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Topological Aspects
of
Classical and Quantum
(2+1)-dimensional Gravity

(Thesis)

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ABSTRACT

Abstract text is extremely faint and illegible. It appears to be a summary of the thesis content, but the specific details are not discernible.

This thesis is based on the following papers;

A.Hosoya and J.Soda, Mod. Phys. Lett. A4 (1989) 2539,

J.Soda, to be published in Prog. Theor. Phys. Vol.83 No.4 (April),

Y.Fujiwara and J.Soda,

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ABSTRACT

In order to understand (3+1)-dimensional gravity, (2+1)-dimensional gravity is studied as a toy model. Our emphasis is on its topological aspects, because (2+1)-dimensional gravity without matter fields has no local dynamical degrees of freedom. Starting from a review of the canonical ADM formalism and York's formalism for the initial value problem, we will solve the evolution equations of (2+1)-dimensional gravity with a cosmological constant in the case of $g = 0$ and $g = 1$, where g is the genus of Riemann surface. The dynamics of it is understood as the geodesic motion in the moduli space. This remarkable fact is the same with the case of (2+1)-dimensional pure gravity and seen more apparently from the action level. Indeed we will show the phase space reduction of (2+1)-dimensional gravity in the case of $g = 1$. For $g \geq 2$, unfortunately we are not able to explicitly perform the phase space reduction of (2+1)-dimensional gravity due to the complexity of the Hamiltonian constraint equation. Based on this result, we will attempt to incorporate matter fields into (2+1)-dimensional pure gravity. The linearization and mini-superspace methods are used for this purpose. By using the linearization method, we conclude that the transverse-traceless part of the energy-momentum tensor affects the geodesic motion. In the case of the Einstein-Maxwell theory, we observe that the Wilson lines interact with the geometry to bend the geodesic motion. We analyze the mini-superspace model of (2+1)-dimensional gravity with the matter fields in the case of $g = 0$ and $g = 1$. For $g = 0$, a wormhole solution is found but for $g = 1$ we can not find an analogous solution. Quantum gravity is also considered and we succeed to perform the phase space reduction of (2+1)-dimensional gravity in the case of $g = 1$ at the quantum level. From this analysis we argue that the conformal rotation is not necessary in the sense that the Euclidean quantum gravity is inappropriate for the full gravity.

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1. Introduction

Our understanding of the universe has been steadily improved. On the basis of standard theory of cosmology,^[1] the inflationary universe scenario was proposed to answer the important cosmological questions and has been developed by many physicists.^[2] In order to make this inflationary universe theory more self-contained, we need quantum cosmology which may possibly give the initial conditions of our universe on the basis of quantum gravity. The Hartle-Hawking proposal^[3] for the quantum state of the universe has been well studied semi-classically, and in some models this leads to inflation. These progresses have been achieved mainly by studying the so-called mini-superspace models, in which the gravitational and matter variables have been reduced to a finite number of degrees of freedom. Obviously this is an oversimplification.^[4] To be more realistic, a thorough understanding of full quantum gravity is necessary. However, our present understanding of quantum gravity is still very poor in spite of the endeavor of many people. The superstring theories^[5] have been investigated as a most promising candidate for it. Indeed, the superstring theories have many attractive features, e.g. a possibly ultraviolet finite theory, unification of all forces etc., which led many young physicists to this field of research. Although we agree to the importance of superstring theory, we must seek various alternatives to achieve the true theory of gravity. For example, the recent work by Ashtekar^[6] revives our interest in the canonical quantization of gravity. Of course, since a pioneering paper by DeWitt^[7] we have pursued this quantization method and have encountered many difficulties.^[8] For instance, to define a quantum theory, it is necessary to construct the Hilbert space at each time slice. Ironically the gravitational theory has the general coordinate invariance, so the time coordinate has no invariant meaning. Hence, even if we quantize the gravitational theory following Dirac,^[9] we cannot properly interpret the wave function which is the solution of the Wheeler-DeWitt equation.^[7] Besides this conceptual problem, there are many technical difficulties. For example, the factor ordering problem of the Wheeler-DeWitt equation is not yet solved in the operator formalism. When we use the functional integral method, the issue whether we should use the Lorentzian or Euclidean formalism is not yet settled. Even if the Euclidean

quantum gravity^[3] is adopted, a new problem arises; that is, divergence from the conformal factor^[10]. As an alternative, an attempt to reduce the phase space before quantization has been made. However we again meet with the difficulty that the Hamiltonian constraint equation is difficult to solve. Moreover the resultant theory would have no general coordinate invariance. As mentioned above, we necessarily have the conceptual and technical difficulties. Then, to get rid of the technical difficulties as much as possible, we take (2+1)-dimensional gravity as a toy model^[11]. By investigating this model, we hope to be able to understand the essential point of quantum gravity.

On the other hand, physicists have long been fascinated by the possibility of processes involving a change in the topology of space^[12]. Recent speculation by Coleman^[13] that topology changing processes have something to do with the vanishing of the cosmological constant made our interest in these processes renew. To attack this problem, (2+1)-dimensional gravity is also advantageous, because a mathematical knowledge about the 2-dimensional compact manifold is available.^[14]

The Einstein gravity in three space-time dimensions exhibits some unusual features, which can be deduced from the properties of the Einstein field equations and the curvature tensor. Einstein's equation for general relativity reads

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}, \quad (1.1)$$

or is expressed in terms of the Ricci tensor as,

$$R_{\mu\nu} = 2\Lambda g_{\mu\nu} + \kappa(T_{\mu\nu} - g_{\mu\nu} T_{\sigma}^{\sigma}). \quad (1.2)$$

Here Λ , κ and $T_{\mu\nu}$ are cosmological constant, the gravitational coupling constant and energy-momentum tensor of the matter fields, respectively. On the other hand, the Riemann tensor can be written in terms of the Ricci tensor as^[15]

$$R_{\mu\nu\lambda\rho} = g_{\mu\lambda} R_{\nu\rho} + g_{\nu\rho} R_{\mu\lambda} - g_{\mu\rho} R_{\nu\lambda} - g_{\nu\lambda} R_{\mu\rho} - \frac{1}{2}R(g_{\mu\lambda}g_{\nu\rho} - g_{\mu\rho}g_{\nu\lambda}). \quad (1.3)$$

Using eq.(1.2) in the right hand side of eq.(1.3) shows that the *local* geometry is completely determined by the matter distribution and the cosmological constant

Λ . This means that there are no dynamical degrees of freedom corresponding to gravitational waves. In regions which are free of matter, space-time is *locally* flat ($\Lambda = 0$), de-Sitter ($\Lambda > 0$) or anti-de-Sitter ($\Lambda < 0$), depending on the value of the cosmological constant. The lack of dynamics in three dimensional Einstein gravity can also be seen from the canonical point of view by counting degrees of freedom. The two dimensional spatial metric and its conjugate each contains three algebraically independent components. Of these six components, one is used to specify the choice of space-like hypersurfaces, while the other two are used to specify coordinates on these two dimensional hypersurfaces. Finally, there are three initial value constraints which completely determine the remaining components. Thus, one might consider that there is nothing left in the Einstein gravity in three space-time dimensions. However, many researchers have studied the Einstein gravity in three space-time dimensions, because of the existence of the global effects in the three dimensional Einstein gravity. Straruskiewicz was the first to give geometrical arguments, showing that a three dimensional space-time with a point source is obtained by removing a "wedge" from Minkowski space and identifying points across the wedge.^[16] Deser, Jackiw and 't Hooft have confirmed this description by explicitly solving the three dimensional Einstein equations with $\Lambda = 0$ for an arbitrary number of static point masses.^[17]

In this thesis we will investigate another global aspect in the three dimensional Einstein gravity. That is, the global deformation of the spatial manifold. Although there exists no graviton, which represents the local deformation of the spatial manifold, there exists the global deformation of the spatial manifold. At first sight, one might consider that we are about to study a very peculiar model which has nothing in common with the physical (3+1)-dimensional gravity. However it should be noted that the global deformation of the spatial manifold is most important even in the case of (3+1)-dimensional gravity. The similar situation can be observed in the Maxwell theory. The Maxwell theory in two space-time dimensions is the topological field theory which has only the global modes.^[16] As we know, however, it reveals the essential points of the dynamics of the global modes of the Maxwell theory theory in 4 dimensions. Therefore we can expect to obtain the important information from our model.

This global aspect of (2+1)-dimensional gravity is also the point in the noble approach by Witten,^[19] which is similar to Ashtekar's formulation of (3+1)-dimensional gravity.^[6] There the zweibein and spinor connection are treated as independent variables. The Einstein-Hilbert action turns out to be a purely topological Chern-Simons term.^[20] Once one formulates (2+1)-dimensional gravity as the Chern-Simons theory,^[21] the dynamical variables become $ISO(2,1)$ flat connections whose number is finite. As we previously emphasized, in the case of (2+1)-dimensional pure gravity, only the global modes or topological modes are important. Witten has extracted this aspects elegantly. We naturally expect that the conventional approach to (2+1)-dimensional gravity also exhibits the global or topological aspect, though the relation between the two approaches is not obvious. In this thesis we shall mainly study (2+1)-dimensional gravity using the standard ADM method.^[22] In this approach, the space-time is pictured as a foliation of space-like manifolds and the deformation process along the time-like direction is formulated as a Hamiltonian system. From this point of view, the geometrical meaning of the dynamical variables is clear. Therefore when we wish to consider the topology changing phenomena, the canonical approach has an advantage to visualize the process.

The conventional method for (2+1)-dimensional gravity is reviewed in Sec.2. Using this formulation, Hosoya and Nakao^[23] discovered the fact that the dynamics of the Einstein gravity becomes the geodesic motion in the moduli space in the case of $g = 1$, where g is the genus of a Riemann surface. Moncrief^[25] independently analyzed this system as an initial value problem and concluded that there exists a unique solution of the constraint equations in the case of $g \geq 1$. These are the subjects of Sec.3 , Sec.4 and Sec.5. In Sec.3 we review York's formalism which is important to understand the canonical structure of gravity. In Sec.4 we will solve the evolution of the geometry in (2+1)-dimensional gravity with a cosmological constant. In Sec.5 the essence of the dynamics of (2+1)-dimensional pure gravity is revealed by phase space reduction. Up to this stage, only the pure gravity will be considered as a first step to a realistic quantum cosmological model. In the standard approach to quantum gravity, it is straightfoward to incorporate matter fields. It is at this point that new approach to quantum gravity such as Witten's formu-

lation will meet with the difficulty because the theory is formulated as background independent way, although the new approach has great successes in pure gravity. This is one of the reasons why we greatly focus on the conventional approach. As a modest step, we shall analyze the linearized gravity in (2+1)-dimensions and consider the matter effects on the geodesic motion perturbatively. These are discussed in Sec.6. To reveal another aspect of gravity coupled with matter fields, a mini-superspace approach is used in Sec.7. In Sec.8 quantum gravity is discussed using a functional integral method. The final section is devoted to discussions of various issues which we must solve to reach the final goal. Mathematical tools is explained in Appendix A. We present the field theory in topologically non-trivial space as a semi-classical theory in Appendix B. To complement the main text, other approaches to (2+1)-dimensional gravity are reviewed in Appendix C.

2. ADM Canonical Formalism

Although the whole analysis in this section is merely a recapitulation of the well-known results in (3+1)-dimensional gravity and its straightforward adaptation to the (2+1)-dimensional gravity, we shall start with the canonical ADM formalism to make this paper self-contained.

The canonical theory begins with the following decomposition of the metric tensor;

$$\begin{aligned} ds^2 &= g_{\mu\nu} dx^\mu dx^\nu \\ &= -N^2 dt^2 + h_{ij} (dx^i + N^i dt)(dx^j + N^j dt), \end{aligned} \tag{2.1}$$

where μ, ν range over 0,1,2 and i, j range over 1,2. Here the lapse function N and the shift vector N^i are not dynamical. Rather the lapse function determines the foliation of spatial manifolds in the whole space-time and the shift vector serves the coordinate choice on each time slice. This decomposition of the metric leads us to the reformulation of the general relativity as the initial value problem and the Cauchy problem.

Given the Einstein-Hilbert action,

$$S = \int \sqrt{-g} R^{(3)} d^3x, \quad (2.2)$$

there is a standard prescription for obtaining a Hamiltonian formulation. Using the (2+1) decomposition of the metric (2.1), we obtain

$$S = \int N \sqrt{h} (K_{ij} K^{ij} - K^2 + R^{(2)}), \quad (2.3)$$

where $K_{ij} = \frac{1}{2N}(\dot{h}_{ij,0} - N_{ij} - N_{ij})$ is the extrinsic curvature and $K = K_{ij} h^{ij}$ is its trace. $R^{(3)}$ and $R^{(2)}$ denote the three and two dimensional scalar curvatures, respectively. The stroke indicates the covariant derivative with respect to the spatial metric h_{ij} . Here we have discarded the surface term, because we shall concentrate on the (2+1)-dimensional space-time $M = R \times \Sigma$ where Σ is a compact closed orientable two manifold. The canonical conjugate momentum π^{ij} to h_{ij} is given by

$$\pi^{ij} = \sqrt{h} (K^{ij} - h^{ij} K). \quad (2.4)$$

The ADM action for Einstein's theory of gravity takes the form,

$$S = \int d^3x \{ \pi^{ij} \dot{h}_{ij} - NH - N_i H^i \}, \quad (2.5)$$

$$H = \frac{1}{\sqrt{h}} (\pi^{ij} \pi_{ij} - \pi^2) - \sqrt{h} R^{(2)}, \quad (2.6)$$

$$H^i = -2\pi^{ik}, \quad (2.7)$$

with $\pi = \pi_i^i = -\sqrt{h} K_i^i$. Here, as indicated in the above, the lapse function N and the shift vector N^i come into the action as the Lagrange multipliers. Note that the Hamiltonian constraint can be rewritten as

$$H = G_{ijkl} \pi^{ij} \pi^{kl} - \sqrt{h} R^{(2)}, \quad (2.8)$$

$$G_{ijkl} = \frac{1}{2} h^{-\frac{1}{2}} (h_{ik} h_{jl} + h_{il} h_{jk} - 2h_{ij} h_{kl}) \quad (2.9)$$

This tensor, G_{ijkl} , is the so-called supermetric on superspace^[7]. This metric has the signature $(-, +, +)$ in the 3-dimensional metric space.

