SUBRECURSIVE HIERARCHIES

AND

PROVABLY COMPUTABLE FUNCTIONS

IN FORMAL THEORIES OF ARITHMETIC

（帰納的関数の部分階層と数論の形式理論での証明可能な計算可能関数）

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CHAPTER 1

INTRODUCTION

The study of computable number-theoretic functions has produced two important notions for classifying them: one is that of *subrecursive hierarchies*, and the other one is that of *provably computable functions* in formal theories of arithmetic.

Subrecursive hierarchies have been developed in recursion theory (cf.Löb and Wainer[31], Wainer[48], Cichon and Wainer[5]). Each hierarchy consists of a sequence $\{f_\alpha\}$ of unary computable functions indexed with ordinals, in such a way that f_β *dominates* f_α (i.e.,

$$f_\alpha(x) < f_\beta(x)$$

for sufficiently large x) if $\alpha < \beta$. Computable functions are classified by this notion of domination.

On the other hand, *provably computable functions* are introduced in proof theory (cf. Kreisel[27], Kino[25], Buchholz and Wainer[4]). In a given formal theory T of arithmetic, we say that a computable function f is provably computable in T if the total-definedness of f, (or equivalently, termination of the algorithm for computing f) is provable in T.

In the present dissertation, we will study these two classifications of computable functions, i.e., subrecursive hierarchies and provably computable functions in formal theories of arithmetic, and clarify the relation between them.

A typical example of subrecursive hierarchies is obtained by a sequence $\{F_n\}_{n\in\mathbb{N}}$ of unary computable functions, indexed with n in the set $\mathbb{N}$ of all natural numbers (=$\{0,1,2,...\}$), which is defined as follows:

$$F_0(x) = x+1;$$

$$F_{n+1}(x) = F_n^{\,x+1}(x).$$

Here, the superscript $x+1$ means $(x+1)$-times *iteration* of F_α (i.e., if $f:\mathbb{N} \longrightarrow \mathbb{N}$, then $f^0(x) = x$, and $f^{n+1}(x) = f(f^n(x))$). Grzegorczyk(cf.Rose[36]) showed that each F_n is *primitive recursive* (cf.Definition 2.1.1), and any primitive recursive function f is *dominated* by F_n for some $n \in \mathbb{N}$, i.e., there is a number $m \in \mathbb{N}$ such that if $m < \max(x_1,\ldots,x_k)$ then

$$f(x_1,\ldots,x_k) < F_n(\max(x_1,\ldots,x_k)).$$

Then, we can measure a given primitive recursive function f by $n \in \mathbb{N}$, where n is the least integer such that F_n dominates f. Hence, the sequence $\{F_n\}_{n\in\mathbb{N}}$ classifies the set of all primitive recursive functions.

In order to extend this to a hierarchy $\{F_\alpha\}_{\alpha<I}$ indexed by ordinals less than a countable ordinal I, we consider an assignment of a sequence $\{\alpha[x]\}_{x\in\mathbb{N}}$ for each limit ordinal $\alpha < I$ which satisfies the following (a) and (b):

(a) $\alpha[0] < \alpha[1] < \cdots < \alpha[n] < \alpha[n+1] < \cdots < \alpha;$

(b) $\sup_{x\in\mathbb{N}} \alpha[x] = \alpha.$

We call this $\{\alpha[x]\}_{x\in\mathbb{N}}$ a *fundamental sequence* for α. Then, we define $\{F_\alpha\}_{\alpha<I}$ by transfinite induction on α as follows:

$$F_0(x) = x+1;$$

$$F_{\alpha+1}(x) = F_\alpha^{\,x+1}(x);$$

$$F_\alpha(x) = F_{\alpha[x]}(x) \quad \text{if } \alpha \text{ is a limit ordinal.}$$

We call $\{F_\alpha\}_{\alpha\in I}$ the *fast-growing hierarchy* (or *extended Grzegorczyk hierarchy*) for I.

For the first limit ordinal ω (=$\{0,1,2,\ldots\}$), we assign a fundamental sequence $\{\omega[x]\}_{x\in\mathbb{N}}$ by

$$\omega[x] = x \quad \text{for every} \quad x \in \mathbb{N}.$$

Then the function $F_\omega(x)$ $(= F_x(x))$ becomes a variant of famous Ackermann's function, which is computable but is *not* primitive recursive.

For the ordinal ε_0, Schwichtenberg[38] and Wainer[48] introduced a so-called standard system of fundamental sequences. The ordinal ε_0 is defined by the least α such that $\alpha = \omega^\alpha$, or

$$\varepsilon_0 = \sup_k \left.\omega^{\omega^{\cdot^{\cdot^{\cdot^{\omega}}}}}\right\} k\ \omega\text{'s}.$$

Each ordinal $0 < \alpha < \varepsilon_0$ is written in its Cantor normal form as

$$\alpha = \omega^{\alpha_1} + \omega^{\alpha_2} + \cdots + \omega^{\alpha_n},$$

where $\alpha > \alpha_1 \geqq \alpha_2 \geqq \cdots \geqq \alpha_n$. We define $\{\alpha[x]\}_{x\in\mathbb{N}}$ as follows:

if $\alpha_n = \beta+1$, then $\alpha[x] = \omega^{\alpha_1} + \cdots + \omega^{\alpha_{n-1}} + \omega^\beta\cdot x$;

if α_n is limit, then $\alpha[x] = \omega^{\alpha_1} + \cdots + \omega^{\alpha_{n-1}} + \omega^{\alpha_n[x]}$.

Then, they showed independently that $\{F_\alpha\}_{\alpha<\varepsilon_0}$ defined by this system classifies the set of ordinal recursive functions of finite order by Kreisel[27] (which we call here α-ordinal recursive functions for $\alpha < \varepsilon_0$), in such a way that, for each ordinal recursive function f, f is dominated by F_α for some $\alpha < \varepsilon_0$.

Provably computable functions are defined as the functions whose total-definedness can be provable in a given formal theory containing basic arithmetic (cf.Kino[25], Kreisel[27]). From Kleene's normal form theorem (cf.Kleene[26]), we can represent any computable function $f:\mathbb{N}^k \longrightarrow \mathbb{N}$ by a primitive recursive predicate A and function g so that

$$f(x_1,\ldots,x_k) = g(\mu y A(x_1,\ldots,x_k,y))$$

where $\mu y A(\ldots y)$ is the *minimization operator* which means the least y such that $A(\ldots y)$. Hence the formula

$$\forall x_1 \cdots \forall x_k \exists y A(x_1,\ldots,x_k,y)$$

expresses that $f(x_1,\ldots,x_k)$ is defined for every $x_1,\ldots,x_k$. This means the total-definedness of f. We say that f is *provably computable* in a theory T if

$$\forall x_1 \cdots \forall x_k \exists y A(x_1,\ldots,x_k,y) \text{ is provable in } T.$$

For the case of Peano arithmetic PA, we have the axioms of mathematical induction:

$$A(0) \wedge \forall x (A(x) \longrightarrow A(x+1)) \longrightarrow \forall x A(x),$$

where A is any formula of the language of arithmetic. Kreisel [27] showed that the set of all provably computable functions in PA is equal to the set of ordinal recursive functions of finite order. As we have mentioned above, the latter set can be classified by the fast-growing hierarchy $\{F_\alpha\}_{\alpha<\varepsilon_0}$. Hence, all functions provably computable in PA can also be classified by this hierarchy.

In 1977, Paris and Harrington[35] discovered a finite combinatorial statement PH which is *undecidable* in PA, i.e., neither PH nor $\neg PH$ are provable in PA. The statement PH is a variant of the finite Ramsey theorem.

Gödel's incompleteness theorem says the existence of undecidable statements in PA. The statement PH is the first example of finite combinatorial undecidable statements.

Ketonen and Solovay[24] gave an alternative proof of the undecidability of PH in PA, by establishing the equivalence of the Ramsey statement with the statement that F_{ε_0} is totally defined. Ono and Kadota[33] studied the relation between $\{F_\alpha\}_{\alpha<\varepsilon_0}$ and provably computable functions in *PA* in detail, and showed the provability and unprovability results on *PH* as applications (cf. Kadota and Ono[23], Kadota[16]).

Each sequence $\{f_\alpha\}_{\alpha<I}$ of computable functions with which we are concerned is defined by transfinite induction on α. In particular, for a limit α, it is defined by a fundamental sequence $\{\alpha[x]\}_{x\in\mathbb{N}}$ for α as

$$f_\alpha(x) = f_{\alpha[x]}(x).$$

Hence, in order to study subrecursive hierarchies, we need to investigate the systems of fundamental sequences. Schmidt[37] introduced the notion of built-upness on the systems to show that the hierarchy determined by a given fundamental sequences has the following properties.

Increase: Each f_α is strictly increasing.

Domination: If $\alpha < \beta < I$, then f_α is dominated by f_β.

Kadota and Aoyama[22] extended this to the notion of (n)-built-upness which can be applied a wider class of fundamental sequences (cf. also Aoyama and Kadota[1], Kadota[17], Kadota and Aoyama[21]).

In recent years, much attention has been paid to the relation between the fast-growing hierarchy $\{F_\alpha\}_{\alpha<I}$ and the *slow-growing* hierarchy $\{G_\alpha\}_{\alpha<I}$ which is defined as follows:

$$G_0(x) = 0;$$

$$G_{\alpha+1}(x) = G_\alpha(x) + 1;$$

$$G_\alpha(x) = G_{\alpha[x]}(x) \quad \text{if } \alpha \text{ is a limit ordinal.}$$

For the ordinal ω, the function G_ω is merely the identity one, since $G_\omega(x) = G_x(x) = x$. Compare this with the fact that F_ω is not primitive recursive. Now, the following problem arises:

> Is there an ordinal α so that the function G_α catches up with the function F_α? If there is such an ordinal α, give the minimum one.

The answer was given by Girard[12],[13]. Then he used the slow-growing hierarchy as an important tool for the study of the theory named Π^1_2-*logic*, introduced by him. From the results of Girard, Wainer[49],[50] gave such a minimum α, which is named τ and called a *subrecursive inaccessible*. Kadota[19],[20] studied this ordinal τ and gave a precise proof of the fact that τ is a minimum subrecursive inaccessible, by showing that $\{F_\alpha\}_{\alpha<\tau}$ has the increase and the domination properties considered above.

Wainer[49],[50] also stated that the fast-growing hierarchy $\{F_\alpha\}_{\alpha<\tau}$ classifies all provably computable functions in ID_n for every $n \in \mathbb{N}$, where ID_n is the theory of n-times iterated inductive definitions(cf.Buchholz[3]). Kadota[18] modified τ and introduced τ'. Then, he showed the similar results on $\{F_\alpha\}_{\alpha<\tau'}$ by using the proof-theoretic method developed by Buchholz[3].

In Chapter 2, we give basic notions on subrecursive hierarchies and provably computable functions, which are used through this dissertation. We summarize the results on the fast-growing hierarchy $\{F_\alpha\}_{\alpha<\varepsilon_0}$ and provably computable functions in Peano arithmetic PA. We also state the unprovability result in PA of the strong Ramsey statement.

In Chapter 3, we develop a basic theory on systems of fundamental sequences for treating subrecursive hierarchies more generally. We introduce the notion of (n)-built-upness ($n \in \mathbb{N}$) on the systems and study the increase and the domination properties of a sequences $\{f_\alpha\}_{\alpha<l}$ of number-theoretic functions. This notion is used in the later chapters.

In Chapter 4, we show the classifications of provably computable functions in fragments PA_n of PA by means of the fast-growing hierarchy up to ω_n. This result is a refinement of that in Chapter 2. We prove the provability and the unprovability results in PA_n of strong Ramsey statements. Then, we extend these results to provably Δ_m-functions.

In Chapter 5, we study the relation between the slow-growing and fast-growing hierarchies. We prove that the ordinal τ is minimum subrecursive inaccessible by showing that the system of fundamental sequences of τ is (3)-built-up. Then, we modify τ and introduce τ', and show the classification of provably computable functions in ID_n by means of the fast-growing hierarchy up to τ'.

In Chapter 6, we discuss some problems on subrecursive hierarchies and provably computable functions in formal theories of arithmetic. We also discuss some applications of our results.

CHAPTER 2

SUBRECURSIVE HIERARCHIES AND FORMAL THEORIES

In this chapter, we give some basic notions and results on subrecursive hierarchies and provably computable functions which are used throughout this dissertation.

To classify computable functions, we consider the following two approaches: one is to classify them by means of subrecursive hierarchies, and the other one is to classify them by means of the notion of provably computable functions.

In Section 2.1, we give basic definitions and facts on the fast-growing hierarchy $\{F_\alpha\}_{\alpha<\varepsilon_0}$, and state the relation with ordinal recursive functions by Wainer[48].

In Section 2.2, we summarize the notions on the provably computable functions in Peano arithmetic PA, and give the relation with the fast-growing hierarchy $\{F_\alpha\}_{\alpha<\varepsilon_0}$.

In Section 2.3, we state the unprovability result in PA of the strong Ramsey statement given by Paris and Harrington[35] using the results of Ketonen and Solovay[24].

2.1 Fast-growing hierarchy

Some of the essentials in classifying computable functions by subrecursive hierarchies are given as follows.

Let $\mathbb{N}$ be the set of all natural numbers $(=\{0,1,2,...\})$ and $f:\mathbb{N} \longrightarrow \mathbb{N}$ be a function. Let $f^m:\mathbb{N} \longrightarrow \mathbb{N}$ be the *iteration* of f m-times. More precisely, we define it by

$$f^0(x) = x \quad \text{and} \quad f^{m+1}(x) = f(f^m(x)).$$

Let us consider f^{x+1} for a given f. For example:

if $f(x) = x+1$, then $f^{x+1}(x) = 2x+1$;

if $f(x) = 2x$, then $f^{x+1}(x) = 2^{x+1}\cdot x$;

if $f(x) = 2^x$, then $f^{x+1}(x) = \left.2^{2^{\cdot^{\cdot^{2^x}}}}\right\}$ $x+1$ 2's.

Consider a sequence $\{F_n\}_{n\in\mathbb{N}}$ of unary number-theoretic functions defined inductively as follows:

$$F_0(x) = x + 1;$$
$$F_{n+1}(x) = F_n^{\,x+1}(x).$$

Then, the following relations hold: for all $x \in \mathbb{N}$,

$F_1(x) = 2x+1$; $F_2(x) \geqq 2^x$; $F_3(x) \geqq \left.2^{2^{\cdot^{\cdot^{2^x}}}}\right\}$ $x+1$ 2's.

Grzegorczyk showed in 1953(cf.Rose[36]) that all primitive recursive functions can be classified as shown in the following definition and proposition:

DEFINITION 2.1.1(*Primitive recursive functions*). The set of all *primitive recursive functions* is the smallest one of number-theoretic functions which contains the *zero* 0 (where $0(x) = 0$), the *successor* S (where $S(x) = x+1$), the *projections* I_i^k (where $I_i^k(x_1,\ldots,x_k) = x_i$ for $1 \leqq i \leqq k$), and is closed under

substitution: $f(\underline{x}) = g(h_1(\underline{x}),\ldots,h_k(\underline{x}))$;

primitive recursion: $f(0,\underline{x}) = g_1(\underline{x})$,
$f(y+1,\underline{x}) = g_2(y,\underline{x},f(y,\underline{x}))$

where $\underline{x}$ is a finite sequence of numbers $x_1,\ldots,x_n$. A predicate $R(x_1,\ldots,x_k)$ is *primitive recursive* if its *representing function* K_R is primitive recursive, where K_R is defined by

$$K_R(x_1,\ldots,x_k) = \begin{cases} 0 & \text{if } R(x_1,\ldots,x_k) \\ 1 & \text{if } \neg R(x_1,\ldots,x_k). \end{cases}$$

Let $f:\mathbb{N}^k \longrightarrow \mathbb{N}$ and $g:\mathbb{N} \longrightarrow \mathbb{N}$ be functions. We say that f is *dominated* by g when there is a number $m \in \mathbb{N}$ such that if $m < \max(x_1,\ldots,x_k)$ then

$$f(x_1,\ldots,x_k) < g(\max(x_1,\ldots,x_k)).$$

PROPOSITION 2.1.2(cf.Rose[36]). The following properties hold:

(a) For each $n \in \mathbb{N}$, the function F_n is primitive recursive.

(b) For each primitive recursive function $f:\mathbb{N}^k \longrightarrow \mathbb{N}$, there is an $n \in \mathbb{N}$ such that f is dominated by F_n. □

From this proposition, we can classify all primitive recursive functions by means of $\{F_n\}_{n\in\mathbb{N}}$ as follows.

DEFINITION 2.1.3(*Elementary closure*). Let C be a set of number-theoretic functions. The *elementary closure* of C, denoted by $\mathcal{E}(C)$, is the smallest set which contains all functions in C, the zero, successor, projections, and is closed under substitution and the following *limited primitive recursion*:

$$f(0,\underline{x}) = g_1(\underline{x});$$
$$f(y+1,\underline{x}) = g_2(y,\underline{x},f(y,\underline{x}));$$
$$f(\underline{x}) \leqq g_3(\underline{x}).$$

Each function in $\mathcal{E}(C)$ is *elementary recursive in* C. If $C = \{f\}$, then $\mathcal{E}(C)$ is written as $\mathcal{E}(f)$. The set $\mathcal{E}(F_n)$ is written as $\mathcal{F}_n$.

Since any function in $\mathcal{F}_n$ is dominated by F_{n+1}, we have the following relation:

$$\mathcal{F}_0 \subsetneqq \mathcal{F}_1 \subsetneqq \cdots \subsetneqq \mathcal{F}_n \subsetneqq \mathcal{F}_{n+1} \subsetneqq \cdots.$$

Moreover, from Proposition 2.1.2, it is easy to see that $\bigcup_{n\in\mathbb{N}} \mathcal{F}_n$

is equal to the set of all primitive recursive functions. Hence $\{F_n\}_{n\in\mathbb{N}}$ gives a classification of all primitive recursive functions.

We are now considering the problem how we can classify computable functions in a larger set. For this problem, it is natural to consider extension of $\{F_n\}_{n\in\mathbb{N}}$ to $\{F_\alpha\}_{\alpha<I}$ where I is an countable ordinal, analogously to the above discussion.

For the definition and basic notions of ordinals, see e.g., Levy[30]. We identify the set $\mathbb{N}$ with the first infinite ordinal ω (i.e., identify $n \in \mathbb{N}$ with $n < \omega$).

Let I be a countable ordinal and let $\mathrm{Lim}(I)$ be the set of all limit ordinals less than I. As we considered in Chapter 1, we consider here $P:\mathrm{Lim}(I) \longrightarrow I^\omega$ which assigns a sequence $\{\alpha[x]\}_{x\in\mathbb{N}}$ for each limit $\alpha < I$, which satisfies the following conditions:

(a) $\alpha[0] < \alpha[1] < \cdots < \alpha[n] < \alpha[n+1] < \cdots < \alpha$;

(b) $\sup_{x\in\mathbb{N}} \alpha[x] = \alpha$.

Then, we call $\{\alpha[x]\}_{x\in\mathbb{N}}$ a *fundamental sequence* for α, and this assignment P a *system of fundamental sequences for* I.

From this notion, we can extend $\{F_n\}_{n\in\mathbb{N}}$ to $\{F_\alpha\}_{\alpha<I}$. We fix a system of fundamental sequences for an countable ordinal I.

DEFINITION 2.1.4. The *first-growing* hierarchy (or *extended Grzegorczyk hierarchy*) $\{F_\alpha\}_{\alpha<I}$ is defined inductively as:

$$F_0(x) = x+1;$$

$$F_{\alpha+1}(x) = F_\alpha^{\,x+1}(x);$$

$$F_\alpha(x) = F_{\alpha[x]}(x) \quad \text{if } \alpha \text{ is limit.}$$

We notice here that F_α depends on the choice of fundamental sequences for α. We stated in Chapter 1 the definition of *standard* system of fundamental sequences for ε_0. In the case of this standard system, the following lemma holds:

LEMMA 2.1.5. Let P be the standard system of fundamental sequences. Then the following properties hold:

(a) For every $\alpha < \varepsilon_0$, F_α is strictly increasing.

(b) If $\alpha < \beta < \varepsilon_0$, then F_α is dominated by F_β.

This lemma says that the fast-growing hierarchy up to ε_0 defined by the standard system of fundamental sequences has the *increase* and the *domination* properties mentioned in Chapter 1. The proof of this lemma will be given in Chapter 3 in a more general situation.

Next, we state the relation between the fast-growing hierarchy up to ε_0 defined by the standard system of fundamental sequences and α-ordinal recursive functions for $\alpha < \varepsilon_0$. The set of α-ordinal recursive functions is an extension of that of primitive recursive functions, which has been studied by Kreisel[27] and are called ordinal recursive functions of finite order by him. We give a definition of this class, following Wainer[48]:

Define the ordinal $\omega_n(m)$ for $n,m \in \mathbb{N}$ inductively by

$$\omega_0(m) = m, \qquad \omega_{n+1}(m) = \omega^{\omega_n(m)}.$$

We write ω_n for $\omega_n(1)$. For each $0 < k \in \mathbb{N}$, $<_k$ denotes the primitive recursive well-ordering on $\mathbb{N}$ of order-type ω_k. For the precise definition, see §3 of Wainer[48]. For each $x \in \mathbb{N}$, $\mathrm{ord}_n(x)$ is the ordinal represented by x in the well-ordering $<_n$ and conversely, for each ordinal $\alpha < \omega_n$, $\mathrm{num}_n(\alpha)$ is the unique $x \in \mathbb{N}$ such that $\mathrm{ord}_n(x) = \alpha$.

Let $\alpha < \varepsilon_0$ and n be the smallest integer such that $\alpha < \omega_n$.

DEFINITION 2.1.6(*α-ordinal recursive functions*). The set of *α-ordinal recursive functions* $\mathcal{U}(\alpha)$ is the smallest one which contains all primitive recursive functions and is closed under substitution and the following *unnested α-recursion*:

$$f(0,\underline{u}) = g_0(\underline{u});$$
$$f(x,\underline{u}) = g_1(x,\underline{u},f(h(x,\underline{u}),\underline{u})) \quad \text{if } 0 <_n x,$$

where $h(x,\underline{u}) <_n x$ whenever $0 <_n x <_n \mathrm{num}_n(\alpha)$, and $h(x,\underline{u}) = 0$ otherwise.

