdots @_i m_r @_i m_b.$$

In order to find chain-connected regions m_a and m_b, the transformation of plane drawing may be required. A method of transforming a plane drawing to another plane drawing has been presented by [Mayeda4 85] as follows:

Type 1: When a subgraph g of a graph is connected to the rest of the graph by one vertex, then this subgraph can be drawn inside of any region whose boundary contains the vertex.

Type 2: When a subgraph g is connected to the rest of the graph by either one or two vertices, then reversing g at the vertices (rotating 180^o).

For example, applying Type 1 to change location of subgraph g_1 in a graph in Fig.5.8(a) will result a graph in Fig.5.8(b). Also rotating a

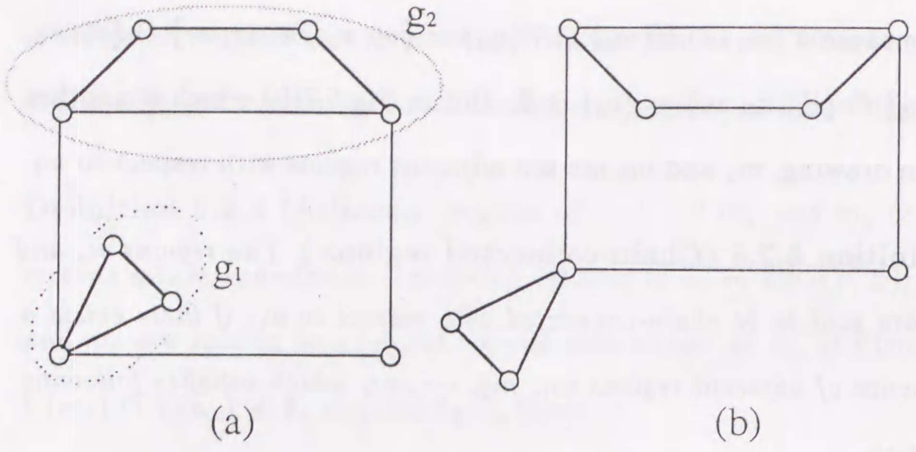


Figure 5.8: Applying planar transformation Type 1 and 2.

subgraph g_2 180^0 in a graph in Fig.5.8(a) by planar transformation of Type 2 will give a graph in Fig.5.8(b).

For a plane drawing of $G(V, E)$, if any two regions containing vertices of wild-component w_i are chain-connected with respect to w_i, there exists at least one structure of w_i which can be drawn on $D[G(V, E)]$ without crossing edges.

Suppose a W-graph contains only one wild-component w_1. Then,

Theorem 5.2.2 *Suppose there exists planar drawing $D[G(V, E)]$ where regions containing vertices of w_i are m_1, m_2, $\cdots$, m_k. If and only if any two regions m_a and m_b $(1 \leq a, b \leq k)$ are chain-connected with respect to w_1, $D[G(V, E)] \cup t_a^{(i)}$ is planar.*

Proof: Two regions m_a and m_b are chain-connected with respect to w_i so that $m_a @_1\ m_1 @_1\ m_2 @_1 \cdots @_1\ m_r @_1\ m_b$ hold. By Lemma 5.2.1, the vertices of a wild-component in a region can be connected by a planar structure in the region. Suppose any two vertices v_a and v_b of w_1 are in $V(m_a)$ and $V(m_b)$, respectively. Since m_a and m_b are chain-connected with respect to w_1, there exist an inner path between v_a and v_b passing the common vertices of these chain-connected regions $m_a, m_1, m_2, \cdots, m_r, m_b$. Hence, there exists at least one structure of w_1 which can be drawn on $D[G(V, E)]$ without crossing edges as shown in Fig.5.9.

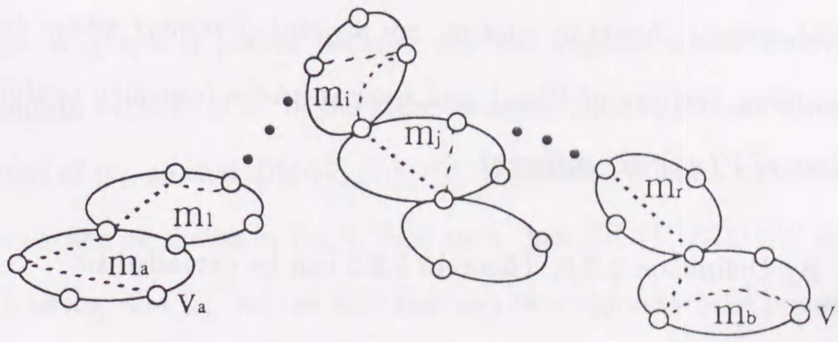


Figure 5.9: A structure of $t_a^{(1)}$ through chain-connected regions

If $D[G(V, E)] \cup t_a^{(1)}$ is planar, there exists a planar structure connecting all vertices of w_1. For any two regions m_p and m_q which contain vertices v_p and v_q of w_1, respectively, there is a path between v_p and v_q in the planar structure by Definition 2.1.1. If the path passes through the regions in sequence of m_p, m_1, m_2, $\cdots$, m_r, m_q, it is clear that two neighborhoods of these regions m_s and m_t satisfy $V(m_s) \cap V(m_t) \cap V(w_1) \neq \emptyset$. Therefore, any two regions in the sequence are chain-connected with respect to w_1. ■