The evolution equations obtained from (2.5) are

$$\frac{\partial h_{ij}}{\partial t} = \frac{2N}{\sqrt{h}}(\pi_{ij} - h_{ij}\pi) + N_{i|j} + N_{j|i}, \quad (2.10)$$

$$\begin{aligned} \frac{\partial \pi^{ij}}{\partial t} = & \frac{N}{2\sqrt{h}}h^{ij}(\pi^{kl}\pi_{kl} - \pi^2) - \frac{2N}{\sqrt{h}}(\pi^{ik}\pi_k^j - \pi^{ij}\pi) + \sqrt{h}(N^{ij} - h^{ij}N_{|k}^k) \\ & + (N^k\pi^{ij})_{|k} - \pi^{ki}N_{|k}^j - \pi^{kj}N_{|k}^i. \end{aligned} \quad (2.11)$$

For later convenience we further rewrite these equations as follows;

$$\begin{aligned} \frac{\partial h_{ij}}{\partial t} &= -NK h_{ij} - 2N\hat{K}_{ij} + \nabla_i N_j + \nabla_j N_i, \\ \frac{\partial \hat{K}_{ij}}{\partial t} &= -2N\hat{K}_i^k \hat{K}_{kj} - (\nabla_i \nabla_j - \frac{1}{2}h_{ij}\Delta)N + (\hat{K}_i^k \nabla_k N_j + \hat{K}_j^k \nabla_k N_i - N^k \nabla_k \hat{K}_{ij}), \\ \frac{\partial K}{\partial t} &= -\Delta N + (R + K^2)N + N^k \nabla_k K, \end{aligned} \quad (2.12)$$

where \hat{K}_{ij} is the traceless part of the extrinsic curvature. Combining the constraint equations,

$$H = 0, \quad (2.13)$$

$$H^i = 0, \quad (2.14)$$

with the evolution equations (2.10) and (2.11), we obtain a complete set of Einstein's equations equivalent to the covariant expression $R_{\mu\nu} = 0$. The equations (2.13) and (2.14) are nothing but the Gauss-Coddazzi equations giving the necessary and sufficient conditions for the embedding of a hypersurface with the second fundamental form K_{ij} in a locally flat space-time. In Dirac's terminology,^[9] (2.13) and (2.14) are the first class constraints which form the following algebra:

$$\begin{aligned} \{H(x), H(\acute{x})\} &= g^{ab}(x)H_a(x)\partial_b\delta(x, \acute{x}) - (x \longrightarrow \acute{x}), \\ \{H_a(x), H(\acute{x})\} &= H(x)\partial_a\delta(x, \acute{x}), \\ \{H_a(x), H_b(\acute{x})\} &= H_b(x)\partial_a\delta(x, \acute{x}) - (x \longrightarrow \acute{x}, a \longrightarrow b). \end{aligned}$$

Here the curly brackets imply the Poisson brackets.

The canonical approach to general relativity can be understood as a deformation theory of the Riemannian manifold Σ . The deformation process is governed by the evolution equations (2.10) and (2.11). We ask the question: what is deformed? Let $Riem(\Sigma)$ be the space of Riemannian metrics on Σ and $Diff(\Sigma)$ the group of diffeomorphisms of Σ . In the coordinate language, a "point" of $Riem(\Sigma)$ is determined by giving three functions $h_{ij}(x)$ of two variables $(x) = (x_1, x_2)$, being subject to the constraints $h_{ij} = h_{ji}$ and $h = det h_{ij} > 0$. Each element of $Diff(\Sigma)$ maps $Riem(\Sigma)$ into itself by the transformation law for covariant tensors and hence $Diff(\Sigma)$ acts as a transformation group on $Riem(\Sigma)$. The orbit of each point of $Riem(\Sigma)$ under the action of $Diff(\Sigma)$, the gauge orbit, is identified with a point in the superspace $S(\Sigma)$. In other words, as the general relativity is a kind of gauge theory, the momentum constraint (2.14) generates the gauge transformation which is nothing but $Diff(\Sigma)$. Therefore a formal definition of the superspace is given by

$$S(\Sigma) = \frac{Riem(\Sigma)}{Diff(\Sigma)}. \quad (2.15)$$

The deformation process may be viewed in this superspace whose point is a 2-geometry. As is well known in quantum cosmology, however, 2-geometry is the carrier of information about time.^[26] This time is understood as the gauge degree of freedom whose transformation is generated by the Hamiltonian constraint (2.13). This extra variable is identified with the conformal factor. Then what we would like to know is a deformation of conformal Riemannian manifolds which is characterized by $\tilde{h}_{ij} = h^{-\frac{1}{2}} h_{ij}$ in (2+1)-dimensions. This is the so-called "conformal superspace" which is defined by the superspace modulo conformal mappings $Conf(\Sigma)$,

$$\tilde{S}(\Sigma) = \frac{Riem(\Sigma)}{Diff(\Sigma) \times Conf(\Sigma)}. \quad (2.16)$$

In the case of orientable closed compact spaces, *i.e.* Riemann surfaces, this conformal superspace is nothing but the moduli space for Riemann surfaces. For example, in the case of genus $g = 1$, the shaded part in Fig. 1 represents the moduli space.

3. York's Formalism

In this chapter we shall review York's formalism^[26] which is useful to investigate the canonical quantization of general relativity. In ordinary particle mechanics we are able to freely specify the initial conditions which determine its dynamical evolution. There is, however, a class of systems which have some constraints for initial values. These systems are called constrained systems. The general relativity is included in this category. It is not trivial to give a set of initial data which satisfy the constraints. It is at this stage that York's method works. Before entering the general relativity, let us recall the electromagnetism as a simple illustration. In flat or in curved Riemannian spaces one can uniquely decompose an arbitrary vector potential or one-form into a sum of exact, co-exact and harmonic forms. Physically, this procedure leads to the identification of the true canonical degrees of freedom of the electromagnetic field and to the identification of the gauge, or non-dynamical variables. Especially harmonic one-forms represent the global structure of gauge fields which reflects the cohomological structure of space. The spirit of York's method lies on this line. He gave a conformally invariant, orthogonal, covariant decomposition of symmetric tensors on a positive definite Riemannian manifolds into transverse-traceless, longitudinal, and pure trace parts. This decomposition enables us to set the initial-value problem of general relativity as a system of three second-order elliptic equations for three unknown functions.

First we note that the TT-decomposition of a symmetric tensor ψ^{ab} is defined by

$$\psi^{ab} = \psi_{TT}^{ab} + \psi_L^{ab} + \psi_{Tr}^{ab}, \quad (3.1)$$

where the longitudinal part is

$$\psi_L^{ab} = \nabla^a W^b + \nabla^b W^a - h^{ab} \nabla_c W^c = (LW)^{ab}, \quad (3.2)$$

and the trace part is

$$\psi_{Tr}^{ab} = \frac{1}{2} h^{ab} \psi, \quad \psi = h_{cd} \psi^{cd}. \quad (3.3)$$

Our next task is to determine the conformal property of this decomposition that turns out to be essential for York's method. A space conformally related to (M, h)

is (\bar{M}, \bar{h}) , where

$$\bar{h}_{ab} = e^{\phi} h_{ab}. \quad (3.4)$$

Therefore, we have for the connection coefficients,

$$\bar{\Gamma}_{bc}^a = \Gamma_{bc}^a + \frac{1}{2}(\delta_b^a \phi_{,c} + \delta_c^a \phi_{,b} - h_{ab} \phi_{,a}), \quad (3.5)$$

with $\phi(x)$ an arbitrary real scalar function. Using this formula, we can easily show that

$$e^{-2\phi} \psi_{TT}^{ab} \quad (3.6)$$

becomes the transverse-traceless tensor on \bar{M} . Thus by identifying this 2-tensor density with the canonical momentum $\bar{\pi}^{ab}$, we can solve the momentum constraint on the orbit of a conformally equivalent class as we shall see below.

The initial value problem is to construct a space-like Riemannian two manifold (\bar{M}, \bar{h}) and a symmetric tensor density of weight one, $\bar{\pi}^{ab}$, such that

$$\bar{\pi}_{|b}^{ab} = 0, \quad (3.7)$$

$$\bar{h}^{-\frac{1}{2}}(\bar{\pi}_{ab} \bar{\pi}^{ab} - \bar{\pi}^2) - \bar{h}^{\frac{1}{2}} \bar{R} = 0, \quad (3.8)$$

where the covariant derivative is defined with respect to \bar{h}_{ab} and \bar{R} is the scalar curvature of (\bar{M}, \bar{h}) . The conformal approach to this problem is to solve (3.7) in a conformally invariant manner, then to choose the conformal factor ϕ in such a way as to satisfy (3.8).

In general, we can perform the following orthogonal decomposition.

$$\bar{\pi}^{ab} = \bar{\sigma}^{ab} + \bar{h}^{\frac{1}{2}}(\bar{L}W)^{ab} + \frac{1}{2}\bar{h}^{\frac{1}{2}}\bar{h}^{ab}\tau, \quad (3.9)$$

where $\bar{\sigma}^{ab}$ represents a transverse-traceless tensor density and $\tau = \bar{h}^{-\frac{1}{2}}\bar{\pi}$. The

momentum constraint (3.7) tells us that W^a must satisfy

$$\bar{\nabla}_b(\bar{L}W)^{ab} = -\frac{1}{2}\bar{\nabla}^a\tau, \quad (3.10)$$

or equivalently the two elliptic equations,

$$\nabla_b(LW)^{ab} + \nabla_b\phi(LW)^{ab} = -\frac{1}{2}\nabla^a\tau. \quad (3.11)$$

On the other hand, the Hamiltonian constraint takes the form,

$$-\Delta\phi + R = h^{-1}\sigma^{ab}\sigma_{ab}e^{-\phi} + 2h^{-\frac{1}{2}}\sigma^{ab}(LW)_{ab} + [(LW)^{ab}(LW)_{ab} - \frac{1}{2}\tau^2]e^{\phi}. \quad (3.12)$$

These elliptic equations, (3.11) and (3.12), are not easy to study in general. As we can choose the lapse function freely, from eq.(2.12) we can also choose τ arbitrarily. So we consider the simplest choice of slicing, i.e.,

$$\tau = \text{constant over the space.} \quad (3.13)$$

This is called York's time slice. In this case, the vector elliptic equations become trivial and the scalar one reduces to

$$-\Delta\phi + R = h^{-1}\sigma^{ab}\sigma_{ab}e^{-\phi} - \frac{1}{2}\tau^2e^{\phi}. \quad (3.14)$$

The existence, uniqueness and linearization stability of initial data of this equation has already been established in the literature [24].

As an illustration,^[24] we shall apply York's method to the cases that the space manifold M is a 2-dimensional compact manifold without boundary. It is a well-known fact that such manifolds are classified into its topological equivalence classes. We can classify them by its genus g . Main attention is focused on the two cases: (1) $g = 0$ or a sphere and (2) $g = 1$ or a torus. In fact they are the only cases for which the initial-value problem can be explicitly solved. For $g \geq 2$, it is not easy to explicitly reduce the phase space of (2+1)-dimensional gravity to the physical one. This causes a technical difficulty to work out a quantum theory of the higher genus surface.

First of all, we borrow some known facts from mathematics. The space of the second rank transverse-traceless tensors Σ_{ab} or "holomorphic quadratic differentials" is locally isomorphic to R^2 for $g = 1$ and R^{6g-6} for $g \geq 2$. We further observe that Σ_{ab} is constant everywhere on a torus ($g = 1$) while Σ_{ab} is simply zero on a sphere ($g = 0$). For a higher genus case, Σ_{ab} must have $4g - 4$ zero points somewhere on M which are the origin of the difficulty for solving the initial value problem for $g \geq 2$. Now it is easy to solve the constraints for $g = 0$ and $g = 1$.

(1) sphere $S^2(g = 0)$

Take as our starting h_{ab} a standard metric induced on the surface of a constant radius sphere embedded in the 3-dimensional Euclidean space. The scalar curvature R for this metric is set equal to 1. Our starting Σ_{ab} must be zero as mentioned above. Then the Hamiltonian constraint equation becomes

$$\Delta\phi - 1 - \frac{1}{2}\tau^2 e^\phi = 0. \quad (3.15)$$

This equation has no solution. This means that we cannot foliate the Minkowski space by 2-spheres. Inclusion of a cosmological constant will change the situation as will be shown in the next chapter.

(2) torus $T^2(g = 1)$

We construct a torus by identifying the two pairs of opposite sides of a square whose coordinates (x, y) are as shown in Fig.2. The starting metric can be taken as

$$h_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (3.16)$$

on a torus. Of course, in general, our starting metric for solving the initial value problem can be taken

$$h_{ab} = \frac{\beta}{\eta} \begin{pmatrix} \xi^2 + \eta^2 & \xi \\ \xi & 1 \end{pmatrix} \quad (3.17)$$

in the coordinates in Fig.2. Note that we can always transform this metric to the form (3.16) with an adequate orthogonal transformation and a coordinate rescaling.