As in Definition 2.1.3, we write $\mathcal{F}_\alpha$ for $\mathcal{E}(F_\alpha)$ when $\alpha < \varepsilon_0$. Wainer[48] showed the following proposition:

PROPOSITION 2.1.7. For each ordinal α such that $0 < \alpha < \varepsilon_0$,

$$\mathcal{U}(\omega^\alpha) = \bigcup_{\beta<\alpha\cdot\omega} \mathcal{F}_\beta .$$

In particular, if $n > 0$, then

$$\bigcup_{m<\omega} \mathcal{U}(\omega_n(m)) = \bigcup_{\beta<\omega_n} \mathcal{F}_\beta . \qquad \square$$

REMARK. The set $\bigcup_{m<\omega} \mathcal{F}_m$ is that of all primitive recursive functions. By this theorem, this is also equivalent to the set $\bigcup_{m<\omega} \mathcal{U}(\omega^m)$, since $\omega_1(m) = \omega^m$.

2.2 Provable computability

In this section, we define the notion of provably computable functions in formal theories of arithmetic.

From Church's thesis, the set of computable functions is equivalent to the set of recursive functions. The set of recursive functions is defined as the smallest one which contains the projections, addition $+$, multiplication $\cdot$, and representing function $K_<$ of $<$ (see Definition 2.1.1), and is closed under substitution and the following *minimalization*:

$$f(\underline{x}) = \mu y(g(\underline{x},y) = 0) \quad \text{if } \forall\underline{x}\exists y(g(\underline{x},y) = 0).$$

Here, $\mu y(\ldots y\ldots)$ means the least number y such that $(\ldots y\ldots)$, and $\underline{x}$ denotes the sequence $x_1,\ldots,x_k$.

By minimalization, we can generate a new recursive function f under the condition that the predicate

$$\forall\underline{x}\exists y(g(\underline{x},y) = 0).$$

The truth of this formula guarantees the total-definedness of f. However, in order to know that this formula is true, we must *prove* this formula in some way. That is, the proof must be carried out in some formal theory. By formalizing this situation, we define provably computable functions.

Here, we sketch some basic notions on formal theories of arithmetic. For more precise definitions, see e.g., Shoenfield [42] and Takeuti[46,Chapter 2]. We consider Peano arithmetic *PA*, which formalizes classical number-theory and is defined as follows.

The language $\mathcal{L}(PA)$ is the first-order one whose non-logical symbols are the constant 0, the function symbols S (successor), + and ·, and the predicate symbols = and <.

The non-logical axioms of *PA* are as follows:

PA1. $\neg(Sx = 0)$.

PA2. $Sx = Sy \longrightarrow x = y$.

PA3. $x + 0 = x$.

PA4. $x + Sy = S(x+y)$.

PA5. $x \cdot 0 = 0$.

PA6. $x \cdot Sy = (x \cdot y) + x$.

PA7. $\neg(x < 0)$.

PA8. $x < Sy \leftrightarrow x < y \lor x = y$.

PA9. $x < y \lor x = y \lor y < x$.

PA10. *Mathematical inductions*:

$$A(0) \land \forall x(A(x) \longrightarrow A(Sx)) \longrightarrow \forall x A(x),$$

where A is any formula of $\mathcal{L}(PA)$, and A is called an *induction formula*.

The logical system of *PA* is the first-order classical logic with equality axioms.

In *PA*, we can treat only elementary number-theoretic statements, but *PA* is strong enough to prove them (cf.Simpson[43]). Actually, the theory of primitive recursive functions can be translated into *PA* (cf.Shoenfield[42,Section 8.1] and Takeuti [46,Proposition 10.6]). Hence we will assume that $\mathcal{L}(PA)$ contains the function symbols for primitive recursive functions and *PA* contains their defining equations for axioms. Also, we will assume that $\mathcal{L}(PA)$ contains predicate symbols for some primitive recursive predicates and *PA* contains their defining formulas for axioms.

However, for PA, the following Gödel's incompleteness theorem holds (cf.Gödel[14], cf.also Takeuti[47] for details).

PROPOSITION 2.2.1(*Gödel's incompleteness theorem*). For any axiomatized extension T of PA, if T is consistent, then for some sentence A, neither A nor $\neg A$ is provable in T. Moreover the formula Cons(T) which asserts the consistency of T is not provable in PA. □

Now we define provably computable functions. By the normal form theorem (cf.Kleene[26]), there are a primitive recursive function U and a primitive recursive predicate T_n for $n \in \mathbb{N}$ such that, for any computable function $f:\mathbb{N}^n \longrightarrow \mathbb{N}$, there is an $e \in \mathbb{N}$ (which is called a Gödel number of f) such that

(a) $\forall \underline{x} \exists y T_n(e, \underline{x}, y)$;

(b) $f(\underline{x}) = U(\mu y T_n(e, \underline{x}, y))$.

Here, the predicate $\forall \underline{x} \exists y T_n(e, \underline{x}, y)$ expresses the total-definedness of the function f as we mentioned above. For the technical reason, we will fix a canonical construction for the predicate T_n (e.g., Kleene[26], Shoenfield[42,Section 7.4]). Thus, we give the following definition. Let T be a formal theory of arithmetic which contains PA. We write $\bar{e}$ for the *numeral* of e defined by SS··S0 with e occurrences of S.

DEFINITION 2.2.2(*Provably computable functions*). A computable function f is *provably computable* in T if the formula

$$\forall \underline{x} \exists y T_n(\bar{e}, \underline{x}, y)$$

is provable in T, where e is a Gödel number of f and T_n in this formula is the predicate symbol which expresses the predicate T_n.

Next, we consider the classification of provably computable functions by means of the fast-growing hierarchy up to ε_0. First we notice the following result of Kreisel[27].

PROPOSITION 2.2.3. Let f be a computable function. Then, f is provably computable in PA if and only if f belongs to $\mathcal{U}(\alpha)$

for some $\alpha < \varepsilon_0$. □

Then the following proposition is obtained immediately from Proposition 2.2.3, and Proposition 2.1.7.

PROPOSITION 2.2.4. Let f be a computable function. Then, f is provably computable in *PA* if and only if f belongs to $\mathcal{F}_\alpha$ for some $\alpha < \varepsilon_0$. □

This proposition shows that provably computable functions in *PA* can be classified by the hierarchy $\mathcal{F}_\alpha$ $(\alpha < \varepsilon_0)$.

2.3 Undecidable statements

In 1977, Paris and Harrington[35] showed that a strong version of finite Ramsey theorem is true but unprovable in *PA*. To explain this result, let us define some notations.

For a set $A \subset \mathbb{N}$ and an $n \in \mathbb{N}$, define

$$A^{[n]} = \{B \subseteq A \mid \mathrm{card}(B) = n\},$$

where card(B) is the number of elements in B. Let f be a function from $A^{[n]}$ to a set X. Then, a set $H \subseteq A$ is *homogeneous* for f if f is constant on $H^{[n]}$. A set $H \subseteq \mathbb{N}$ is *large* if

$$\mathrm{card}(H) \geq \min(H),$$

where $\min(H)$ is the smallest element of H. For any $k, m \in \mathbb{N}$, $[k, m]$ is the set $\{x \in \mathbb{N}: k \leq x \leq m\}$. For $c, k, m, n \in \mathbb{N}$, the predicate, which we call *the Ramsey relation*,

$$[k, m] \underset{*}{\longrightarrow} (n+1)^n_c$$

means that for every $f:[k, m]^{[n]} \longrightarrow \{0, 1, \ldots, c-1\}$, there is a set $H \subset [k, m]$ such that

(a) $\mathrm{card}(H) \geq n + 1$,

(b) H is homogeneous,

(c) H is large.

Here, we remark that the Ramsey relation is primitive recursive. Hence, it is expressed by a predicate symbol in *PA* (cf. the definition of *PA*). We define the formula *PH* as

$$\forall x \forall z \forall w \exists y([x, y] \underset{*}{\longrightarrow} (z+1)^{z}_{w}).$$

The following proposition can be shown by using infinite Ramsey theorem (cf. Paris and Harrington[35, p.1135]).

PROPOSITION 2.3.1. *PH* is true (i.e., for every $c, k, n \in \mathbb{N}$, there is an $m \in \mathbb{N}$ such that $[k, m] \underset{*}{\longrightarrow} (n+1)^{n}_{c}$ holds). □

By this proposition, we define a recursive function $\sigma_{n,c}$ by

$$\sigma_{n,c}(x) = \mu y([x, y] \underset{*}{\longrightarrow} (n+1)^{n}_{c}).$$

Then, the following lemma can be easily shown.

LEMMA 2.3.2. If $c \leqq c'$, $k \leqq k'$, then $\sigma_{n,c}(k) \leqq \sigma_{n,c'}(k')$. □

Ketonen and Solovay[24] showed the following proposition:

PROPOSITION 2.3.3.(a) Let $n > 1$, $c > 1$ and $x > 3$. Then,

$$\sigma_{n,c}(x) \leqq F_{\omega_{n-2}(c+5)}(x).$$

(b) Let $n > 1$. For any increasing function f, f is dominated by $\sigma_{n,c}$ for some c if and only if f is dominated by F_{α} for some $\alpha < \omega_{n-1}$. □

By using this proposition, Ketonen and Solovay[24] proved the unprovability of *PH* in *PA*, by a proof-theoretic method. We will show this in Section 4.2 as a corollary (Corollary 4.2.2) of our study on provably computable functions. Here, we state this statement as a proposition, and give a proof by using Theorem 4.1.2.

PROPOSITION 2.3.4. *PH* is unprovable in *PA*.

Proof. Assume contrarily that *PH* is provable in *PA*. Then, by Theorem 4.1.2, the function $\lambda z.\lambda w.\lambda x.\sigma_{z,w}(x)$ is provably computable in PA. Hence, it belongs to $\mathcal{U}(\alpha)$ for some $\alpha < \varepsilon_0$ by Proposition 2.2.3. From Proposition 2.1.7, it belongs also to $\mathcal{F}_\beta$ for some $\beta < \varepsilon_0$. Then, it is dominated by $F_{\beta+1}$ since every function in $\mathcal{F}_\beta$ is dominated by $F_{\beta+1}$. Therefore, by Proposition 2.3.2, it is dominated by the function $\lambda x.\sigma_{n,c}(x)$ for some $n,c \in \mathbb{N}$, and hence there is m such that $\sigma_{z,w}(x) < \sigma_{n,c}(\max(z,w,x))$ for all z,w,x such that $m < \max(z,w,x)$. Put $z = n$, $w = c$, and $x = \max(n,c,m+1)$. Then we have $\max(z,w,x) = x$. Thus, the relation

$$\sigma_{z,w}(x) < \sigma_{n,c}(\max(z,w,x)) = \sigma_{z,w}(x)$$

is led to the contradiction. □

We will study this undecidable sentence of PA in more detail in Chapter 4. There, we will give a refinement of this argument by considering it in some fragments of PA.

CHAPTER 3

BUILT-UP SYSTEMS OF FUNDAMENTAL SEQUENCES

In Chapter 2, we considered the fast-growing hierarchy up to ε_0 as a tool for classifying a subclass of computable functions or provably computable functions in *PA*.

In this chapter, we consider more general situation on the hierarchy of number-theoretic functions. This consideration is useful for classifying large subclasses of number-theoretic functions, because our general theory is applicable to the study on such classification by hierarchies. The results of this chapter were obtained by Kadota and Aoyama[21].

In Section 3.1, we introduce (n)-built-up systems of fundamental sequences. Then, we study properties such as increase and domination of the fast-growing hierarchies defined by (n)-built-up systems.

In Section 3.2, we clarify relations between conditions on systems of fundamental sequences considered in the literature, comparing with (n)-built-up systems. We examine the results in Section 3.1 under some conditions weaker than (n)-built-upness.

In Section 3.3, we study the existence problem on systems of fundamental sequences for the first uncountable ordinal Ω, under the conditions which are considered in Section 3.2.

3.1 Growing hierarchies on (n)-built-up systems

In Chapter 2, we studied the fast-growing hierarchy up to ε_0 defined by the standard system of fundamental sequences. Here we study sequences of unary number-theoretic functions defined by transfinite induction such as the fast-growing hierarchy.

Let l be a countable ordinal and let P be a system of fundamental sequences for l (as for the definition of systems, cf.

Chapter 2.1). For a sequence $\{f_\alpha\}_{\alpha<I}$ of number-theoretic functions, we consider the following conditions:

(f1) f_0 is strictly increasing;

(f2) $f_{\alpha+1}$ is defined from f_α so that, if f_α is strictly increasing, then $f_{\alpha+1}$ is also strictly increasing,

$$f_\alpha(0) \leqq f_{\alpha+1}(0) \text{ and } f_\alpha(x) < f_{\alpha+1}(x) \text{ for } x > 0;$$

(f3) $f_\alpha(x) = f_{\alpha[x]}(x)$ for all $x \in \mathbb{N}$, if α is a limit,

where $\alpha[x]$ is the x-th element of a fundamental sequence for α.

For a given system P for I, we can define sequences $\{f_\alpha\}_{\alpha<I}$ which satisfy the above conditions (f1)-(f3), as the following examples show:

EXAMPLES 3.1.2.(a) The fast-growing hierarchy $\{F_\alpha\}_{\alpha<I}$ (cf. Section 2.1) satisfies the conditions (f1)-(f3), e.g., we obtain (f2) as follows. It is easily shown that $x < F_\alpha(x)$ for all $x \in \mathbb{N}$ by induction on $\alpha \in I$. Hence, if F_α is strictly increasing, then $F_\alpha{}^n(x) < F_\alpha{}^n(x+1)$ and thus

$$F_{\alpha+1}(x) = F_\alpha{}^{x+1}(x) < F_\alpha{}^{x+1}(x+1) < F_\alpha{}^{x+2}(x+1) = F_{\alpha+1}(x+1).$$

Therefore (f2) is obtained.

(b) The *Hardy hierarchy* $\{H_\alpha\}_{\alpha<I}$ is defined inductively as:

$$H_0(x) = x; \quad H_{\alpha+1}(x) = H_\alpha(x+1);$$
$$H_\alpha(x) = H_{\alpha[x]}(x) \quad \text{for limit } \alpha.$$

This hierarchy satisfies the conditions (f1)-(f3). This can be easily shown by induction on α.

Now, we are concerned with the problem that under what conditions on systems of fundamental sequences, $\{f_\alpha\}_{\alpha<I}$ satisfying (f1)-(f3) have the increase and domination properties:

Increase: For each $\alpha \in I$, f_α is strictly increasing.

Domination: For each $\alpha < \beta \in I$, then f_α is dominated by f_β.

Concerning this problem, Schmidt[37] introduced built-up systems of fundamental sequences and proved that if a system is built-up, then $\{f_\alpha\}_{\alpha<I}$ has the increase and domination properties. However, there are some important systems used in the literature which are not built-up. In particular, the standard system for ε_0 is not built-up. Then, Kadota and Aoyama[22] introduced a stronger notion of built-upness, called (n)-built-upness for each $n \in \mathbb{N}$ so that it can be applicable to a wide class of systems, and studied the increase and domination properties of the fast-growing hierarchy determined by (n)-built-up systems.

In the remaining of this section, we will explain these results of Kadota and Aoyama[22].

DEFINITION 3.1.3. Let $n \in \mathbb{N}$ and let P be a system for I. The relation $\xrightarrow[n]{}$ on I is the transitive closure of

(a) $\alpha+1 \xrightarrow[n]{} \alpha$;

(b) $\alpha \xrightarrow[n]{} \alpha[n]$ for a limit ordinal α.

The relation $\alpha \overset{}{\underset{n}{\Longrightarrow}} \beta$ means $\alpha \xrightarrow[n]{} \beta$ or $\alpha = \beta$.

In other word, $\alpha \xrightarrow[n]{} \beta$ means that there is a sequence $\{\gamma_i \mid 0 \leqq i \leqq k, 0<k\}$ such that $\gamma_0=\alpha$, $\gamma_k=\beta$, and $\gamma_i[n]=\gamma_{i+1}$ if γ_i is limit or $\delta = \gamma_{i+1}$ if γ_i is $\delta+1$ for $0 \leqq i < k$.

Now, we define (n)-built-up systems as follows (cf.Kadota and Aoyama[22]):

DEFINITION 3.1.4((n)-*built-up systems*). A system P for I is (n)-*built-up* if

$$\alpha[x+1] \xrightarrow[n]{} \alpha[x]$$

for any limit $\alpha < I$ and $x \in \mathbb{N}$.

In particular, (0)-built-up systems for I in our sense are just the same as built-up systems for I in Schmidt's sense [37]. We will show the following theorem by Kadota and Aoyama[22, Theorem 3.1].

THEOREM 3.1.5. If $\{f_\alpha\}_{\alpha<I}$ satisfies the conditions (f1), (f2) and (f3), and if P is (1)-built-up, then the following properties hold:

(a) f_α is strictly increasing for each $\alpha \in I$.

(b) If $\alpha \xrightarrow[m]{} \beta$ and $m > 0$, then $f_\beta(m) \leqq f_\alpha(m)$ and $f_\beta(x) < f_\alpha(x)$ for $m < x \in \mathbb{N}$.

(c) If $\beta < \alpha$, then f_β is dominated by f_α.

Before proving this theorem, we show a lemma (cf.Kadota and Aoyama[22,Lemma 2.3]).

LEMMA 3.1.6. Let $n \in \mathbb{N}$ and let P be an (n)-built-up system for I. Then the following properties hold:

(a) If $\alpha \xrightarrow[m]{} \beta$ and $m,n \leqq s \in \mathbb{N}$, then $\alpha \xrightarrow[s]{} \beta$.

(b) If $\beta < \alpha$, then $\alpha \xrightarrow[m]{} \beta$ for some $m \geqq n$.

Proof. We show this by induction on α. (a) Assume $\alpha \xrightarrow[m]{} \beta$ and $m,n \leqq s$. Then $\alpha[m] \xRightarrow[m]{} \beta$. Since P is (n)-built-up, $\alpha[s] \xRightarrow[n]{} \alpha[m]$. Hence $\alpha[s] \xRightarrow[s]{} \alpha[m] \xRightarrow[s]{} \beta$ by the induction hypothesis. Therefore $\alpha \xrightarrow[s]{} \beta$. (b) Assume $\beta < \alpha$. Then $\beta \leqq \alpha[m_0]$ for some $m_0 \geqq n$. Hence $\alpha[m_0] \xRightarrow[k]{} \beta$ for some $k \geqq n$ by the induction hypothesis. If we put $m = \max(m_0, k)$, then $\alpha \xrightarrow[m]{} \alpha[m_0] \xRightarrow[m]{} \beta$ by (a). Hence $\alpha \xrightarrow[m]{} \beta$. □

By this lemma, if the system P is (n)-built-up and $n < k$, then P is (k)-built-up. Then, we can prove Theorem 3.1.5 as follows:

Proof of Theorem 3.1.5. First, we show (a) and (b) by induction on α.

Case 1. $\alpha = 0$: (a) holds by (f1). (b) is trivial.

Case 2. $\alpha = \gamma + 1$: (a) By the induction hypothesis, f_γ is strictly increasing, so is f_α from (f2). (b) If $\alpha \underset{m}{\longrightarrow} \beta$, then $\gamma \underset{m}{\Longrightarrow} \beta$. By the induction hypothesis and (f2), $f_\beta(m) \leqq f_\gamma(m) \leqq f_\alpha(m)$ and $f_\beta(x) \leqq f_\gamma(x) < f_\alpha(x)$ for $m < x$.

Case 3. α is limit: (a) Since P is (1)-built-up, $\alpha[x+1] \underset{1}{\longrightarrow} \alpha[x]$. So, $f_\alpha(x+1) = f_{\alpha[x+1]}(x+1) \geqq f_{\alpha[x]}(x+1) > f_{\alpha[x]}(x) = f_\alpha(x)$ from the induction hypothesis. (b) If $\alpha \underset{m}{\longrightarrow} \beta$, then $\alpha[m] \underset{m}{\Longrightarrow} \beta$. By the induction hypothesis, $f_\beta(m) \leqq f_{\alpha[m]}(m) = f_\alpha(m)$ for $m > 0$. Moreover, since P is (1)-built-up, $\alpha[x] \underset{1}{\longrightarrow} \alpha[m]$ for $x > m$. Hence $\alpha[x] \underset{m}{\longrightarrow} \alpha[m]$ by Lemma 3.1.6. Thus, $f_\beta(x) \leqq f_{\alpha[m]}(x) < f_{\alpha[x]}(x) = f_\alpha(x)$ for $x > m \geqq 1$ by the induction hypothesis.

We show (c). If $\beta < \alpha$, then $\alpha \underset{m}{\longrightarrow} \beta$ for some $m > 0$ by Lemma 3.1.6. By (b), f_β is dominated by f_α. □

The following proposition says that Theorem 3.1.5 can be applied to the standard system of fundamental sequences for ε_0 (cf. Ketonen and Solovay[24]).

PROPOSITION 3.1.7. The standard system of fundamental sequences for ε_0 is (1)-built-up.

To prove this proposition, recall that the standard system is defined as follows: For a limit $\alpha < \varepsilon_0$, we write α to the Cantor normal form:

$$\alpha = \omega^{\alpha_1} + \cdots + \omega^{\alpha_{k-1}} + \omega^{\alpha_k} \quad (\alpha > \alpha_1 \geqq \cdots \geqq \alpha_k).$$

Then we define $\alpha[x]$ for $x \in \mathbb{N}$ as follows:

If $\alpha_k = \beta + 1$, then $\alpha[x] = \omega^{\alpha_1} + \cdots + \omega^{\alpha_{k-1}} + \omega^\beta \cdot x$.

If α_k is limit, then $\alpha[x] = \omega^{\alpha_1} + \cdots + \omega^{\alpha_{k-1}} + \omega^{\alpha_k[x]}$.