It should be noticed that if two regions m_a and m_b whose boundaries contain vertices of $V(w_i)$ are not chain-connected with respect to w_i for every plane drawings of $G(V, E)$, it can be seen that $\Omega_w(V, E, W)$ is non-planar by Theorem 5.2.2.

Definition 5.2.6 (Disjonit wild-components) *In a W-graph, two wild-compo- nents w_i and w_j are disjoint if regions whose boundary contains vertices of $V(w_i)$ and regions whose boundary contains vertices of $V(w_j)$ are different.*

By Definition 5.2.6, Theorem 5.2.2 can be extended to:

Corollary 5.2.1 *Suppose a W-graph contains wild-components w_1, w_2, $\cdots$, w_k and any two of which are disjoint. There exist at least one planar drawing $D[G(V, E)]$ such that any two of w_1, w_2, $\cdots$, w_k satisfies Theorem 5.2.2, then the W-graph is planar.* ■

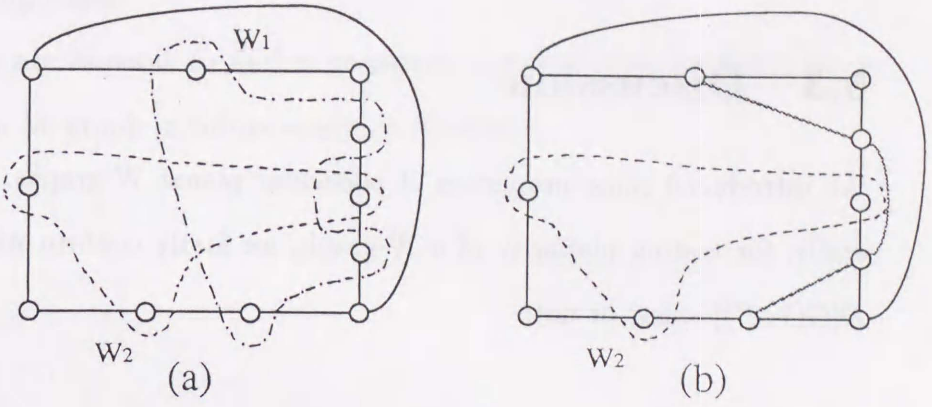


Figure 5.10: (a)A W-graph,(b)plane drawing of $D[G(V,E)] \cup t_a^{(1)} \cup t_a^{(2)}$

Example 5.2.2 A given W-graph $\Omega_w(V, E, W)$ containing two wild-components is shown as Fig.5.10(a). By Theorem 5.2.2, it can be seen that the W-graph is planar because any two regions whose boundaries contain vertices of w_1 in the plane drawing $D[G(V,E)]$ are chain-connected of w_1, so that $D[G(V,E)] \cup t_a^{(1)}$ is planar. When a structure of w_1 is chosen as shown in Fig.5.10(b) such that $D[G(V,E) \cup t_a^{(1)}]$ is a plane drawing. For w_2, we can find that any two regions whose boundaries contain the vertices of w_2 in Fig.5.10(b) are chain-connected of w_2. Hence, $D[G(V,E)] \cup t_a^{(1)} \cup t_a^{(2)}$ is planar. □

However, there is other example of an non-planar W-graph shown as Fig.5.7. The any two boundaries containing vertices of a wild-component are not chain-connected under any plane drawings. Hence, the W-graph is non-planar.

5.3 Discussion

We introduced some properties of particular planar W-graphs. Generally, for testing planarity of a W-graph, we firstly confirm whether $D[G(V, E)]$ exists or not.