Of course the boundary condition in the new coordinates is different from the one in the original coordinates. However, because the evolution equations are local, after solving the evolution we can transform the metric back into the original coordinate system. Thus we can concentrate only on the simplest case. The scalar curvature R for this metric h_{ab} vanishes. As we already observed the traceless transverse tensor Σ_{ab} is constant on a torus, and so is

$$M = \Sigma_{ab} \Sigma^{ab} = 2(\Sigma_{xx}^2 + \Sigma_{xy}^2). \quad (3.18)$$

We write $M = 2m^2$ where a parameter m is a constant. The Hamiltonian constraint now becomes

$$\Delta\phi + 2m^2 e^{-\phi} - \frac{1}{2}\tau^2 e^\phi = 0 \quad (3.19)$$

Appealing to Moncrief's theorem,^[25] we find that a trivial solution,

$$e^\phi = \frac{2m}{\tau}, \quad (3.20)$$

is the unique solution. Using these initial data we can easily solve the evolution equations and find the geodesic motion in the conformal superspace.^[23]

4. Evolution of the Geometry

In this section, we shall explicitly solve the evolution equations of (2+1)-dimensional gravity with a cosmological constant.^[24] As we learned how to set up the initial values, it is easy to obtain the initial data on a sphere and on a torus.

(1) time evolution of a sphere

From now on, it is convenient to write the time evolution equations for (h_{ab}, Σ^{ab}) , not for (h_{ab}, K^{ab}) where Σ_{ab} is the traceless part of K_{ab} . Because we

have adopted York's slice, Σ_{ab} becomes the transverse-traceless from the outset. By some manipulations we can rearrange the time evolution eqs.(2.10) and (2.11) as:

$$\begin{aligned}\frac{\partial h_{ab}}{\partial t} &= -N\tau h_{ab} - 2N\Sigma_{ab} + \nabla_a N_b + \nabla_b N_a, \\ \frac{\partial \Sigma_{ab}}{\partial t} &= -2Nh^{cd}\Sigma_{ac}\Sigma_{bd} - (\nabla_a \nabla_b - \frac{1}{2}h_{ab}\nabla_c \nabla^c)N \\ &\quad + (\Sigma_i^k \nabla_k N_j + \Sigma_j^k \nabla_k N_i - N^k \nabla_k \Sigma_{ij}).\end{aligned}\quad (4.1)$$

The relation between the York's time τ and the parameter t is deduced from eq.(2.12) as

$$\frac{\partial \tau}{\partial t} = -\Delta N + (\Sigma_{ab}\Sigma^{ab} + \frac{1}{2}\tau^2 - 2\Lambda)N + N^a \nabla_a \tau \quad (4.2)$$

Because we take $\tau = \text{constant}$ on each spacelike hypersurface, the last term drops out.

First we choose the lapse function and the shift vector. We set the shift vector equal to zero. This choice implies that our initial coordinate frame is the comoving frame. Secondly the lapse function is set equal to one. Note that our time slice is not exactly the same as York's time slice (see ref.[25]), though $\tau = K$ is still constant on the spacelike hypersurface.

Because of our choice of the lapse function and the shift vector, the time evolution equations are cast into a simple form,

$$\frac{\partial h_{ab}}{\partial t} = -\tau h_{ab}. \quad (4.3)$$

The transverse-traceless part of the extrinsic curvature Σ_{ab} must be zero as we mentioned earlier. We also have

$$\frac{d\tau}{dt} = \frac{\tau^2 - 4\Lambda}{2}. \quad (4.4)$$

The Hamiltonian constraint reads

$$\Delta\phi - 1 - (\frac{1}{2}\tau^2 - 2\Lambda)e^\phi = 0. \quad (4.5)$$

A trivial solution of this equation is a constant ϕ .

$$e^\phi = \frac{2}{-\tau^2 + 4\Lambda}. \quad (4.6)$$

Note that $-\tau^2 + 4\Lambda$ must be positive. This condition implies that the time t decreases as τ increases. So we rewrite eq.(4.4) by reversing the direction of time $t \rightarrow -t$ so that

$$\frac{d\tau}{dt} = \frac{-\tau^2 + 4\Lambda}{2}. \quad (4.7)$$

Integrating eq.(4.7), the time coordinate transformation between t and τ is

$$\tau = 2a \frac{\sinh at}{\cosh at}, \text{ or } t = \frac{1}{2a} \log \frac{2a + \tau}{2a - \tau}, \quad (4.8)$$

where $a = \sqrt{\Lambda}$. As τ runs from τ_0 to $2a$, t increases from t_0 (the corresponding to τ_0) to ∞ . Inserting eq.(4.8) into eq.(4.3) (the direction of t reversed),

$$\frac{\partial h_{ab}}{\partial t} = 2a \frac{\sinh at}{\cosh at} h_{ab}. \quad (4.9)$$

Assuming a homogeneous solution $h_{ab} = A(t)h_{ab}(t = t_0)$, we can immediately see that the scale factor $A(t)$ is $(1/a^2) \cosh^2 at$. The resulting 3-geometry is

$$ds^2 = -dt^2 + \frac{1}{a^2} \cosh^2 at (d\theta^2 + \sin^2 \theta d\phi^2), \quad (4.10)$$

where (θ, ϕ) is the ordinary polar coordinates on the sphere. This is the well known de-Sitter solution.

(2) time evolution of a torus

As in the previous section, we fix the gauge as $N = 1$ and $N^a = 0$. The latter choice means that our coordinate frame (x,y) is a comoving frame. So all the information of the geometry of the torus is contained in the 2-metric only.

Because of our lapse function and shift vector choice, the time evolution equations can be cast into a simple form,

$$\begin{aligned}\frac{\partial h_{ab}}{\partial t} &= -\tau h_{ab} - 2\Sigma_{ab}, \\ \frac{\partial \Sigma_{ab}}{\partial t} &= -2h^{cd}\Sigma_{ac}\Sigma_{bd},\end{aligned}\tag{4.11}$$

with

$$\frac{d\tau}{dt} = \tau^2 - 4\Lambda.\tag{4.12}$$

Integrating eq.(4.12) we obtain

$$\tau = -2a \frac{\cosh 2at}{\sinh 2at}, \text{ or } t = \frac{1}{4a} \log \frac{\tau - 2a}{\tau + 2a},\tag{4.13}$$

where $a = \sqrt{\Lambda}$ and Λ is henceforth supposed to be positive. As τ runs from τ_0 to ∞ , t goes from $t_0 (< 0)$ to 0. Notice that the range of t is $t \leq 0$.

Eliminating Σ_{ab} from the above time evolution equations, we can obtain a non-linear second order differential equation which contains the metric only. That is

$$\frac{\partial^2 h_{ab}}{\partial t^2} - h^{cd} \frac{\partial h_{ac}}{\partial t} \frac{\partial h_{bd}}{\partial t} + 2a \frac{\cosh 2at}{\sinh 2at} \frac{\partial h_{ab}}{\partial t} - 4a^2 h_{ab} = 0.\tag{4.14}$$

Now let us look for a solution of this equation in the diagonal form,

$$h_{ab} = \begin{pmatrix} A(t) & 0 \\ 0 & B(t) \end{pmatrix}.\tag{4.15}$$

Substituting eq.(4.15) into eq.(4.14), we have

$$\frac{d^2 A}{dt^2} - \frac{1}{A} \left(\frac{dA}{dt} \right)^2 + 2a \frac{\cosh 2at}{\sinh 2at} A \frac{dA}{dt} - 4a^2 A^2,\tag{4.16}$$

and a similar equation for B .

We turn to the initial velocity of the metric. From eq.(4.11) or the definition of the extrinsic curvature;

$$\frac{\partial h_{ab}}{\partial t} = -2(\Sigma_{ab} + \frac{1}{2}h_{ab}\tau), \quad (4.17)$$

so that we have the initial velocity as

$$\begin{aligned} \frac{\partial h_{xx}}{\partial t} &= -2(\Sigma_{xx} + \frac{1}{2}h_{xx}\tau) \\ &= -2(m \cos \theta + \frac{1}{2}\beta\tau_0), \\ \frac{\partial h_{yy}}{\partial t} &= -2(-m \cos \theta + \frac{1}{2}\beta\tau_0), \\ \frac{\partial h_{xy}}{\partial t} &= -2m \sin \theta. \end{aligned} \quad (4.18)$$

To keep the metric diagonal all the time, we need $\dot{h}_{xy} = 0$, i.e. $\theta = 0$, or π . Choosing $\theta = 0$, the initial velocities for A and B are written as

$$\begin{aligned} \frac{\partial h_{xx}}{\partial t} &= \frac{dA}{dt} = -2(m + \frac{1}{2}\beta\tau_0) \\ &= 2\beta a \frac{\cosh 2at_0 + 1}{\sinh 2at_0}, \\ \frac{\partial h_{yy}}{\partial t} &= \frac{dB}{dt} = 2\beta a \frac{\cosh 2at_0 + 1}{\sinh 2at_0}, \end{aligned} \quad (4.19)$$

where the initial values for A and B are $A = B \equiv \beta$.

Now we are ready to solve the time evolution eq.(4.16). Noting that this can be rewritten as

$$\frac{d}{dt} \left(\frac{1}{A} \frac{dA}{dt} \sinh 2at \right) = 4a^2 \sinh 2at, \quad (4.20)$$

we can integrate it to get

$$\frac{1}{A} \frac{dA}{dt} = 2a \frac{\cosh 2at + c_1}{\sinh 2at}, \quad (4.21)$$

where c_1 is an integration constant. c_1 is determined to be 1 by using the initial

conditions (4.19). Integrating eq.(4.21) once more, we obtain the solution for A .

$$A = c_2 \sinh^2 at, \quad (4.22)$$

where c_2 is an integration constant. Again using the initial condition, we have

$$c_2 = \frac{\beta}{\sinh^2 at_0}. \quad (4.23)$$

$B(t)$ can be solved in a similar way. Thus we arrive at the solutions:

$$\begin{aligned} A &= \beta \left(\frac{\sinh at}{\sinh at_0} \right)^2, \\ B &= \beta \left(\frac{\cosh at}{\cosh at_0} \right)^2. \end{aligned} \quad (4.24)$$

Because eqs.(4.11) and (4.12) defines a well-posed Cauchy problem, we can convince ourselves that this diagonal solution is the unique solution that satisfies the initial conditions for $\theta = 0$.

The resulting 3-geometry is

$$ds^2 = -dt^2 + \beta \left[\left(\frac{\sinh at}{\sinh at_0} \right)^2 dx^2 + \left(\frac{\cosh at}{\cosh at_0} \right)^2 dy^2 \right]. \quad (4.25)$$

An interesting feature of this result is the ratio of the lengths of the two cycles,

$$\frac{\sinh at}{\sinh at_0} / \frac{\cosh at}{\cosh at_0} = \frac{\tanh at}{\tanh at_0}. \quad (4.26)$$

Recall that t goes from $t_0 < 0$ to 0. In the limit $t \rightarrow -\infty$, the above ratio asymptotically tends to a constant $-1/\tanh at_0$, while the overall scale factor increases exponentially. This simple solution is certainly the geodesic in the moduli space. The general geodesics are obtained by considering the general solutions $\theta \neq 0$ ^[23].

5. Phase Space Reduction

We have been discussing the dynamics of (2+1)-dimensional pure gravity from the ADM canonical point of view. In the analysis, York's time slicing has made the problem tractable. It is remarkable that the dynamics reduces to that of finite degrees of freedom in the limited cases, $g = 0$ and $g = 1$. Although our analysis of the solution have revealed the interesting properties of (2+1)-dimensional gravity, it may be more appropriate to consider this problem at the action level. In the electromagnetic case, in the Hamiltonian formalism, we first solve the initial value problem. Then we study the dynamics by solving the evolution equation. It becomes clearer to see this process at the action level. In the processes, the orthogonal decomposition of an arbitrary vector into a sum of transverse, longitudinal and global modes is essentially used. In the gravitational case, when we attempt to reduce the phase space at the action level, we must also heavily use York's method. From now on we shall perform phase space reduction of (2+1)-dimensional gravity.^[27] Before going through this procedure, we have to keep in mind the limitation of our method. In higher genus cases, as will be shown later, we have a difficulty for the reduction of phase space because of the complexity of the Hamiltonian constraint equation. If we can get rid of this difficulty which has a similarity with that of (3+1)-dimensional gravity, we would gain some insight into the (3+1)-dimensional gravity. In the spherical topology case, it is trivial to reduce the phase space. Thus we shall concentrate our attention on the toroidal case for a while. The phase space action of (2+1)-dimensional gravity takes the form,

$$S = \int d^3x \{ \pi^{ij} \dot{h}_{ij} - NH - N_i H^i \}, \quad (5.1)$$

or in terms of the extrinsic curvature,

$$S = \int d^3x \sqrt{h} [(K^{ij} - K h^{ij}) \dot{h}_{ij} - N (K_{ij} K^{ij} - K^2 - R^{(2)}) + 2N_i (K^{ij} - K h^{ij})_{|j}]. \quad (5.2)$$

In York's slice, it is convenient to decompose the extrinsic curvature, K^{ij} , into the

traceless part and the trace part as follows

$$K^{ij} = \Sigma^{ij} + \frac{1}{2}h^{ij}K. \quad (5.3)$$

Then the action becomes

$$S = \int d^3x \sqrt{h} [(\Sigma^{ij} - \frac{1}{2}Kh^{ij})\dot{h}_{ij} - N(\Sigma_{ij}\Sigma^{ij} - \frac{1}{2}K^2 - R^{(2)}) + 2N_i(\Sigma^{ij} - \frac{1}{2}Kh^{ij})_{|j}]. \quad (5.4)$$

It is at this stage that we use York's slice, $\tau = -K = \text{constant}$ over the spatial manifold. We should remark that this gauge condition implies the spatial constancy of the lapse function N (see eq.(2.12)). Therefore we can rewrite the action as

$$S = \int d^3x [\sqrt{h}\Sigma^{ij}\dot{h}_{ij} + \tau\dot{\sqrt{h}} - N\sqrt{h}(\Sigma_{ij}\Sigma^{ij} - \frac{1}{2}\tau^2 - R^{(2)}) + 2N_i\Sigma_{|j}^{ij}]. \quad (5.5)$$