Then the following lemma can be shown by induction on α.

LEMMA 3.1.8. (a) For each $n \in \mathbb{N}$ and $\alpha > 0$, $\alpha \underset{n}{\longrightarrow} 0$.
(b) If $\alpha \leqq \gamma$ and $\alpha \underset{n}{\longrightarrow} \beta$ for some $n \in \mathbb{N}$, then $\gamma + \alpha \underset{n}{\longrightarrow} \gamma + \beta$.
(c) If $\alpha \underset{n}{\longrightarrow} \beta$ for some $n \in \mathbb{N}$, then $\omega^\alpha \underset{n}{\longrightarrow} \omega^\beta$. □

Proof of Proposition 3.1.7. For a limit α, write $\alpha = \gamma +$

ω^{α_k} in the above Cantor normal form where $\gamma = \omega^{\alpha_1} + \cdots + \omega^{\alpha_{k-1}}$. Then we show $\alpha[x+1] \underset{1}{\longrightarrow} \alpha[x]$ for $x \in \mathbb{N}$ by induction on α. We have the following two cases:

(i) If $\alpha_k = \beta + 1$, then $\alpha[x+1] = \gamma + \omega^{\beta}\cdot(x+1) \underset{1}{\longrightarrow} \gamma + \omega^{\beta}\cdot x = \alpha[x]$ by Lemma 3.1.8(a) and (b).

(ii) If α_k is limit, then $\alpha[x+1] = \gamma + \omega^{\alpha_k[x+1]} \underset{1}{\longrightarrow} \gamma + \omega^{\alpha_k[x]} = \alpha[x]$ by the induction hypothesis that $\alpha_k[x+1] \underset{1}{\longrightarrow} \alpha_k[x]$ since $\alpha_k < \alpha$ and by Lemma 3.1.8(c). □

From this proposition and Theorem 3.1.5, the proof of Lemma 2.1.5 can be obtained, which states that the fast-growing hierarchy up to ε_0 defined by the standard system has the increase and domination properties.

Next, we consider an extension of Theorem 3.1.5 where P is $(n+1)$-built-up for some $n \in \mathbb{N}$. We can prove the following Theorem 3.1.9 which is a relativization of Theorem 3.1.5(cf. Kadota and Aoyama[22,p.361]) by the same way as Theorem 3.1.5.

Consider the following conditions for each $n \in \mathbb{N}$:

(f1)$_n$ f_0 is strictly increasing after n (i.e., $f_0(x) < f_0(x+1)$ for $n \leqq x$).

(f2)$_n$ If f_α is strictly increasing after n, then so is $f_{\alpha+1}$ and $f_\alpha(n) \leqq f_{\alpha+1}(n)$, $f_\alpha(x) < f_{\alpha+1}(x)$ for $n < x$.

(f3)$_n$ $f_\alpha(x) = f_{\alpha[x]}(x)$ for $n \leqq x$ if α is a limit ordinal.

THEOREM 3.1.9. Let $n \in \mathbb{N}$. If $\{f_\alpha\}_{\alpha<I}$ satisfies conditions (f1)$_n$,(f2)$_n$ and (f3)$_n$, and if P is $(n+1)$-built-up, then the following holds:

(a) f_α is strictly increasing after n for each $\alpha \in I$.

(b) If $\alpha \underset{m}{\longrightarrow} \beta$ and $m > n$, then $f_\beta(m) \leqq f_\alpha(m)$ and $f_\beta(x) < f_\alpha(x)$ for $m < x \in \mathbb{N}$.

(c) If $\alpha < \beta$, then f_α is dominated by f_β. □

The case that $n = 0$ of this theorem is just the same as

Theorem 3.1.5.

EXAMPLES 3.1.10.(a) The fast-growing hierarchy $\{F_\alpha\}_{\alpha<I}$ satisfies the conditions $(f1)_n$, $(f2)_n$ and $(f3)_n$ for every $n \in \mathbb{N}$ as in Examples 3.1.2.

(b) The Hardy hierarchy $\{H_\alpha\}_{\alpha<I}$ satisfies the conditions $(f1)_n$, $(f2)_n$ and $(f3)_n$ for every $n \in \mathbb{N}$ as in Examples 3.1.2.

EXAMPLE 3.1.11. The slow-growing hierarchy $\{G_\alpha\}_{\alpha<I}$ is defined as follows:

$$G_0(x) = 0;$$
$$G_{\alpha+1}(x) = G_\alpha(x) + 1;$$
$$G_\alpha(x) = G_{\alpha[x]}(x) \quad \text{for a limit ordinal } \alpha.$$

This hierarchy does not satisfy (f1). In fact $G_k(x) = k$ for $k<\omega$. However, we can prove the following proposition by the same way as Theorems 3.1.5 and 3.1.9.

PROPOSITION 3.1.12. If the system P is $(n+1)$-built-up for some $n \in \mathbb{N}$, then the following results on $\{G_\alpha\}_{\alpha<I}$ hold:

(a) $G_\alpha(x) \leqq G_\alpha(x+1)$ for $n \leqq x$ for every $\alpha < I$.

(b) If $\alpha \xrightarrow[m]{} \beta$ and $m > n$, then $G_\beta(m) \leqq G_\alpha(m)$ and $G_\beta(x) < G_\alpha(x)$ for $x > m$.

(c) If $\alpha < \beta$, then G_α is dominated by G_β. □

3.2 Conditions on systems of fundamental sequences

In this section, we clarify relations between several conditions for systems of fundamental sequences, and study the increase and domination properties of $\{f_\alpha\}_{\alpha<I}$ determined by these systems.

Let I be a countable ordinal. We defined in the preceding section that the system P of fundamental sequences for I is

(n)-built-up if

$$\alpha[x+1] \xrightarrow[n]{} \alpha[x]$$

for any limit $\alpha < I$ and $x \in \mathbb{N}$. In the literature, several conditions other than (n)-built-upness are studied:

DEFINITION 3.2.1. Let P be a system for I.

(a)(Aoyama and Kadota[1]) P is *(n)-diagonal-built-up* if

$\alpha[x+1] \xrightarrow[x+n]{} \alpha[x]$ for any limit $\alpha < I$ and $x \in \mathbb{N}$.

(b)(cf.Löb and Wainer[31]) P is *LW* if $\alpha[1] \xrightarrow[1]{} \alpha[0]$ and

$\alpha[x+1] \xrightarrow[x]{} \alpha[x]$ for any limit $\alpha < I$ and $0 < x$.

(c)(Dennis-Jones and Wainer[8]) P is *structured* if

$\alpha[x+1] \xRightarrow[x+1]{} \alpha[x] + 1$ for any limit $\alpha < I$ and $x \in \mathbb{N}$.

(In Kadota and Aoyama[22], this is said to be *nice*.)

(c)(Zemke[51]) P is *normed* if it has a *norm* $N: I \longrightarrow \mathbb{N}$ which satisfies the following conditions (N1)-(N3):

(N1) $N(0) = 0$;

(N2) $N(\alpha) < N(\alpha+1)$;

(N3) $N(\alpha[x]) < N(\alpha[x+1])$ for any limit $\alpha < I$ and $x \in \mathbb{N}$.

(d)(Zemke[51]) P is *regulated* if it is normed and it satisfies

$\alpha[N(\beta)] \geqq \beta$ for $\beta < \alpha < I$.

As the case of (n)-built-upness, we can show the following lemma which states elementary properties for our conditions:

LEMMA 3.2.2. Let P be a system for I.

(a) If P is either *LW* or (k)-diagonal-built-up where $k = 0$ or 1, and $\alpha \xrightarrow[m]{} \beta$, then $\alpha \xrightarrow[s]{} \beta$ for $m < s$.

(b) If P is (0)-diagonal-built-up and $\alpha \xrightarrow[n]{} \beta$, then $\alpha \xRightarrow[n+1]{} \beta+1$.

Proof. We prove by induction on α. (a) Assume $\alpha \xrightarrow[m]{} \beta$ and $m < s$. Then $\alpha[m] \xRightarrow[m]{} \beta$. If P is (k)-diagonal-built-up for $k = 0$ or 1, then $\alpha[s] \xrightarrow[s+k-1]{} \cdots \xrightarrow[m+k]{} \alpha[m]$. By the induction hypothesis, $\alpha[s] \xrightarrow[s]{} \alpha[m] \xRightarrow[s]{} \beta$. Therefore $\alpha \xrightarrow[s]{} \beta$. If P is *LW*, the proof is similar to this case of (1)-diagonal-built-up systems.

(b) Case 1. $\alpha = 0$: Trivial. Case 2. $\alpha = \gamma + 1$: If $\alpha \xrightarrow[n]{} \beta$, then

$\gamma \underset{n}{\Longrightarrow} \beta$. By the induction hypothesis, $\alpha \underset{n+1}{\Longrightarrow} \beta + 1$. Case 3. α is limit: If $\alpha \underset{n}{\longrightarrow} \beta$, then $\alpha[n] \underset{n}{\Longrightarrow} \beta$. Since P is (0)-diagonal-built-up, $\alpha[n+1] \underset{n}{\longrightarrow} \alpha[n]$. By the induction hypothesis, $\alpha[n+1] \underset{n+1}{\Longrightarrow} \alpha[n]+1$. If $\alpha[n] = \beta$, then the conclusion holds. If $\alpha[n] \underset{n}{\longrightarrow} \beta$, then by (a) and the induction hypothesis, $\alpha[n+1] \underset{n+1}{\longrightarrow} \alpha[n] \underset{n+1}{\Longrightarrow} \beta+1$. Therefore, $\alpha \underset{n+1}{\Longrightarrow} \beta+1$. □

Using this lemma and Lemma 3.1.6, we can show the following theorem by Kadota and Aoyama[22,Theorem 2.4].

THEOREM 3.2.3. Let P be a system for I.
(a) If P is (n)-built-up, then P is $(n+1)$-built-up.
(b) If P is (n)-built-up, then P is (n)-diagonal-built-up.
(c) If P is (1)-diagonal-built-up and $n > 1$, then P is (n)-diagonal-built-up.
(d) If P is (1)-built-up, then P is LW.
(e) If P is (0)-diagonal-built-up, then P is LW and structured.
(f) If P is LW or structured, then P is (1)-diagonal-built-up. □

Next, we show a theorem on relations between (n)-built-up systems and regulated systems by using the following proposition which is shown by Kadota and Aoyama[22] (cf.Schmidt[37]).

PROPOSITION 3.2.4. Let P be a system for I, and let $n \in \mathbb{N}$. The following three conditions are equivalent:
(a) P is (n)-built-up.
(b) P satisfies the Bachmann property B[n] which is defined as follows: If $\alpha[x] < \mu \leqq \alpha[x+1]$, then $\alpha[x] \leqq \mu[n]$, for limit $\alpha \in I$ and $x \in \mathbb{N}$.
(c) P satisfies the property that for limit $\alpha \in I$ and $x \in \mathbb{N}$, if $\lambda[x] < \mu \leqq \lambda[x+1]$, then $\mu \underset{n}{\longrightarrow} \lambda[x]$. □

Kadota and Aoyama[22,p.359] showed the following Theorem and Corollaries on the regulated systems:

THEOREM 3.2.5. Let P be a system for I and let $N(\alpha) = |\{\beta \in I\colon \alpha \underset{n}{\longrightarrow} \beta\}|$. If P is (n)-built-up, then N is a norm on I

such that $\alpha[N(\beta)] \geqq \beta$ whenever $\alpha > \beta$.

(Hence, if P is (n)-built-up, then P is regulated.) □

COROLLARY 3.2.6. Let P be a regulated system for I. If $\alpha > \beta$, then $\alpha \xrightarrow[N(\beta)]{} \beta$. □

COROLLARY 3.2.7. Let P be either regulated or (1)-diagonal-built-up. If $\alpha > \beta$, then $\alpha \xrightarrow[m]{} \beta$ for some $m \in \mathbb{N}$. □

Let (n)-*BU*, (n)-*DBU*, *REG*, *STR* and *LW* be the class of all (n)-built-up, (n)-diagonal-built-up, regulated, structured and *LW* systems for I, respectively. By Theorems 3.2.3 and 3.2.5, we can obtain the following diagram. Here, for two classes S and S', $S \longrightarrow S'$ means that S' contains S. Moreover, each arrow means that S' contains S properly(see the following example).

$$\begin{array}{ccccccc}
REG & \cdots\longleftarrow & (n)\text{-}BU & \longleftarrow\cdots\longleftarrow & (1)\text{-}BU & \longleftarrow & (0)\text{-}BU \\
 & & \Big\downarrow & & \downarrow & & \downarrow \\
 & & & & LW & \longleftarrow & (0)\text{-}DBU \\
 & & \Big\downarrow & & \downarrow & & \downarrow \\
 & & (n)\text{-}DBU & \longleftarrow & (1)\text{-}DBU & \longleftarrow & STR \\
 & & (n>1) & & & &
\end{array}$$

EXAMPLES 3.2.8. Let $f:\mathbb{N} \longrightarrow \mathbb{N}$ and $I = \omega\cdot\omega+1$. The following system P for I gives the proof of the properness of the above propositions. P:

$$\omega\cdot\omega[x] = \omega\cdot x, \qquad \omega\cdot(m+1)[x] = \begin{cases} \omega\cdot m + x & \text{if } x \geqq f(m,n) \\ x & \text{otherwise.} \end{cases}$$

(a) Let n > 0 and $f(m,n) = n$. Then, P is (n)-built-up, but it is not $(n-1)$-built-up. In particular, if $n = 1$, then P is *LW* but it is not (0)-diagonal-built-up.

(b) Let $f(m,n) = m+n$. Then, P is (n)-diagonal-built-up, but it is not (n)-built-up. For $n > 0$, it is not $(n-1)$-diagonal-built-up. In particular, if $n = 1$, then P is structured but it is not LW. If $n = 0$, then P is LW but it is not (1)- built-up.

(c) We consider the system P' for I which is defined by modifying P such that

$$\omega\cdot\omega[x] = \omega\cdot x,\quad \omega\cdot(m+1)[x] = \begin{cases} \omega\cdot m+x-(m+1) & \text{if } x \geqq m+1 \text{ and } m > 0 \\ x & \text{otherwise.} \end{cases}$$

Then P' is (1)-diagonal-built-up but it is not structured.

The following theorem is a result of Kadota and Aoyama[22, Theorem 3.1] which shows that the condition (1)-built-upness in Theorem 3.1.5 can be weakened.

THEOREM 3.2.9. If $\{f_\alpha\}_{\alpha<I}$ satisfies conditions (f1)-(f3) and the system P for I is either LW or structured, then the following holds:

(a) f_α is strictly increasing for each $\alpha \in I$.

(b) If $\alpha \underset{m}{\longrightarrow} \beta$ and $m > 0$, then $f_\beta(m) \leqq f_\alpha(m)$ and $f_\beta(x) < f_\alpha(x)$ for $m < x \in \mathbb{N}$.

(c) If $\beta < \alpha$, then f_β is dominated by f_α. □

3.3 Existence problems

In this section, we study the existence of systems of fundamental sequences for all countable limit ordinals, which possess some natural conditions considered in the preceding section.

Let Ω be the first uncountable ordinal and let $\mathrm{Lim}(\Omega)$ be the set of all countable limit ordinals. Then, we say that $P{:}\mathrm{Lim}(\Omega) \longrightarrow \Omega^\omega$ which assigns a fundamental sequence for any countable limit ordinal is a *system* of fundamental sequence for Ω (or a *system* for Ω).

In [37], Schmidt showed the following results on the problem whether there is a built-up (i.e.,(0)-built-up) system of fundamental sequences for all countable limit ordinals.

(a) There *is* a built-up system for any initial segment I of countable ordinals, but

(b) there is *no* built-up system for Ω.

Here, we prove another two theorems on this problem. One is on regulated systems for Ω and the other is on (0)-diagonal-built-up systems for Ω. The latter case is essentially different from the result (b), i.e. there *is* a (0)-diagonal-built-up system for Ω. All the following results are proved in Kadota and Aoyama[22, Section 4].

THEOREM 3.3.1. There is no regulated system for Ω. (Hence, for any $n < \omega$, there is no (n)-built-up system for Ω.)

Proof. We show that there is no regulated system for Ω, i.e., there is no system for Ω such that

(*) for all $\beta < \Omega$, there is an m (depending only on β) such that, for any α, if $\beta < \alpha < \Omega$, then $\alpha[m] \geq \beta$.

Assume there is such a system P. Then, for each n, the function $f_n:\Omega \longrightarrow \Omega$ defined by $f_n(\alpha) = \alpha[n]$ is *regressive* (i.e., $f_n(\alpha) < \alpha$ for all $\alpha > 0$). Hence, there are an $A_n \subset \Omega$ of order type Ω and a $\beta_n < \Omega$ such that $f_n(\alpha) = \beta_n$ for all $\alpha \in A_n$ (cf. Levy[30] p.154, Theorem 4.41). We define $(\sup_{n<\omega} \beta_n)+1 = \beta$ and α_n = the least α of $A_n \cap \{\alpha<\Omega \mid \beta<\alpha\}$. This contradicts (*), since $\alpha_n = \alpha[n] = \beta_n < \beta$. □

On the contrary, we can show the following theorem, whose proof is suggested by M. Hanazawa. It can be proved in *ZF* set theory with the axiom of choice.

THEOREM 3.3.2. There is a (0)-diagonal-built-up system of fundamental sequences for Ω.

Proof. Firstly, we prove the following claim:

CLAIM. Let α be a countable limit ordinal and P be a (0)-diagonal-built-up system for α such that $(\lambda + \omega)[x] = \lambda + x$ for all λ with $\lambda + \omega < \alpha$ and $x \in \mathbb{N}$. Then, there is a (0)-diagonal-built-up system P' for $\alpha+1$ such that $P'(\beta) = P(\beta)$ for all $\beta < \alpha$, and $(\lambda + \omega)[x] = \lambda + x$ for all λ with $\lambda + \omega \leqq \alpha$ and $x \in \mathbb{N}$.

Proof of Claim. We define P' such that $P'(\alpha) = \{\alpha[x]\}_{x\in\mathbb{N}}$ and $P'(\beta) = P(\beta)$ for $\beta < \alpha$. where $\{\alpha[x]\}_{x\in\mathbb{N}}$ is a fundamental sequence for α defined as follows:

Case 1. α is of the form $\beta + \omega$. Then, $\alpha[x]$ is $\beta + x$.

Case 2. α is not of the form $\beta + \omega$. Then, there is a sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ such that $\lambda_0 < \lambda_1 < \cdots < \alpha$, $\lim_{x\in\mathbb{N}} \lambda_x = \alpha$ and $\lambda_x + \omega < \lambda_{x+1}$ for all $x \in \mathbb{N}$. Since P is (0)-diagonal-built-up, for every i, there is an x_i such that for all $m \geqq x_i$, $\lambda_{i+1} \xrightarrow[m]{} \lambda_i + \omega$. We define a sequence of numbers $\{m_n\}_{n\in\mathbb{N}}$:

$$m_0 = 0, \quad m_{i+1} = \max(x_i, m_i+1) \text{ for } i > 0.$$

Then, $\{\alpha[x]\}_{x\in\mathbb{N}}$ is defined as follows:

$$\alpha[0] = \lambda_0,$$

$$\alpha[m_i+1] = \lambda_i \text{ for } i > 0,$$

$$\alpha[m_{i-1}+j+1] = \lambda_{i-1} + j + 1, \text{ where } m_{i-1} + j < m_i \text{ for } i > 0.$$

We prove this claim as follows.

If $x = m_i$ for some $i > 0$, then $\alpha[x+1] = \lambda_i \xrightarrow[m_i]{} \lambda_{i-1}+\omega \xrightarrow[m_i]{} \lambda_{i-1}+m_i \xRightarrow[m_i]{} \lambda_{i-1} + (m_i - m_{i-1}) = \alpha[m_i] = \alpha[x]$. So $\alpha[x+1] \xrightarrow[x]{} \alpha[x]$. Otherwise, $\alpha[x+1] = \alpha[x]+1 \xrightarrow[x]{} \alpha[x]$. Hence, P' is (0)-diagonal-built-up. The proof of this claim is completed.

By our claim, we can prove there is a sequence $\{P_\alpha\}_{\alpha<\Omega}$ so that every P_α is (0)-diagonal-built-up for α and if $\beta < \alpha < \Omega$, then P_β is a restriction of P_α (i.e., $P_\beta(\gamma) = P_\alpha(\gamma)$ for $\gamma < \beta$). We define P_Ω by putting $P_\Omega(\alpha) = P_{\alpha+1}(\alpha)$ for $\alpha < \Omega$. Then, P_Ω is (0)-diagonal-built-up system for Ω. □

From this theorem and Theorem 3.2.8, we can prove the following corollary.

COROLLARY 3.3.3. If a sequence $\{f_\alpha\}_{\alpha<\Omega}$ satisfies (f1)-(f3) (see Example 3.1.2) and is defined by a (0)-diagonal-built-up systems for Ω (cf. Theorem 3.3.2), then it has the increase and the domination properties.

Finally, we show a result on the problem whether any unary function $g:\mathbb{N} \longrightarrow \mathbb{N}$ is dominated by some f_α in $\{f_\alpha\}_{\alpha<\Omega}$. The following is proved in *ZF* set theory with the axiom of choice and the continuum hypothesis *CH*.

COROLLARY 3.3.4. There is a (0)-diagonal-built-up system for Ω which satisfies that for each sequence $\{f_\alpha\}_{\alpha<\Omega}$ with (f1)-(f3) and for each $g:\mathbb{N} \longrightarrow \mathbb{N}$, g is dominated by f_α for some $\alpha < \Omega$.

