Universal Reversible Cellular Automata in Which Counter Machines Are Concisely Embedded (With Movies)

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Abstract

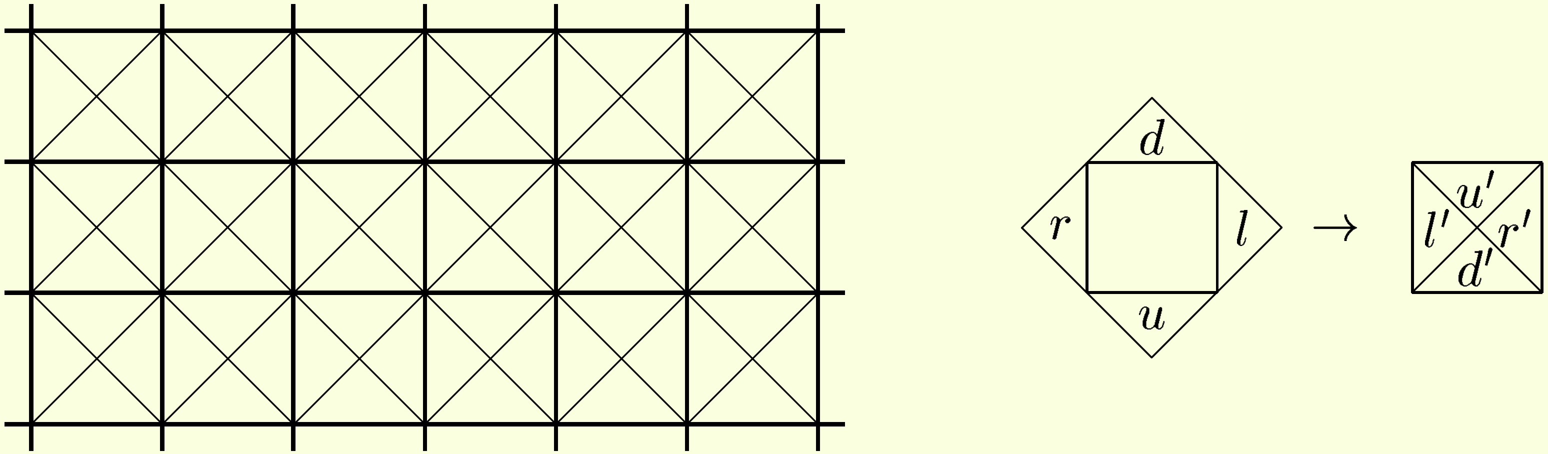
A reversible cellular automaton (RCA) is regarded as a mathematical model for spatiotemporal phenomena having physical reversibility. In spite of the strong constraint of reversibility, even very simple RCAs have universal computing ability. Here, we show two models of RCAs P_3 and P_4 in which any reversible counter machine (RCM) can be embedded in a finite configuration concisely. Since an RCM with two counters is known to be computation-universal, universality of these RCAs are also concluded. We show how computing is performed in these RCAs using examples of movies attached in this slide file.

Reversible Cellular Automaton (RCA)

- It is a CA whose global transition function is one-to-one.
- It is considered as an abstract model for describing spatiotemporal phenomena having physical reversibility.
- It is also studied as a model of reversible computing (see e.g., [Morita, 2008], [Morita, 2011]).
- We use the framework of a partitioned cellular automaton (PCA) to design RCAs.
- A PCA is a subclass of a usual CA.
- It is known that the global transition function of a PCA is one-to-one iff its local transition function is one-to-one.
- Hence, a PCA makes it easy to design an RCA.

Partitioned Cellular Automaton (PCA)

- In a 2-dimensional 4-neighbor PCA, the cellular space, and the form of a local transition function is as below:



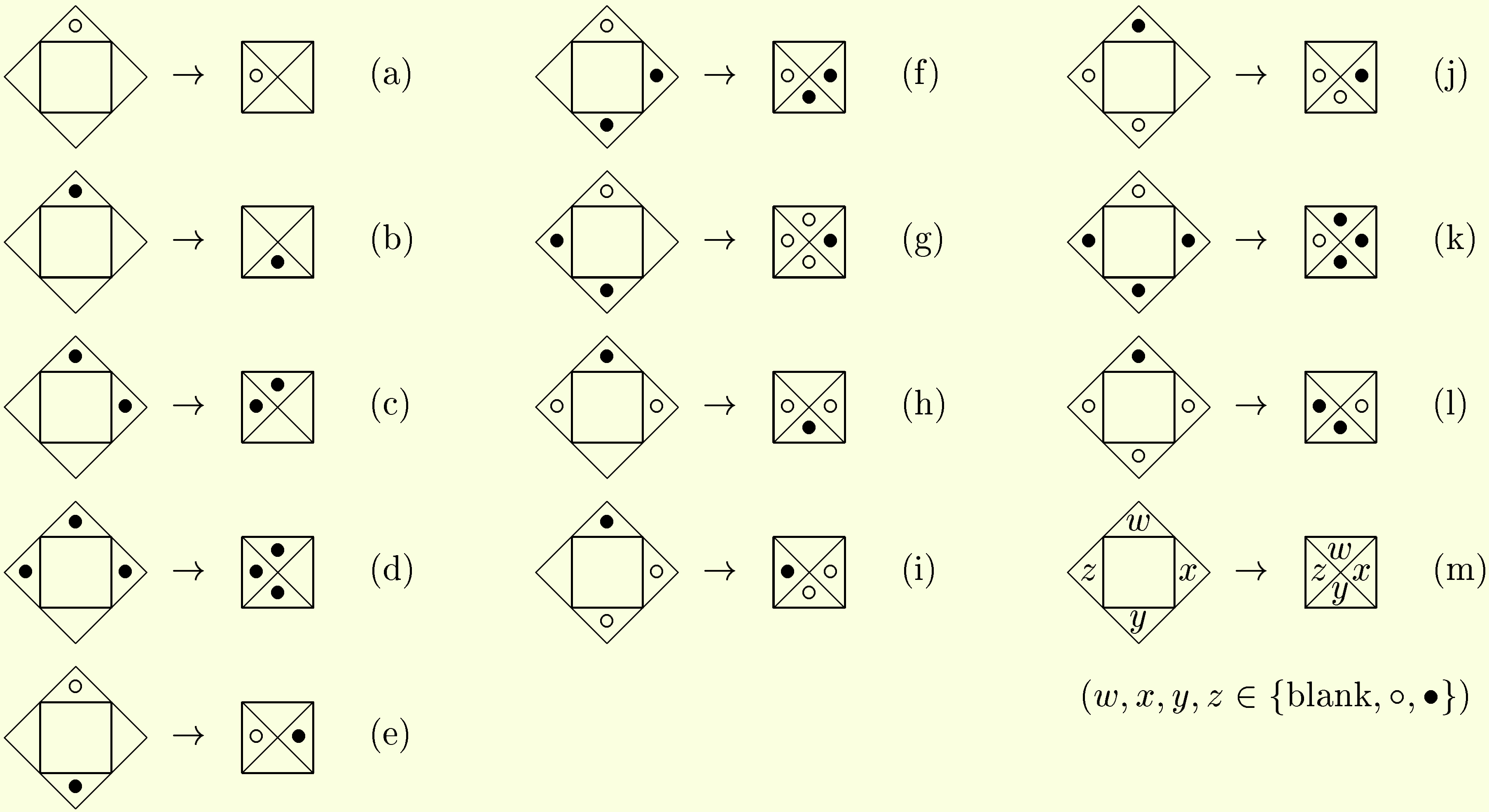
- Each square cell is divided into 4 triangular parts, where each part has its own set of states.
- The local transition function determines the next state (u', r', d', l') of each cell by the tuple (u, r, d, l) of the present states of the nearest parts of the 4 neighboring cells.
- The local function is applied to all the cells synchronously.