In this form, we can easily solve the momentum constraint equation by expanding Σ^{ij} in terms of the basis of the quadratic differentials $\phi^{(\alpha)ij}$,

$$\Sigma^{ij} = \sum_{(\alpha)} p_{(\alpha)} \phi^{(\alpha)ij} / 2v, \quad (5.6)$$

with $v = \int d^2x \sqrt{h}$. The deformation of h_{ij} is represented as

$$\frac{\partial h_{ij}}{\partial t} = \sum_{(\alpha)} \frac{\partial \rho^{(\alpha)}}{\partial t} \mu_{(\alpha)i}^l h_{lj} + \text{diffeo.} \quad (5.7)$$

This equation defines the Teichmüller parameters $\rho^{(\alpha)}$ and the corresponding Beltrami differentials $\mu_{(\alpha)i}^l$. We substitute the expansions (5.6) and (5.7) for Σ^{ij} and $\frac{\partial h_{ij}}{\partial t}$ into the action (5.5) in the phase space. Due to the special gauge $N = N(t)$ that we chose, the final form of the action becomes

$$S = \int dt [\sum_{(\alpha)} p_{(\alpha)} \frac{\partial \rho^{(\alpha)}}{\partial t} + \tau \frac{\partial v}{\partial t} - \tilde{N} (\sum_{(\alpha)} p_{(\alpha)} p_{(\beta)} g^{(\alpha)(\beta)} - v^2 \tau^2)]. \quad (5.8)$$

where

$$v = \int d^2x \sqrt{h}, \quad (5.9)$$

$$\tilde{N} = N/2v.$$

Here

$$g^{(\alpha)(\beta)} = \int d^2x \sqrt{h} \phi_{ij}^{(\alpha)} \phi_{kl}^{(\beta)} h^{ik} h^{jl} / 2v \quad (5.10)$$

is the Weil-Petersson metric.^[28] From this result, the geodesic motion in the conformal superspace is apparent. This result is significant, because we have many technical utilities about the geodesic motion at our hand. In our case, the conformal superspace is the compact negative constant curvature space, then the geodesic motion is necessarily chaotic according to the standard argument about dynamical systems.^[29]

It may be appropriate at this point to show that the reduction of phase space for (2+1)-dimensional gravity in the case of $g \geq 2$ is complicated. Using York's slice and the momentum constraint, it is possible to obtain the action,

$$S = \int d^3x [\sqrt{h} \Sigma^{ij} \dot{h}_{ij} - K \dot{\sqrt{h}} - N \sqrt{h} (\Sigma_{ij} \Sigma^{ij} - \frac{1}{2} K^2 - R^{(2)})]. \quad (5.11)$$

It is at this stage that we encounter the difficulty that the lapse function N is necessarily a function of the spatial coordinates. Due to this fact, we cannot extract the standard kinetic term for the global modes. To get rid of this spatial coordinate dependence from the $N \sqrt{h} \Sigma_{ab} \Sigma^{ab}$ term, we introduce a conformal factor Ω as

$$N = \nu(t) \Omega(t, \mathbf{x}), \quad (5.12)$$

$$h_{ab} = \Omega(t, \mathbf{x}) \hat{h}_{ab}.$$

The conformal transformations of Σ^{ab} and R are

$$\Sigma^{ab} = \Omega^{-2} \hat{\Sigma}^{ab}, \quad (5.13)$$

$$R = \frac{1}{\Omega} (\hat{R} - \hat{\Delta} \log \Omega).$$

We rewrite (5.11) as

$$\begin{aligned}
S &= \int d^3x [\sqrt{\hat{h}} \hat{\Sigma}^{ij} \dot{\hat{h}}_{ij} - K(\Omega \sqrt{\hat{h}}) - \nu \sqrt{\hat{h}} \hat{\Sigma}_{ij} \hat{\Sigma}^{ij} \\
&\quad + \frac{1}{2} \nu \Omega^2 K^2 \sqrt{\hat{h}} + \nu \Omega (\hat{R} - \hat{\Delta} \log \Omega) \sqrt{\hat{h}}] \\
&= \int d^3x [\sqrt{\hat{h}} \hat{\Sigma}^{ij} \dot{\hat{h}}_{ij} - \nu \sqrt{\hat{h}} \hat{\Sigma}_{ij} \hat{\Sigma}^{ij} + \frac{1}{2} \nu \Omega^2 K^2 \sqrt{\hat{h}} + \nu \Omega (\hat{R} - \hat{\Delta} \log \Omega) \sqrt{\hat{h}}] \\
&\quad - \int dt K \dot{v} + \int \nu \beta \left(\int d^2x \sqrt{\hat{h}} \Omega - v \right).
\end{aligned} \tag{5.14}$$

The last term is the Lagrange multiplier term which constrains Ω as

$$\int d^2x \sqrt{\hat{h}} \Omega = v. \tag{5.15}$$

From (5.14) it is obvious that Ω under the constraint (5.15) is not dynamical so that we can eliminate it from the action by using the Euler equation,

$$\Omega K^2 + \hat{R} - \hat{\Delta} \log \Omega - \frac{1}{\Omega} \hat{\Delta} \Omega + \beta = 0. \tag{5.16}$$

What we have to do is to solve this equation and to put the result into the action. The solution of eq.(5.16) depends on K, β , and \hat{R} . Then we must solve the Hamiltonian constraint to further reduce the phase space. It is at this point that the complexity mentioned before arises because of the momentum dependence of its solution. Thus if we want to reduce the phase space to the physical one, we encounter a formidable but technical difficulty.

6. Linearized Gravity

The main purpose of this chapter is to analyze the effects of matter fields on the geodesic motion.^[30] Our standing point is the following. We pick up an arbitrary point in the conformal superspace and look at the infinitesimal neighborhood of the point. From this point of view, the geodesic motion is a straight line. The effects of matter fields can also be easily seen. To do this we shall start from the linearized theory of gravity and then incorporate the matter fields perturbatively. Our strategy is to adopt the approximation in which gravity is assumed to be "weak". In the context of general relativity this means that the space-time metric is nearly flat. The criterion of the weak gravity does not seem to apply to our case, because the Einstein equation in (2+1)-dimensions implies that the space-time is locally flat. Globally, however, there are "topological degrees of freedom" to be taken into account.

Let us start with analyzing the pure linearized gravity. For the moment, we simply assume that the deviation, $h_{\mu\nu}$, of the actual space-time metric

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad (6.1)$$

from the flat metric $\eta_{\mu\nu}$ is "small". We mean by "linearized gravity" the approximation to general relativity which is obtained by substituting equation (6.1) for $g_{\mu\nu}$ in the Einstein-Hilbert action and retaining only the terms quadratic in $h_{\mu\nu}$. The result is given by

$$S = \int d^3\mathbf{x} (\Gamma_{\lambda}^{\sigma\nu} \Gamma_{\nu\sigma}^{\lambda} - \Gamma_{\nu}^{\lambda\nu} \Gamma_{\sigma\lambda}^{\sigma}), \quad (6.2)$$

where $\Gamma_{\sigma\mu\nu} = 1/2(h_{\sigma\mu,\nu} + h_{\sigma\nu,\mu} - h_{\mu\nu,\sigma})$. This action is also rewritten as

$$S = \int d^3\mathbf{x} [K_{ij} K^{ij} - K^2 + \Gamma_{ijk} \Gamma_{kji} - \Gamma_{ikk} \Gamma_{jji} + n/2(h_{kk,ii} - h_{ik,ik})], \quad (6.3)$$

where $K_{ij} = \Gamma_{0ij}$, $n = h_{00}$ and $N_i = h_{0i}$. To cast this action into a first order form,

let us define the momentum,

$$\pi^{ij} = K\eta^{ij} - K^{ij}. \quad (6.4)$$

Then, we get

$$S = \int d^3x \{ \pi^{ij} \dot{h}_{ij} - [\pi_{ij} \pi^{ij} - \pi^2 + \Gamma_{ijk} \Gamma_{kji} - \Gamma_{ikk} \Gamma_{jji} + 2N_i \pi^{ij} + n/2(h_{kk,ii} - h_{ik,ik})] \}. \quad (6.5)$$

Our next task is to solve the constraint equations,

$$\begin{aligned} \pi_j^j &= 0, \\ h_{kk,ii} - h_{ik,ik} &= 0. \end{aligned} \quad (6.6)$$

In contrast to the full gravity, the constraint equations are the linear equations which are easily solvable. To solve them, we use the following decomposition;

$$\begin{aligned} \pi^{ij} &= \pi_{TT}^{ij} + (LV)^{ij} + 1/2\eta^{ij}\pi, \\ h_{ij} &= h_{ij}^{TT} + (LW)_{ij} + 1/2\eta_{ij}h, \end{aligned} \quad (6.7)$$

where $(LV)^{ij} = \partial^i V^j + \partial^j V^i - \eta^{ij} \partial_k V^k$. Here h_{ij}^{TT} and π_{TT}^{ij} represent transverse-traceless parts of h_{ij} and π^{ij} , respectively. The general solution of the constraint equations are

$$\begin{aligned} \pi^{ij} &= \pi_{TT}^{ij} - \partial^i \partial^j (\pi/\Delta) + \eta^{ij} \pi, \\ h_{ij} &= h_{ij}^{TT} - 1/2\eta_{ij}h + \epsilon_{jk} \partial_i \partial^k \phi + \epsilon_{ik} \partial_j \partial^k \phi, \end{aligned} \quad (6.8)$$

where ϕ is an arbitrary function. Inserting these into the action, we obtain the well-known action,^[31]

$$S = \int d^3x \{ \pi_{TT}^{ij} \dot{h}_{ij}^{TT} - [\pi_{ij}^{TT} \pi_{TT}^{ij} + \frac{1}{4}(h_{jk,i}^{TT})^2] \}. \quad (6.9)$$

Note that the Hamiltonian is positive definite, so we can quantize this system consistently. Starting from this fact, Hartle and Schleich showed the naturalness of the conformal rotation prescription by Gibbons, Hawking and Perry. Now let us go

back to the classical analysis. In the topologically trivial space nothing happens. Let us consider the torus case. There h_{ij}^{TT} and π_{TT}^{ij} are spatially constant. So we get the following action by using the Beltrami differentials for the background geometry,

$$\begin{aligned} S &= \int dt \int d^2x [\pi_{TT}^{ij} \dot{h}_{ij}^{TT} - \pi_{ij}^{TT} \dot{\pi}_{TT}^{ij}] \\ &= \int dt [p_\alpha \dot{q}^\alpha - \frac{g^{\alpha\beta}}{2v} p_\alpha p_\beta]. \end{aligned} \quad (6.10)$$

Note that the Weil-Petersson metric $g^{\alpha\beta}$ does not depend on q^α in contrast to the full gravity. This fact is understood as follows. As we regard the deviation from the background metric as small, we are on the tangent space at some point in the conformal superspace. As a consequence the geodesic motion is along a straight line. This sounds natural, because any geodesic trajectory is locally straight. At this point we would like to emphasize that the clear-cut result for the case of $g = 1$ heavily depends on the constancy of h_{ij}^{TT} . We cannot expect that the Teichmüller motion is the geodesic motion in the conformal superspace in the case of $g \geq 2$.

We are now in a position to discuss the effects of matter fields on geodesic motion. Our action to consider is

$$S = \int d^3x \sqrt{g} R - \frac{1}{4} \int d^3x \sqrt{g} g^{\mu\lambda} g^{\nu\rho} F_{\mu\nu} F_{\lambda\rho}. \quad (6.11)$$

To analyze this system, we assume that the Maxwell fields are sufficiently small so that quartic terms are negligible. Then the full action reduces to

$$\begin{aligned} S &= \int d^3x (\Gamma_\lambda^{\sigma\nu} \Gamma_{\nu\sigma}^\lambda - \Gamma_\nu^{\lambda\sigma} \Gamma_{\sigma\lambda}^\nu) - \frac{1}{4} \int d^3x F_{\mu\nu} F^{\mu\nu} + \int d^3x \frac{1}{2} h^{\sigma\mu} T_{\sigma\mu} \\ &= \int d^3x \{ \pi^{ij} \dot{h}_{ij} - [\pi_{ij} \dot{\pi}^{ij} - \pi^2 + \Gamma_{ijk} \Gamma_{kji} - \Gamma_{ikk} \Gamma_{jji}] \\ &\quad + N_i (2\pi_{ij}^i - T_{0i}) + n/2 (h_{kk,ii} - h_{ik,ik} + T_{00}) + \frac{1}{2} h_{ij} T_{ij} \} \\ &\quad + \int d^3x [\pi^i \dot{A}_i - \frac{1}{2} (\pi^i \pi_i + \frac{1}{2} F^{ij} F_{ij}) + A_0 \partial_i \pi^i]. \end{aligned} \quad (6.12)$$

The perturbation term can be written as

$$h_{ij} T_{ij} = h_{ij}^{TT} T_{ij}^{TT} + h_{ij}^L T_{ij}^L + h_{ij}^T T_{ij}^T. \quad (6.13)$$