Proof. By *CH*, we take a sequence $\mathcal{G} = \{g_\alpha\}_{\alpha<\Omega}$ of all unary number-theoretic functions. Then, we can get a new sequence $\mathcal{H} = \{h_\alpha\}_{\alpha<\Omega}$ of unary number-theoretic functions by defining that:

$$h_\alpha(0) = g_\alpha(0) + 1,$$

$$h_\alpha(x+1) = \max\{h_\alpha(x),\ g_\alpha(x+1)\} + 1$$

for $\alpha < \Omega$ and $x \in \mathbb{N}$. We can easily show that h_α is strictly increasing, and dominates g_α. Moreover, we get a sequence $\mathcal{H}' = \{h'_\alpha\}_{\alpha<\Omega}$ of unary number-theoretic functions as follows:

$$h'_\alpha(x) = h_\alpha(x) + 1 \quad \text{for } \alpha = 0 \text{ or } \alpha \text{ is a limit},$$

$$h'_{\alpha+1}(x) = \max\{h'_\alpha(x), h_{\alpha+1}(x)\} + 1$$

for $x \in \mathbb{N}$. Then, the function $h'_{\alpha+x}(x)$ of x dominates the function $h'_{\alpha+n}(x)$ of x for every $n < \omega$.

Now, we get a (0)-diagonal-built-up system for Ω from Theorem 3.3.2 by modifying $(\lambda+\omega)[x] = \lambda+x$ in its Claim to:

$$(\lambda+\omega)[x] = \lambda + h'_{\lambda+x}(x),$$

and $\alpha[x] = \beta+x$ in Case 1 of the Claim to:

$$\alpha[x] = \beta + h'_{\beta+x}(x).$$

Let $\{f_\alpha\}_{\alpha<\Omega}$ be any sequence which satisfies (f1)-(f3). Then, for limit λ, we have

$$f_{\lambda+\omega}(x) = f_{\lambda+h'_{\lambda+x}(x)}(x) \geqq f_\lambda(x) + h'_{\lambda+x}(x)$$

for $x > 0$ by (f2). By this relation, we can show this theorem as follows. For a given $g:\mathbb{N} \longrightarrow \mathbb{N}$, there is $\alpha < \Omega$ such that $g = g_\alpha$. Hence g is dominated by h'_α. On the other hand, we can express $\alpha = \lambda+n$, where λ is a limit or 0 and $n < \omega$. Since the function

$h'_{\lambda+x}(x)$ of x dominates this h'_α from the argument above,

$$f_{\lambda+\omega}(x) \geqq f_\lambda(x) + h'_{\lambda+x}(x) \geqq h'_{\lambda+x}(x) > h'_{\lambda+n}(x)$$

for sufficiently large x. Thus, we have that g is dominated by the function $f_{\lambda+\omega}$. This completes the proof. □

We have shown the existence of a (0)-diagonal-built-up (hence *LW* and structured) system of fundamental sequences for Ω. Hence, the sequences $\{f_\alpha\}_{\alpha<\Omega}$ which satisfies (f1)-(f3) and is defined by (0)-diagonal-built-up systems for Ω have the increase and domination properties. On the other hand, by Theorem 3.3.1, there is *no* (n)-built-up system for Ω for any n. Hence, in order to treat sequences $\{f_\alpha\}_{\alpha<\Omega}$ indexed with all countable ordinals, the conditions such as (0)-diagonal-built-upness, LW-ness and structuredness should be considered.

However, as we shall see in the following chapters, (n)-built-upness is useful for treating a subrecursive hierarchy which consists of sequence $\{f_\alpha\}_{\alpha<I}$ indexed with all ordinals less than a countable ordinal I.

CHAPTER 4

PROVABLY COMPUTABLE FUNCTIONS IN PEANO ARITHMETIC

We have already shown in Chapter 2 that the set of all provably computable functions in Peano arithmetic PA can be classified by the fast-growing hierarchy up to ε_0 using the fact that any function provable computable in PA is dominated by F_α for some α.

In this chapter, we study this characterization in detail. Then, we analyze the unprovability result of undecidable finite combinatorial statement PH.

In Section 4.1, we introduce fragments PA_n of PA for $n \in \mathbb{N}$ and prove that the set of all provably computable functions in PA_n can be classified by the fast-growing hierarchy up to ω_n for $n \geqq 1$. This result was proved by Ono and Kadota[33,Section 3].

In Section 4.2, we give the provability and unprovability results on finite combinatorial statements $PH(n)$ following Ono and Kadota[33,Section 4].

In Section 4.3, we give the relativization results of those in Sections 4.1 and 4.2, which were studied by Kadota[16].

4.1 Provable computability

In this section, we will introduce some fragment PA_n of PA for each $n > 0$, and study provably computable functions in it. Then, we will prove that the set of all provably computable functions in PA_n can be classified by the fast-growing hierarchy up to ω_n. This result gives a refinement of Proposition 2.3.4.

Our formal theory PA of Peano arithmetic has been defined in Section 2.2. As we mentioned in Section 2.2, we assume that $\mathcal{L}(PA)$ contains the symbols for primitive recursive functions and predicates. For convenience, we will use the same letters to express these functions or predicates and to express the symbols in $\mathcal{L}(PA)$ which express them.

We abbreviate the formulas $\forall x(x \leqq t \supset A(x))$ and $\exists x(x \leqq t \wedge A(x))$ (where t does not contain x) to $\forall x \leqq t A(x)$ and $\exists x \leqq t A(x)$, respectively, and these types of quantifiers are called *bounded quantifiers*. A formula is called *bounded* if it contains only bounded quantifiers as quantifiers. Any bounded formula is both a Π_0-formula and a Σ_0-formula. A formula A is a Π_{m+1}-formula if it is of the form $\forall x_1 \cdots \forall x_k B$ with a Σ_m-formula B, and A is a Σ_{m+1}-formula if it is of the form $\exists x_1 \cdots \exists x_k C$ with a Π_m-formula C.

For each $0 < n \in \mathbb{N}$, the formal theory PA_n is defined from PA by restricting the induction formulas of the mathematical induction to formulas containing at most n quantifiers. Then we define provable computable functions in PA_n in the same way as the case of PA as follows (cf.Definition 2.2.2).

DEFINITION 4.1.1(*Provably computable functions in* PA_n). For each $n > 0$, a computable function $f:\mathbb{N}^k \longrightarrow \mathbb{N}$ is said to be *provably computable in* PA_n if there exists a Gödel number e of f such that the formula

$$\forall \underline{x} \exists y T_k(\bar{e}, \underline{x}, y)$$

is provable in PA_n.

Now, we state our main theorem of this section (cf.Ono and Kadota[33,Theorem 3.1]). For a formula $R(x_1, \ldots, x_k, y)$ of $\mathcal{L}(PA)$, the predicate $(R(\bar{x}_1, \cdots, \bar{x}_k, \bar{y})$ is true) is often abbreviated by $R(\bar{x}, \cdots, \bar{x}_k, \bar{y})$, for convenience.

THEOREM 4.1.2. Let $n \geqq 1$. Then, the following conditions (a)-(d) are equivalent:

(a) f is provably computable in PA_n.

(b) There are a primitive recursive function g and a bounded formula R such that

$f(x_1, \cdots, x_k) = g(x_1, \cdots, x_k, \mu y R(\bar{x}_1, \cdots, \bar{x}_k, \bar{y}))$;

$\forall \underline{x} \exists y R(\underline{x}, y)$ is provable in PA_n.

(c) $f \in \bigcup_{m<\omega} \mathcal{U}(\omega_n(m)))$
(i.e., f is $\omega_n(m)$-ordinal recursive for some $m < \omega$).

(d) $f \in \bigcup_{\alpha<\omega_n} \mathcal{F}_\alpha$
(i.e., f is elementary recursive in F_α for some $\alpha < \omega_n$).

This theorem shows that a refinement of the result given in Theorem 2.3.4 which says that the set of all provably recursive functions in PA is classified by the fast-growing hierarchy up to ε_0. We will give the proof of Theorem 4.1.2 in the following.

Clearly, (a) implies (b) by the definition. As we stated in Proposition 2.1.7, we have that (c) implies (d). Hence, we will show that (d) implies (a), and that (b) implies (c), to complete the proof of Theorem 4.1.2.

We first show that (4) in Theorem 4.1.2 implies (1). Notice here that we can show easily the following lemma (cf. Kino[25, Section 3]).

LEMMA 4.1.3. Let $n > 0$. The class of all provably computable functions in PA_n contains the zero, successor and projection functions and is closed under substitution and primitive recursion. □

Hence, every primitive recursive function is provably computable in PA_1.

We will make use of the following primitive recursive functions. Let $<\cdot,\cdot>$ be the function defined by

$$<x,y> = \frac{1}{2}((x+y)^2+3x+y).$$

Then, $<\cdot,\cdot>$ is a bijection from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$. We can define projection functions $(\cdot)_1$, $(\cdot)_2$, satisfying that

(a) $<(z)_1,(z)_2> = z$,

(b) $(<x,y>)_1 = x$ and $(<x,y>_2) = y$,

for all $x,y,z \in \mathbb{N}$. As for the detail of these functions, see Davis[7,Chapter 3].

Recall that in Section 2.1, for each $n > 0$, we took primitive recursive well-ordering $<_n$ on $\mathbb{N}$, which is of order-type ω_n and has the least element 0. For each $x \in \mathbb{N}$, define $\mathrm{ord}_n(x)$ to be the ordinal represented by x in the ordering $<_n$ and for each ordinal $\alpha < \varepsilon_0$, define $\mathrm{num}_n(\alpha)$ to be the natural number x such that $\mathrm{ord}_n(x) = \alpha$. We introduce a primitive recursive predicate $\mathrm{lim}_n(x)$ and a primitive recursive function $\mathrm{pr}_n(x)$ by

$$\mathrm{lim}_n(x) \text{ if and only if } \mathrm{ord}_n(x) \text{ is a limit number.}$$

$$\mathrm{pr}_n(x) = \begin{cases} \mathrm{num}_n(\beta) & \text{if } \mathrm{ord}_n(x) = \beta + 1 \\ 0 & \text{otherwise.} \end{cases}$$

By using the ordering $<_n$, we can define another ordering $<^*_n$ on $\mathbb{N}$ by the condition that

$$x <^*_n y \text{ if and only if}$$
$$(x)_2 <_n (y)_2 \text{ or } ((x)_2 = (y)_2 \text{ and } (x)_1 < (y)_1),$$

where $<$ is the usual order on $\mathbb{N}$. It is easy to see that $<^*_n$ is a primitive recursive well-ordering of order-type $\omega \cdot \omega_n$. As usual, $x \leqq_n y$ means that $x <_n y$ or $x = y$.

In [41], Shirai obtained the provability and the unprovability results of transfinite induction in fragments of Peano arithmetic, by examining into the Gentzen's proof[11] in detail. For our present purpose, we refer to his results in the following specialized form. For each $x \in \mathbb{N}$, $\bar{x}$ denotes the numeral of x.

PROPOSITION 4.1.4. Let $\alpha < \omega_n$ for $n > 1$. Then,

(a) $\forall y[\forall x(x <^*_n y \to \varepsilon(x)) \to \varepsilon(y)] \to \forall u \forall v(v \leqq_n \overline{\mathrm{num}_n(\alpha)} \to \varepsilon(\langle u, v\rangle))$

is provable in PA_{n-1}, where $\varepsilon(z)$ is a new predicate symbol;
(b) in particular, if $A(z)$ is a Π_2-formula, then

$$\forall y[\forall x(x <^*_n y \to A(x)) \to A(y)] \to \forall u \forall v(v \leqq_n \mathrm{num}_n(\alpha) \to A(\langle u, v\rangle))$$

is provable in PA_n. □

Notice here that the set $\{x \in \mathbb{N} \mid (x)_2 \leqq_n \mathrm{num}_n(\alpha)\}$ is an initial segment of the well-ordering $<^*_n$, which is of order-type $\omega \cdot \alpha$ $(<\omega_n)$, when $n > 1$ and $\alpha < \omega_n$.

Next, we introduce a ternary function h by

$$h(u,v,x) = F^{u+1}_{\mathrm{ord}_n(v)}(x),$$

where F_α's are of the fast-growing hierarchy up to ε_0. Clearly, h is a computable function. Let e be a Gödel number of h. Then we show the following lemma.

LEMMA 4.1.5. Let $\alpha < \omega_n$ for $n > 1$. Then,

$$\mathrm{v} \leqq_n \overline{\mathrm{num}_n(\alpha)} \rightarrow \forall \mathrm{x} \exists \mathrm{y} T_3(\bar{e},\mathrm{u},\mathrm{v},\mathrm{x},\mathrm{y})$$

is provable in PA_n.

Proof. Let $W(\mathrm{z})$ be the Π_2-formula $\forall \mathrm{x} \exists \mathrm{y} T_3(\bar{e},(\mathrm{z})_1,(\mathrm{z})_2,\mathrm{x},\mathrm{y})$. We first show that the formula

(a) $\qquad \forall \mathrm{u}(\mathrm{u} <^*_n \mathrm{v} \rightarrow W(\mathrm{u})) \rightarrow W(\mathrm{v})$

is provable in PA_1. Suppose first that $(\mathrm{v})_1 = 0$. If $(\mathrm{v})_2 = 0$, then $W(\mathrm{v})$ is provable in PA_1 and hence (a) is also provable in it. Next, we assume that

(b) $\qquad (\mathrm{v})_1 = 0 \wedge 0 <_n (\mathrm{v})_2 \wedge \neg \mathrm{lim}((\mathrm{v})_2)$.

Then, $\langle \mathrm{x},\mathrm{pr}((\mathrm{v})_2)\rangle <^*_n \mathrm{v}$ is provable in PA_1. Therefore

$$\forall \mathrm{u}(\mathrm{u} <^*_n \mathrm{v} \longrightarrow W(\mathrm{u})) \longrightarrow W(\langle \mathrm{x},\mathrm{pr}((\mathrm{v})_2)\rangle)$$

is provable in PA_1. On the other hand,

$$\exists \mathrm{y} T_3(\bar{e},\mathrm{x},\mathrm{pr}((\mathrm{v})_2),\mathrm{x},\mathrm{y}) \longrightarrow \exists \mathrm{y} T_3(\bar{e},0,(\mathrm{v})_2,\mathrm{x},\mathrm{y}),$$

i.e.,

$$W(\langle \mathrm{x},\mathrm{pr}((\mathrm{v})_2)\rangle) \longrightarrow W(\mathrm{v})$$

is also provable in PA_1, since we can effectively construct the computation for the input $(0,(\mathrm{v})_2,\mathrm{x})$ from the computation for the input $(\mathrm{x},\mathrm{pr}((\mathrm{v})_2),\mathrm{x})$. Hence, (a) is provable in PA_1 under the assumption (b). Similarly, we can show that (a) is provable in PA_1 under the assumption that $(\mathrm{v})_1 = 0 \wedge \mathrm{lim}((\mathrm{v})_2)$ or $(\mathrm{v})_1 > 0$. Combining these facts, we can deduce that (a) is provable in PA_1. Now, taking $W(\mathrm{z})$ for $A(\mathrm{z})$ in Proposition 4.1.4(b), we obtain that

$$\mathrm{v} \leqq_n \overline{\mathrm{num}(\alpha)} \rightarrow \forall \mathrm{x} \exists \mathrm{y} T_3(\bar{e},\mathrm{u},\mathrm{v},\mathrm{x},\mathrm{y})$$

is provable in PA_n. □

The following proposition says that Kleene's iteration theorem can be proved in PA_1, which is shown by Ono and Kadota [33, Lemma 3.7].

PROPOSITION 4.1.6. For each $i \in \mathbb{N}$, there exists a primitive recursive function s^i such that

$$\forall \underline{x} \forall y T_{i+m}(\bar{c}, \bar{k}_1, \ldots, \bar{k}_i, \underline{x}, y) \longleftrightarrow \forall \underline{x} \forall y T_m(\overline{s^i(c, k_1, \ldots, k_i)}, \underline{x}, y)$$

is provable in PA_1 for every $c, k_1, \ldots, k_i \in \mathbb{N}$. □

By this proposition, we have that

$$\forall x \forall y T_3(\bar{e}, \bar{m}, \bar{k}, x, y) \longleftrightarrow \forall x \forall y T_1(\overline{s^2(e, m, k)}, x, y)$$

is provable in PA_1 for every $m, k \in \mathbb{N}$. Clearly, $s^2(e,m,k)$ is a Gödel number of the function $h(m,k,x)$ $(= F^{m+1}_{\mathrm{ord}_n(k)}(x))$ of x. Then we show the following lemma(cf.Ono and Kadota[33, Lemma 3.9]).

LEMMA 4.1.7. For $n > 0$, if $\alpha < \omega_n$ then F_α is provably computable in PA_n.

Proof. If $n = 1$, then F_α is primitive recursive. Hence it is provably computable in PA_1 (see Lemma 4.1.3). Suppose that $n > 1$. By Lemma 4.1.5, the formula $\forall x \exists y T_3(\bar{e}, 0, \overline{\mathrm{num}_n(\alpha)}, x, y)$ is provable in PA_n. By Proposition 4.1.6, the formula

$$\forall x \exists y T_1(\overline{s^2(e, 0, \mathrm{num}_n(\alpha))}, x, y)$$

is also provable in PA_n, where $s^2(e,0,\mathrm{num}_n(\alpha))$ is a Gödel number of F_α, since $\mathrm{ord}_n(\mathrm{num}_n(\alpha) = \alpha$. Thus, F_α is provably computable in PA_n. □

Thus, we have completed the proof of that (d) of Theorem 4.1.2 implies (b) of Theorem 4.1.2.

Now, it remains to show that (b) in Theorem 4.1.2 implies (c). From Corollary 12.16 of Takeuti[46], we immediately have the following proposition (cf. Ono and Kadota[33, Lemma 3.3]).

PROPOSITION 4.1.8. Let $n > 0$. Suppose that $R(x,y)$ is a Π_0-formula such that $\forall x \exists y R(x,y)$ is provable in PA_n. Then, the function f defined by

$$f(x) = \mu y R(\bar{x}, \bar{y})$$

is $\omega_n(m)$-ordinal recursive for some $m < \omega$. □

Then, we have the following corollary:

COROLLARY 4.1.9. Let $n > 0$. Suppose that $R(\underline{x},y)$ is a bounded formula and the formula $\forall \underline{x} \exists y R(\underline{x},y)$ is provable in PA_n. Then, the function f defined by

$$f(x_1, \cdots, x_k, y) = \mu y R(\bar{x}_1, \cdots, \bar{x}_k, \bar{y})$$

is $\omega_n(m)$-ordinal recursive for some $m < \omega$. (Thus, (b) in Theorem 4.1.2 implies (c) in Theorem 4.1.2.) □

Proof. First, we remark that we can assume that the sequence $\underline{x}$ of variables consists of only one variable x. To see this remark, we assume that $\forall x_1 \forall x_2 \exists y R(x_1, x_2, y)$ is provable in PA_n, as an example. We put $R'(x,y) \longleftrightarrow R((x)_1, (x)_2, y)$ and $f'(x) = \mu y R'(\bar{x}, \bar{y})$. Then, R' is also a bounded formula and $\forall x \exists y R'(x,y)$ is provable in PA_n. We have that $f(x_1, x_2) = f'(\langle x_1, x_2 \rangle)$. Since the function $\langle \cdot, \cdot \rangle$ is primitive recursive, f is $\omega_n(m)$-ordinal recursive if so is f'. Thus, by iterating this argument, we can assume that $\underline{x}$ consists of only one variable.

We also remark that for any bounded formula A, there is a predicate symbol p of $\mathcal{L}(PA)$ such that

$$A(x) \longleftrightarrow p(x)$$

is provable in PA_1. Hence, we can prove this corollary from Proposition 4.1.8. □

Thus, we have completed our proof of Theorem 4.1.2.

4.2 Undecidable combinatorial statements

In Section 2.3, we studied a finite combinatorial statement *PH*, which is shown to be unprovable in Peano arithmetic *PA* by Paris and Harrington[35]. Here we analyze this statement in fragments of *PA* by using Theorem 4.1.2. We defined in Section 2.3 the formula *PH*:

$$PH \equiv \forall w \forall x \forall z \exists y([x,\ y] \underset{*}{\longrightarrow} (w+1)^{w}_{z}).$$

By Proposition 2.3.1, *PH* is true. We defined also a computable function $\sigma_{n,c}$ for $n, c \in \mathbb{N}$ by

$$\sigma_{n,c}(k) = \mu y([k,y] \underset{*}{\longrightarrow} (n+1)^{n}_{c}).$$

Here, we define the formula $PH(n)$ for each $n \in \mathbb{N}$.

$$PH(n) \equiv \forall x \forall z \exists m([x,\ y] \underset{*}{\longrightarrow} (\bar{n}+1)^{\bar{n}}_{z}).$$

The Ramsey relation $[k,\ m] \underset{*}{\longrightarrow} (n+1)^{n}_{c}$ can be *represented* by a bounded formula $P(w,x,z,y)$ of $\mathcal{L}(PA)$, i.e., $P(\bar{w},\bar{x},\bar{z},\bar{y})$ is true if and only if $[x,\ y] \underset{*}{\longrightarrow} (w+1)^{w}_{z}$ for all $w,x,z,y \in \mathbb{N}$. We must pay attention to the fact that there are many ways of expressing the Ramsey relation by formulas. Here, for each fixed n, we say that a formula $P(x,z,y)$ which *represents the Ramsey relation* if $P(\bar{x},\bar{z},\bar{y})$ is true if and only if $[x,y] \underset{*}{\longrightarrow} (n+1)^{n}_{z}$ for all $x,z,y \in \mathbb{N}$. Then, we prove the following theorem (cf.Theorem 4.5 of Ono and Kadota[33]).

THEOREM 4.2.1. Let $n > 1$. If $P(x,z,y)$ is a bounded formula which represents the Ramsey relation, then the formula

$$\forall x \forall z \exists y P(x,z,y)$$

is not provable in PA_{n-1}.

Proof. Suppose that $\forall x \forall z \exists y P(x,z,y)$ is provable in PA_{n-1}. Then $\forall u \exists y P(u,u,y)$ is also provable in PA_{n-1}. Let us define a function γ_n by $\gamma_n(u) = \mu y P(\bar{u},\bar{u},\bar{y})$, i.e., $\gamma_n(u) = \sigma_{n,u}(u)$. Then, γ_n is elementary recursive in F_β for some $\beta < \omega_{n-1}$ by Theorem 4.1.2. So γ_n is dominated by $F_{\beta+1}$. Thus, γ_n is dominated by $\sigma_{n,c}$ for some c by Proposition 2.3.3. Hence, there is $k \in \mathbb{N}$ such that for every $u \geqq k$,

(a) $\sigma_{n,u}(u) = \gamma_n(u) < \sigma_{n,c}(u)$.

