WEIGHTED ORLICZ-RIESZ CAPACITY OF BALLS

YOSHIHIRO MIZUTA, TAKAO OHNO, AND TETSU SHIMOMURA

(Communicated by Tatiana Toro)

ABSTRACT. Our aim in this paper is to estimate the weighted Orlicz-Riesz capacity of balls.

1. INTRODUCTION AND STATEMENT OF RESULTS

Several versions of capacities for Orlicz-Riesz spaces have appeared in research papers, for example those by Aissaoui and Benkirane [5], Kuznetsov [14], Mizuta [17], Adams and Hurri-Syrjänen [3, 4], and Joensuu [12]. The notion of capacity offers a standard way to characterize exceptional sets and is indispensable to an understanding of the local behavior of functions in Orlicz-Riesz spaces. Various capacity estimates also play an important role in the study of solutions to partial differential equations.

Recently the authors [8] gave an estimate of the Orlicz-Riesz capacity of balls as an extension of Adams and Hurri-Syrjänen [4] and Joensuu [12]. An estimate of the weighted Sobolev capacity of balls can be found e.g. in Heinonen, Kilpeläinen and Martio [9].

A positive measurable function w on $\mathbf{R}^n$ is called an A_p weight (written as $w \in A_p$) if there exists a positive constant C_p such that

$$\left(\frac{w(B)}{|B|}\right) \left(\frac{w^{1/(1-p)}(B)}{|B|}\right)^{p-1} \leq C_p \quad (< \infty)$$

for all balls B, where $1 < p < \infty$, $|\cdot|$ denotes the n-dimensional Lebesgue measure and

$$w(B) = \int_B w(y) \, dy.$$

As an example, we have that the function $w(x) = |x|^\delta$ is an A_p weight if and only if $-n < \delta < n(p-1)$. It is well known that, for an A_p weight w, the corresponding measure w is doubling, that is, $w(2B) \leq cw(B)$ for all balls $B = B(x, r)$; here the constant c depends only on n, p and C_p and $2B$ stands for the enlarged ball $B(x, 2r)$. For these and other fundamental properties of A_p weights, see, for example, Heinonen, Kilpeläinen and Martio [9].

For $0 < \alpha < n$ and a locally integrable function f on $\mathbf{R}^n$, we define the Riesz potential $I_\alpha f$ of order α by

$$I_\alpha f(x) = \int_{\mathbf{R}^n} |x - y|^{\alpha-n} f(y) \ dy.$$

In the present paper, we treat functions f satisfying an Orlicz condition with an A_p weight ω:

$$\text{(1.1)} \qquad \int_{\mathbf{R}^n} \varphi_p(|f(y)|) \omega(y) \ dy < \infty.$$

Here $\varphi_p(r)$ is a positive nondecreasing function on the interval $(0, \infty)$ of the form

$$\varphi_p(r) = r^p \varphi(r),$$

where $p > 1$ and $\varphi(r)$ is a positive quasi-increasing function on $(0, \infty)$ which is of logarithmic type; that is, there exists $c_1 > 0$ such that

$$\text{(φ1)} \qquad c_1^{-1} \varphi(r) \le \varphi(r^2) \le c_1 \varphi(r) \qquad \text{whenever } r > 0.$$

We say that $\varphi(r)$ is quasi-increasing if there exists $c > 0$ such that

$$\varphi(s) \le c\varphi(t) \qquad \text{whenever } 0 < s < t.$$

We set

$$\varphi_p(0) = 0,$$

because we will see from (φ4) below that

$$\lim_{r \to 0+} \varphi_p(r) = 0;$$

see [19, p. 205].

For an open set $G \subset \mathbf{R}^n$, we denote by $L^{\varphi_p, \omega}(G)$ the family of all locally integrable functions f on G such that

$$\int_G \varphi_p(|f(y)|) \omega(y) \ dy < \infty,$$

and define

$$\|f\|_{\varphi_p, \omega, G} = \inf \left\{ \lambda > 0 : \int_G \varphi_p(|f(y)|/\lambda) \omega(y) \ dy \le 1 \right\}.$$

This is a quasi-norm in $L^{\varphi_p, \omega}(G)$. For $E \subset G$, the relative $(\alpha, \varphi_p, \omega)$-capacity is defined by

$$B_{\alpha, \varphi_p, \omega}(E; G) = \inf \int_G \varphi_p(|f(y)|) \omega(y) \ dy,$$

where the infimum is taken over all functions f such that $f = 0$ outside G and

$$I_\alpha f(x) \ge 1 \qquad \text{for all } x \in E$$

(cf. Adams and Hedberg [2], Meyers [15], Ziemer [22] and the first author [16, 17]).