Solving the constraint equations, we can obtain the longitudinal and the trace parts of h_{ij} which are represented by matter fields. Because T_{ij}^L and T_{ij}^T are already quadratic of matter fields, its contribution to eq.(6.13) is higher order effects which we disregard in our approximation scheme. Finally retaining the relevant parts only, we obtain

$$\begin{aligned} S &= \int dt \int d^2x [\pi_{TT}^{ij} \dot{h}_{ij}^{TT} - \pi_{ij}^{TT} \pi_{TT}^{ij} + \frac{1}{2} h_{ij}^{TT} T_{TT}^{ij}] \\ &= \int dt [p_\alpha \dot{q}^\alpha - \frac{g^{\alpha\beta}}{2v} p_\alpha p_\beta + q^\alpha F_\alpha], \end{aligned} \quad (6.14)$$

where $F_\alpha = \int d^2x \mu_\alpha^{ij} T_{ij}$. This is our main result.^[30] The geodesic motion in the conformal superspace is deviated by the transverse-traceless parts of the energy momentum tensor, *i.e.* its global mode part. Note that the final formula need not assume specific matter fields. The reason why we concentrate on the Maxwell fields is the existence of global modes of Maxwell fields on torus.^[18] Using the basis in Appendix B, the gauss law constraint is solved as

$$E^i = \epsilon^{ij} \partial_j \phi + \{ \pi_{(1)} \eta^{(1)i} + \pi_{(2)} \eta^{(2)i} \}, \quad (6.15)$$

and the vector potential decomposes to

$$A_i = \partial_i \chi - \epsilon_{ij} \partial^j \left(\frac{B}{\Delta} \right) + \{ a^{(1)} \xi_{(1)i} + a^{(2)} \xi_{(2)i} \}, \quad (6.16)$$

where Δ is the Laplacian and η and ξ are dual basis of the harmonics (see Appendix.B). Using this result, we can obtain the spatial part of the energy momentum tensor as

$$\begin{aligned} T_{ij} &= \partial_i \phi \partial_j \phi - \frac{1}{2} \eta_{ij} \partial_k \phi \partial^k \phi + \frac{1}{2} \eta_{ij} \pi_\alpha \eta_l^\alpha \epsilon_{lk} \partial^k \phi + \frac{1}{2} \eta_{ij} \pi_\alpha \pi_\beta \eta_k^\alpha \eta_l^\beta \\ &\quad - \pi_\alpha \eta_i^\alpha \epsilon_{jk} \partial^k \phi - \pi_\alpha \eta_j^\alpha \epsilon_{ik} \partial^k \phi - \pi_\alpha \pi_\beta \eta_i^\alpha \eta_j^\beta \\ &\quad + 2 \partial_i \partial_j \left(\frac{B}{\Delta} \right) B - 2 \partial_i \partial^l \left(\frac{B}{\Delta} \right) \partial_j \partial^l \left(\frac{B}{\Delta} \right) + \eta_{ij} \partial_k \partial^l \left(\frac{B}{\Delta} \right) \partial_k \partial^l \left(\frac{B}{\Delta} \right) - \frac{1}{2} \eta_{ij} B^2. \end{aligned} \quad (6.17)$$

We pick up the most interesting terms *i.e.*, those which contain the global modes

π_α quadratically,

$$T_{ij} = \frac{1}{2} \eta_{ij} \pi_\alpha \pi_\beta \eta_k^\alpha \eta_k^\beta - \pi_\alpha \pi_\beta \eta_i^\alpha \eta_j^\beta, \quad (6.18)$$

and substitute this expression into the formula (6.14). The result is

$$S = \int dt [p_\alpha \dot{q}^\alpha - \frac{g^{\alpha\beta}}{2v} p_\alpha p_\beta + q^\alpha \pi_\alpha \pi_\beta C_\alpha^{\beta\gamma}], \quad (6.19)$$

$$C_\alpha^{\beta\gamma} = \int d^2 x \mu_\alpha^{ij} (\frac{1}{2} \eta_{ij} \eta_k^\beta \eta_k^\gamma - \eta_i^\beta \eta_j^\gamma).$$

In the case of the Einstein-Maxwell theory, the Wilson loop degrees π_α bend the trajectory of the global degrees of gravity. This is understood as follows: The Wilson loop winds the non-trivial cycles of the torus, then the free motion of the torus is disturbed by its tension.

7. Mini-superspace

In the previous chapter, we appealed to a perturbative method to grasp some feeling of classical dynamics of (2+1)-dimensional gravity. Our aim in this chapter is to see another aspect of (2+1)-dimensional gravity coupled with matter fields using a minisuperspace method. Let us begin with the Einstein-Maxwell theory,

$$S = \int d^3 x \sqrt{-g} R^{(3)} - \frac{1}{4} \int d^3 x \sqrt{-g} F_{\mu\nu} F^{\mu\nu}. \quad (7.1)$$

The ADM (2+1)-decomposition is easily carried out as follows;

$$S = \int dt \int d^2 x \{ \pi^{ij} \dot{h}_{ij} + \pi^i \dot{A}_i - NH - N^i H_i + A_0 \partial_i \pi^i \},$$

$$H = \frac{1}{\sqrt{h}} [\pi^{ij} \pi_{ij} - \pi^2] - \sqrt{h} R^{(2)} + \frac{1}{2\sqrt{h}} \pi^i \pi_i + \frac{1}{4} \sqrt{h} F_{ij} F^{ij}, \quad (7.2)$$

$$H_i = -2\pi_{ij}^j + \pi^j F_{ij}.$$

For the time being, we assume the space has spherical topology. Then we can solve

the constraint equation,

$$\partial_i \pi^i = 0, \quad (7.3)$$

and the solution is given by

$$\pi^i = -h\epsilon^{ij}\partial_j\phi. \quad (7.4)$$

The vector potential decomposes to

$$A_i = \partial_i\chi + \sqrt{h}\epsilon_i^l\partial_l\left(-\frac{1}{\Delta}\left(\frac{B}{\sqrt{h}}\right)\right) \quad (7.5)$$

where ϕ, χ are the arbitrary scalar functions and the magnetic field is given by $B = F_{12}$. After inserting these formula into the action, we obtain

$$S = \int dt \int d^2x \left\{ \pi^{ij}\dot{h}_{ij} + B\dot{\phi} - N\left[\frac{1}{\sqrt{h}}(\pi^{ij}\pi_{ij} - \pi^2) - \sqrt{h}R^{(2)}\right] + \frac{1}{2\sqrt{h}}B^2 + \frac{1}{2}\sqrt{h}h^{ij}\partial_i\phi\partial_j\phi\right\} - N^i[-2\pi_{ij}^j + B\partial_i\phi]. \quad (7.6)$$

This action is apparently equivalent to that of the Einstein-scalar theory. Now we are going to study a wormhole solution of this action. The condition that the electric field is zero and the magnetic field is constant over the 2-sphere is equivalent the homogeneity of the scalar field ϕ . Under this condition, we can adopt the minisuperspace approach. There we can see the essential features of the whole theory probably except for topology changing phenomena.*

Putting the following ansatz of the metric,

$$ds^2 = -N^2(t)dt^2 + a^2(t)(d\theta^2 + \sin^2\theta d\phi^2), \quad (7.7)$$

we obtain the action,

$$S = \frac{1}{16\pi G} \int dt \left\{ 2\left(N - \frac{\dot{a}^2}{N}\right) - 2\Lambda Na^2 + 8\pi G \frac{a^2}{N} \dot{\phi}^2 \right\}. \quad (7.8)$$

* In quantum gravity this is a serious disadvantage. However we now study the classical aspects of gravity, hence we postpone this problem until we study quantum gravity.

Extremization of (7.8) yields the equations of motion,

$$\begin{aligned} \dot{a}^2 + 1 - \Lambda a^2 - 4\pi G a^2 \dot{\phi}^2 &= 0, \\ \ddot{a} - \Lambda a + 4\pi G a \dot{\phi}^2 &= 0, \\ \frac{d}{dt}(a^2 \dot{\phi}) &= 0. \end{aligned} \quad (7.9)$$

The solution of these Lorentzian equations is

$$a(t) = H^{-1} \left[\frac{1}{2} (1 - \sqrt{(1 - 4m^2 H^2) \cosh 2Ht}) \right]^{\frac{1}{2}}, \quad (7.10)$$

and this leads to an Euclidean wormhole solution,

$$a(\tau) = H^{-1} \left[\frac{1}{2} (1 - \sqrt{(1 - 4m^2 H^2) \cos 2H\tau}) \right]^{\frac{1}{2}}. \quad (7.11)$$

This solution is the same as the one obtained by Hosoya and Ogura in the Einstein-Maxwell theory.^[32]

Next we are going to seek a similar solution in the case of toroidal space. Our parametrization of 2-metric in the case of $g = 1$ is the following,

$$h_{ij} = \frac{d^2}{\eta} \begin{pmatrix} \xi^2 + \eta^2 & \xi \\ \xi & 1 \end{pmatrix}, \quad (7.12)$$

where d , ξ and η depend only on t . From this parametrization, it is easy to calculate the Einstein-Hilbert action,

$$S = \frac{1}{16\pi G} \int dt \left\{ -\frac{2}{N} (\dot{d}^2 - \frac{1}{4} d^2 \frac{\dot{\xi}^2 + \dot{\eta}^2}{\eta^2}) - 2\Lambda N d^2 + 8\pi G \frac{d^2}{N} \dot{\phi}^2 \right\}. \quad (7.13)$$

From this action, we can read off the DeWitt metric^[7] for the minisuperspace. The conformal superspace part of this metric is nothing but the Weil-Petersson metric. For simplicity let us further reduce our models by setting $\xi = 0$. As we mentioned in Sec.4, general solutions can be obtained from this simple case. Therefore this simplification never loses any generality.

From this minisuperspace approach we can reproduce all of the previous results. For example, in the case of pure gravity, we obtain the metric,

$$ds^2 = -dt^2 + t^2 dx^2 + dy^2, \quad (7.14)$$

where a periodic boundary condition is imposed on x, y . This solution coincides with the one which is obtained by Hosoya and Nakao.^[23] The geodesic motion,

$$\eta(t) = t, \quad (7.15)$$

is a part of their results. However it is easy to reproduce all of the geodesic motions by including ξ .^[24]

Inclusion of the cosmological constant is interesting from a cosmological point of view. The solution is simply

$$ds^2 = -dt^2 + \frac{1}{H} \sinh^2 Ht dx^2 + \frac{1}{H} \cosh^2 Ht dy^2, \quad (7.16)$$

and the moduli,

$$\eta(t) = \tanh Ht, \quad (7.17)$$

asymptotically stops its motion at $\eta = 1$ as $t \rightarrow \infty$. Finally we shall examine whether there exist wormhole solutions in the case of $g = 1$, or not. The equation of motion are the following;

$$d^2 - \frac{1}{4} d^2 \frac{\dot{\eta}^2}{\eta^2} - \Lambda d^2 - 4\pi G d^2 \dot{\phi}^2 = 0, \quad (7.18)$$

$$\frac{d^2}{dt^2} d + \frac{1}{4} d \frac{\dot{\eta}^2}{\eta^2} - \Lambda d + 4\pi G d \dot{\phi}^2 = 0, \quad (7.19)$$

$$\frac{d}{dt} (d^2 \frac{\dot{\eta}^2}{\eta^2}) + d^2 \frac{\dot{\eta}^2}{\eta^3} = 0, \quad (7.20)$$

$$\frac{d}{dt} (d^2 \dot{\phi}) = 0. \quad (7.21)$$

Inserting eq.(7.20) and eq.(7.21) with constants of integration, m and α , we obtain a single equation,

$$d^2 - \frac{\alpha^2}{d^2} - \Lambda d^2 - \frac{4\pi G m^2}{d^2} = 0. \quad (7.22)$$

The solution of this equation is qualitatively similar to the previous one, eq.(7.17). Therefore the wormhole solution in the case of $g = 1$ does not exist.

8. Quantum Gravity

It was a great success to discover the quantum mechanics which was a conceptually revolutionary theory. Now no one doubts that the quantum theory is a correct theory which account for the laws of nature. Indeed, the electro-weak theory or the Weinberg-Salam theory, shows us the surprising agreement with the experimental results. Quantum chromodynamics is also considered to be a correct theory for strong interaction. Thus these theories are called the standard theory. In our universe, however, there exists the gravitational force as well as the ones mentioned above. The standard theory of gravitation is the general relativity which is not only mathematically beautiful but also explains many phenomena in our universe. Therefore we may say that we have found all of the theories. There are, however, two unsatisfactory points. One of them is based on the ideology that the unified theory of nature should exist. Another reason is more serious one that the lack of the conceptual consistency between the standard theory which is a quantum theory and the general relativity which is a classical theory. So we have to incorporate the quantum theory into the general relativity to get a conceptually consistent theory. As a preliminary step to quantization, we need to put Einstein's equations for the gravitational field into the Hamiltonian form. The Hamiltonian form forces us to specify a physical state at a certain time. In the usual quantum theory, the physical state is determined by the Schrödinger equation. In our case, the corresponding equations are

$$H\psi = 0, \quad (8.1)$$

$$H_1\psi = 0. \quad (8.2)$$

The equation (8.2) simply expresses that ψ must be invariant under the spatial diffeomorphism. To get ψ to satisfy this equation is thus not difficult, while to satisfy eq.(8.1) is considerably difficult because it is a functional differential equation. We might be able to circumvent this situation by fixing the gauge before quantization. The complete fixation of the gauge degree of freedom is, however, non-trivial in the case of general relativity, because the Hamiltonian constraint includes a quadratic form of the momentum. To what extent can we fix the gauge degree at the classical level? In Sec.5, we have explicitly reduced the phase space by using York's time slicing. Our proposal is that we should start to quantize from the action (5.8) using functional methods. Functional methods are particularly useful in the development of theories which have certain invariances, such as gauge theories or parametrized theories, because they allow these invariances to be displayed explicitly. One expects these methods to be especially useful in the search for a quantum theory of gravity, which has invariances of both types. Indeed, Euclidean functional integrals for amplitudes have been proposed^[3] as the fundamental starting point of a quantum gravitational theory, an idea which has many noble consequences. This program immediately encounters a difficulty. The Euclidean Einstein action is not positive definite and the path integral will diverge. As Gibbons et al.^[10] showed, the Euclidean functional integrals can be made convergent by a conformal rotation. There is, however, no direct analog of the conformal rotation in the ordinary gauge theories such as electrodynamics. The actions of gauge theories are typically positive semi-definite when expressed in terms of the natural Euclidean variables. In view of this lack of analogy between Einstein gravitational theories and ordinary gauge theories, it would be helpful to have a more physically sound motivation for the Euclidean gravitational integrals in their conformally rotated form. Fortunately we know that a conformal rotation is needed to construct the Euclidean functional integrals of linearized gravity which is well defined when expressed in terms of its physical degrees of freedom. Hartle and Schleich have shown^[33] that the conformally rotated linearized Euclidean functional integral for a quantum amplitude can be deduced from the functional integral for that amplitude expressed in terms of the physical degrees of freedom. Their strategy is the following. Beginning with the classical theory expressed in its manifestly gauge invariant form, one first isolates

the physical degrees of freedom and expresses the dynamics in terms of them. One next formulates the quantum theory as functional integrals in the physical variables weighted by the appropriate physical action. Finally, one introduces a certain number of integrations over the redundant variables to recover the manifest invariance expressed in the full set of variables. The resulting parametrized functional integral is equivalent to those originally given in terms of the physical variables.