Let d be $\max\{c+1,k\}$. Then, by (a),

(b) $\sigma_{n,d}(d) < \sigma_{n,c}(d)$,

which contradicts Lemma 2.3.2. Therefore, $\forall x \forall z \exists y P(x,z,y)$ is not provable in PA_{n-1}. □

In this theorem, the formula $\forall x \forall z \exists y P(x,z,y)$ is interpreted as $PH(n)$ in the standard sense. Hence, from this theorem, we sometimes say informally that $PH(n)$ is not provable PA_{n-1} for $n > 1$. The following result follows immediately from Theorem 4.2.1 (cf. Proposition 2.3.4).

COROLLARY 4.2.2. The formula $\forall w \forall x \forall z \exists y([x, y] \underset{*}{\longrightarrow} (w+1)^{w}_{z})$ is not provable in PA for any bounded formula representation of Ramsey relation. □

We prove the following theorem, which is in some sense stronger but in another sense more restricted than the previous theorem (cf. Theorem 4.7 of Ono and Kadota[33]).

THEOREM 4.2.3. For $n \geqq 2$, $\forall x \forall z \exists y([x, y] \underset{*}{\longrightarrow} (\bar{n}+1)^{\bar{n}}_{z})$ is provable in PA_n, but not provable in PA_{n-1} in the following sense: For each $n \geqq 2$, there exists a Σ_1-formula $P(x,z,y)$ which represents the Ramsey relation such that,

$$\forall x \forall z \exists y P(x,z,y)$$

is provable in PA_n, but not provable in PA_{n-1}.

Proof. We can show similarly to Theorem 4.2.1. From Proposition 2.3.3, we can obtain that

$$\sigma_{n,z}(x) \leqq F_{\omega_{n-2}(<x,z>+7)}(<x,z>+7) = F_{\omega_{n-1}}(<x,z>+7)$$

since $<x,z> \geqq x, z$. Hence we have

$$\sigma_{n,z}(z) = \mu y \leqq F_{\omega_{n-1}}(<x,z>+7)(R(x,z,y)),$$

where R denotes the Ramsey relation $[x,y] \underset{*}{\longrightarrow} (n+1)^{n}_{z}$. Define a function j by

$$j(x,z,v) = \begin{cases} \mu y R(x,z,y) & \text{if} \quad \exists y \leqq U(v)\ R(x,z,y) \\ 0 & \text{otherwise.} \end{cases}$$

Then, j is primitive recursive. From Theorem 4.1.2, the function $F_{\omega_{n-1}}$ is provably computable in PA_n. Hence, we have a Gödel number e of the function $F_{\omega_{n-1}}$ such that $\forall x \exists y T_1(\bar{e},x,y)$ is provable in PA_n (cf.Section 4.1). Since

$$F_{\omega_{n-1}}(\langle x,z\rangle+7) = U(\mu v T_1(e,\langle x,z\rangle+7,v)),$$

$\sigma_{n,x}(z) = j(x,z,\mu v T_1(e,\langle x,z\rangle+7,v))$. Now, we will define a Σ_1-formula $P(x,z,y)$ by

$$P(x,z,y) \equiv \exists v(T_1(\bar{e},\langle x,z\rangle+7,v) \wedge \forall u<v \neg T_1(\bar{e},\langle x,z\rangle+7,u) \wedge j(x,z,v) = y).$$

Then we can easily show that P represents the Ramsey relation and $\forall x \forall z \exists y P(x,z,y)$ is provable in PA_n.

It can be easily seen that $P(x,z,y)$ is of the form $\exists v P'(x,z,y,v)$ where P' is bounded. Let Q be the formula

$$\forall x \forall z \exists w P'(x,z,(w)_1,(w)_2).$$

Then Q is also provable in PA_n, since so is $\forall x \forall z \exists y P(x,z,y)$. Now, we assume that $\forall x \forall z \exists y P(x,z,y)$ is provable in PA_{n-1}. Then, Q is also provable in PA_{n-1}. Similarly to the proof of Theorem 4.2.2, we define function γ'_n by

$$\gamma'_n(u) = \mu w P'(u,u,(w)_1,(w)_2).$$

Then, since Q is provable in PA_{n-1}, γ'_n is elementary recursive in F_β for some $\beta < \omega_{n-1}$ by Theorem 4.1.2. So γ'_n is dominated by $F_{\beta+1}$. Thus, γ'_n is dominated by $\sigma_{n,c}$ for some c by Proposition 2.3.3. Here we can assume that $c \geqq 2$, by Lemma 2.3.2. Hence, there exists a k such that for every $u \geqq k$,

(a) $\qquad \sigma_{n,u}(u) = (\gamma'_n(u))_1 \leqq \gamma'_n(u) < \sigma_{n,c}(u).$

Let d be $\max\{c+1,k\}$. Then, by (a)

(b) $\qquad \sigma_{n,d}(d) < \sigma_{n,c}(d).$

Thus, we are led to a contradiction, by (b) and Lemma 2.3.2. Therefore, $\forall x \forall z \exists y P(x,z,y)$ is not provable in PA_{n-1}. □

We notice here that the formula $\forall x \forall z \exists y([x,y] \underset{*}{\longrightarrow} (\bar{n}+1)^{\bar{n}}_{z})$ is not provable in PA_n for *some* Σ_0-representation of the Ramsey relation, contrary to Theorem 4.2.4. This can be shown as follows: Let $P(x,z,y)$ be any Σ_0-formula representing the Ramsey relation and $\mathrm{Prov}_n(u,v)$ be a Σ_0-formula representing the provability predicate for PA_n in the canonical way. More precisely, $\mathrm{Prov}_n(\ulcorner P \urcorner, \ulcorner A \urcorner)$ means the provability of a formula A in PA_n with a proof P, where $\ulcorner Z \urcorner$ is the Godel number of Z. Then,

$$P(x,z,y) \wedge \neg\mathrm{Prov}_n(x, \ulcorner 0=1 \urcorner)$$

is also a Σ_0-formula representing the Ramsey relation, since for each m, $\neg\mathrm{Prov}_n(\bar{m}, \ulcorner 0=1 \urcorner)$ is true. On the other hand, since

$$\forall x \forall z \exists y (P(x,z,y) \wedge \neg\mathrm{Prov}_n(x, \ulcorner 0=1 \urcorner))$$

implies the consistency of PA_n, it is not provable in PA_n.

4.3 Relativized hierarchies

We are concerned here with an extended version of Theorem 4.1.2 in Section 4.1, which gives a characterization of provably Δ_m-functions in PA_n for $n \geqq m \geqq 1$. This characterization theorem is studied by Kadota[16].

First, we will consider a relation between the relativized ordinal recursive hierarchy and relativized fast-growing hierarchy. Let $f:\mathbb{N} \longrightarrow \mathbb{N}$ be a strictly increasing function. Let I be a countable ordinal and P a system of fundamental sequences for I. Then, we define the fast-growing hierarchy relativized by f as follows:

$$F^{(f)}_0(x) = x + 1;$$

$$F^{(f)}_{\alpha+1}(x) = \left(F^{(f)}_{\alpha}\right)^{f(x)+1}(x);$$

$$F^{(f)}_{\alpha}(x) = F^{(f)}_{\alpha[f(x)]}(x) \qquad \text{if } \alpha \text{ is limit.}$$

Let Φ be a set of number-theoretic functions which satisfies the following property(#):

$$(\#)\begin{cases}\text{(a) } \Phi \text{ contains the identity function, and} \\ \text{(b) for every } g_1,\ g_2 \in \Phi, \text{ there is a unary strictly} \\ \quad\text{increasing function } f \text{ such that for} \\ \quad\text{every } x \in \mathbb{N},\ \max(g_1(x), g_2(x)) \leqq f(x).\end{cases}$$

Now, we consider the standard system of fundamental sequences for ε_0, and define the following classes of functions by relativizing the corresponding classes in Section 2.1.

DEFINITION 4.3.1. $\mathcal{F}^{\Phi}$ is the smallest set of functions containing all functions in Φ, all functions $F_{\beta}^{(f)}$ for each $\beta \leqq \alpha$ and each unary strictly increasing $f \in \Phi$, the zero, successor and projection functions, which is closed under substitution and limited recursion.

DEFINITION 4.3.2. Let $\alpha < \varepsilon_0$ and n the least number such that $\alpha < \omega_n$. Then, $\mathcal{U}^{\Phi}(\alpha)$ is the smallest set of functions containing all functions in Φ and all primitive recursive functions, which is closed under substitution and unnested α-recursion.

In [16], Kadota showed the following theorem, which is a relativization of Proposition 2.1.7.

THEOREM 4.3.3. Let $n > 0$ and Φ a class of functions with the property (#). Then,

$$\bigcup_{\alpha<\omega_n} \mathcal{F}_{\alpha}^{\Phi} = \bigcup_{\alpha<\omega_n} \mathcal{U}^{\Phi}(\omega^{\alpha}). \qquad \square$$

Next, we introduce extended language $\mathcal{L}^{(i)}$ of $\mathcal{L}(PA)$ for each $i \in \mathbb{N}$ inductively as follows:

We write $\mathcal{L}^{(0)}$ for $\mathcal{L}(PA)$. Let $i > 0$. Then, we assume that $\mathcal{L}^{(i-1)}$ is defined. For each formula $A(\underline{x},y)$ of $\mathcal{L}^{(i-1)}$ whose free variables are in $\underline{x}$,y, we define the function f_A as follows:

$$f_A(x_1,\ldots,x_k) = \begin{cases}\mu y A(\bar{m}_1,\ldots,\bar{m}_k,\bar{y}) & \text{if } \exists y A(\bar{m}_1,\ldots,\bar{m}_k,y) \text{ is true} \\ 0 & \text{otherwise.}\end{cases}$$

For each such formula A of $\mathcal{L}^{(i-1)}$, we consider a new function symbol $\bar{f}_A$ whose interpretation on $\mathbb{N}$ is f_A. Then, we define

$$\mathcal{L}^{(i)} = \mathcal{L}^{(i-1)} \cup \{\bar{f}_A \mid A(\underline{x},y) \text{ is a bounded formula of } \mathcal{L}^{(i-1)}\}.$$

Next, for each $i \geqq 1$, we will define functions $\delta_i:\mathbb{N} \longrightarrow \mathbb{N}$ and $\psi_i:\mathbb{N} \longrightarrow \mathbb{N}$. Let $\mathrm{Tr}_{i-1}(z,x,y)$ be the Σ_{i-1}-truth formula of $\mathcal{L}(PA)$ for Σ_{i-1}-formulas of $\mathcal{L}(PA)$ with two fixed variables, i.e., for every Σ_{i-1}-formula $A(x,y)$ of $\mathcal{L}(PA)$ which has its Gödel number e,

$$\mathrm{Tr}_{i-1}(\bar{e},x,y) \longleftrightarrow A(x,y)$$

is provable in PA_1(cf. Takeuti[46, Proposition 14.1]).

DEFINITION 4.3.4($\delta_i:\mathbb{N} \longrightarrow \mathbb{N}$, $\psi_i:\mathbb{N} \longrightarrow \mathbb{N}$ for $i > 0$). The functions δ_1 and ψ_1 are both identity functions. For $i > 1$,

$$Q_i(u,z) \equiv \forall w \leqq u \forall x \leqq u(\exists y \mathrm{Tr}_{i-1}(w,x,y) \longrightarrow \exists y < z \mathrm{Tr}_{i-1}(w,x,y)));$$

$$\delta_i(u) = \mu z Q_i(\bar{u},\bar{z}); \qquad Q_i^*(u,z) \equiv Q_i(u,z) \wedge \forall x < u \neg Q_i(u,z);$$

$$W_i(u,z) \equiv \exists y(\forall x \leqq u Q_i^*(x,y) \wedge z = y+u);$$

$$\psi_i(u) = \mu z W_i(\bar{u},\bar{z}).$$

A formula A of $\mathcal{L}(PA)$ is Δ_k *in* PA_n if there are a Σ_k-formula B and a Π_k-formula C such that $(A \leftrightarrow B)\wedge(A \leftrightarrow C)$ is provable in PA_n. Then, we have the following lemma(cf.Lemma 3.5 of Kadota [16]).

LEMMA 4.3.5. Let $i > 1$.

(a) Q_i is Δ_i in PA_1 and the function δ_i dominates f_A for each Σ_{i-1}-formula $A(\underline{x},y)$ of $\mathcal{L}(PA)$.

(b) W_i is Δ_i in PA_1, the function ψ_i dominates δ_i and it is strictly increasing. □

DEFINITION 4.3.6. Let $i > 0$.

(a) F_0 is the set of all primitive recursive functions.

(b) F_i is $\mathrm{F}_{i-1} \cup \{f_A \mid A(\underline{x},y)$ is a bounded formula of $\mathcal{L}^{(i-1)}\}$.

(c) Φ_i is the set of all functions elementary recursive in $\{\psi_i\} \cup \mathrm{F}_{i-1}$.

We give the definition of provably Δ_m-functions in the fragments of PA.

DEFINITION 4.3.7. Let $k,n > 0$. A function f is *provably* Δ_k *in* PA_n if there is a Δ_k-formula $A(\underline{x},y)$ in PA_n such that

(a) $f(x_1,\ldots,x_r) = \mu y A(\bar{x}_1,\ldots,\bar{x}_r,\bar{y})$ for all $x_1,\ldots,x_r \in \mathbb{N}$;

(b) $\forall\underline{x}\exists y A(\underline{x},y)$ is provable in PA_n.

Then, we have the following theorem shown by Kadota [16, Theorem 4.1], which gives a characterization of provably Δ_m-functions within relativized ordinal recursive functions. This theorem is a relativization of Theorem 4.1.2 in Section 4.1.

THEOREM 4.3.8. Let $n \geqq 1$ and $m \geqq 1$. Let Φ be the set Φ_m defined above. Then, the following are equivalent:

(a) f is provably Δ_m in PA_{n+m-1}.

(b) $f \in \bigcup_{m<\omega} \mathcal{U}^{\Phi}(\omega_n(m))$.

(c) $f \in \bigcup_{\alpha<\omega_n} \mathcal{F}^{\Phi}_{\alpha}$. □

Now, we extend the combinatorial statements which are studied in Section 4.2, and give some provability and unprovability results in fragments of Peano arithmetic by using Theorem 4.3.8.

Let $f:\mathbb{N} \longrightarrow \mathbb{N}$ be a strictly increasing function. For $c,k,m,n \in \mathbb{N}$, the predicate

$$[k,\ m] \xrightarrow[*]{f} (n+1)^n_c$$

means that, for every function $g:[k,m]^{[n]} \longrightarrow \{0,1,\ldots,c-1\}$, there is $H \subseteqq [k,\ m]$ such that

(a) $\mathrm{card}(H) \geqq n + 1$;

(b) H is homogeneous (i.e., g is constant on $H^{[n]}$);

(c) H is f-*large*, i.e., $f(\min(H)) \leqq \mathrm{card}(H)$.

Then, we can prove the following proposition similarly to Proposition 2.3.1.

PROPOSITION 4.3.9. For a strictly increasing $f:\mathbb{N} \longrightarrow \mathbb{N}$ and for each $c,k,n \in \mathbb{N}$, there is $m \in \mathbb{N}$ such that the following holds.

$$[k,\ m] \xrightarrow[*]{f} (n+1)^n_c. \qquad \square$$

From this proposition, we define a computable function as follows: For $c,k,n \in \mathbb{N}$,

$$\sigma^{(f)}_{n,c}(k) = \mu y([k,y] \xrightarrow[*]{f} (n+1)^n_c).$$

Now, we say that the relation $[k,\ m] \xrightarrow[*]{f} (n+1)^n_c$ is *f-Ramsey relation*. We consider representations of f-Ramsey relation in $\mathcal{L}(PA)$. For a given function $f:\mathbb{N} \longrightarrow \mathbb{N}$, let $\bar{f}$ be a new unary function symbol whose interpretation on $\mathbb{N}$ is f. Then, we can represent f-Ramsey relation by using a bounded formula $P(\mathrm{w,x,z,y};\bar{f})$ of the language $\mathcal{L}(\mathrm{PA})+\{\bar{f}\}$, i.e., $P(\bar{w},\bar{x},\bar{z},\bar{y};\bar{f})$ is true if and only if $[x,y] \xrightarrow[*]{f} (w+1)^w_z$ for all $w,x,z,y \in \mathbb{N}$. If f is defined by minimalization of a Δ_m-formula $R(\mathrm{x,y})$ in PA_n, i.e., $f(x) = \mu y R(\bar{x},\bar{y})$ for all $x \in \mathbb{N}$, then f-Ramsey relation can be represented by a Δ_m-formula, since we can replace the formulas of the form $\mathrm{p}(\bar{f}(\mathrm{x}))$ by the Δ_m-formula $\exists \mathrm{y}(R^*(\mathrm{x,y}) \wedge \mathrm{p(y)})$ (or $\forall \mathrm{z}(R^*(\mathrm{x,z}) \rightarrow \mathrm{p(z)})$), where p is a predicate symbol in $\mathcal{L}(PA)$ and $R^*(\mathrm{x,y}) \equiv R(\mathrm{x,y}) \wedge \forall \mathrm{z<y} \neg R(\mathrm{x,y})$.

Then, we have the following (see Kadota[16, Theorem 5.4 and Theorem 5.6):

THEOREM 4.3.10. Let $m > 0$ and let us denote ψ for ψ_m.

(a) Let $n > 0$. For some Δ_m representation of ψ-Ramsey relation in PA_{n+m-1}, the formula

$$\forall \mathrm{z} \forall \mathrm{x} \exists \mathrm{y}([\mathrm{x},\ \mathrm{y}] \xrightarrow[*]{\psi} (\bar{n}+1)^{\bar{n}}_{\mathrm{z}})$$

is provable in PA_{n+m-1}.

(b) Let $n > 1$. For any Δ_m representation of ψ-Ramsey relation in PA_{n+m-2}, the formula

$$\forall \mathrm{z} \forall \mathrm{x} \exists \mathrm{y}([\mathrm{x},\ \mathrm{y}] \xrightarrow{\psi} (\bar{n}+1)^{\bar{n}}_{\mathrm{z}})$$

is not provable in PA_{n+m-2}. $\square$

This theorem shows a relativization of the results given in Section 4.2.

CHAPTER 5

THE FAST AND SLOW GROWING HIERARCHIES AND INDUCTIVE DEFINITIONS

In this chapter, we consider the slow-growing hierarchy, opposing it to the fast-growing one, and study the relation between them. Actually, we study the ordinal τ where G_τ of the slow-growing hierarchy catches up with F_τ of the fast-growing hierarchy. This ordinal τ is called a subrecursive inaccessible ordinal (or s-inaccessible, for short).

In Section 5.1, we summarize the definition and results on the s-inaccessible ordinal.

In Section 5.2, we introduce the term structure of the ordinal notation, and show the strong normalizability of the structure. We also show that the ordinal τ is (3)-built-up.

In section 5.3, we introduce an ordinal τ', which is a variant of τ, and show that the fast-growing hierarchy up to τ' classifies the set of all provably computable functions in the theory of finitely iterated inductive definitions $ID_{<\omega}(=\bigcup_{n\in\mathbb{N}} ID_n)$.

5.1 Fast-growing versus slow-growing

Let I be a countable ordinal. We say that an ordinal $\alpha < I$ is *subrecursive inaccessible* (or *s-inaccessible*) if the slow-growing hierarchy $\{G_\beta\}_{\beta<\alpha}$ *catches up with* the fast-growing hierarchy $\{F_\beta\}_{\beta<\alpha}$ at α, i.e., for some $p \in \mathbb{N}$,

$$G_\alpha(x) < F_\alpha(x) \leqq G_\alpha(x+1)$$

for all $x > p$.

In this section, we define a tree-ordinal τ following Wainer[49], and show that τ is a minimum s-inaccessible by assuming the collapsing theorem and (3)-built-upness of τ, which will be proved in the following sections.

Here, we will consider countable ordinals as infinitary terms. These ordinals are called countable tree-ordinals. Each fundamental sequence of a countable limit ordinal will be considered as a tree-ordinal. We will use the symbol Ω for the set of countable tree-ordinals which is the same for the set of countable ordinals, since we will pay attention to the systems of fundamental sequences in this chapter.

DEFINITION 5.1.1(*Tree-ordinals* Ω). The set Ω of the countable *tree-ordinals* consists of the infinitary terms generated inductively by:

(a) $0 \in \Omega$;

(b) if $\alpha \in \Omega$, then $\alpha+1 \in \Omega$;

(c) if $\alpha_x \in \Omega$ for all $x \in \mathbb{N}$, then $(\alpha_x)_{x\in\mathbb{N}} \in \Omega$.

(In the case of (c), the term $(\alpha_x)_{x\in\mathbb{N}}$ is called a *limit*, and $\alpha[x]$ denotes α_x.)

We define the less than relation $<$ on Ω as the transitive closure of

(a) $\alpha < \alpha + 1$ for all $\alpha \in \Omega$; and
(b) $\alpha[x] < \alpha$ for each limit $\alpha \in \Omega$ and $x \in \mathbb{N}$.

We remark that the notion of tree-ordinals includes that of systems of fundamental sequences. More precisely, for each system P for I, each limit ordinal $\alpha < I$ and its fundamental sequence $\{\alpha[x]\}_{x\in\mathbb{N}}$, we can identify α with $(\alpha[x])_{x\in\mathbb{N}} \in \Omega$.

Next, we define the fast-growing $\{F_\alpha\}_{\alpha\in\Omega}$ and slow-growing $\{G_\alpha\}_{\alpha\in\Omega}$ hierarchies inductively as follows:

$$F_0(x) = x+1; \qquad G_0(x) = 0;$$
$$F_{\alpha+1}(x) = F_\alpha^{x+1}(x); \qquad G_{\alpha+1}(x) = G_\alpha(x) + 1;$$
$$F_\lambda(x) = F_{\lambda[x]}(x); \qquad G_\lambda(x) = G_{\lambda[x]}(x),$$

where λ is a limit.