The Reversible PCA (RPCA) P_3

- P_3 is an RPCA in which any reversible counter machine can be embedded [Morita, et al., 2002], [Ogiro, Morita, 2002].
- Each part of a cell has the state set {blank, ∘, •}, and hence the cell has $3^4 = 81$ states.
- The local transition function is specified by 13 schemes of local transition rules shown in the next page.
- Here, we assume rotation symmetry, hence each scheme represents 4 rules that are obtained by rotating both sides of the scheme by 90, 180, and 270 degrees.
- There is no pair of distinct rules whose right-hand sides are the same. Therefore the local transition function is one-to-one, and thus P_3 is a reversible CA.

The Local Transition Function of P_3

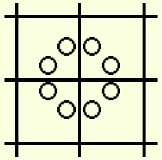
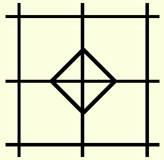
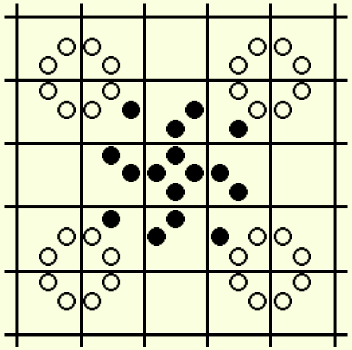
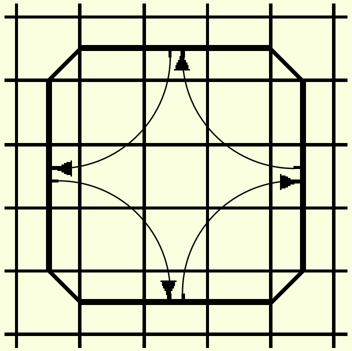
- It is specified by the following rule schemes, where (m) represents 33 rules not specified by (a)–(l).



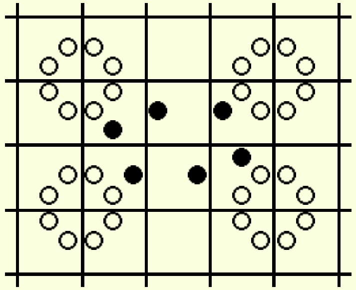
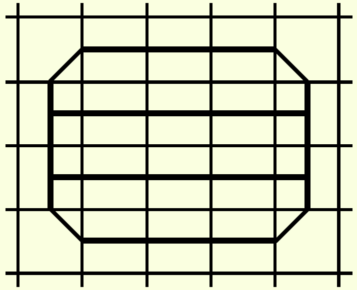
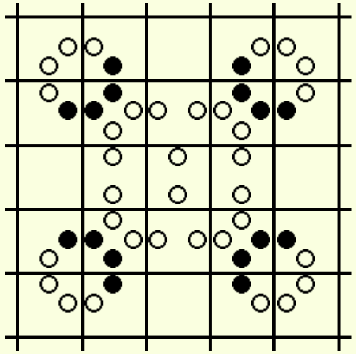
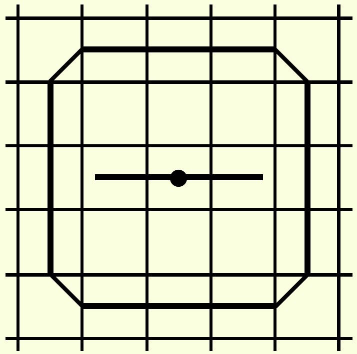
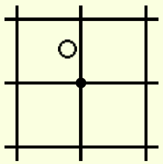
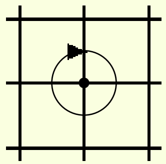
$(w, x, y, z \in \{\text{blank}, \circ, \bullet\})$

Five Basic Elements in P_3 (1)

- We give 5 kinds of elements for processing a signal in P_3.
- They are realized as local patterns of states of cells in the P_3 space.

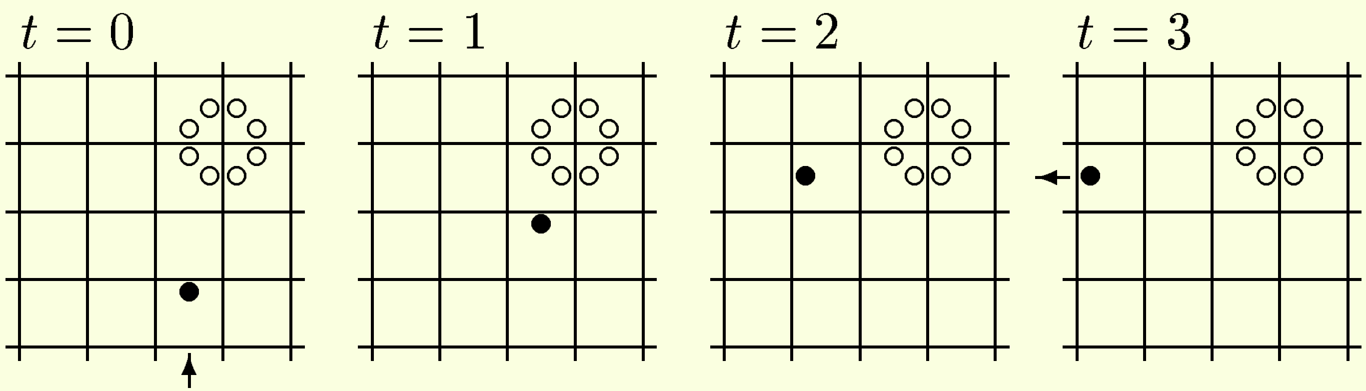
Element name	Pattern	Symbolic notation
LR-turn element		
R-turn element		

Five Basic Elements in P_3 (2)

Element name	Pattern	Symbolic notation
Reflector		
Rotary element		
Position marker		