The essence of their arguments may be illustrated by a simple quantum mechanical model. The configuration space of the model consists of the physical degree of freedom, $q(t)$, and two variables $\phi(t)$ and $\lambda(t)$ which represent the redundant variables. The Lagrangian is given by a sum of the Lagrangian for the physical degree of freedom and the Lagrangian for the redundant variables,

$$L = \frac{1}{2}\dot{q}^2 - V(q) + \frac{1}{2}\mu(\dot{\phi} - \lambda)^2. \quad (8.3)$$

This exhibits a simple model of gauge invariance. Actually the total Lagrangian is invariant under gauge transformations,

$$\begin{aligned} \phi(t) &\rightarrow \phi(t) + \Lambda(t), \\ \lambda(t) &\rightarrow \lambda(t) + \dot{\Lambda}(t). \end{aligned} \quad (8.4)$$

Let us study this model in its Hamiltonian form.^[9] Reflecting the fact that λ is not a dynamical variable, there is a primary constraint on the system,

$$p_\lambda = \frac{\partial L}{\partial \lambda} = 0. \quad (8.5)$$

One finds that the Hamiltonian corresponding to (8.3) is

$$H = \frac{1}{2}p^2 + V(q) + \frac{1}{2\mu}\pi^2 + \lambda\pi, \quad (8.6)$$

where $\pi = \mu(\dot{\phi} - \lambda)$ and $p(t)$ is the conjugate momentum to $q(t)$. Since the system

has a gauge invariance, the secondary constraint exists:

$$\{p_\lambda, H\} = \pi = 0. \quad (8.7)$$

On the constraint surface, the Hamiltonian becomes

$$H_{phys} = \frac{1}{2}p^2 + V(q). \quad (8.8)$$

We have explicitly reduced the dynamical degrees of freedom in this model to the physical ones. We can now proceed to construct quantum amplitudes as sums over histories of physical variables. The transition amplitude or the propagator is then given by

$$\int dpdq \exp i \int dt [p\dot{q} - H_{phys}]. \quad (8.9)$$

One can readily introduce the integrations over the redundant variables to recover the manifest invariance expressed in the full set of variables. The result is

$$\int dpdq d\pi d\phi d\lambda \delta(\Phi(\phi)) \left| \frac{\delta\Phi}{\delta\phi} \right| \exp i \int dt [p\dot{q} + \pi\dot{\phi} - \frac{1}{2}p^2 - V(q) - \frac{1}{2\mu}\pi^2 - \lambda\pi]. \quad (8.10)$$

Or equivalently we have

$$\int dq d\phi d\lambda \delta(\Phi(\phi)) \left| \frac{\delta\Phi}{\delta\phi} \right| \exp i \int dt [\frac{1}{2}\dot{q}^2 - V(q) + \frac{1}{2}\mu(\dot{\phi} - \lambda)^2], \quad (8.11)$$

where integration over the momentum variables is performed.

To make clear the logic of Hartle-Schleich, we shall follow the same procedure as the Euclidean method. The Euclidean phase space path integral has the form

$$\int dpdq \exp \int d\tau [ip\dot{q} - \frac{1}{2}p^2 - V(q)]. \quad (8.12)$$

If we wish to recover the original invariance, the following manipulation would be necessary

$$\int dpdq d\pi d\phi d\lambda \delta(\Phi(\phi)) \left| \frac{\delta\Phi}{\delta\phi} \right| \exp \int d\tau [ip\dot{q} + i\pi\dot{\phi} - \frac{1}{2}p^2 - V(q) \pm \frac{1}{2\mu}\pi^2 - i\lambda\pi] \quad (8.13)$$

where we do not specify the sign of the kinetic term of the redundant variable. If μ has a positive sign, we choose a negative sign for the kinetic term and then we

obtain an expression of Euclidean configuration space path integral;

$$\int dq d\phi d\lambda \delta(\Phi(\phi)) \left| \frac{\delta\Phi}{\delta\phi} \right| \exp - \int d\tau \left[\frac{1}{2} \dot{q}^2 + V(q) + \frac{1}{2} \mu (\dot{\phi} - \lambda)^2 \right]. \quad (8.14)$$

However, if μ has a negative sign, we would reach a different expression

$$\int dq d\phi d\lambda \delta(\Phi(\phi)) \left| \frac{\delta\Phi}{\delta\phi} \right| \exp - \int d\tau \left[\frac{1}{2} \dot{q}^2 + V(q) - \frac{1}{2} \mu (\dot{\phi} - \lambda)^2 \right]. \quad (8.15)$$

This is nothing but the simplified version of the notorious conformal rotation. The key criterion of choosing the sign is the convergence of the Gaussian integration. In the case of $\mu < 0$, the action which results from the Euclidean path integral in terms of the physical degrees of freedom has the same gauge invariance with the original action, however it differs from its original form with $t = -i\tau$ (see eq.(8.3)).

In the case of the linearized Einstein gravity, one can demonstrate the naturalness of the conformal rotation in a similar way. Having obtained some experience, we are going to discuss the role of the conformal rotation in the context of (2+1)-dimensional gravity in which we do not use the linearization approximation. Before proceeding to our main subject, one more exercise is necessary. In a topologically non-trivial situation that we wish to study, the reduced phase space of the system is not so easy to identify and we must be careful to precisely define the path integral.

As an illustration, we shall perform this program in the case of (1+1)-dimensional Maxwell theory on a circle. Starting from the action,

$$\begin{aligned} S &= -\frac{1}{4} \int dt dx F_{\mu\nu} F^{\mu\nu} \\ &= \int dt \int dx \left[E\dot{A} - \frac{1}{2} E^2 + A_0 \partial E \right], \end{aligned} \quad (8.16)$$

one can isolate the physical degrees of freedom by solving the Gauss law constraint,

$$\partial E = 0. \quad (8.17)$$

Its solution is given by $E = p(t)$. Further we impose the gauge condition,

$$\partial A = 0. \quad (8.18)$$

The solution becomes $A = q(t)$. Consequently we obtain the physical action as

$$S = \int dt [p\dot{q} - \pi R p^2]. \quad (8.19)$$

where R is the radius of the circle. Let us quantize this system by path integration,

$$\sum_{winding} \int Dp Dq \exp i \int dt [p\dot{q} - \pi R p^2]. \quad (8.20)$$

Here the summation over the winding number arises due to the spatial topology. To make the invariance of the system manifest, we must add the redundant variables using the identity,

$$1 = \int d\chi \delta(\Delta\chi) |det\Delta|. \quad (8.21)$$

The expression (8.19) becomes

$$\int Dp Dq d\chi d\eta \delta(\Delta\chi) \delta(\Delta\eta) |det\Delta|^2 \exp i \int dt dx [p\dot{q} - \frac{1}{2} p^2]. \quad (8.22)$$

We can rewrite this expression by adding zero to the action,

$$\begin{aligned} & \int Dp Dq d(\partial\chi) d(\partial\eta) \delta(\Delta\chi) \delta(\Delta\eta) |det\Delta| \\ & \times \exp i \int dt dx [(p + \partial\eta)(\dot{q} + \partial\dot{\chi}) - \frac{1}{2}(p + \partial\eta)^2], \end{aligned} \quad (8.23)$$

and by setting $A = q + \partial\chi$ and $E = p + \partial\eta$, we obtain

$$\int DA DE \delta(\partial A) \delta(\partial E) |det\{\partial A, \partial E\}| \exp i \int dt dx [E\dot{A} - \frac{1}{2} E^2]. \quad (8.24)$$

In the above example, the space-time topology is the cylinder. It causes some complications in the path integration. That is, the summation over the winding numbers.

In the case of the torus universe, we do not know how to functionally integrate the reduced action due to the complexity of the integration regions.

The formal path integral representation of the torus universe is given by

$$\int Dp_\alpha D\rho^\alpha D\tau Dv DN \exp i \int dt [p_\alpha \dot{\rho}^\alpha + \tau \dot{v} - N[g^{\alpha\beta} p_\alpha p_\beta - \tau^2 v^2]]. \quad (8.25)$$

It is, however, difficult to perform the path integration, because the integration region is very complicated. Alternatively we shall start from the Wheeler-DeWitt equation,

$$\left[\frac{\partial^2}{\partial s^2} - y^2 \left\{ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right\} \right] \psi = 0. \quad (8.26)$$

Then we define the path integral representation by its solution. It is our starting point to recover the full symmetry of the theory. Separating variables, we can easily find a solution of eq.(8.26) as

$$u_\nu^{(n)}(s, x, y) = \sqrt{y} K_{i\nu}(2\pi | n | y) e^{2\pi i n x} e^{-iEs}, \quad (n=\text{integer}) \quad (8.27)$$

with $\nu = \sqrt{E^2 - \frac{1}{4}}$.

Here K is the modified Bessel function which approaches zero exponentially when its argument goes to infinity. The number n has to be an integer due to the periodicity $x \rightarrow x + 1 \in SL(2, Z)$. Of course (8.27) itself is not invariant under the full $SL(2, Z)$. We have to superpose (8.27) so that the result satisfies the $SL(2, Z)$ invariance;

$$U_\nu(s, x, y) = \sum_{n \neq 0} \rho_\nu(n) \sqrt{y} K_{i\nu}(2\pi | n | y) e^{2\pi i n x} e^{-iEs}. \quad (8.28)$$

The coefficients $\rho_\nu(n)$ have not been analytically given and the discrete eigenvalues ν are known only numerically. However, their properties are fairly well studied by mathematicians in number theory.^[35] The automorphic function (8.28) is called the Maass form. Note that we have excluded $n = 0$ from the sum (8.28), because we have the boundary condition $U_\nu(s, x, y) \rightarrow 0$ as $y \rightarrow 0$. That is, we demand the singular universe has no chance to appear. The idea behind this boundary condition is similar to Hartle and Hawking's.^[31] In a sense, the singularity of the space-time is circumvented in quantum cosmology of the torus universe.

Here we assume that the path integral expression (8.25) contains the above mentioned contents. As we have finished to define the path integral representation (8.25), we shall proceed to recover the original symmetry of (2+1)-dimensional pure gravity. Now we are about to perform an identical rewriting of the path integral. Therefore we shall reverse the process here. We shall start from the path integral,^[36]

$$\int DK^{ij} Dh_{ij} \delta(H) \delta(H_i) \delta(K - k(t)) | \det\{K, H\} | \exp i \int dt d^2x \sqrt{h} (K^{ij} - Kh^{ij}) \dot{h}_{ij}. \quad (8.29)$$

First we notice that the following orthogonal decomposition is useful,

$$\begin{aligned} DK^{ij} &= D\Sigma^{ij} D(LW)^{ij} DK \\ &= | \det L^\dagger L |^{\frac{1}{2}} D\Sigma^{ij} DW^i DK, \end{aligned} \quad (8.30)$$

and

$$Dh_{ij} = | \det L^\dagger L |^{\frac{1}{2}} D\tilde{h}_{ij} Dv^i D\phi. \quad (8.31)$$

The path integral becomes

$$\begin{aligned} &\int D\Sigma^{ij} D\hat{h}_{ij} Dk(t) D\phi \delta(H^\phi) | \det\{K, H^\phi\} | \\ &\quad \times \exp i \int dt d^2x [\sqrt{h} \Sigma^{ij} \dot{h}_{ij} - K \frac{d}{dt} (\sqrt{h} e^\phi)], \end{aligned} \quad (8.32)$$

where the integration over W^i is done and the gauge volume element Dv^i is factored out. Here

$$H^\phi = e^{-2\phi} \Sigma^{ij} \Sigma_{ij} - \frac{1}{2} K^2 - e^{-\phi} (-\Delta\phi + R^{(2)}). \quad (8.33)$$

Using the identities,

$$\delta(H^\phi) = \int DE(t) \delta(H^\phi - E(t)) \delta(\int d^2x \sqrt{h} e^\phi H^\phi), \quad (8.34)$$

and

$$1 = \int Dv(t) \delta(v - \int d^2x \sqrt{h} e^\phi), \quad (8.35)$$

we obtain the expression,

$$\int D\Sigma^{ij} D\hat{h}_{ij} Dk(t) Dv(t) D\phi DE(t) \delta(H^\phi - E(t)) \delta\left(\int d^2x \sqrt{\hat{h}} e^\phi H^\phi\right) \\ \times \delta\left(v - \int d^2x \sqrt{\hat{h}} e^\phi v\right) | \det\{K, H^\phi\} | \exp i \int dt d^2x \left[\sqrt{h} \Sigma^{ij} \dot{h}_{ij} - K \frac{d}{dt} (\sqrt{h} e^\phi) \right]. \quad (8.36)$$

Performing the integration of ϕ and $E(t)$, we get

$$\int D\Sigma^{ij} D\hat{h}_{ij} Dk(t) Dv(t) Dn(t) v | \det\{K, H^\phi\} | \\ \exp i \int dt d^2x \left[\sqrt{h} \Sigma^{ij} \dot{h}_{ij} - K \frac{d}{dt} (\sqrt{h} e^\phi) - n \sqrt{h} e^\phi H^\phi \right], \quad (8.37)$$

where ϕ is supposed to be the solution of the inhomogeneous part of the Hamiltonian constraint equation. After the simple transformation of variables and the expansion by global basis, we reach the final result,

$$\int Dp_\alpha D\rho^\alpha D\tau Dv Dn | \det\{K, H^\phi\} | \exp i \int dt [p_\alpha \dot{\rho}^\alpha + \tau \dot{v} - n [g^{\alpha\beta} p_\alpha p_\beta - \tau^2 v^2]]. \quad (8.38)$$

Now we are in a position to discuss the role of the conformal rotation. The conclusion which we have reached is that the conformal rotation seems irrelevant in quantum gravity, at least in the case of our model. To say more precisely, it is necessary to remark that our insertion of the delta function is meaningless in the Euclidean region and furthermore our standing point was the Wheeler-DeWitt equation that has no solution in the Euclidean region when we perform the conformal rotation. We conclude that the introduction of the conformal rotation is a complete illusion in the case of the full gravity from our analysis.