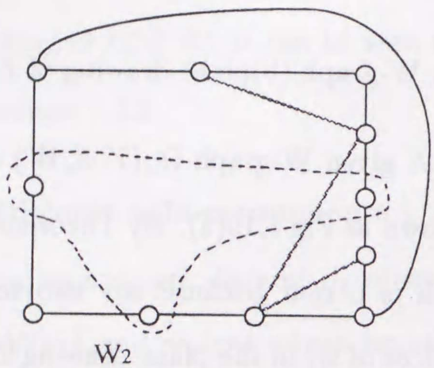


Figure 5.11: An example

It must be point out that there exist many planar drawing of $G(V, E)$ when $G(V, E)$ is planar. For testing whether a W-graph is planar or not by Theorem 5.2.2, we must check each wild-component one by one. It is difficult that we not only need to choose a suitable $D[G(V, E)]$ but also provide a proper tree $t_a^{(i)}$ for wild-component w_i so that $D[G(V, E) \cup t_a^{(i)}]$ can aid to check next wild-component. As an example, the W-graph in Fig.5.10(a) is planar. However, when we give a tree to w_1 as shown in

Fig.5.11, $D[G(V,E)\cup t_a^{(1)}]\cup t_a^{(2)}$ can not be drawn on a plane without crossing edges.

We are hopeful to find a necessary and sufficient condition for a planar W-graph in future study on W-graphs.

Chapter 6

Conclusions

In this dissertation, a new graph model containing unspecified edges, called a W-graph, has been presented. Because of existence of unspecified edges in wild components, a W-graph is a partially defined graph where we know that the structure of each wild-component is a tree but it is unspecified. In other words, except we know that there exists one and only one inner path between any two vertices in a wild-component, there are no other information available in a wild-component.

The reason why we introduce a W-graph is because there exist some partially defined systems arising in routing problem and communication net and so on. To describe such partially defined systems by an ordinary graph is impossible since the relation of edge and vertex in an ordinary graph must be specified. It therefore needs to introduce new graph model to satisfy these actual systems. Another reason that we study W-graphs is because W-graphs are partially known graphs and it is important to discuss the unknown part with limited known informations. The third reason is that W-graphs have many interesting and useful properties which can be provided without specifying the structure of each wild-components.

The main properties of W-graphs have been discussed from two aspects in this dissertation, one is in a W-graph (Chapter 3) and the other is between a W-graph and its derived graphs (Chapter 4). We summarize the main points of usefulness and results in this dissertation as follows:

1. W-circuits and W-cutsets can be defined in a W-graph though there are unspecified tree-structures.
2. A set of W-circuits (W-cutsets) including an empty set in a W-graph is an Abelian group under the W-ring sum operation of W-circuits (W-cutsets).
3. Matrix representation which is a convenient way of representing a W-graph algebraically are presented and a set of linearly independent W-circuits and W-cutsets can be obtained from a fundamental W-circuit matrix and a fundamental W-cutset matrix, respectively.

4. When a W-graph becomes a derived graph, W-trees become trees of the derived graph and a W-circuit (W-cutset) can become one and only one circuit (cutset) or edge disjoint unions of circuits (cutsets) of the derived graph. Particularly, when some W-circuits (W-cutsets) in a W-graph are linearly independent, we can obtain linearly independent circuits (cutsets) or edge disjoint unions of circuits (cutsets) of a derived graph from these W-circuits (W-cutsets) by Transformation Γ (Transformation Θ).

5. The relations between any two of derived graphs are established with respect to circuits and cutsets.

6. An approach and suggestions on routing problems by a way of W-graphs have been proposed though it is not a complete work in this dissertation. We wish to introduce and verity a W-graph as a new model to be able to be applied to this field. The planarities of some particular W-graphs have been discussed.

Finally, future research on W-graphs is briefly shown as follows:

- The property of a planar W-graph is very important not only in theories but also in applications. In order to test whether a W-graph is planar or not, a necessary and sufficient condition should be solved.

- Since the number of all derived graphs corresponding to a W-graph is very huge, to clear what common properties in all derived graphs is useful in graph theory.

- Comparing with the properties of W-graphs and ordinary graphs is useful for developing the theories of W-graphs. As an example, we known that for a W-graph there are no dual graph as those in an ordinary graph because the regions in a W-graph are unspecified. However, it is possible to define something similar to a dual graph in a W-graph because these exist planar W-graphs.

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Index of Definitions

adjacent region of w_i, 120

boundary of a region, 119

connected W-graph, 25

closed train, 33

chain-connected regions, 121

created vertex, 111

derived graph, 79

disjoint wild-component, 123

exhaustive W-cutset matrix, 60

exhaustive W-circuit matrix, 70

exhaustive W-circuit matrix, 70

fundamental W-cutset matrix, 66

fundamental W-circuit matrix, 71

general property, 27

inner path of w_i, 23

net, 108

planar W-graph, 116

plane drawing, 118

restricted property, 27

reference vertex of w_i, 50

specified tree $t_a^{(i)}$), 78

terminal set, 36

transformation Γ, 90

transformation Θ, 99

via, 107

wild-component, 22

W-graph, 24

W-circuit, 30

W-ring sum of W-circuit, 37

W-cutset, 42

W-ring sum of W-cutsets, 44

W-incidence set, 51

W-incidence matrix, 53

W-tree, 59