The relation $\xrightarrow[n]{}$ on Ω for each $n \in \mathbb{N}$ are defined by the transitive closure of

(a) $\alpha + 1 \underset{n}{\longrightarrow} \alpha$ for each $\alpha \in \Omega$, and

(b) $\alpha \underset{n}{\longrightarrow} \alpha[n]$ for each limit $\alpha \in \Omega$.

This relation $\underset{n}{\longrightarrow}$ can be identified with the relation $\underset{n}{\longrightarrow}$ in Chapter 3. We also define the relation $\underset{n}{\Longrightarrow}$ on Ω for each $n \in \mathbb{N}$ similarly to Chapter 3 as follows: For $\alpha, \beta \in \Omega$, $\alpha \underset{n}{\Longrightarrow} \beta$ is $\alpha \underset{n}{\longrightarrow} \beta$ or $\alpha = \beta$.

We define the notion of (n)-built-upness for $n \in \mathbb{N}$ defined as follow: The subset $\Omega^{(n)\text{-bu}} \subset \Omega$ of *(n)-built-up* tree-ordinals is defined by the set of all $\alpha \in \Omega$ satisfying that:

$$\lambda[x+1] \underset{n}{\longrightarrow} \lambda[x] \quad \text{for any limit } \lambda \leqq \alpha \text{ and } x \in \mathbb{N}.$$

As in Chapter 3, we can prove the following theorem.

PROPOSITION 5.1.2. Assume $\alpha \in \Omega^{(p)\text{-bu}}$ for some $p \in \mathbb{N}$. Then the following holds:

(a) $F_\alpha(x) < F_\alpha(x+1)$ and $G_\alpha(x) \leqq G_\alpha(x+1)$ for $p \leqq x+1$.

(b) If $\alpha \underset{n}{\longrightarrow} \beta$ for $p \leqq m$, then $F_\beta(x) < F_\alpha(x)$ and $G_\beta(x) < G_\alpha(x)$ for $x > m$. □

Next, we say that $\alpha \in \Omega$ is a *subrecursive inaccessible* (or *s-inaccessible* for short) if the following property holds: For some $m \in \mathbb{N}$,

$$F_\alpha(x) \leqq G_\alpha(x+1)$$

for all $x > m$.

Then, we show the following lemma and proposition(cf.Wainer [50]). For $n \in \mathbb{N}$, the tree-ordinal $0+1+ \cdots +1$ for n times 1's is said to be *finite* and is denoted by n.

LEMMA 5.1.3. For $p \in \mathbb{N}$ and $\alpha \in \Omega^{(p)\text{-bu}}$, the following holds:

(a) For all $x > p$, $G_\alpha(x) < F_\alpha(x)$.

(b) If α is an *s-inaccessible*, then α is a limit and G_α dominates every F_β with $\beta < \alpha$.

Proof. (a) We can show by induction on α. (b) Assume α is an s-inaccessible. Clearly, α cannot be 0. Moreover, α cannot be of the form $\beta+1$, since for any $\beta+1 \in \Omega^{(p)\text{-bu}}$ and $x > \max(p,1)$,

$G_{\beta+1}(x+1) = G_{\beta}(x+1)+1 \leqq F_{\beta}(x+1) \leqq F_{\beta}(F_{\beta}(x)) < F_{\beta}{}^{x+1}(x) = F_{\beta+1}(x)$.

Hence α must be a limit. Assume $\beta < \alpha$. Then $\beta+1 < \alpha$ since α is a limit, and then we can see that for some $m > p$, $\alpha \xrightarrow[m]{} \beta+1$. Hence $F_{\beta}(x+1) < F_{\beta}{}^{x+1}(x) = F_{\beta+1}(x) < F_{\alpha}(x) \leqq G_{\alpha}(x+1)$. □

PROPOSITION 5.1.4. Let $p \in \mathbb{N}$ and $\alpha \in \Omega^{(p)\text{-bu}}$ satisfy that

$$G_{\alpha[n+1]} = F_{\alpha[n]}$$

for each $n \in \mathbb{N}$. Then α is s-inaccessible and, if $\alpha[0]$ is finite, then no $\beta < \alpha$ is s-inaccessible.

Proof. If $G_{\alpha[n+1]} = F_{\alpha[n]}$ for each n, then

$$F_{\alpha}(x) = F_{\alpha[x]}(x) = G_{\alpha[x+1]}(x) \leqq G_{\alpha[x+1]}(x+1) = G_{\alpha}(x+1)$$

and hence α is s-inaccessible. If $\alpha[0]$ is finite and $\beta < \alpha$ were s-inaccessible, then $\alpha[0] < \beta$, since β is limit. So $\alpha[n] < \beta \leqq \alpha[n+1]$ for some n. For sufficient large x, $\alpha[n+1] \xRightarrow[x]{} \beta$, and hence

$$G_{\alpha[n+1]}(x) = F_{\alpha[n]}(x) < G_{\beta}(x) \leqq G_{\alpha[n+1]}(x).$$ □

Now, we define the minimum s-inaccessible ordinal τ following Wainer[49].

DEFINITION 5.1.5. For each $n \in \mathbb{N}$, the set Ω_n of *higher level tree-ordinals* are defined by induction similarly to the case of Ω:

(a) $0 \in \Omega_n$.

(b) If $\alpha \in \Omega_n$, then $\alpha+1 \in \Omega_n$.

(c) If $\alpha_{\gamma} \in \Omega_n$ for all $\gamma \in \Omega_k (k<n)$, then $(\alpha_{\gamma})_{\gamma\in\Omega_k} \in \Omega_n$.

(In the case of (c), the term $(\alpha_{\gamma})_{\gamma\in\Omega_k}$ is called a *limit*, and and $\alpha[\gamma]$ denotes α_{γ}.)

From this definition the sets Ω_0 and Ω_1 can be identified with $\mathbb{N}$ and Ω, respectively. Similarly to the case of Ω, we define the relation $<$ on Ω_n as the transitive closure of (a) $\alpha < \alpha+1$, and (b) $\alpha[\gamma] < \alpha$ for each limit $\alpha = (\alpha[\gamma])_{\gamma\in\Omega_k}$ and $\gamma \in \Omega_k$. We also define the set theoretic height $|\alpha|$ of $\alpha \in \Omega_n$ inductively as (a) $|0| = 0$, (b) $|\alpha+1| = |\alpha|+1$, and (c) $|(\alpha[\gamma])_{\gamma\in\Omega_k}| = \sup\{|\alpha[\gamma]| \mid \gamma\in\Omega_k\}$.

DEFINITION 5.1.6. For each $n \in \mathbb{N}$, the function

$$\varphi_n : \Omega_{n+1}\times\Omega_n \longrightarrow \Omega_n,$$

which is called the *level n fast-growing hierarchy*, is defined inductively by

(a) $\varphi_n(0,\beta) = \beta+1$;

(b) $\varphi_n(\alpha+1,\beta) = \varphi_n{}^{\beta}(\alpha,\varphi_n(\alpha,\beta))$;

(c) $\varphi_n(\lambda,\beta) = (\varphi_n(\lambda[\gamma],\beta))_{\gamma\in\Omega_k}$ for $\lambda = (\lambda[\gamma])_{\gamma\in\Omega_k}$ $(k < n)$,

(d) $\varphi_n(\lambda,\beta) = \varphi_n(\lambda[\beta],\beta)$ for $\lambda = (\lambda[\gamma])_{\gamma\in\Omega_n}$,

where $\varphi_n{}^{\beta}$ denotes the iteration β-times of φ_n, i.e.,if

$$\psi : \Omega_{n+1}\times\Omega_n \longrightarrow \Omega_n,$$

then $\psi^0(\alpha,\beta) = \beta$, $\psi^{\delta+1}(\alpha,\beta) = \psi(\alpha,\psi^{\delta}(\alpha,\beta))$,

$\psi^{\lambda}(\alpha,\beta) = (\psi^{\lambda[\gamma]}(\alpha,\beta))_{\gamma\in\Omega_m}$ for $\lambda = (\lambda[\gamma])_{\gamma\in\Omega_m}$.

Note that, in the case $n = 0$, $\varphi_0(\alpha,\beta) = F_{\alpha}(\beta)$ for $\alpha \in \Omega_1$ and $\beta \in \Omega_0 (= \mathbb{N})$. For each $k < n \in \mathbb{N}$, we define $w_k \in \Omega_n$ by

$$w_k = (\gamma)_{\gamma\in\Omega_k},$$

i.e., $w_k[\gamma] = \gamma$. The tree-ordinals w_0 and w_k for each $k > 0$ has its set-theoretic height ω and the k-th uncountable cardinal, respectively.

DEFINITION 5.1.7. For each $n \in \mathbb{N}$, the set $\mathcal{T}_n$ $(\subset \Omega_n)$ of *named tree-ordinals* is defined inductively by:

(a) $0, 1, w_0, w_1, \cdots, w_{n-1} \in \mathcal{T}_n$;

(b) $\mathcal{T}_k \subseteq \mathcal{T}_n$ for $k < n$;

(c) if $\alpha \in \mathcal{T}_{n+1}$ and $\beta,\gamma \in \mathcal{T}_n$, then $\varphi_n{}^{\gamma}(\alpha,\beta) \in \mathcal{T}_n$.

For a fixed $x \in \mathbb{N}$, the function co ($= co_x$) which collapses each $\mathcal{T}_{n+1}$ to $\mathcal{T}_n$ is defined by:

$$co(0) = 0, \quad co(1) = 1, \quad co(w_0) = x, \quad co(w_{k+1}) = w_k,$$

$$co(\varphi_{k+1}{}^{\gamma}(\delta,\xi)) = \varphi_k{}^{co(\gamma)}(co(\delta),co(\xi)), \quad co(\varphi_0{}^{\gamma}(\delta,\xi)) = \varphi_0{}^{\gamma}(\delta,\xi).$$

The well-definedness of this function can be proved by using Theorem 5.2.4 and Lemma 5.2.5.

THEOREM 5.1.8(*Collapsing Theorem*). Let $x \in \mathbb{N}$, $\alpha \in \mathcal{T}_2$ and $\beta \in \mathcal{T}_0$. Then,

$$G_{\varphi_1(\alpha,\beta)}(x) = F_{co(\alpha)}(G_\beta(x)).$$

Hence, in particular, if α is generated in $\mathcal{T}_2$ without reference to w_0 then, as $G_{w_0}(x) = x$, we have $G_{\varphi_1(\alpha,w_0)} = F_{co(\alpha)}$.

Proof. We will prove in Section 5.2. □

DEFINITION 5.1.9. The tree-ordinal $\tau = (\tau[x])_{x\in\mathbb{N}}$ is defined as follows: $\tau[0] = 3$;

$$\tau[n+1] = \varphi_1(\ldots\varphi_n(\varphi_{n+1}(3,w_n),w_{n-1}),\ldots,w_0) \quad \text{for } n > 0.$$

THEOREM 5.1.10. τ is a minimal s-inaccessible.

Proof. From the results of Section 5.2, τ is (3)-built-up. Then, we can prove this theorem by using Proposition 5.1.4 and Theorem 5.1.8(Collapsing Theorem). □

5.2 The collapsing theorem and (3)-built-upness

In this section, we will prove Theorem 5.1.8(Collapsing Theorem) and that the tree-ordinal τ is (3)-built-up, which were used in Section 5.1. First, we prove the strong normalization theorem shown by Kadota[20]. We introduce term structures $<\bar{\mathcal{T}}_n, NT_n, \cdot[\cdot], \longrightarrow>$ by considering each element in $\mathcal{T}_n$ as a finitary term and each defining equation of φ_n as a rewrite (or reduction) rule of the terms. Let $\bar{0}$, $\bar{1}$, $\bar{w}_0$, $\bar{w}_1$, ...; $\bar{\varphi}_0$, $\bar{\varphi}_1$, ... be formal symbols.

DEFINITION 5.2.1. For each $n \in \mathbb{N}$, the set $\bar{\mathcal{T}}_n$ of *terms* is defined inductively by:

(a) $\bar{0}, \bar{1}, \bar{w}_0, \bar{w}_1, \ldots, \bar{w}_{n-1} \in \bar{\mathcal{T}}_n$;

(b) $\bar{\mathcal{T}}_k \subseteq \bar{\mathcal{T}}_n$ for $k < n$;

(c) if $a \in \bar{\mathcal{T}}_{n+1}$ and $b, c \in \bar{\mathcal{T}}_n$, then $\bar{\varphi}_n{}^c(a,b) \in \bar{\mathcal{T}}_n$.

Naturally, terms in $\bar{\mathcal{T}}_n$ are interpreted as tree-ordinals by the function ord: $\bar{\mathcal{T}}_n \longrightarrow \mathcal{T}_n$ such that

(a) $\mathrm{ord}(\bar{0}) = 0$, $\mathrm{ord}(\bar{1}) = 1$, $\mathrm{ord}(\bar{w}_k) = w_k$;

(b) $\mathrm{ord}(\bar{\varphi}_n{}^c(a,b)) = \varphi_n{}^{\mathrm{ord}(c)}(\mathrm{ord}(a),\mathrm{ord}(b))$.

ABBREVIATIONS. $\bar{\varphi}_n(a,b) = \bar{\varphi}_n{}^{\bar{1}}(a,b)$; $b+1 = \bar{\varphi}_n(\bar{0},b)$.

DEFINITION 5.2.2. The sets NT_n of *normal terms* in $\bar{\mathcal{T}}_n$; $\mathrm{dom}(a) \in \{\phi, \{\bar{0}\}, \bar{\mathcal{T}}_0, \ldots, \bar{\mathcal{T}}_{n-1}\}$ and $a[z]$ for $a \in NT_n$, $z \in \mathrm{dom}(a)$ are defined inductively as follows:

(N1) $\bar{0} \in NT_n$; $\mathrm{dom}(\bar{0}) = \phi$.

(N2) $\bar{1} \in NT_n$; $\mathrm{dom}(\bar{1}) = \{\bar{0}\}$, $\bar{1}[\bar{0}] = \bar{0}$.

(N3) $\bar{w}_k \in NT_k$ for $k < n$; $\mathrm{dom}(\bar{w}_k) = \bar{\mathcal{T}}_k$, $\bar{w}_k[z] = z$.

(N4) $NT_k \subseteq NT_n$ for $k < n$.

(N5) Let $a \in NT_{n+1}$, $b, c \in NT_n$ and $A = \bar{\varphi}_n{}^c(a,b)$. Then, $A \in NT_n$ if one of the following holds:

  (a) $c = \bar{1}$ and $a = \bar{0}$ (i.e., $A = b+1$).
  In this case, define $\mathrm{dom}(A) = \{\bar{0}\}$, $A[z] = b$.

  (b) $\mathrm{dom}(c) = \bar{\mathcal{T}}_k$ for $k < n$.
  In this case, define $\mathrm{dom}(A) = \mathrm{dom}(c)$, $A[z] = \bar{\varphi}_n{}^{c[z]}(a,b)$.

  (c) $c = \bar{1}$ and $\mathrm{dom}(a) = \bar{\mathcal{T}}_k$ for $k < n$; $\mathrm{dom}(A) = \mathrm{dom}(a)$,
  $A[z] = \bar{\varphi}_n(a[z],b)$.

Next, we define term rewriting system S (see e.g., Dershowitz[9] as for the definition) so that, for every term in $\mathcal{T}_n$ which is not normal, some rewrite rule in S is applied to it. Its rewrite rules are as follows: For normal terms a, b, c,

(R1) $\bar{\varphi}_n{}^{\bar{0}}(a,b) \longrightarrow b$; (R2) $\bar{\varphi}_n(\bar{1},b) \longrightarrow \bar{\varphi}_n{}^{b}(\bar{0},\bar{\varphi}_n(\bar{0},b))$;

(R3) $\bar{\varphi}_n(a+1,b) \longrightarrow \bar{\varphi}_n{}^{b}(a,\bar{\varphi}_n(a,b))$;

(R4) $\bar{\varphi}_n{}^{c+1}(a,b) \longrightarrow \bar{\varphi}_n(a,\bar{\varphi}_n{}^{c}(a,b))$;

(R5) $\bar{\varphi}_n(a,b) \longrightarrow \bar{\varphi}_n(a[b],b)$ if $\mathrm{dom}(a) = \mathcal{T}_n$.

PROPOSITION 5.2.3. For every $a \in \mathcal{T}_n$, $a \in NT_n$ if and only if there is no $b \in \mathcal{T}_n$ such that $a \xrightarrow{1} b$ (where $a \xrightarrow{1} b$ means that b is obtained from a by a single application of some rule of S).

Proof. We can prove by induction on the length of a. □

Kadota[20, Theorem 1] showed the following theorem.

THEOREM 5.2.4(*Strong normalization theorem*). Every term a in $\mathcal{T}_n$ is *strongly normalizable* (i.e., there is no infinite sequence such that $a \xrightarrow{1} a_1 \xrightarrow{1} a_2 \xrightarrow{1} \cdots$). □

Now, we introduce a function $\overline{co}(= \overline{co}_x)$ for a fixed $x \in \mathbb{N}$, which represents the function co (in the collapsing theorem) on the terms as follows:

(a) $\overline{co}(\bar{0}) = \bar{0}$, $\overline{co}(\bar{1}) = \bar{1}$, $\overline{co}(\bar{w}_0) = \bar{x}$, $\overline{co}(\bar{w}_{k+1}) = \bar{w}_k$,

(b) $\overline{co}(\bar{\varphi}_{k+1}{}^{c}(a,b)) = \bar{\varphi}_k{}^{\overline{co}(c)}(\overline{co}(a),\overline{co}(b))$ and

$\overline{co}(\bar{\varphi}_0{}^{c}(a,b)) = \bar{\varphi}_0{}^{c}(a,b)$,

where $\bar{x}$ is the *numeral* of x (i.e., if $x = 0$, then $\bar{x} = \bar{0}$; if $x = y+1$, then $\bar{x} = \bar{\varphi}_0(\bar{0},\bar{y})$ $(= \bar{y}+1)$).

LEMMA 5.2.5. Let $a \in \mathcal{T}_n$. Then, the following hold.

(a) If $a = b+1$ for some b, then $\overline{co}(a) = \overline{co}(b)+1$.

(b) If $a \in NT_n$ and $\mathrm{dom}(a) = \mathcal{T}_0$, then $\overline{co}(a[\bar{x}]) = \overline{co}(a)$ and

$\mathrm{ord}(a[\bar{x}]) = \mathrm{ord}(a)[x]$ for $x \in \mathbb{N}$.

(c) If $a \in NT_n$ and $\mathrm{dom}(a) = \bar{\mathcal{T}}_k$ for some $k > 0$, then

$$\mathrm{ord}(a[b]) = \mathrm{ord}(a)[\mathrm{ord}(b)] \text{ and}$$

$$\mathrm{ord}(\overline{co}(a[b])) = \mathrm{ord}(\overline{co}(a))[\mathrm{ord}(\overline{co}(b))] \text{ for } b \in \mathrm{dom}(a).$$

(d) If $a \xrightarrow{1} b$, then $\mathrm{ord}(a) = \mathrm{ord}(b)$ and

$$\mathrm{ord}(\overline{co}(a)) = \mathrm{ord}(\overline{co}(b)).$$

Proof. We can prove by induction on the length of a. □

LEMMA 5.2.6. If $x \in \mathbb{N}$ and $a \in \bar{\mathcal{T}}_1$, then

$$G_{\mathrm{ord}(a)}(x) = \mathrm{ord}(\overline{co}(a)).$$

Proof. From the strong normalization theorem, the proof is proceeded by transfinite induction on a over the well-founded ordering << (where << on $\bar{\mathcal{T}}_n$ is defined as the transitive closure of (a) $b[z] << b$ for normal b with $z \in \mathrm{dom}(b)$, (b) $d << b$ for nonnormal b with $b \xrightarrow{1} d$).

Case 1. $a = \bar{0}$. This case is trivial.

Case 2. $a \in NT_1$ and $\mathrm{dom}(a) = \{\bar{0}\}$. Then, $a = \bar{1}$ or $b+1$ for some $b \in \bar{\mathcal{T}}_1$. If $a = \bar{1}$, then the assertion is trivial. If $a = b+1$, then

$$G_{\mathrm{ord}(a)}(x) = G_{\mathrm{ord}(b)}(x)+1 = \mathrm{ord}(\overline{co}(b))+1 = \mathrm{ord}(\overline{co}(a))$$

by the induction hypothesis and Lemma 5.2.5(a).

Case 3. $a \in NT_1$ and $\mathrm{dom}(a) = \bar{\mathcal{T}}_0$. By Lemma 5.2.5(b) and the induction hypothesis,

$$G_{\mathrm{ord}(a)}(x) = G_{\mathrm{ord}(a[\bar{x}])}(x) = \mathrm{ord}(\overline{co}(a[\bar{x}])) = \mathrm{ord}(\overline{co}(a)).$$

Case 4. $a \xrightarrow{1} b$ for some b. By Lemma 5.2.5(d) and the induction hypothesis,

$$G_{\mathrm{ord}(a)}(x) = G_{\mathrm{ord}(b)}(x) = \mathrm{ord}(\overline{co}(b)) = \mathrm{ord}(\overline{co}(a)).$$ □

Proof of Theorem 5.1.8 (Collapsing Theorem). For $a \in \bar{\mathcal{T}}_2$ and $b \in \bar{\mathcal{T}}_1$, we have

$$\overline{co}(\bar{\varphi}_1(a,b)) = \bar{\varphi}_0(\overline{co}(a),\overline{co}(b))$$

and hence $\mathrm{ord}(\overline{co}(\bar{\varphi}_1(a,b)) = \varphi_0(\mathrm{ord}(\overline{co}(a)),\mathrm{ord}(\overline{co}(b)))$. Thus,

$$G_{\varphi_1(\mathrm{ord}(a),\mathrm{ord}(b))}(x) = G_{\mathrm{ord}(\bar{\varphi}_1(a,b))}(x)$$

$$= \mathrm{ord}(\overline{co}(\bar{\varphi}_1(a,b))) \quad \text{by Lemma 5.2.6,}$$

$$= \varphi_0(\mathrm{ord}(\overline{co}(a)),\mathrm{ord}(\overline{co}(b)))$$

$$= F_{\mathrm{ord}(\overline{co}(a))}(\mathrm{ord}(\overline{co}(b)))$$

$$= F_{\mathrm{ord}(\overline{co}(a))}(G_{\mathrm{ord}(b)}(x))$$

by Lemma 5.2.6. For given $\alpha \in \mathcal{T}_2$ and $\beta \in \mathcal{T}_1$, we choose a and b above such that (a) $\mathrm{ord}(a) = \alpha$, $\mathrm{ord}(\overline{co}(a)) = co(\alpha)$, and (b) $\mathrm{ord}(b) = \beta$. (We can choose such a and b since the elements of $\mathcal{T}_n$ are constructed by the same way as to the element in $\bar{\mathcal{T}}_n$). This completes the proof. □

Next, we prove that τ is (3)-built-up. This completes the proof of Theorem 5.1.10 that τ is a minimal s-inaccessible. First, we remark that the following proposition holds:

PROPOSITION 5.2.7. Let $\alpha \in \mathcal{T}_n$ and $\alpha = (\alpha[\gamma])_{\gamma\in\Omega_m}$. Then, $\alpha[\gamma] \in \mathcal{T}_n$ for every $\gamma \in \mathcal{T}_m$.