9. Conclusion

In this thesis, the whole attention is paid for the topological aspects of (2+1)-dimensional gravity in the conventional ADM approach. First we shall summarize our analysis. As is shown in Sec.2, the ADM canonical formalism of (2+1)-dimensional gravity is parallel to the one of (3+1)-dimensional gravity. However, as (2+1)-dimensional gravity has a special property; i.e. the space-time is locally flat and there are no local gravitational wave modes, we can expect some technical advantages in the analysis of the global aspects of gravity. Indeed we have succeeded to reduce the phase space of (2+1)-dimensional gravity in the case of $g = 1$. It is beautifully formulated as the geodesic motion in the conformal superspace. In this analysis, York's slicing is essentially important. We have also made the difficulty for $g \geq 2$ cases apparent. It is necessary to incorporate matter fields into (2+1)-dimensional gravity for discussing cosmological significance of the global modes. For this purpose the linearization method is used and we have concluded that the transverse-traceless part of energy-momentum tensor bends the geodesic motion. In a special example, the Einstein-Maxwell theory, we observed that the Wilson line interacts with the geometry and modifies the geodesic motion. As an alternative approach we used the minisuperspace method. In the case of pure gravity with toroidal and spherical topologies for the spatial surfaces, we reproduced the same results with the one of the full gravity. For the spherical topology, we incorporated matter fields and derived a wormhole solution which is related to the Coleman theory ^[13]. For the toroidal topology, we have not found a wormhole solution. As we succeeded to identify the physical variables in the case of $g = 1$, we can quantize this system. We studied the formal path integral representation of the wavefunction of universe.

What have we learned from our model analysis? At least, in the case of the torus universe we have completely analyzed its classical and quantum structures. However, we could not make concrete statements about $g \geq 2$ universes. We think that this discrepancy between $g = 1$ and $g \geq 2$ is the key for understanding (3+1)-dimensional gravity. The main difficulty was the complexity of the Hamiltonian constraint equation. In the case of $g = 1$, we have explicitly solved the momen-

tum constraint equation and the inhomogeneous part of the Hamiltonian constraint equation, hence the quantization becomes tractable. As the result reduces to the well-known relativistic particle system, we can construct the Hilbert space. However, the probability density is not positive definite, which might lead us to the third quantization of gravity. The most important observation which we made from our analysis is that the Euclidean quantum gravity is inappropriate. Specifically we showed that the conformal rotation cannot be justified.

What about the topology changing phenomena? All we can say concerning to the Coleman theory is that the only spherical topology is permitted as the spatial section of the wormhole solution in (2+1)-dimensions. We have some speculations about the topology changing phenomena itself. There are at least two approaches to this issue. One of which is the summation over the histories method and another is the third quantization of the gravity. The former one is used in the string theories which are beautifully formulated as the geometrical theory.^[37] In the case of quantum gravity, we also expect that it has a nice geometrical structure that is helpful to calculate the topology changing amplitudes. Indeed, Witten proposed that the topology changing amplitudes can be calculated as the Ray-Singer analytic torsion.^[38] However, as we have not yet understood the relation between the Witten theory and the conventional theory, we do not understand in what sense he says the topology is changing. If we succeed to understand the relation between the two approaches, the relation between the Ashtekar theory and the conventional theory in (3+1)-dimensions will be clarified readily. Once this relation is established, the topology changing problem reduces to a mathematical problem. On the other hand, the latter one, i.e. the third quantization of the gravity, is attractive, because we can use the technique of the field theory. The problem is how to construct the action of the theory. We have no basic principle to construct the action. Once the action is given, the calculation of the topology changing amplitudes is rather straightforward.

Finally we would like to mention the relation between the conventional ADM canonical formalism and the new formalism, i.e. the relation between the conventional ADM canonical formalism and Witten's formalism in (2+1)-dimensions and

the relation between the conventional ADM canonical formalism and Ashtekar's formalism in (3+1)-dimensions. It is this parallelism between (2+1)-dimensions and (3+1)-dimensions that is useful for understanding (3+1)-dimensional quantum gravity by studying (2+1)-dimensional quantum gravity. Classically, the essential difference between the conventional ADM canonical formalism and the new formalism is that the new formalism permits a degenerate metric. Moving to quantum gravity, for instance in (2+1)-dimensions, this difference makes quantum gravity renormalizable and finite. In a sense, the new formalism may reveal the new phase which must be realized in the Planckian region. Our future task is to connect the new theory in the Planckian region with the ordinary theory in the low energy stage. Anyway (2+1)-dimensional gravity will be an illuminating playground to understand (3+1)-dimensional gravity.

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APPENDIX A

One of the reasons why we study (2+1)-dimensional gravity is the existence of a lot of mathematical results for 2-dimensional manifold. Especially, when concentrating on the closed orientable compact manifold, we know many useful facts. Here we briefly summarize these results. Topologically the 2-dimensional closed manifold can be completely classified by the Euler number,

$$\chi = \frac{1}{4\pi} \int d^2x \sqrt{h} R. \quad (\text{A1})$$

The genus g of the Riemann surface is related to the Euler number as $\chi = 2 - 2g$. Fixing the genus, we have different surfaces which are not related by diffeomorphism

and the Weyl transformation. To characterize these different manifolds called the Riemann surfaces, the concept of the moduli is useful. We now fix the topology, i.e., the number of handles of the spatial manifold M . An infinitesimal deformation δg_{mn} of a metric is a symmetric two-tensor, and the natural norm for δg_{mn} is

$$\|\delta h_{ij}\|^2 = \int d^2x (ch^{ij}h^{kl} + h^{ik}h^{jl})\delta h_{ij}\delta h_{kl}. \quad (\text{A2})$$

We ask whether all modes of g_{mn} can be gauged away with the help of diffeomorphism. There remains the conformal mode and it is a classic theorem of Gauss that in any simply connected patch on the surface the metric can indeed be made conformally Euclidean by diffeomorphism. Whenever the topology is non-trivial, however, the diffeomorphisms of different patches need not match and there may be topological obstructions. To see this, we note that the action of diffeomorphism on the metric is given by

$$\delta h_{ij} = (L\delta v)_{ij} + (\nabla^k \delta v_k)h_{ij}, \quad (\text{A3})$$

where the operator L sends vectors into symmetric traceless two-tensors,

$$(L\delta v)_{ij} = \nabla_i \delta v_j + \nabla_j \delta v_i - h_{ij} \nabla^k \delta v_k, \quad (\text{A4})$$

and describes the traceless piece of the deformation coming from diffeomorphism by the vector field δv^m . Thus the only metric deformations δg_{mn} that are not gotten by diffeomorphism and are not related to the conformal modes are in $(\text{Range } L)^\perp$. this means that any diffeomorphism is given by the decomposition orthogonal under eq.(A.2);

$$\{\delta h_{ij}\} = \{\delta\sigma h_{ij}\} \oplus \{\text{Range } L\} \oplus \{\text{Ker } L^\dagger\} \quad (\text{A5})$$

where the action of P_1^\dagger on symmetric traceless two-tensors is given by

$$(L^\dagger \delta h)_m = -2\nabla^n \delta h_{ij} \quad (\text{A6})$$

and we have used the result that we have the identification

$$(\text{Range } L)^\perp = \text{Ker } L^\dagger \quad (\text{A7})$$

The way to determine the number of zero modes of these operators is to appeal to

an index theorem, which gives the difference between the number of zero modes of the operator and its adjoint in terms of a topological invariant. In the present case, zero modes of L are just conformal Killing vectors, the topological invariant is the Euler characteristic, and the index theorem reduces to the following version of the Riemann-Roch theorem:

$$\dim Ker L - \dim Ker L^\dagger = 3\chi(M). \quad (A8)$$

For the sphere, the conformal Killing transformations form the $SL(2, C)$ so that $\dim Ker L = 6$. For the torus, it is the group of translations that has dimension 2. For higher genus, there are no conformal Killing vectors on a surface without boundary. Thus we conclude that

$$\dim Ker L^\dagger = \begin{cases} 0, & g = 0 \\ 2, & g = 1 \\ 6g - 6, & g \geq 2 \end{cases} \quad (A9)$$

Elements of $Ker L$ are called real quadratic differentials or moduli deformations.

APPENDIX B

As a preliminary attempt to quantum gravity, the field theory in curved space-time has been investigated. In this thesis we have stressed on the topological aspects of quantum gravity. Therefore we shall present here a quantum field theory in topologically non-trivial space as a semi-classical theory of the quantum gravity in which non-trivial topology is incorporated.

Let us start from the Maxwell theory as an illustration. Given the action in the curved space,

$$\begin{aligned} S &= -\frac{1}{4} \int d^D x \sqrt{g} F_{\mu\nu} F^{\mu\nu} \\ &= \int dt \int_{\Sigma} d^{D-1} x (\pi^i \dot{A}_i - \frac{1}{2} (\frac{1}{\sqrt{h}} \pi_i \pi^i + \frac{1}{2} \sqrt{h} F_{ij} F^{ij}) + A_0 \partial_i \pi^i), \end{aligned} \quad (B1)$$

the Gauss law constraint becomes

$$\partial_i \pi^i = 0, \quad (\text{B2})$$

where h_{ij} is the spatial metric on Σ and $h = \det(h_{ij})$. Eq.(B2) is written as

$$\delta\left(\frac{\pi}{\sqrt{h}}\right) = 0, \quad (\text{B3})$$

where δ is a co-derivative in space Σ . In general this constraint eq.(B3) is solved as^[34]

$$\pi = \sqrt{h} \delta \phi + \sum_{(\alpha)} p_{(\alpha)} \eta^{(\alpha)} \sqrt{h}, \quad (\text{B4})$$

where ϕ is some 2-form in space and $\eta^{(\alpha)} \in H^1(M)$. Owing to Hodge's decomposition theorem, the vector potential is written as

$$A = d\chi + \delta\omega + \sum_{(\alpha)} q^{(\alpha)} \xi_{(\alpha)}, \quad (\text{B5})$$

where $\xi_{(\alpha)}$ are the dual of $\eta^{(\alpha)}$, i.e. $\int \eta^{(\alpha)} \wedge \star \xi_{(\beta)} = \delta_{(\beta)}^{(\alpha)}$ and ω is also a 2-form. The theory becomes trivial in the (1+1)-dimensional Minkowski space, since there exists no 2-form ω in space. Now as Hodge's decomposition is orthogonal, the gauge degree of freedom, $d\chi$, decouples. So eq.(B1) becomes

$$S = \int dt \left\{ \int \delta\phi \wedge \star \delta\omega - \frac{1}{2} \int (\delta\phi \wedge \star \delta\phi + \tilde{F} \wedge \star \tilde{F}) + \sum p_{(\alpha)} \dot{q}^{(\alpha)} - \sum \frac{1}{2} g^{(\alpha)(\beta)} p_{(\alpha)} p_{(\beta)}, \right\} \quad (\text{B6})$$

where $g^{(\alpha)(\beta)} = \int \eta^{(\alpha)} \wedge \star \eta^{(\beta)}$ and \tilde{F} means spatial components of F . We have separated the global modes from the local fluctuation modes. Then the canonical quantization can be carried out completely in a standard way.

As a demonstration we shall study the (2+1)-dimensional Maxwell theory on $T^2 \times R$, where T^2 represents a torus, in greater detail. In this theory, the action is given by

$$S = \frac{1}{2} \int dt \int_T d^2x (E^i \dot{A}_i - \frac{1}{2} (E_i E^i + B^2) + A_0 \partial_i E^i), \quad (\text{B7})$$

where $B = F_{12}$. Here we take the flat metric on the torus.