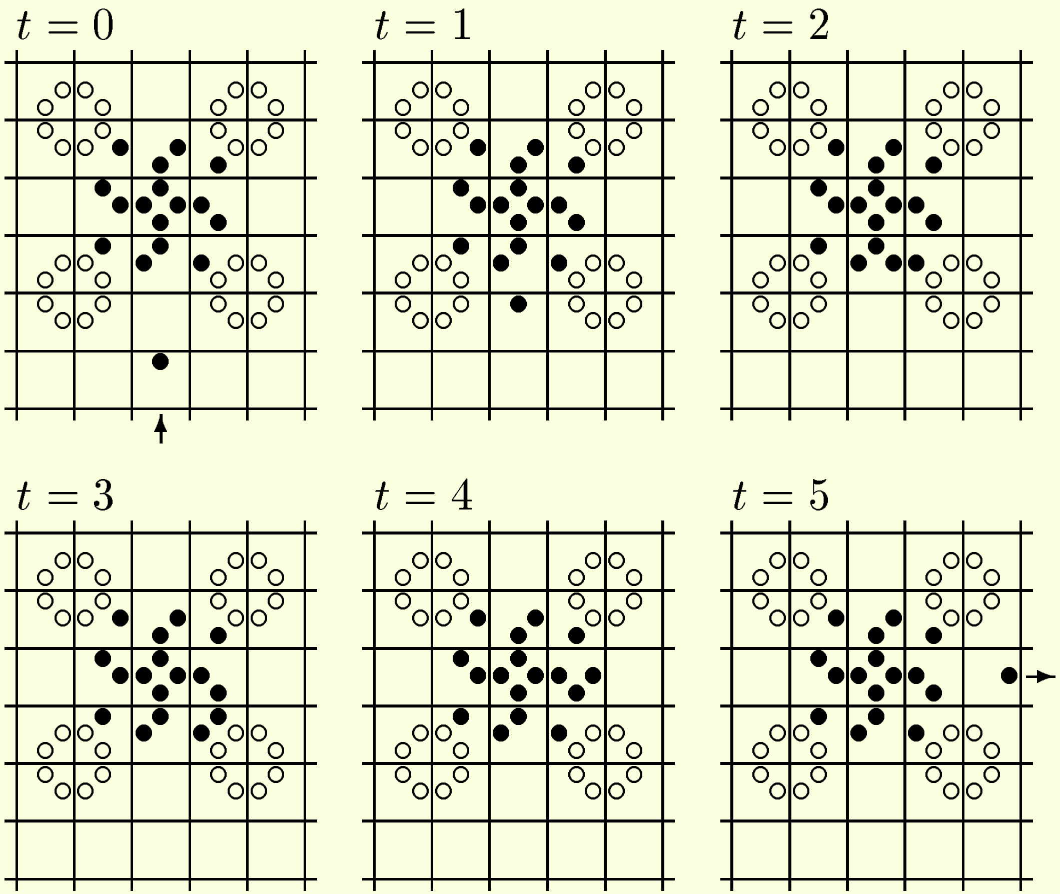
LR-Turn Element in P_3

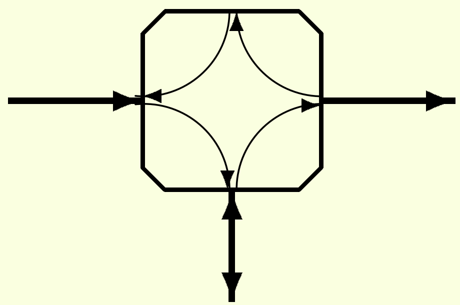
- A signal is realized by a single $\bullet$ in P_3. If no obstacle exists, it goes straight ahead in the P_3 space.
- An LR-turn element is for turning the moving direction of a signal $\bullet$ to the right or left.
- The figure below shows a left turn.



R-Turn Element in P_3

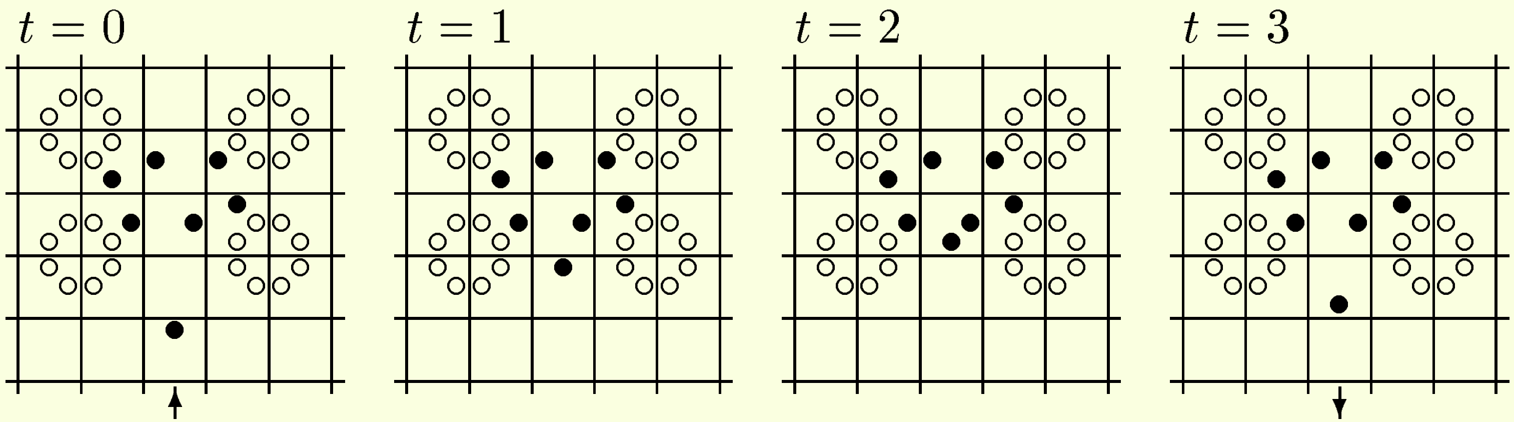
- This element is for turning a signal • to the right. It is necessary for splitting the output and the input from/to a bidirectional signal line as shown in the right figure.





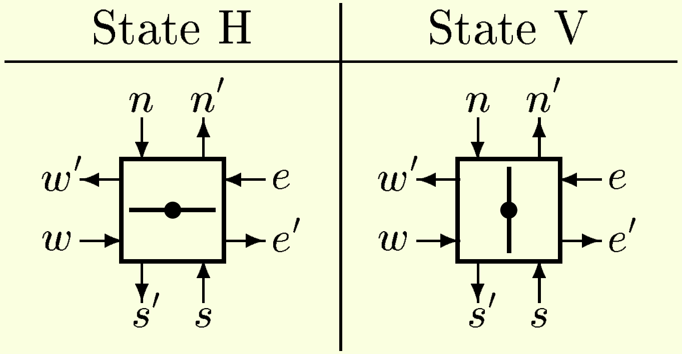
Reflector in P_3

- It reflects a signal • as shown below. It can be simulated by several R-turn elements, but we can reduce the size of a configuration of a counter machine by using reflectors.