Our first aim in the present note is to give an estimate of the modular capacity $B_{\alpha, \varphi_p, \omega}$ of open balls $B(x, r)$ centered at x of radius r, as an extension of Adams and Hurri-Syrjänen [4, Theorem 2.11], Joensuu [12], Heinonen, Kilpeläinen and Martio [9, Theorems 2.18 and 2.19] and the authors [8, Theorem A]. In fact, our first theorem is stated in the following.

Theorem A (cf. [16, Lemma 7.3]). *Suppose $p > 1$ and $\omega \in A_p$. For $R > 0$, there exists a constant $A > 0$ such that*

$$A^{-1} \left(\int_r^R \{t^{-\alpha p} \varphi(t^{-1}) \omega(B(x,t))\}^{1/(1-p)} \, dt/t \right)^{1-p} \leq B_{\alpha,\varphi_p,\omega}(B(x,r); B(0,R))$$

$$\leq A \left(\int_r^R \{t^{-\alpha p} \varphi(t^{-1}) \omega(B(x,t))\}^{1/(1-p)} \, dt/t \right)^{1-p}$$

whenever $B(x,r) \subset B(0, R/4)$.

We write $f \sim g$ if there exists a constant A so that $A^{-1}g \leq f \leq Ag$.

Example 1.1. Let $\omega(x) = |x|^\delta$ and $\varphi(t) = (\log(e+t))^\beta$.

(1) If $\alpha p - n < \delta < n(p-1)$, then

$$\int_r^R \{t^{-\alpha p} \varphi(t^{-1}) \omega(B(0,t))\}^{1/(1-p)} \, dt/t \sim r^{(-\alpha p + n + \delta)/(1-p)} (\log(e+1/r))^{\beta/(1-p)}$$

for $0 < r < R/2 < 1$. In this case,

$$B_{\alpha,\varphi_p,\omega}(B(0,r); B(0,R)) \sim r^{-\alpha p + n + \delta} (\log(e+1/r))^\beta.$$

(2) If $\alpha p - n = \delta$ and $\beta < p - 1$, then

$$\int_r^R \{t^{-\alpha p} \varphi(t^{-1}) \omega(B(0,t))\}^{1/(1-p)} \, dt/t \sim (\log(e+1/r))^{\beta/(1-p)+1}$$

for $0 < r < R/2 < 1$. In this case,

$$B_{\alpha,\varphi_p,\omega}(B(0,r); B(0,R)) \sim (\log(e+1/r))^{\beta+1-p}.$$

(3) If $\alpha p - n = \delta$ and $\beta = p - 1$, then

$$\int_r^R \{t^{-\alpha p} \varphi(t^{-1}) \omega(B(0,t))\}^{1/(1-p)} \, dt/t \sim \log(e+(\log(e+1/r)))$$

for $0 < r < R/2 < 1$. In this case,

$$B_{\alpha,\varphi_p,\omega}(B(0,r); B(0,R)) \sim (\log(e+(\log(e+1/r))))^{-(p-1)}.$$

(4) If $\alpha p - n > \delta > -n$ or $\alpha p - n = \delta$ and $\beta > p - 1$, then

$$B_{\alpha,\varphi_p,\omega}(\{0\}; B(0,1)) > 0.$$

Next we are concerned with the norm capacity. For $E \subset G$, we define

$$C_{\alpha,\varphi_p,\omega}(E; G) = \inf \|f\|_{\varphi_p,\omega,G},$$

where the infimum is taken over all functions f such that $f = 0$ outside G and

$$I_\alpha f(x) \geq 1 \qquad \text{for all } x \in E.$$

Theorem B. *Suppose $p > 1$ and $\omega \in A_p$. For $R > 0$, there exists a constant $A > 0$ such that*

$$A^{-1} \left(\int_r^R \{t^{-\alpha p} \varphi(t^{-1}) \omega(B(x,t))\}^{1/(1-p)} \, dt/t \right)^{(1-p)/p} \leq C_{\alpha, \varphi_p, \omega}(B(x,r); B(0,R))$$

$$\leq A \left(\int_r^R \{t^{-\alpha p} \varphi(t^{-1}) \omega(B(x,t))\}^{1/(1-p)} \, dt/t \right)^{(1-p)/p}$$

whenever $B(x,r) \subset B(0, R/4)$.