Let us recall the several elementary facts about the Riemann surfaces.^[14] In the complex notation, the Abelian differential ω satisfies the periodicity;

$$\oint_a \omega = 1, \quad \oint_b \omega = \tau = \tau_1 + i\tau_2, \quad (\text{B8})$$

and the Riemann relation,

$$\frac{i}{2} \int \omega \wedge \bar{\omega} = \text{Im} \tau = \tau_2, \quad (\text{B9})$$

where a and b represent homology cycles for the two different directions of a torus. In terms of the real harmonics,

$$\omega = \alpha + i * \alpha, \quad (\text{B10})$$

eq.(B8) means

$$\begin{aligned} \oint_a \alpha &= 1, & \oint_a * \alpha &= 0, \\ \oint_b \alpha &= \tau_1, & \oint_b * \alpha &= \tau_2, \end{aligned} \quad (\text{B11})$$

and also eq.(B9) implies

$$\int \alpha \wedge * \alpha = \tau_2. \quad (\text{B12})$$

We shall take the basis as

$$\begin{aligned} \xi_{(1)} &= \alpha, & \xi_{(2)} &= * \alpha, \\ \eta^{(1)} &= \frac{\alpha}{\tau_2}, & \eta^{(2)} &= \frac{* \alpha}{\tau_2}. \end{aligned} \quad (\text{B13})$$

The orthonormality relation, $\int \xi_{(\alpha)} \wedge * \eta^{(\beta)} = \delta_{(\alpha)}^{(\beta)}$, holds and the metric becomes

$$g^{(\alpha)(\beta)} = \int \eta^{(\alpha)} \wedge * \eta^{(\beta)} = \frac{1}{\tau_2} \delta^{(\alpha)(\beta)}. \quad (\text{B14})$$

Note that

$$\oint_a A = q^{(1)}, \quad \oint_b A = q^{(1)}\tau_1 + q^{(2)}\tau_2. \quad (\text{B15})$$

Using these basis, eq.(B13), the Gauss law constraint is solved as

$$E^i = \epsilon^{ij}\partial_j\phi + \{p_{(1)}\eta^{(1)i} + p_{(2)}\eta^{(2)i}\}, \quad (\text{B16})$$

and the vector potential decomposes to

$$A_i = \partial_i\chi - \epsilon_{ij}\partial^j\left(\frac{B}{\Delta}\right) + \{q^{(1)}\xi_{(1)i} + q^{(2)}\xi_{(2)i}\}, \quad (\text{B17})$$

where Δ is the Laplacian. Inserting these eqs.(B16) and (B17) into eq.(B7), we obtain

$$S = \int dt \int d^2x \left\{ B\dot{\phi} - \frac{1}{2}(B^2 + \partial_i\phi\partial^i\phi) + \sum \int dt (p_{(\alpha)}\dot{q}^{(\alpha)} - \frac{1}{2}g^{(\alpha)(\beta)}p_{(\alpha)}p_{(\beta)}) \right\}, \quad (\text{B18})$$

where $g^{(\alpha)(\beta)} = \frac{1}{Im\tau}\delta^{(\alpha)(\beta)}$ and $\mathbf{q} = (q^1, q^2)$ is on the torus generated by $\mathbf{e}_1 = (1, -\frac{\tau_1}{\tau_2})$ and $\mathbf{e}_2 = (0, \frac{1}{\tau_2})$. Noticing the fact that the dual lattice of the torus is again the lattice with basis $\mathbf{e}_1^* = (1, 0)$ and $\mathbf{e}_2^* = (\tau_1, \tau_2)$. As a result,^[18] the transition kernel of the Wilson variables is

$$G(q'', q') = \sum_{\mathbf{m}} \exp i[\mathbf{m}(\mathbf{q}'' - \mathbf{q}') - \frac{t'' - t'}{Im\tau}\mathbf{m}^2], \quad (\text{B19})$$

where $\mathbf{m} = (m, -m\frac{\tau_2}{\tau_1} + n\tau_2)$. Here we have discarded the effects of a scalar field ϕ , since we are interested in the topological effects. It is easy to verify that this kernel satisfies the Schrödinger equation by direct calculation. By varying the moduli parameters, the deformation of the theory is described. This observation is crucial when we consider the coupling with gravity. In the (2+1)-dimensional Einstein gravity the only dynamical degrees of freedom are moduli parameters of space-like surfaces. As shown in ref.[23], the dynamics of the moduli parameters is reduced to a quantum mechanics of a relativistic particle in a curved space with the Weil-Petersson metric. Therefore the gravitational coupling with the Maxwell field manifests itself through $g^{(\alpha)(\beta)} = \frac{1}{Im\tau}\delta^{(\alpha)(\beta)}$.

Let us remark on the other cases of the 2-space topology. In the case of sphere, there exists no harmonics. Thus there exists nothing for the global mode. In the higher genus ($g > 1$) case, the arguments are parallel to the torus case. In the complex notation, the Abelian differential ω_i satisfies the periodicity;

$$\oint_{a_i} \omega_j = \delta_{ij}, \quad \oint_{b_i} \omega_j = \tau_{ij}, \quad (\text{B20})$$

and the Riemann relation,

$$\frac{i}{2} \int \omega_i \wedge \bar{\omega}_j = \text{Im} \tau_{ij}, \quad (\text{B21})$$

where a_i and b_i represent canonical homology cycles for the Riemann surface. In terms of the real harmonics,

$$\omega_i = \alpha_i + i * \alpha_i, \quad (\text{B22})$$

eq.(B20) means

$$\begin{aligned} \oint_{a_i} \alpha_j &= \delta_{ij}, & \oint_{a_i} * \alpha_j &= 0, \\ \oint_{b_i} \alpha_j &= \text{Re} \tau_{ij}, & \oint_{b_i} * \alpha_j &= \text{Im} \tau_{ij}, \end{aligned} \quad (\text{B23})$$

and also eq.(B21) implies

$$\int \alpha_i \wedge * \alpha_j = \text{Im} \tau_{ij}. \quad (\text{B24})$$

We shall take the basis as

$$\begin{aligned} \xi_i &= \alpha_i & 1 \leq i \leq g, \\ \xi_i &= * \alpha_i & g+1 \leq i \leq 2g, \\ \eta^i &= \text{Im} \tau_{ij}^{-1} \alpha_j & 1 \leq i \leq g, \\ \eta^i &= \text{Im} \tau_{ij}^{-1} * \alpha_j & g+1 \leq i \leq 2g. \end{aligned} \quad (\text{B25})$$

The orthonormality relation, $\int \xi_i \wedge^* \eta^j = \delta_i^j$, holds and the metric becomes

$$g^{ij} = \int \eta^i \wedge^* \eta^j = \text{Im} \tau_{ij}^{-1} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (\text{B26})$$

Thus we can obtain all of the necessary information for analyzing the Maxwell theory on a Riemann surface with arbitrary genus.

Until this stage we have considered the topological aspects of field theory through the first cohomology H^1 of the space. However, other cohomologies come into play in physics by considering a field theory with a p-form potential which has a gauge symmetry. Let us start with the action,

$$S = -\frac{1}{2} \frac{1}{(p+1)!} \int_M F \wedge^* F, \quad (\text{B27})$$

where $F = dA^p$ and $A^p = A_{\mu_1 \dots \mu_p} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p}$. From the variational principle we get the equation of motion $\delta F = 0$. Due to the gauge invariance these include the constraint equations,

$$\tilde{\delta} E^p = 0, \quad (\text{B28})$$

where $\tilde{\delta}$ represents the coderivative in space. Solving the constraints eq.(B28) as

$$E^p = \tilde{\delta} \omega^{p+1} + \sum p_\alpha \eta^\alpha, \quad (\text{B29})$$

we can proceed with the same arguments as given in the case of the 1-form potential. If $D = p + 1$, then the $p + 1$ form in space does not exist. Therefore the theory is purely topological in this case. In general the excitation modes and the topological modes can coexist but decouple in the action. Caution is necessary about the excitation modes. The $p+1$ form ω^{p+1} has ambiguity in the form $\tilde{\delta} \omega^{p+2}$ and ω^{p+2} has ambiguity $\tilde{\delta} \omega^{p+3}$, etc.

APPENDIX C

The main claim made by Witten^[19] is that (2+1)-dimensional gravity can be reformulated as the Chern-Simons theory in (2+1)-dimensions which is exactly soluble at both the classical and quantum levels. Let us first explain this fact. In the tetrad formalism, the Einstein-Hilbert action becomes

$$I = \frac{1}{2} \int \epsilon^{ijk} \epsilon_{abc} e_i^a (\partial_j \omega_k^{bc} - \partial_k \omega_j^{bc} + [\omega_j, \omega_k]^{bc}), \quad (C1)$$

where e_i^a and ω_i^a are the vierbein and the spin connection respectively. Here we denote the space-time indices by i, j, k and the Lorentz indices by a, b, c . Before we ask whether gravity in (2+1)-dimensions is equivalent to $ISO(2, 1)$ gauge theory with a Chern-Simons interaction, we should ask whether there exists an invariant and a non-degenerate metric on the Lie algebra of $ISO(2, 1)$. The magic of $d = 3$ is the very existence of such a metric. The commutation relations of $ISO(2, 1)$ take the form,

$$\begin{aligned} [J_a, J_b] &= \epsilon_{abc} J^c, \\ [J_a, P_b] &= \epsilon_{abc} P^c, \\ [P_a, P_b] &= 0. \end{aligned} \quad (C2)$$

Here we replaced J^{ab} with $J^a = \frac{1}{2} \epsilon^{abc} J_{bc}$. The invariant quadratic form of interest is then

$$\langle J_a, P_b \rangle = \delta_{ab}, \quad \langle J_a, J_b \rangle = \langle P_a, P_b \rangle = 0. \quad (C3)$$

Let us use these formulas and construct a gauge theory for the group $ISO(2, 1)$. The gauge field is a Lie-algebra-valued one form,

$$A_i = e_i^a P_a + \omega_i^a J_a. \quad (C4)$$

An infinitesimal gauge parameter is expressed as $u = \rho^a P_a + \tau^a J_a$, with ρ^a and τ^a being infinitesimal parameters. The variation of A_i under a gauge transformation

is

$$\delta A_i = -D_i u, \quad (C5)$$

where by definition,

$$D_i u = \partial_i u + [A_i, u]. \quad (C6)$$

In terms of the vierbein and the spin connection, we arrive at the transformation laws;

$$\begin{aligned} \delta e_i^a &= -\partial_i \rho^a - \epsilon^{abc} e_{ib} \tau_c - \epsilon^{abc} \omega_{ib} \rho_c, \\ \delta \omega_i^a &= -\partial_i \tau^a - \epsilon^{abc} \omega_{ib} \tau_c. \end{aligned} \quad (C7)$$

Now we calculate the curvature tensor,

$$\begin{aligned} F_{ij} = [D_i, D_j] &= P_a (\partial_i e_j^a - \partial_j e_i^a + \epsilon^{abc} (\omega_{ib} e_{jc} + e_{ib} \omega_{jc})) \\ &\quad + J_a (\partial_i \omega_j^a - \partial_j \omega_i^a + \epsilon^{abc} \omega_{ib} \omega_{jc}). \end{aligned} \quad (C8)$$

Using these expressions, we can evaluate the Chern-Simons action;

$$I_{CS} = \frac{1}{2} \int \text{Tr} (A \wedge dA + \frac{2}{3} A \wedge A \wedge A). \quad (C9)$$

The result is

$$I_{CS} = \int \epsilon^{ijk} e_{ia} (\partial_j \omega_k^a - \partial_k \omega_j^a + \epsilon_{abc} \omega_j^b \omega_k^c). \quad (C10)$$

This is obviously equivalent to the Einstein-Hilbert action.

Constructing a canonical formalism, the phase space is easily determined. The physical phase space of (2+1)-dimensional gravity is the same as the moduli space of flat $ISO(2, 1)$ connections whose dimension is $6g - 6 (g \geq 2)$. Witten discussed^[19] using his formulation, the renormalizability and unitarity. He also calculated the topology-changing amplitudes which are essentially Ray-Singer analytic torsion.^[36]

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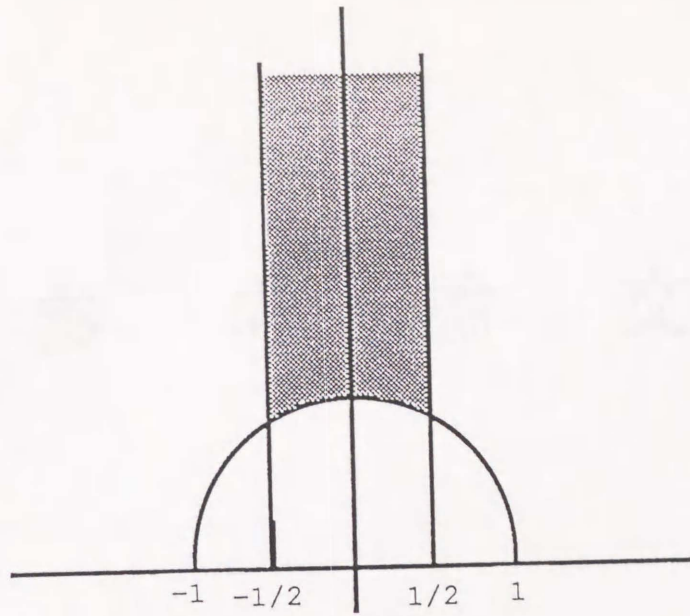


Fig. 1

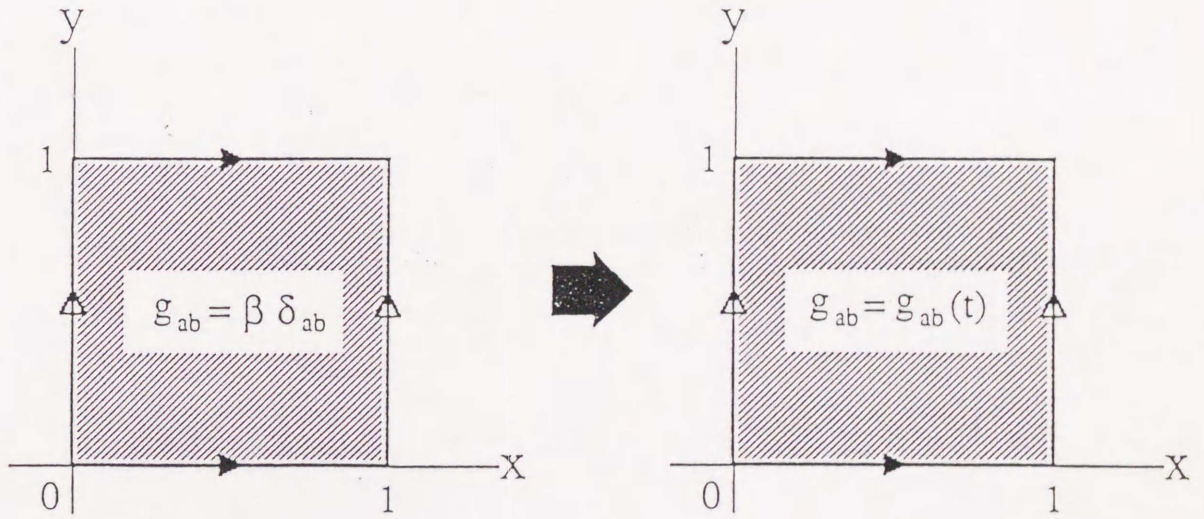


Fig.2