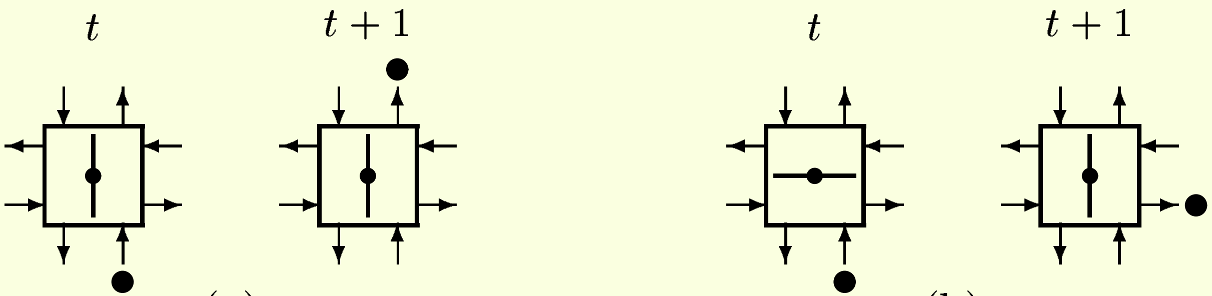


Rotary Element

- It is a universal 2-state reversible logic element with 4 input lines and 4 output lines [Morita, 2001].



- Its operation is very easy to understand. The figure below shows the cases where the directions of the incoming signal and the bar are (a) parallel, and (b) orthogonal.

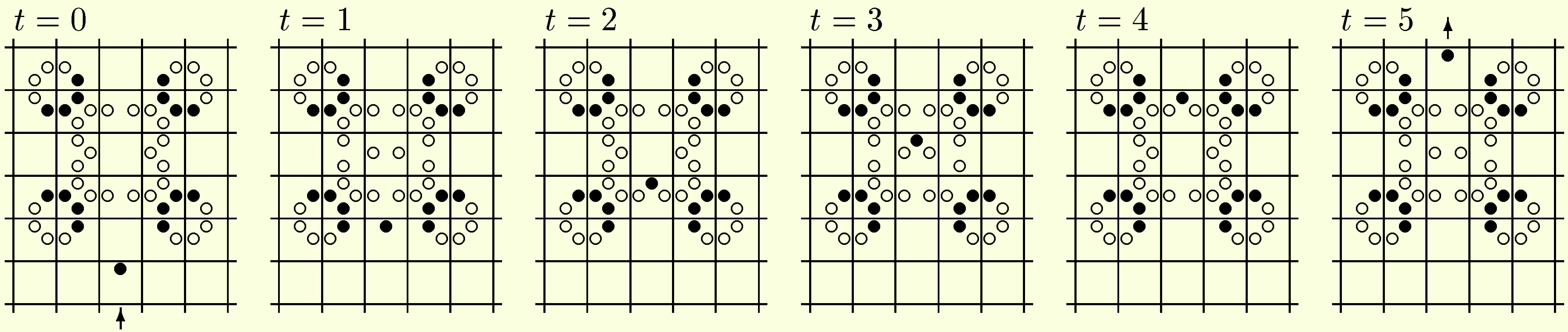


(a) (b)

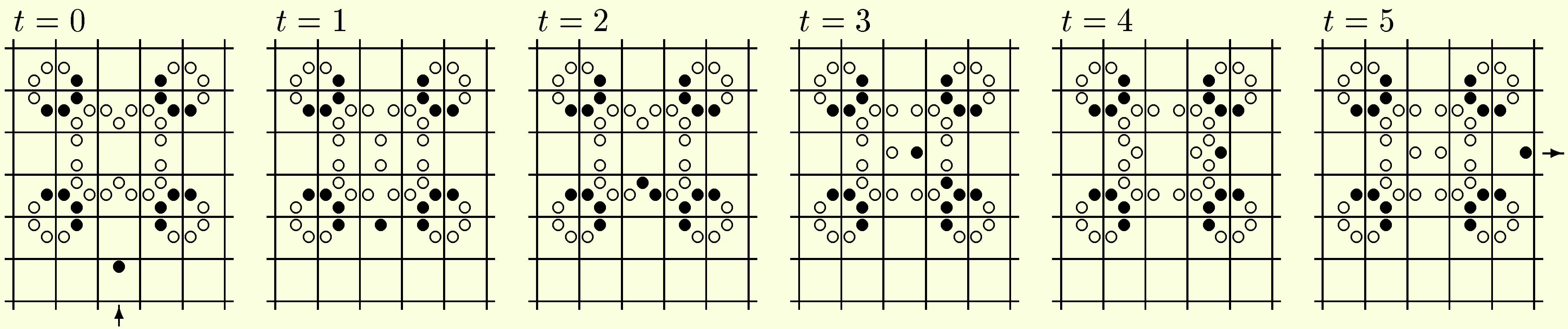
Rotary Element in P_3

- A rotary element is simulated in P_3 as below.

(a) The directions of the signal and the bar are parallel.

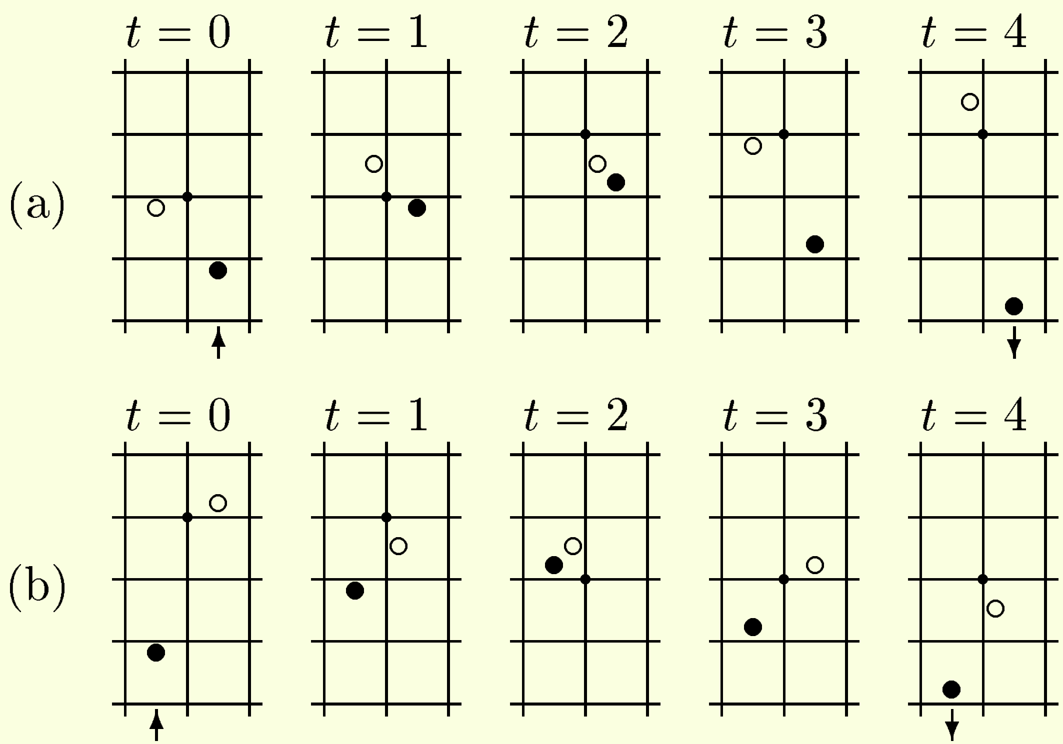


(b) The directions of the signal and the bar are orthogonal.