In view of Theorems A and B, we have the following result, which extends the results by Adams and Hurri-Syrjänen [4], Joensuu [12] and the authors [8].

Corollary 1.2. *Suppose $p > 1$ and $\omega \in A_p$. For $R > 0$, there is a constant $A > 0$ such that*

$$A^{-1} B_{\alpha, \varphi_p, \omega}(B(x,r); B(0,R))^{1/p}$$
$$\leq C_{\alpha, \varphi_p, \omega}(B(x,r); B(0,R)) \leq A B_{\alpha, \varphi_p, \omega}(B(x,r); B(0,R))^{1/p}$$

whenever $B(x,r) \subset B(0, R/4)$.

For further related results, we refer the reader to Adams [1], Adams and Hurri-Syrjänen [3], Edmunds and Evans [7], Kilpeläinen [13] and Mizuta and Shimomura [19, 20, 21].

Throughout this paper, let A denote various constants independent of the variables in question and let $A(a, b, \cdots)$ be a constant that depends on $a, b, \cdots$.

2. Proof of Theorem A

First we collect properties which follow from condition $(\varphi 1)$ (see [17], [19, Lemma 2.3], [18, Section 7]).

$(\varphi 2)$ φ satisfies the doubling condition; that is, there exists $c_2 > 1$ such that

$$c_2^{-1} \varphi(r) \leq \varphi(2r) \leq c_2 \varphi(r) \qquad \text{whenever } r > 0.$$

$(\varphi 3)$ For each $\gamma > 0$, there exists $c_3 = c_3(\gamma) \geq 1$ such that

$$c_3^{-1} \varphi(r) \leq \varphi(r^\gamma) \leq c_3 \varphi(r) \qquad \text{whenever } r > 0.$$

$(\varphi 4)$ For each $\gamma > 0$, there exists $c_4 = c_4(\gamma) \geq 1$ such that

$$s^\gamma \varphi(s) \leq c_4 t^\gamma \varphi(t) \qquad \text{whenever } 0 < s < t.$$

$(\varphi 5)$ For each $\gamma > 0$, there exists $c_5 = c_5(\gamma) \geq 1$ such that

$$t^{-\gamma} \varphi(t) \leq c_5 s^{-\gamma} \varphi(s) \qquad \text{whenever } 0 < s < t.$$

$(\varphi 6)$ If φ and ψ are positive monotone functions on $[0, \infty)$ satisfying $(\varphi 1)$, then for each $\gamma > 0$, there exists a constant $c_6 = c_6(\gamma) \geq 1$ such that

$$c_6^{-1} \varphi(r) \leq \varphi(r^\gamma \psi(r)) \leq c_6 \varphi(r) \qquad \text{whenever } r > 0.$$

Let us begin with an upper estimate for the modular $B_{\alpha, \varphi_p, \omega}$-capacity of balls.

Theorem 2.1. *Suppose $p > 1$ and ω is a positive locally integrable function on $\mathbf{R}^n$. Then there exists a constant $A > 0$ such that*

$$B_{\alpha,\varphi_p,\omega}(B(0,r); B(0,R)) \le A \left(\int_{2r}^{2R} \{t^{-\alpha p}\varphi(t^{-1})\omega(B(0,t))\}^{1/(1-p)} \, dt/t \right)^{1-p}$$

whenever $0 < r < R/2 < \infty$.

Proof. For $r > 0$, consider the function

$$f_r(y) = |y|^{-\alpha}$$

for $r < |y| < 2r$ and $f_r = 0$ elsewhere. If $x \in B(0,r)$ and $y \in B(0,2r) \setminus B(0,r)$, then $|x - y| < 3r$, so that

$$I_\alpha f_r(x) \ge (3r)^{\alpha-n} \int_{B(0,2r) \setminus B(0,r)} |y|^{-\alpha} \, dy = A_1$$

with a constant $A_1 = A_1(\alpha, n) > 0$.