Proof. For a given $\alpha = (\alpha[\gamma])_{\gamma\in\Omega_m} \in \mathcal{T}_n$, there is a normal term $a \in \bar{\mathcal{T}}_n$ such that $\mathrm{ord}(a) = \alpha$ by Lemma 5.2.5(d) and the strong normalization theorem. We fix such a term $a \in \bar{\mathcal{T}}_n$ with the minimal length. We can prove this proposition by induction on the length of this term a for α. □

It follows from this proposition that we can use transfinite induction on the tree-ordinals in $\mathcal{T}_n$ over the ordering <.

DEFINITION 5.2.8. For each $k \in \mathbb{N}$, the relation $\xrightarrow[k]{}$ on the set $\mathcal{T}_n$ for each $n \in \mathbb{N}$ is defined inductively as follows:

$\alpha \xrightarrow[k]{} \beta$ if $\alpha \neq 0$ and one of the following holds;

(a) $\gamma \overset{=}{\underset{k}{\Longrightarrow}} \beta$ if $\alpha = \gamma + 1$,

(b) $\alpha[k] \overset{=}{\underset{k}{\Longrightarrow}} \beta$ if $\beta = (\beta[x])_{x\in\Omega_0}$,

(c) $\alpha[\gamma] \xrightarrow[k]{} \beta$ for all $\gamma \in \mathcal{T}_m\setminus\{0\}$ if $\beta = (\beta[\gamma])_{\gamma\in\Omega_m}$ $(m>0)$.

where $\delta \underset{k}{\Longrightarrow} \beta$ means that $\delta \underset{k}{\longrightarrow} \beta$ or $\delta = \beta$.

Note that if α, $\beta \in \mathcal{T}_1$, then the relation $\underset{k}{\longrightarrow}$ for each $k \in \mathbb{N}$ is the same as that defined on Ω. Then, the following lemmas and theorem can be proved (see Kadota[18]). Let n, $k \in \mathbb{N}$.

LEMMA 5.2.9. If $\alpha \in \mathcal{T}_{n+1}$, $\beta \in \mathcal{T}_n$ and $\gamma \in \mathcal{T}_n \setminus \{0\}$, then

$$\varphi_n{}^{\gamma}(\alpha, \beta) \underset{k}{\longrightarrow} \beta.$$

LEMMA 5.2.10. Let $\alpha \in \mathcal{T}_{n+1}$ and $\beta, \delta, \gamma \in \mathcal{T}_n$. If $\delta \underset{k}{\longrightarrow} \gamma$, then

$$\varphi_n{}^{\delta}(\alpha, \beta) \underset{k}{\longrightarrow} d_n{}^{\gamma}(\alpha, \beta).$$

LEMMA 5.2.11. Let α, $\gamma \in \mathcal{T}_{n+1}$, $\beta \in \mathcal{T}_n \setminus \{0\}$ and $n > 0$.

If $\alpha \underset{k}{\longrightarrow} \gamma$, then $\varphi_n(\alpha, \beta) \underset{k}{\longrightarrow} \varphi_n(\gamma, \beta)$.

THEOREM 5.2.12. (a) If $\alpha \in \mathcal{T}^+_n$, $\alpha = (\alpha[\xi])_{\xi \in \Omega_m}$, $\gamma, \delta \in \mathcal{T}_m$ and $\gamma \underset{k}{\longrightarrow} \delta$, then

$$\alpha[\gamma] \underset{k}{\longrightarrow} \alpha[\delta].$$

Here the set $\mathcal{T}^+_n$ ($\subseteq \mathcal{T}_n$) for each $n \in \mathbb{N}$ is defined inductively by

(T^+1) $0, 1, w_0, \cdots, w_{n-1} \in \mathcal{T}^+_n$;

(T^+2) $\mathcal{T}^+_k \subseteq \mathcal{T}^+_n$ for $k < n$;

(T^+3) if $\alpha \in \mathcal{T}^+_{n+1}$, $\gamma \in \mathcal{T}^+_n$ and $\beta \in \mathcal{T}^+_n \setminus \{0\}$, then $\varphi_n{}^{\gamma}(\alpha, \beta) \in \mathcal{T}^+_n$.

(b) Each $\alpha \in \mathcal{T}^+_1$ is (k)-built-up for each $k \in \mathbb{N}$.

We remark that (k)-built-upness does not hold for some element in $\mathcal{T}_1$ since, if we put $\alpha = \varphi_1(w_0, 0)$, then $\alpha[x] = \varphi_1(x, 0) = 1$ for all $x \in \mathbb{N}$.

THEOREM 5.2.13. τ is (3)-built-up.

Proof. From the definition of τ, we can show that $\tau[x] \in \mathcal{T}^+_1$ for every $x \in \mathbb{N}$. So, $\tau[x]$ is (0)-built-up. Hence it is sufficient to prove that $\tau[x+1] \xrightarrow[3]{} \tau[x]$. For this, we have

$$\tau[x+1] = \varphi_1(\dots\varphi_x(\varphi_{x+1}(3,w_x),w_{x-1})\dots,w_0)$$
$$\xrightarrow[3]{} \varphi_1(\dots\varphi_x(w_x,w_{x-1})\dots,w_0)$$
$$\xrightarrow[3]{} \varphi_1(\dots\varphi_x(w_0,w_{x-1})\dots,w_0)$$
$$\xrightarrow[3]{} \varphi_1(\dots\varphi_x(3,w_{x-1})\dots,w_0) = \tau[x]$$

since $w_0 \xrightarrow[3]{} 3$. □

5.3 Provable computability

In this section, we summarize the results of Kadota[18] on the classification of provably computable functions in $ID_{<\omega}$ by means of the fast growing hierarchy. Here, $ID_{<\omega}$ is the theory of finitely iterated inductive definitions, which is defined later in this section. In [18], Kadota modified τ, introduced τ' and showed the following three theorems.

THEOREM 5.3.1. $F_{\tau'}(x) \leqq G_{\tau'}(G_{\tau'}(x+1))$ for $x > 3$.

THEOREM 5.3.2. F_α is provably computable in $ID_{<\omega}$ for $\alpha < \tau'$.

THEOREM 5.3.3. If a computable function $f^k:\mathbb{N} \longrightarrow \mathbb{N}$ is provably computable in $ID_{<\omega}$, then f is dominated by F_α for some $\alpha < \tau'$.

As a corollary of the last two theorems, we can immediately prove the following corollary.

COROLLARY 5.3.4. Provably computable functions in $ID_{<\omega}$ are exactly those which are elementary recursive in $\{F_\alpha \mid \alpha < \tau'\}$.

The tree-ordinal τ' is defined by the same way as τ except that the definition (d) of φ_n is replaced by

(d)' $\quad \varphi_n(\lambda,\beta) = \varphi_n(\lambda[z],\beta)$, where $z = \varphi_n(\lambda[1],\beta)$.

We define the formal theory ID_ν for $\nu \in \mathbb{N}$ following

Buchholz[3, Section 4].

By $s,t,t_0,\ldots,$- we denote arbitrary terms of $\mathcal{L}(PA)$. We will use the same symbols i,j,k,m,n,u,v to denote natural numbers and numerals for convenience. A formula of the shape $\mathrm{p}(t_1,\cdots,t_k)$ or $\neg\mathrm{p}(t_1,\cdots,t_k)$, where p is a k-ary predicate symbol of $\mathcal{L}(PA)$, is called an *arithmetic prime formula* (abbreviated by a.p.f.).

Let X be a unary and Y a binary predicate variable. A *positive operator form* is a formula $\mathfrak{A}(\mathrm{X,Y,y,x})$ of $\mathcal{L}(PA)+\{\mathrm{X,Y}\}$ in which only X,Y,y,x occur free and all occurrences of X are positive. The language $\mathcal{L}(ID)$ is obtained from $\mathcal{L}(PA)$ by adding a binary predicate constant $P^{\mathfrak{A}}$ and a 3-ary predicate constant $P^{\mathfrak{A}}_{<}$ for each positive operator form $\mathfrak{A}$.

ABBREVIATIONS. $(t \in P^{\mathfrak{A}}_s) = P^{\mathfrak{A}}_s(t) = P^{\mathfrak{A}}(s,t)$;

$(t \notin P^{\mathfrak{A}}_s) = \neg(t \in P^{\mathfrak{A}}_s)$; $\qquad P^{\mathfrak{A}}_{<s}(t_0,t_1) = P^{\mathfrak{A}}_{<}(s,t_0,t_1)$;

$\mathfrak{A}_s(\mathrm{X,x}) = \mathfrak{A}(\mathrm{X},P^{\mathfrak{A}}_{<s},s,\mathrm{x})$.

The formal theory ID_ν with $\nu \in \mathbb{N}$ is an extension of Peano Arithmetic, formulated in the language $\mathcal{L}(\mathrm{ID})$, by the following axioms:

$(\mathrm{P}^{\mathfrak{A}}.1)$ $\quad \forall\mathrm{y}\forall\mathrm{x}(\mathfrak{A}_{\mathrm{y}}(P^{\mathfrak{A}}_{\mathrm{y}},\mathrm{x}) \longrightarrow \mathrm{x} \in P^{\mathfrak{A}}_{\mathrm{y}})$.

$(\mathrm{P}^{\mathfrak{A}}.2)_{<\nu}$ $\quad \forall\mathrm{x}(\mathfrak{A}_u(A,\mathrm{x}) \longrightarrow A(\mathrm{x})) \longrightarrow \forall\mathrm{x}(\mathrm{x} \in P^{\mathfrak{A}}_u \longrightarrow A(\mathrm{x}))$, for each formula $A(\mathrm{x})$ of $\mathcal{L}(ID)$ and each $u < \nu$.

$(\mathrm{P}^{\mathfrak{A}}.3)$ $\quad \forall\mathrm{y}\forall\mathrm{x}\forall\mathrm{z}(P^{\mathfrak{A}}_{<\mathrm{y}}(\mathrm{x,z}) \longleftrightarrow ((\mathrm{x} < \mathrm{y}) \wedge \mathrm{z} \in P^{\mathfrak{A}}_{\mathrm{x}}))$.

Next, we introduce the infinitary theory $\varphi ID^{\infty}_{<\omega}$, as in Buchholz[3,Section 4]. The theory $\varphi ID^{\infty}_{<\omega}$ shall be formulated in the language $\mathrm{L}(ID)+\{\mathrm{N}\}$ where N is a new unary predicate symbol. This is a technical tool which will help us to keep control over the numerals n occurring in $\exists$-inferences $\mathrm{A}(n) \vdash \exists\mathrm{xA(x)}$ of $\varphi ID^{\infty}_{<\omega}$-derivations. Following Tait[45], we assume all formulas to be in *negation normal form*, i.e., the formulas are built up from atomic and negated atomic formulas by means of $\wedge,\vee,\forall,\exists$. If A is a complex formula we consider $\neg A$ as a notation for the corresponding negation normal form.

Let $\mathcal{L}_{ID}(\mathrm{N})$ be the language $\mathcal{L}(ID)+\{\mathrm{N}\}$. The length $|A|$ of each formula A of $\mathcal{L}_{ID}(\mathrm{N})$ is defined as follows:

(a) $|\mathrm{N}(t)| = |\neg\mathrm{N}(t)| = 0$.

(b) $|A| = 1$ if A is an a.p.f., $P^{\mathfrak{A}}_{s}(t)$ or $\neg P^{\mathfrak{A}}_{s}(t)$.

(c) $|P^{\mathfrak{A}}_{<s}(t_0,t_1)| = |\neg P^{\mathfrak{A}}_{<s}(t_0,t_1)| = 2$.

(d) $|A \wedge B| = |A \vee B| = \max\{|A|,|B|\} + 1$.

(e) $|\forall xA| = |\exists xA| = |A| + 1$.

PROPOSITION 5.3.5. $|\neg A| = |A|$, for each formula A of $\mathcal{L}_{ID}(\mathrm{N})$.

For each $v \in \mathbb{N}$, the set Pos_v of formulas of $\mathcal{L}_{ID}(\mathrm{N})$ is defined as follows:

(a) All formulas of $\mathcal{L}(PA)+\{\mathrm{N}\}$ belong to Pos_v.

(b) All formulas $P^{\mathfrak{A}}_{u}(t)$, $P^{\mathfrak{A}}_{<u}(t_0,t_1)$, $\neg P^{\mathfrak{A}}_{<u}(t_0,t_1)$ with $u \leqq v$ belong to Pos_v.

(c) All formulas $\neg P^{\mathfrak{A}}_{u}(t)$ with $u < v$ belong to Pos_v.

(d) If A and B belong to Pos_v, then the formulas $A \wedge B$, $A \vee B$, $\forall xA$, $\exists xA$ also belong to Pos_v.

REMARK 5.3.6. If $P^{\mathfrak{A}}_{u}(t) \in \mathrm{Pos}_v$, then also $\mathfrak{A}_u(P^{\mathfrak{A}}_{u},t) \in \mathrm{Pos}_v$.

In the following, A,B,C always denote closed formulas of $\mathcal{L}_{ID}(\mathrm{N})$. Γ, Γ', Δ denote finite sets of closed formulas of $\mathcal{L}_{ID}(\mathrm{N})$. We write, e.g., Γ, Δ, A for $\Gamma \cup \Delta \cup \{A\}$. A^{N} denotes the result of restricting all quantifiers in A to N. We define the following:

$$(t \in \mathrm{N}) = \mathrm{N}(t); \quad (t \notin \mathrm{N}) = \neg\mathrm{N}(t).$$

DEFINITION 5.3.7. For $\alpha,\beta \in \Omega_n$ for some $n \in \mathbb{N}$, $\alpha \xrightarrow[\Gamma]{} \beta$ is $\alpha \xrightarrow[k]{} \beta$ where $k = \max(\{3\} \cup \{3n \mid \neg\mathrm{N}(n) \in \Gamma\})$.

PROPOSITION 5.3.8. Let $\alpha,\beta \in \Omega_n$ for some $n \in \mathbb{N}$.

(a) If $\alpha \xrightarrow[\Gamma]{} \beta$ and $\Gamma \subseteqq \Delta$, then $\alpha \xrightarrow[\Gamma]{} \beta$.

(b) If $\alpha \xrightarrow[\Gamma \cup \{0 \notin N\}]{} \beta$, then $\alpha \xrightarrow[\Gamma]{} \beta$.

To define the system $\varphi ID^{\infty}_{<\omega}$, we first define the basic inference rules as follows:

($\wedge$) $A_0, A_1 \vdash A_0 \wedge A_1$.

($\vee$) $A \vdash A \vee B$; $B \vdash A \vee B$.

($\forall^{\infty}$) $(A(n))_{n \in \mathbb{N}} \vdash \forall x A(x)$.

($\exists$) $A(n) \vdash \exists x A(x)$.

(N) $n \in \mathrm{N} \vdash \mathrm{S}n \in \mathrm{N}$.

($P^{\mathfrak{A}}_{<u}$) $P^{\mathfrak{A}}_j(n) \vdash P^{\mathfrak{A}}_{<u}(j,n)$, if $j < u$.

($\neg P^{\mathfrak{A}}_{<u}$) $\neg P^{\mathfrak{A}}_j(n) \vdash \neg P^{\mathfrak{A}}_{<u}(j,n)$, if $j < u$.

Every instance $(A_i)_{i \in I} \vdash A$ of these rules is called a basic inference. If $(A_i)_{i \in I} \vdash A$ is a basic inference with $A \in \mathrm{Pos}_v$, then $A_i \in \mathrm{Pos}_v$ for all $i \in I$. This property will be used in the proof of Lemma 5.3.13.

The system $\varphi ID_{<\omega}{}^{\infty}$ will consist of the language $\mathcal{L}_{ID}(\mathrm{N})$ and a certain derivability relation $\vdash^{\alpha}_{m} \Gamma$ for each $\alpha \in \mathcal{T}^*_n$ for some $n \in \mathbb{N}$ and for each $m \in \mathbb{N}$. This means that Γ is derivable with order $\alpha \in \mathcal{T}^*_n$ and cut degree $m \in \mathbb{N}$. Here, for each $n \in \mathbb{N}$, the set $\mathcal{T}^*_n$ ($\subseteq \mathcal{T}^+_n$, cf. Theorem 5.2.12) are defined inductively as follows:

(a) $\mathcal{T}^*_{-1} = \{0\}$.

(b) $0, 1, w_0, \cdots, w_{n-1} \in \mathcal{T}^*_n$.

(c) $\mathcal{T}^*_k \subseteq \mathcal{T}^*_n$ for $k < n$.

(d) If $\alpha \in \mathcal{T}^*_{n+1}$, $\gamma \in \mathcal{T}^*_n$ and $\beta \in \mathcal{T}^*_n \setminus \mathcal{T}^*_{n-1}$, then $\varphi_n{}^{\gamma}(\alpha, \beta) \in \mathcal{T}^*_n$.

We define $\vdash^{\alpha}_{m} \Gamma$ ($m \in \mathbb{N}$, $\alpha \in \mathcal{T}^*_k$ for some $k \in \mathbb{N}$) inductively as follows. Let $n \in \mathbb{N}$.

(Ax1) $\vdash^{\alpha}_{m} \Gamma, A$ if A is a true a.p.f., $(0 \in \mathrm{N})$ or $\neg P^{\mathfrak{A}}_{<u}(j,n)$ with $u \leq j$.

(Ax2) $\vdash^{\alpha}_{m} \Gamma, \neg A, A$ if A is $(n \in \mathrm{N})$ or $P^{\mathfrak{A}}_u(n)$.

(Bas) If $(A_i)_{i\in I} \vdash A$ is a basic inference with $A \in \Gamma$ and $\vdash^{\alpha}_{m} \Gamma, A_i$ for all $i \in I$, then $\vdash^{\alpha+1}_{m} \Gamma$.

$(P^{\mathfrak{A}}_{u})$ If $\vdash^{\alpha}_{m} \Gamma$, $n \in \mathrm{N} \wedge \mathfrak{A}^{\mathrm{N}}_{u}(\mathrm{P}^{\mathfrak{A}}_{u}, n)$ and $P^{\mathfrak{A}}_{u}n \in \Gamma$, then $\vdash^{\alpha+3}_{m} \Gamma$.

(Cut) If $\vdash^{\alpha}_{m} \Gamma, \neg C$ and $\vdash^{\alpha}_{m} \Gamma, C$ and $|C| < m$, then $\vdash^{\alpha+1}_{m} \Gamma$.

(Ω_{u+1}) If $\alpha = (\alpha[\gamma])_{\gamma\in\Omega_{u+1}}$ and $\vdash^{\alpha[1]}_{m} \Gamma, P^{\mathfrak{A}}_{u}n$ and

$\vdash^{\alpha[z]}_{m} \Delta, \Gamma$ for all $z \in \Omega_{u+1}$ and $\Delta \subseteq \mathrm{Pos}_u$ such that $\vdash^{z}_{1} \Delta, P^{\mathfrak{A}}_{u}n$, then $\vdash^{\alpha+1}_{m} \Gamma$.

$(\longrightarrow)$ If $\vdash^{\beta}_{m} \Gamma$ and $\alpha \xrightarrow[\Gamma]{} \beta$, then $\vdash^{\alpha}_{m} \Gamma$.

Then, the following lemmas and a theorem are proved similarly to Buchholz[3].

LEMMA 5.3.9. (a) If $\vdash^{\alpha}_{m} \Gamma$ and $m \leq k$, $\Gamma \subseteq \Delta \Longrightarrow \vdash^{\alpha}_{k} \Delta$.
(b) If $\vdash^{\alpha}_{m} \Gamma$, then $\vdash^{\gamma+\alpha}_{m} \Gamma$ (where $\gamma+\alpha = \varphi^{\alpha}_{n}(0,\gamma)$).
(c) If $\vdash^{\alpha}_{m} \Gamma$, $0 \notin \mathrm{N}$, then $\vdash^{\alpha}_{m} \Gamma$.

LEMMA 5.3.10(*Inversion*). Let $(A_i)_{i\in \mathrm{I}} \vdash A$ be a basic inference $(\wedge)$, $(\forall^{\infty})$, $(\mathrm{P}^{\mathfrak{A}}_{<\mathrm{u}})$, $(\neg\mathrm{P}^{\mathfrak{A}}_{<\mathrm{u}})$. Then, $\vdash^{\alpha}_{m} \Gamma, A$ implies $\vdash^{\alpha}_{m} \Gamma, A_i$ for all $i \in I$.

LEMMA 5.3.11(*Reduction*). Suppose $\vdash^{\alpha}_{m} \Gamma_0, \neg C$ and $|C| \leqq m$, where C is a formula of the shape $A \vee B$ or $\exists x A(\mathrm{x})$ or $P^{\mathfrak{A}}_{<u}(j,n)$ or $\neg P^{\mathfrak{A}}_{u}(n)$ or a false a.p.f. Then, $\vdash^{\beta}_{m} \Gamma, \mathrm{C}$ implies $\vdash^{\alpha+\beta}_{m} \Gamma_0, \Gamma$.