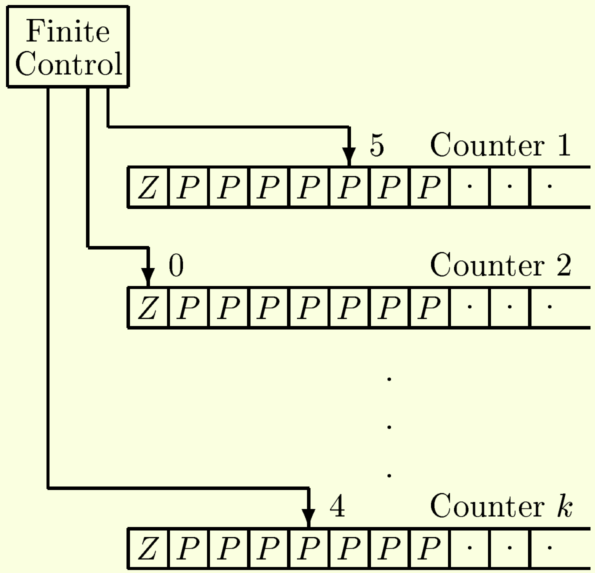
Position Marker in P_3

- A position marker consist of a single $\circ$, which rotates around a certain point if no obstacle exists.
- We can shift the position of a marker by appropriately giving a signal $\bullet$: (a) pushing, and (b) pulling.



Counter Machine (CM)

- It consists of a finite control, and a finite number of counters, each of which can keep a nonnegative integer.
- It is formalized as a multi-tape Turing machine with read-only heads.



- Each head can sense whether it is at the position of zero (Z) or positive (P), and be shifted to the right or left.
- A CM with two counters is universal [Minsky, 1967].

Definition of a CM

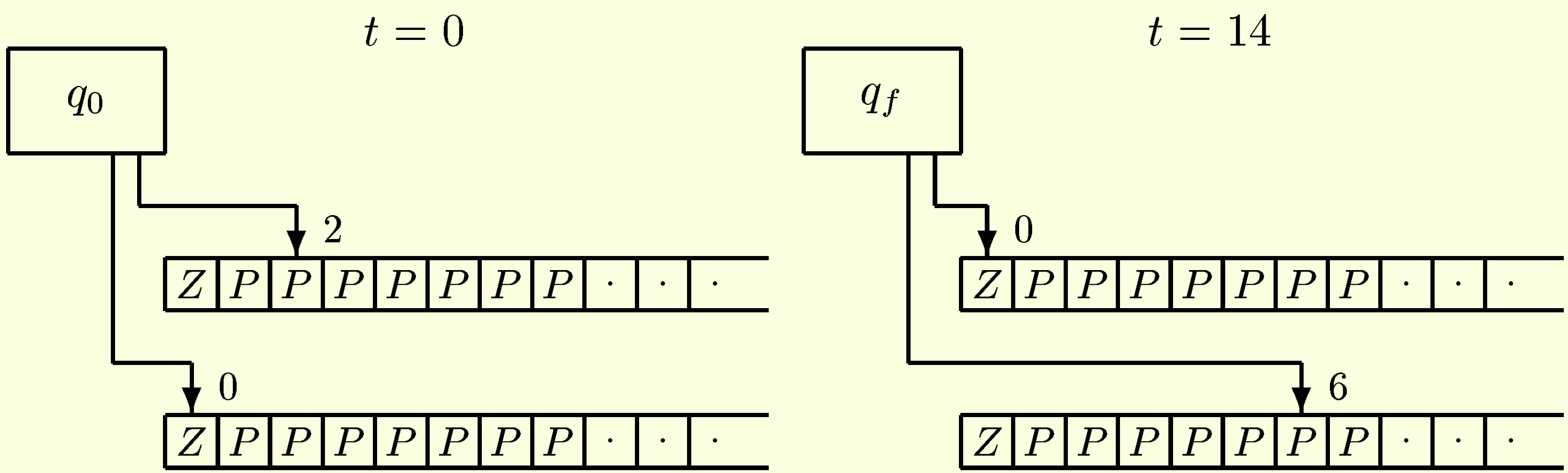
- It is defined by $M = (k, Q, q_0, F, \delta)$, where
  - k is a number of heads,
  - Q is a set of states,
  - $q_0 (\in Q)$ is an initial state,
  - $F (\subset Q)$ is a set of final states, and
  - δ is a move relation.
- δ is a set of quadruples of the form $[p, i, s, q]$ or $[p, i, d, q]$, where $p, q \in Q$, $i \in \{1, \ldots, k\}$, $s \in \{Z, P\}$, and $d \in \{-, 0, +\}$.
- $[p, i, s, q]$ means if the i-th head reads s in the state p, then go to the state q.
- $[p, i, d, q]$ means if M is in the state p, then shift the i-th head to the direction d, and go to the state q.

Determinism and Reversibility of a CM

- A CM $M = (k, Q, q_0, F, \delta)$ is called deterministic iff the following holds.
$$\forall [p,i,x,q],[p',i',x',q'] \in \delta$$
$$[([p,i,x,q] \neq [p',i',x',q'] \ \wedge \ p = p')$$
$$\Rightarrow (i = i' \wedge \ x, x' \in \{Z, P\} \ \wedge \ x \neq x')]$$
- In what follows, we consider only deterministic CMs.
- A CM M is called reversible and denoted by RCM iff the following holds.
$$\forall [p,i,x,q],[p',i',x',q'] \in \delta$$
$$[([p,i,x,q] \neq [p',i',x',q'] \ \wedge \ q = q')$$
$$\Rightarrow (i = i' \wedge \ x, x' \in \{Z, P\} \ \wedge \ x \neq x')]$$
- An RCM with two counters is universal [Morita, 1996].

Example of an RCM M_1 with Two Counters

- Consider $M_1 = (2, Q_1, q_0, \{q_f\}, \delta_1)$.
- $Q_1 = \{q_0, q_1, \ldots, q_5, q_f\}$
- $\delta_1 = \{\ [q_0, 2, Z, q_1],\ [q_1, 2, +, q_2],\ [q_2, 2, +, q_3],\ [q_3, 1, P, q_4],$
  $[q_3, 1, Z, q_f],\ [q_4, 1, -, q_5],\ [q_5, 2, P, q_1]\ \ \}$
- M_1 computes the function $f(n) = 2n+2$, i.e., if an integer n is given to the first counter and starts from the state q_0, then M_1 halts in q_f giving $2n+2$ in the second counter.



Counter Module Embedded in P_3

- A counter module and a finite control module are composed of five basic elements (see [Ogiro, Morita, 2002]).
- A counter module is as below. A non-negative integer is kept by the position markers placed at the right side.