Now let $0 < r < R/2$, and take j_0 such that $2^{j_0+1}r \le R < 2^{j_0+2}r$. For $\{a_j\}$ such that $a_j \ge 0$ and $\sum_{j=0}^{j_0} a_j = 1$, set

$$f = \sum_{j=0}^{j_0} a_j f_{2^j r}/A_1.$$

Then

$$I_\alpha f(x) = A_1^{-1} \sum_{j=0}^{j_0} a_j I_\alpha f_{2^j r}(x) \ge 1$$

for $x \in B(0,r)$. Therefore we have by $(\varphi 2)$ and $(\varphi 3)$ that

$$
\begin{aligned}
B_{\alpha,\varphi_p,\omega}(B(0,r); B(0,R)) &\le \int_{B(0,R)} \varphi_p(f(y))\omega(y) \, dy \\
&= \sum_{j=0}^{j_0} \int_{B(0,R)} [a_j f_{2^j r}(y)/A_1]^p \varphi(a_j f_{2^j r}(y)/A_1)\omega(y) \, dy \\
&\le A_2 \sum_{j=0}^{j_0} a_j^p (2^j r)^{-\alpha p} \varphi((2^j r)^{-1})\omega(B(0, 2^{j+1}r)).
\end{aligned}
$$

Now, letting $K = \sum_{j=0}^{j_0} \{(2^j r)^{-\alpha p}\varphi((2^j r)^{-1})\omega(B(0, 2^{j+1}r))\}^{1/(1-p)}$ and

$$a_j = \frac{\{(2^j r)^{-\alpha p}\varphi((2^j r)^{-1})\omega(B(0, 2^{j+1}r))\}^{1/(1-p)}}{K},$$

we find

$$
\begin{aligned}
B_{\alpha,\varphi_p,\omega}(B(0,r); B(0,R)) &\le A_2 K^{-p} \sum_{j=0}^{j_0} \{(2^j r)^{-\alpha p}\varphi((2^j r)^{-1})\omega(B(0, 2^{j+1}r))\}^{1/(1-p)} \\
&= A_2 K^{1-p}.
\end{aligned}
$$

Here we have

$$
\begin{aligned}
K &\geq A_3 \sum_{j=0}^{j_0} \int_{2^{j+1}r}^{2^{j+3}r} \{t^{-\alpha p}\varphi(t^{-1})\omega(B(0,t))\}^{1/(1-p)} \, dt/t \\
&\geq A_3 \int_{2r}^{2^{j_0+3}r} \{t^{-\alpha p}\varphi(t^{-1})\omega(B(0,t))\}^{1/(1-p)} \, dt/t,
\end{aligned}
$$

so that

$$
K \geq A_3 \int_{2r}^{2R} \{t^{-\alpha p}\varphi(t^{-1})\omega(B(0,t))\}^{1/(1-p)} \, dt/t,
$$

which proves the result. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Corollary 2.2. *If ω is doubling, then there exists a constant $A > 0$ such that*

$$
B_{\alpha,\varphi_p,\omega}(B(x,r); B(0,R)) \leq A \left(\int_r^R \{t^{-\alpha p}\varphi(t^{-1})\omega(B(x,t))\}^{1/(1-p)} \, dt/t \right)^{1-p}
$$

whenever $B(x,r) \subset B(0,R/4)$. In fact,

$$
\begin{aligned}
B_{\alpha,\varphi_p,\omega}(B(x,r); B(0,R)) &\leq B_{\alpha,\varphi_p,\omega}(B(x,r); B(x,R/2)) \\
&\leq A \left(\int_r^R \{t^{-\alpha p}\varphi(t^{-1})\omega(B(x,t))\}^{1/(1-p)} \, dt/t \right)^{1-p}
\end{aligned}
$$

whenever $B(x,r) \subset B(0,R/4)$.

Next we give a lower estimate for the modular $B_{\alpha,\varphi_p,\omega}$-capacity of balls.

Theorem 2.3. *Suppose $p > 1$ and $\omega \in A_p$. For $R > 0$, there exists a constant $A = A(R) > 0$ such that*

$$
B_{\alpha,\varphi_p,\omega}(B(0,r); B(0,R)) \geq A \left(\int_r^R \{t^{-\alpha p}\varphi(t^{-1})\omega(B(0,t))\}^{1/(1-p)} \, dt/t \right)^{1-p}
$$

whenever $0 < r < R/2 < \infty$.