THEOREM 5.3.12(*Cutelimination*). If $\vdash^{\alpha}_{m+1} \Gamma$ and $\alpha \in \mathcal{T}^{*}_{\nu+1}$ for some $\nu \in \mathbb{N}$ and $m > 0$, then $\vdash^{z(k)}_{m} \Gamma$ where

$z(k) = \varphi^{\alpha}_{\nu+1}(1,\varphi_{\nu+1}(1,\varphi^{k}_{\nu+1}(2,\mathrm{w}_{\nu})))$ for each $k \in \mathbb{N}$.

LEMMA 5.3.13(*Collapsing Lemma*). If $\vdash^{\alpha}_{1} \Gamma$ and $\Gamma \subset \mathrm{Pos}_{\nu}$, $\alpha \in \mathcal{T}^{*}_{\nu+2}$, then $\vdash^{z}_{1} \Gamma$ where $z = \varphi_{\nu+1}(\alpha,\mathrm{w}_{\nu})$.

DEFINITION 5.3.14. Let $\mathcal{L}(\mathrm{N})_+$ be the set $\{A \mid A$ is a sentence of $\mathcal{L}(PA)+\{\mathrm{N}\}$ in which N occurs only positively$\}$. For $\Gamma = \{A_1,\ldots,A_n\} \subseteqq \mathcal{L}(\mathrm{N})_+$ and each $k \in \mathbb{N}$, the relation $\models \Gamma(k)$ is defined as

$$\begin{cases} A_1\vee\cdots\vee A_n \text{ is true in the standard model} \\ \text{when N is interpreted as } \{i \in \mathbb{N} \mid 3i < k\}. \end{cases}$$

LEMMA 5.3.15. If $\vdash_1^\alpha i_1 \notin \mathrm{N},\ldots,i_m \notin \mathrm{N}, \Gamma$ for $\alpha \in \mathcal{T}_1^*$ and $\Gamma \subseteqq \mathcal{L}(\mathrm{N})_+$ and $n \geqq \max\{3, 3i_1,\ldots,3i_m\}$, then $\models \Gamma(F_\alpha(n))$.

THEOREM 5.3.16(*Bounding*). If $\vdash_1^\alpha \forall x{\in}\mathrm{N}(\exists y{\in}\mathrm{N}(A^{\mathrm{N}}(x,y)))$, where $0 < \alpha \in \mathcal{T}_1^*$ and $A(x,y)$ a Σ_1-formula of $\mathcal{L}(PA)$, then for each $n > 1$, there is $k \in \mathbb{N}$ such that $k < F_{\alpha+1}(n)$ and $A(n,k)$ is true.

Proof. From the premise, we obtain $\vdash_1^\alpha n \notin \mathrm{N}, \exists y{\in}\mathrm{N}(A^{\mathrm{N}}(n,y))$. Then, we get $\models(\exists y{\in}\mathrm{N}(A^{\mathrm{N}}(n,y))(F_\alpha(\hat{n}))$ for $\hat{n} \geqq \max\{3,3n\}$ by Lemma 5.3.15. Hence for each n, there is a k such that $k < F_\alpha(3n+3)$ and $A(n,k)$ is true. From $3n+3 < 4n+2 = F_1^2(n)$ since $n > 1$. Thus,

$$F_\alpha(3n+3) < F_\alpha(F_1^2(n)) \leqq F_\alpha^3(n) \leqq F_\alpha^{n+1}(n) = F_{\alpha+1}(n)$$

since $\alpha \xrightarrow[1]{} 1$ from $0 < \alpha$. □

In the following, we show that ID_ν for $\nu\in\mathbb{N}$ can be embedded into $\varphi ID^\infty_{<\omega}$. The following results can be proved as Buchholz[3, Section 4] (cf. Kadota [18]).

ABBREVIATION. $k^\sim = \varphi_{\nu+1}^{k+1}(2,w_\nu)$.

LEMMA 5.3.17. $\vdash_0^{k^\sim} \neg A, A$ where $k = |A|$.

LEMMA 5.3.18. $\vdash^z \neg A(0), \neg\forall x{\in}\mathrm{N}(A(x) \longrightarrow A(\mathrm{S}x)), n \notin \mathrm{N}, A(n)$ where $z = (|A|+1)+w_\nu$.

DEFINITION 5.3.19. Let $\mathrm{B}(x)$ be a formula of $\mathcal{L}_{ID}(\mathrm{N})$. For $A \in \mathrm{Pos}_u$, A^* denotes the result of replacing all occurrences of $P_u^{\mathfrak{A}}$ in A by $B(\cdot)$. $\{A_1,\ldots,A_m\}^* = \{A_1^*,\ldots,A_m^*\}$.

PROPOSITION 5.3.20. If $\Gamma_0 \cup \Gamma \subseteqq \mathrm{Pos}_u$, $\alpha \in \mathcal{T}^*_{u+1}$, $u+1 \leqq \nu$, $k = |B|$ and $\vdash^{\alpha}_{1} \Gamma_0, \Gamma$, then

$$\vdash_1^{(k^{\sim}+1)+\alpha} \Gamma_0, \neg(\forall x \in N(\mathfrak{A}^N_u(B,x) \longrightarrow B(x))), \Gamma^*.$$

LEMMA 5.3.21. If $\alpha \in \mathcal{T}^*_{u+1}$, $\Delta \subset \mathrm{Pos}_u$, and $\vdash^{\alpha}_{1} \Delta, \mathrm{P}^{\mathfrak{A}}_u(n)$, then

$$\vdash_1^{(k^{\sim}+1)+\alpha} \Delta, \neg(\forall x \in N(\mathfrak{A}^N_u(B,x) \longrightarrow B(x)), B(n)$$

where $k = |B|$.

LEMMA 5.3.22. Let $z = (|B|+1)+w_{u+1}$. Then,

$$\vdash^z_1 \neg\forall x \in N(\mathfrak{A}^N_u(B,x) \longrightarrow B(x)), \neg P^{\mathfrak{A}}_u n, B(n).$$

PROPOSITION 5.3.23. For a mathematical axiom $A(x_1,\ldots,x_m)$ of ID_ν, there is $k \in \mathbb{N}$ such that $\vdash^{k^{\sim}}_1 A(i_1,\ldots,i_m)^N$ for all $i_1,\ldots,i_m \in \mathbb{N}$.

PROPOSITION 5.3.24. By *PL1*, we denote Tait's calculus for the first-order predicate logic in the language $\mathcal{L}(ID)$. If a set of formulas $\Gamma(x_1,\ldots,x_m)$ is derivable in *PL1*, then there is $k \in \mathbb{N}$ such that

$$\vdash^{k^{\sim}}_0 i_1 \notin N,\ldots,i_m \notin N, \Gamma(i_1,\ldots,i_m) \text{ for all } i_1,\ldots,i_m \in \mathbb{N}.$$

THEOREM 5.3.25. If the sentence A is provable in ID_ν for $\nu \in \mathbb{N}$, then there is $k \in \mathbb{N}$ such that $\vdash^z_k A^N$ where $z = \varphi^k_{\nu+1}(2,w_\nu)$.

Proof. Suppose a closed formula A is provable in ID_ν. Then, $\neg(A_1\wedge\cdots\wedge A_n), A$ is provable in *PL1* where each A_i is the universal closure of an axiom of ID_ν. Hence, there is m such that $\vdash^{\bar{m}}_1 (A_1\wedge\cdots\wedge A_n)^N$ and $\vdash^{\bar{m}}_1 \neg(A_1\wedge\cdots\wedge A_n)^N, A^N$. By a (cut) with the cut formula $(A_1\wedge\cdots\wedge A_n)^N$, we obtain that $\vdash^{\bar{k}} A^N$ for some k. □

THEOREM 5.3.26. If a Π_2-sentence $\forall x\exists y A(x,y)$ for $A \in \Sigma_1$ is provable in ID_ν for $\nu \in \mathbb{N}$, then there is $\alpha < \tau'[\nu+1]$ such that for all $n > 1$, there is k such that $k < F_\alpha(n)$ and $A(n,k)$ is true.

Proof. Suppose $ID_\nu \vdash A$ for A closed. Then, $\vdash^{\alpha}_{k} A^{\mathrm{N}}$ where $\alpha = \varphi^{k}_{\nu+1}(2,w_\nu)$ for some $0 < k \in \mathbb{N}$. If $k > 1$, then by Theorem 5.3.12(*Cutelimination*), $\vdash^{\alpha'}_{k-1} A^{\mathrm{N}}$ where

$$\alpha' = \varphi^{\alpha}_{\nu+1}(1,\varphi_{\nu+1}(1,\varphi^{k}_{\nu+1}(2,w_\nu))) = \varphi^{\alpha}_{\nu+1}(1,\varphi_{\nu+1}(1,\alpha))$$
$$= \varphi_{\nu+1}(2,\alpha) = \varphi^{k+1}_{\nu+1}(2,w_\nu).$$

By iterating this argument, we obtain $\vdash^{\beta}_{1} A^{\mathrm{N}}$ where $\beta = \varphi^{k+m}_{\nu+1}(2,w_\nu)$ for some $m \in \mathbb{N}$. Then, by iterating Lemma 5.3.13(*Collapsing*) we have $\vdash^{\gamma}_{1} A^{\mathrm{N}}$ where $\gamma = \varphi_1(\ldots\varphi_\nu(\varphi^{k+m}_{\nu+1}(2,w_\nu),w_{\nu-1})\ldots,w_0)$. And we have $\gamma < \tau'[\nu+1]$ since

$$\gamma = \varphi_1(\ldots\varphi_\nu(\varphi^{k+m-1}_{\nu+1}(2,\varphi_{\nu+1}(2,w_\nu)),w_{\nu-1})\ldots,w_0)$$
$$< \varphi_1(\ldots\varphi_\nu(\varphi^{w_0}_{\nu+1}(2,\varphi_{\nu+1}(2,w_\nu)),w_{\nu-1})\ldots,w_0)$$
$$< \varphi_1(\ldots\varphi_\nu(\varphi^{\varphi_1(z,w_0)}_{\nu+1}(2,\varphi_{\nu+1}(2,w_\nu)),w_{\nu-1})\ldots,w_0)$$
$$\text{where } z = \varphi_2(\ldots\varphi_\nu(\varphi^{1}_{\nu+1}(2,\varphi_{\nu+1}(2,w_\nu)),w_{\nu-1})\ldots,w_1)$$
$$= \varphi_1(\ldots\varphi_\nu(\varphi^{w_1}_{\nu+1}(2,\varphi_{\nu+1}(2,w_\nu)),w_{\nu-1})\ldots,w_0)$$
$$< \cdots$$
$$< \varphi_1(\ldots\varphi_\nu(\varphi^{w_\nu}_{\nu+1}(2,\varphi_{\nu+1}(2,w_\nu)),w_{\nu-1})\ldots,w_0)$$
$$= \varphi_1(\ldots\varphi_\nu(\varphi_{\nu+1}(3,w_\nu),w_{\nu-1})\ldots,w_0)$$
$$= \tau'[\nu+1].$$

Hence, $\gamma < \tau'[\nu+1] < \tau'$. Thus, $\gamma+1 < \tau'[\nu+1]$. By Theorem 5.3.16 (*Bounding*), for all $n > 1$, there is k such that $k < F_{\gamma+1}(n)$ and $A(n,k)$ is true. □

From this theorem, we can immediately derive Theorem 5.3.1 and Theorem 5.3.2 which were proved by Kadota[18].

CHAPTER 6

DISCUSSIONS

In this chapter, we discuss the following two aspects which come from the present dissertation.

In Section 6.1, we state problems on undecidable finite combinatorial statements in some formal theories of arithmetic. Then, we discuss these problems using our results on the relation between subrecursive hierarchies and provably computable functions.

In Section 6.2, we discuss the meanings and problems of the minimum subrecursive inaccessible ordinal. Then, we consider the relation with the inductive definitions, and state some application to computer science.

6.1 Undecidable statements in theories of arithmetic

In Chapter 2, we showed that a finite combinatorial statement *PH*, which represents strong Ramsey property, is undecidable in Peano arithmetic *PA*, i.e., neither *PH* nor $\neg PH$ is provable in *PA*.

In Chapter 4, we studied this unprovability result in detail by considering the fragments PA_n of *PA*. Here, PA_n is obtained from *PA* by restricting the induction formulas of the mathematical induction to formulas containing at most n quantifiers. This unprovability result is obtained there as follows:

First, we assume that f is a computable function defined by

$$f(\underline{x}) = \mu y R(\underline{x}, y)$$

where R is a primitive recursive predicate. Then, from Theorem 4.1.2, f is provably computable in PA_n if the formula

$$\forall \underline{x} \exists y R(\underline{x}, y)$$

is provable in PA_n, where R is the predicate symbol for the primitive recursive predicate R. This formula expresses the total-definedness of f. In this case, we can prove from Theorem 4.1.2 that for $n > 0$, f is provably computable in PA_n if and only if f is dominated by F_α for some $\alpha < \omega_n$.

Next, we consider the formula $PH(n)$ for every $1 < n \in \mathbb{N}$ by:

$$PH(n) \equiv \forall x \forall z \exists y([x, y] \xrightarrow[*]{} (\bar{n}+1)^{\bar{n}}_{z}),$$

where the relation $[x, y] \xrightarrow[*]{} (n+1)^n_z$ is primitive recursive. This formula $PH(n)$ expresses the total-definedness of the function $\sigma_n(x,z)$ which is defined by:

$$\sigma_n(x,z) = \mu y([x, y] \xrightarrow[*]{} (n+1)^n_z).$$

Then, by Theorem 2.3.2, we obtain the following

(a) The function σ_n is dominated by F_α for some $\alpha < \omega_n$.

(b) The function σ^*_n is *not* dominated by F_α for any $\alpha < \omega_{n-1}$.

By applying the argument above, we immediately obtain the following results:

(c) $PH(n)$ is provable in PA_n.

(d) $PH(n)$ is *not* provable in PA_{n-1}.

This argument gives a method of obtaining the provability and unprovability results for some finite combinatorial statements. This also gives some problems on such statements unprovable in formal theories of arithmetic. In the remaining of this section, we discuss these problems.

In 1982, Friedman, McAloon and Simpson[10] introduced a finite combinatorial statement *FMS* which states some strong Ramsey property as *PH*. Then, they showed that *FMS* is undecidable in the formal theory ATR_0. Here, ATR_0 is the theory of second order arithmetic with arithmetical transfinite recursions as axioms, which is much stronger than *PA*. The formulas $FMS(n)$ and *FMS* are defined by:

$$FMS(n) \equiv \forall x \exists y([x, y] \text{ is } \bar{n}\text{-dense}); \quad FMS \equiv \forall z FMS(z),$$

where the predicate ($[x, y]$ is n-dense) is primitive recursive. The function $\sigma_n^*(x)$ is defined by:

$$\sigma_n^*(x) = \mu y([x, y] \text{ is } n\text{-dense}).$$

They also showed that the set of all provably computable functions in ATR_0 is classified by the fast-growing hierarchy up to Γ_0, where Γ_0 is the proof-theoretic ordinal of ATR_0 which is larger than ε_0.

However, we have not known the detailed relation between the functions σ_n^* and the fast-growing hierarchy up to Γ_0 such as Proposition 2.3.2 which implies (a) and (b) above. Also we have not known fragments of ATR_0 from which we obtain a relation such as (c) and (d). Thus, we now have the following problems:

(e) To prove the detailed relation between σ_n^* and F_α for $\alpha < \Gamma_0$ which implies such as (a) and (b).

(f) To obtain fragments of ATR_0 which correspond to $FMS(n)$, in the sense of (c) and (d).

Concerning these problems, we remark the work of Kurata and Shimoda[29]. They studied the relations among FMS, the reflection principle of ATR_0 for Σ_1-formulas, transfinite induction up to Γ_0 and the large set principle for Γ_0.

There are some other statements which are finite combinatorial, and undecidable in certain formal systems(cf. Buchholz[3], Shelah[39] and Paris[34]). It is interesting to prove the relations between undecidable statements and the fast-growing hierarchies up to some ordinals, which will answer the problems such as (e) and (f).

6.2 Applications of subrecursive hierarchies

In Chapter 5, we said that an ordinal α is subrecursive inaccessible (or s-inaccessible) if the following holds: There is $m \in \mathbb{N}$ such that

$$F_\alpha(x) \leqq G_\alpha(x+1) \quad \text{for all } x > m.$$

Then, we proved that the ordinal τ is a minimum s-inaccessible. Here we consider this result and discuss its meaning informally.

Let I be a countable ordinal and P a system of fundamental sequences for I. Then, we consider any sequence $\{f_\alpha\}_{\alpha<I}$ of unary number-theoretic functions. Here, we recall that $\{f_\alpha\}_{\alpha<I}$ have the domination property if the following holds:

Domination: If $\alpha < \beta$, then f_α is dominated by f_β.

If $\{f_\alpha\}_{\alpha<I}$ has the domination property, we have a bijection from the set of all $\alpha < I$ to the set $\{f_\alpha \mid \alpha<I\}$ by mapping α to f_α (cf. the following figure):

$$\begin{array}{ccc} \vdots & & \vdots \\ \beta & \longrightarrow & f_\beta \\ \vee & & \gg \\ \alpha & \longrightarrow & f_\alpha \\ \vdots & & \vdots \end{array}$$

Here, $f_\alpha << f_\beta$ means that f_α is dominated by f_β. We call this bijection a *coding* for I, and each f_α a *code* of α.

Then, we consider the slow-growing hierarchy $\{G_\alpha\}_{\alpha<I}$. The hierarchy $\{G_\alpha\}_{\alpha<I}$ is defined inductively by:

$$G_0(x) = 0;$$
$$G_{\alpha+1}(x) = G_\alpha(x) + 1;$$
$$G_\alpha(x) = G_{\alpha[x]}(x) \text{ if } \alpha \text{ is a limit ordinal,}$$

where $\{\alpha[x]\}_{x\in\mathbb{N}}$ is the fundamental sequence for α.

If the system of fundamental sequences for I is (n)-built-up for some $n \in \mathbb{N}$, then $\{G_\alpha\}_{\alpha<I}$ has the domination property, by Proposition 3.1.12. Hence, then $\{G_\alpha\}_{\alpha<I}$ gives a coding for I.

In order to see this situation, we consider the standard system of fundamental sequences for ε_0 as an example of systems for I. In this case, we have that

$$G_\omega(x) = x;$$

$$G_{\omega^\omega}(x) = x^x;$$

$$G_{\omega_n}(x) = \left. x^{x^{\cdot^{\cdot^{x}}}} \right\} n\ x\text{'s, where } \omega_n = \left. \omega^{\omega^{\cdot^{\cdot^{\omega}}}} \right\} n\ \omega\text{'s.}$$

This system is (1)-built-up by Proposition 3.1.7. Thus the hierarchy $\{G_\alpha\}_{\alpha<\varepsilon_0}$ has the domination property. Hence $\{G_\alpha\}_{\alpha<\varepsilon_0}$ gives a coding for I, where G_α is a code of $\alpha < \varepsilon_0$.

Next, we consider the computation of the function F_τ for the minimal subrecursive inaccessible ordinal τ. When we compute the value of $F_\tau(x)$ for any input x, we need to have the system of fundamental sequences for τ. More precisely, we need a code of any ordinals $\alpha < \tau$ which has the information on the assignment of the fundamental sequence for α. As we discussed above, $\{G_\alpha\}_{\alpha<\tau}$ produces natural one of such codings.

If we take G_τ for a code of τ, we have the following situation:

We can use G_τ when we compute F_τ.

But, since τ is s-inaccessible, $F_\tau(x) \leqq G_\tau(x+1)$ for sufficiently large x. This means that F_τ is not greater than the function which is used in the computation of F_τ.

This situation suggests the limitation of the construction of new greater functions by means of subrecursive hierarchies.

On the other hand, we showed that the ordinal τ' satisfies the following condition which is similar to s-inaccessibility:

$$F_{\tau'}(x) \leqq G_{\tau'}(G_{\tau'}(x)) \quad \text{for } x > 3.$$

Then, we showed also the following:

(g) F_α is provably computable in $ID_{<\omega}$ for every $\alpha < \tau'$.

(h) For any function f provably computable in $ID_{<\omega}$, there is an $\alpha < \tau$ such that f is dominated by F_α.

Here $ID_{<\omega}$ is the theory of finitely iterated inductive definitions.

The theory $ID_{<\omega}$ contains PA and also contains all arithmetical consequences of ATR_0. In particular, the total-definedness of the function F_{Γ_0} is not provable in ATR_0 but is provable in $ID_{<\omega}$. However, by (h), we can easily show that the total-definedness of $F_{\tau'}$ is not provable in $ID_{<\omega}$. From the argument above and the results (g) and (h), we can say that the following two notions are closely related:

(i) To construct functions by means of subrecursive hierarchies.

(j) To construct mathematical structures by means of inductive definitions.

This observation will suggest the possibility of applying the relation in various fields. In fact, inductive definitions are used in the constructions of many inductive structures which appear in fields of mathematics and computer sciences. In particular, inductive definitions are used quite often in formal language theory. So, we can expect that there will be a lot of important applications of results on subrecursive hierarchies and provably computable functions in these fields, especially in formal language theory.

Finally, we remark a relation between the results of Chapter 5 and proofs using real computers. The collapsing theorem in Chapter 5 was proved first by Wainer[49] in a quite abstract manner. On the other hand, Coquand and Paulin[6] gave a simpler proof of it by using their computer system CC (Calculus of Construction) based on type theory. The proof is, of course, far more constructive than Wainer's, but it lacks mathematical intuitions. Our proof given in Section 5.2 is more constructive than that of Wainer[49], since it is formalized in $ID_{<\omega}$. Hence, the author believes that our proof, which highly relies on the this normalization theorem, will be more understandable than these two and moreover it will clarify the relation between the proof using computers by Coquand and Paulin[6] and the abstract proof by Wainer[48].

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