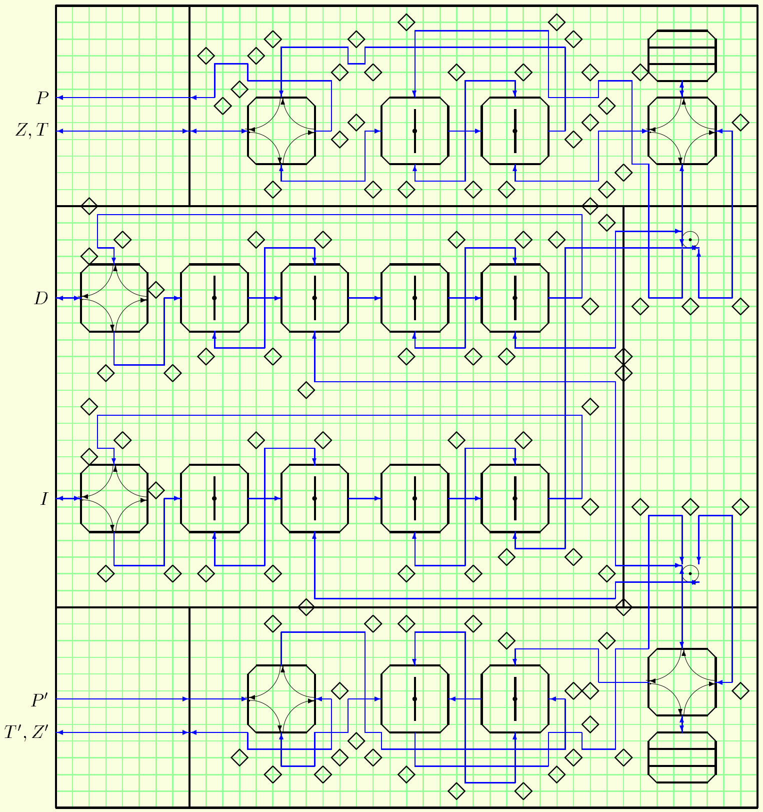
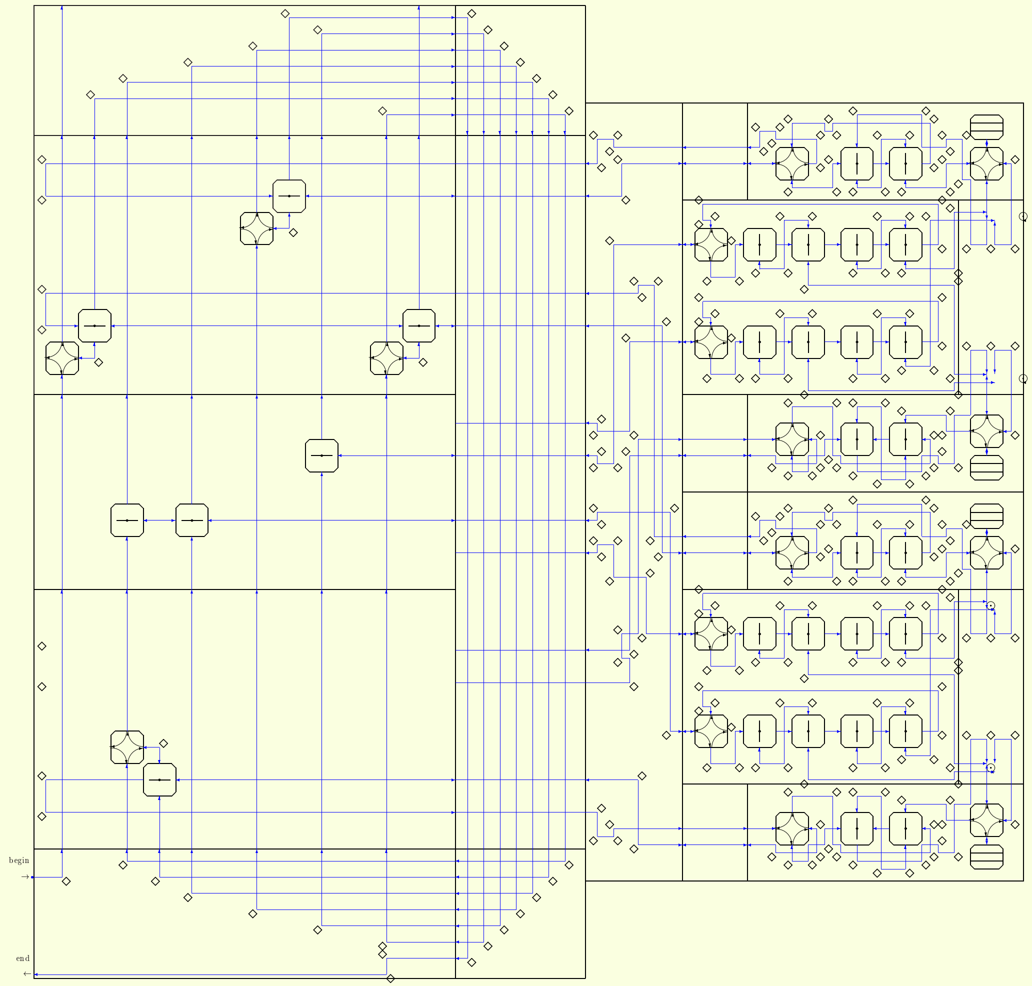
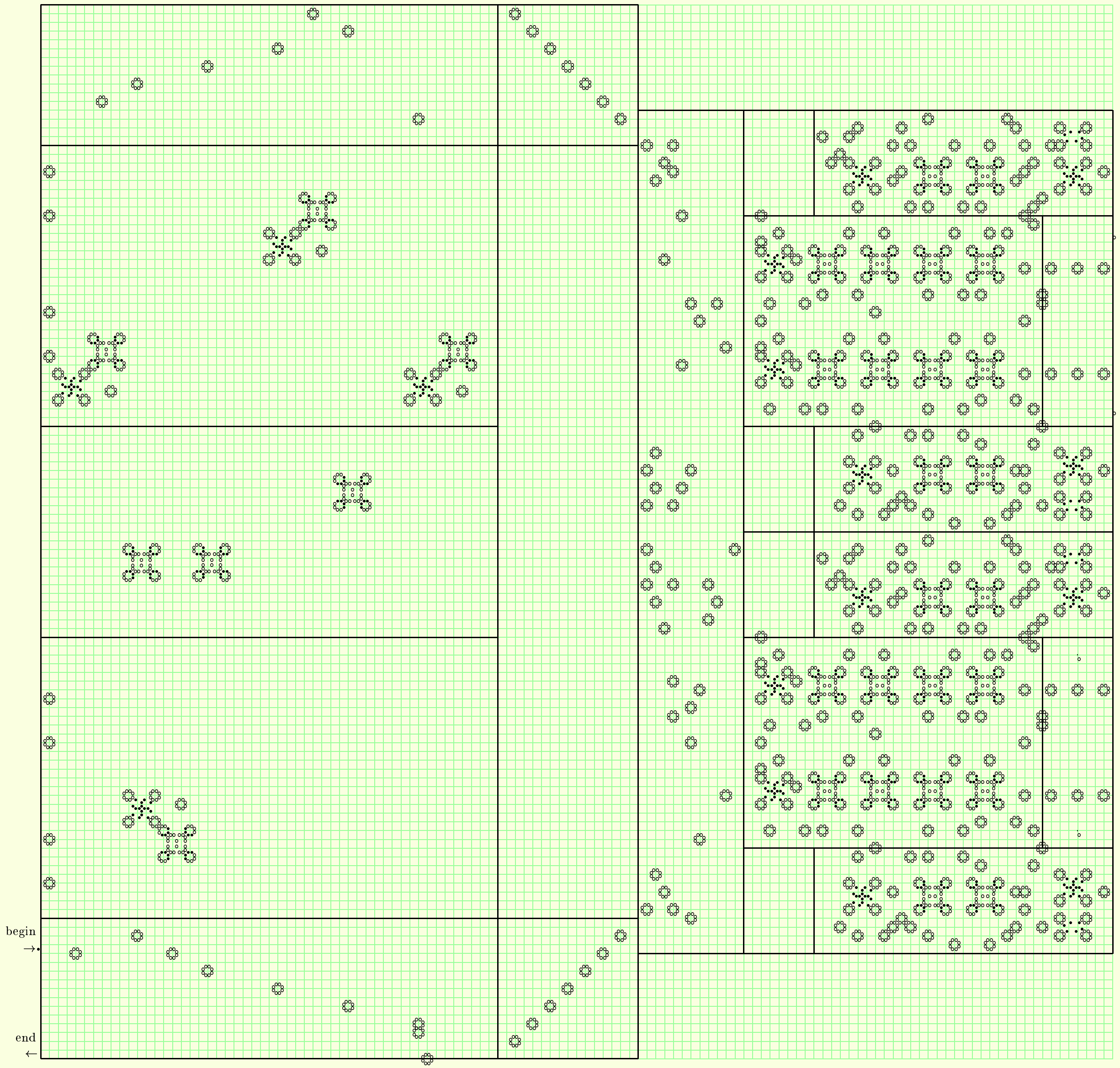


Diagram of the RCM M_1 Embedded in P_3

- A finite control module for M_1 is placed at the left side, and two counter modules are at the right side.

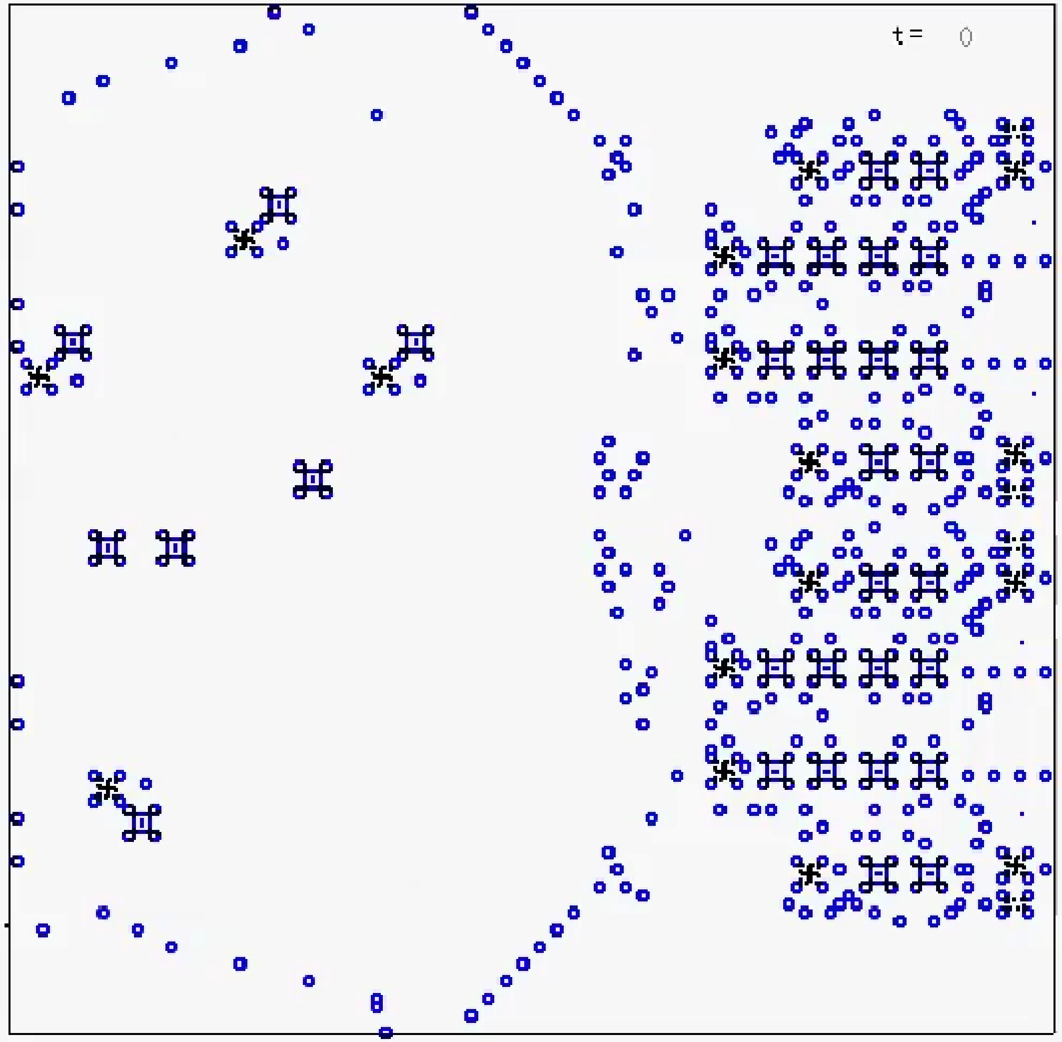


Configuration of P_3 That Realizes M_1



Movie of a Computing Process of M_1 in P_3

(Created by T. Ogiro in 2000)



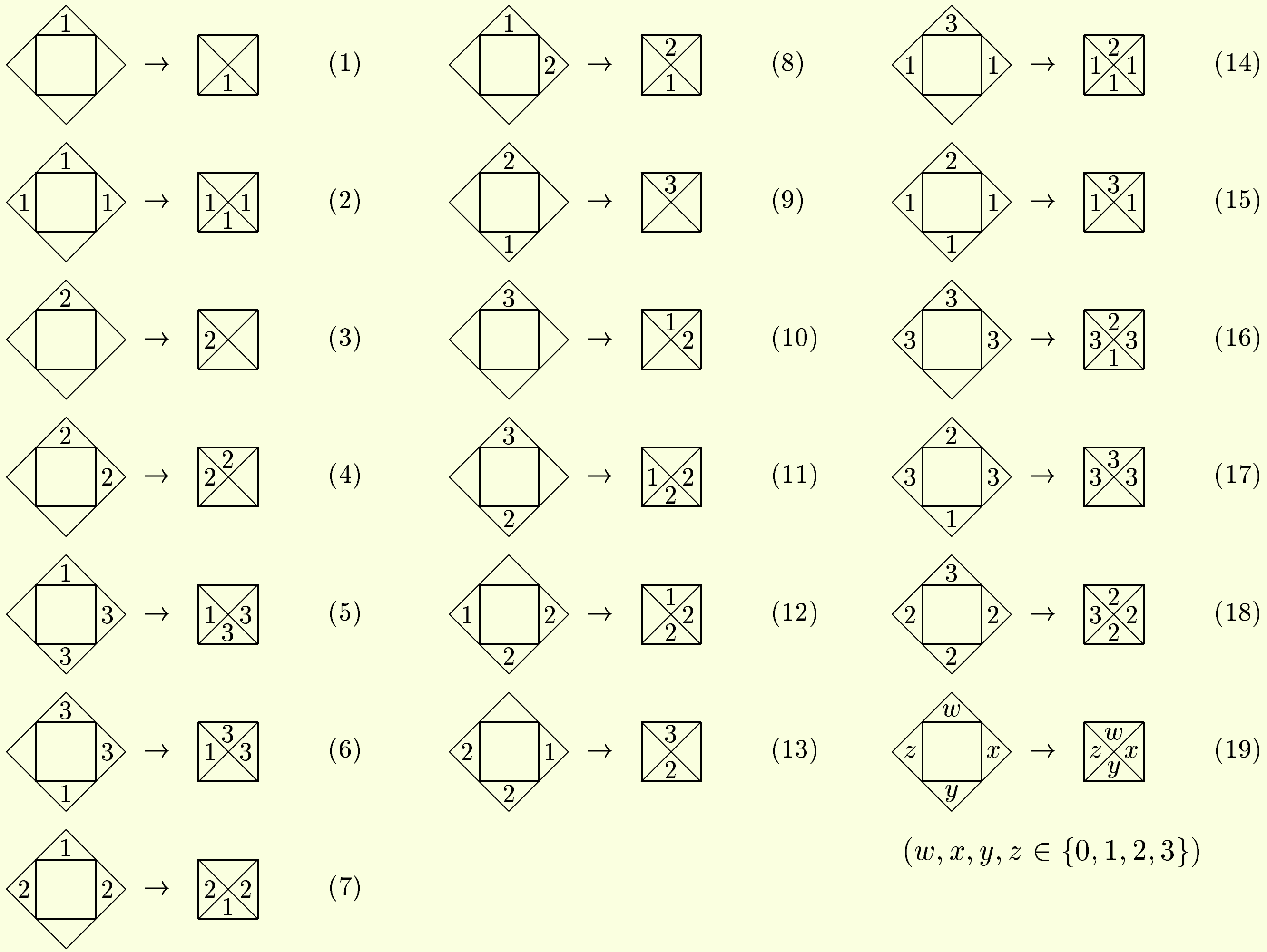
An RCM(2) that computes y = 2x+2

The RPCA P_4

- P_4 is another RPCA that was designed before P_3, and can also embed any RCM in its space [Morita, et al., 2002].
- Each part of a cell has the state set $\{0, 1, 2, 3\}$, and hence the cell has $4^4 = 256$ states.
- Though the number of states of a cell is larger than that of P_3, the size of a configuration realizing an RCM is smaller than in P_3.
- The local transition function is specified by 19 schemes of local transition rules shown in the next page.
- We can give several basic elements in P_4 as in the case of P_3, from which a finite control module and a counter module are constructed.

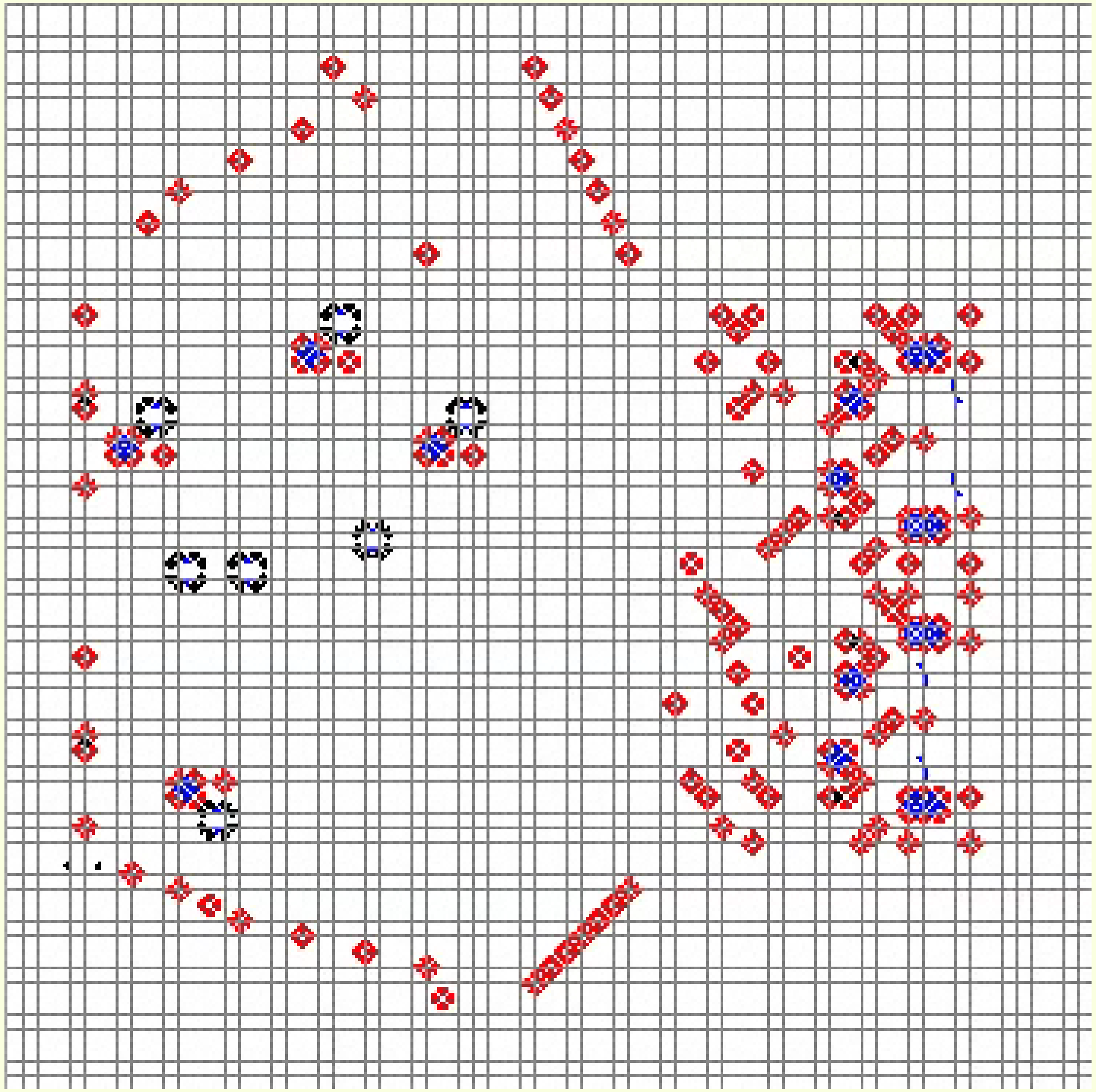
The Local Transition Function of P_4

- It is specified by the following rule schemes, where (19) represents 184 rules not specified by (1)–(18).



Movie of a Computing Process of M_1 in P_4

(Created by Y. Tojima in 1999)



Concluding Remarks

- We gave two simple reversible cellular automata (RCAs) P_3 and P_4.
- Any reversible counter machine can be realized as a finite configuration in P_3 and P_4.
- Hence, they are computationally universal RCAs.

References (1)

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