Proof. For $0 < r < R/2$, take a nonnegative measurable function f on $B(0,R)$ such that

$$
I_\alpha f(x) \geq 1 \qquad \text{for } x \in B(0,r).
$$

Then we have by Fubini's theorem

$$
\begin{aligned}
\int_{B(0,r)} dx &\leq \int_{B(0,r)} I_\alpha f(x) \, dx \\
&\leq \int_{B(0,R)} \left(\int_{B(0,r)} |x-y|^{\alpha-n} \, dx \right) f(y) \, dy \\
&\leq A_1 r^n \int_{B(0,R)} (r+|y|)^{\alpha-n} f(y) \, dy.
\end{aligned}
$$

For $\varepsilon > 0$ and $0 < \delta < \alpha$, we see from Hölder's inequality that

$$\int_{B(0,R)} (r + |y|)^{\alpha-n} f(y) \, dy$$

$$= \int_{\{y \in B(0,R): f(y) > \varepsilon(r+|y|)^{-\delta}\}} (r + |y|)^{\alpha-n} f(y) \, dy$$

$$+ \int_{\{y \in B(0,R): f(y) \leq \varepsilon(r+|y|)^{-\delta}\}} (r + |y|)^{\alpha-n} f(y) \, dy$$

$$\leq \left(\int_{B(0,R)} [(r + |y|)^{\alpha-n} \{\varphi(\varepsilon(r + |y|)^{-\delta})\omega(y)\}^{-1/p}]^{p'} \, dy \right)^{1/p'}$$

$$\times \left(\int_{B(0,R)} \varphi_p(f(y))\omega(y) \, dy \right)^{1/p} + \varepsilon \int_{B(0,R)} (r + |y|)^{\alpha-n-\delta} dy,$$

where $1/p + 1/p' = 1$. Since $\omega \in A_p$, we find by $(\varphi 2)$ and $(\varphi 3)$ that

$$\int_{B(0,R)} [(r + |y|)^{\alpha-n} \{\varphi(\varepsilon(r + |y|)^{-\delta})\omega(y)\}^{-1/p}]^{p'} \, dy$$

$$\leq A_2(\varepsilon) \int_0^R (r + t)^{(\alpha-n)p/(p-1)} \varphi((r + t)^{-1})^{1/(1-p)} \left(\int_{B(0,t)} \omega(y)^{1/(1-p)} \, dy \right) \frac{dt}{t}$$

$$\leq A_3(\varepsilon) \int_0^R (r + t)^{\alpha p/(p-1)} \varphi((r + t)^{-1})^{1/(1-p)} \left(\int_{B(0,r+t)} \omega(y) \, dy \right)^{1/(1-p)} \frac{dt}{t}$$

$$\leq A_3(\varepsilon) \int_r^{2R} \{t^{-\alpha p} \varphi(t^{-1})\omega(B(0,t))\}^{1/(1-p)} \, dt/t$$

$$\leq A_4(\varepsilon) \int_r^R \{t^{-\alpha p} \varphi(t^{-1})\omega(B(0,t))\}^{1/(1-p)} \, dt/t.$$

Thus we derive

$$\int_{B(0,R)} (r + |y|)^{\alpha-n} f(y) \, dy$$

$$\leq A_4(\varepsilon) \left(\int_r^R \{t^{-\alpha p} \varphi(t^{-1})\omega(B(0,t))\}^{1/(1-p)} \, dt/t \right)^{1/p'}$$

$$\times \left(\int_{B(0,R)} \varphi_p(f(y))\omega(y) \, dy \right)^{1/p} + A_5\varepsilon,$$

so that

$$1 \leq A_6(\varepsilon) \left(\int_r^R \{t^{-\alpha p} \varphi(t^{-1})\omega(B(0,t))\}^{1/(1-p)} \, dt/t \right)^{1/p'}$$

$$\times \left(\int_{B(0,R)} \varphi_p(f(y))\omega(y) \, dy \right)^{1/p} + A_7\varepsilon.$$

If $A_7\varepsilon = 1/2$, then we establish

$$B_{\alpha,\varphi_p,\omega}(B(0,r); B(0,R)) \geq A \left(\int_r^R \{t^{-\alpha p}\varphi(t^{-1})\omega(B(0,t))\}^{1/(1-p)} \, dt/t \right)^{1-p},$$

as required. $\square$

Corollary 2.4. *Suppose $p > 1$ and $\omega \in A_p$. For $R > 0$, there exists a constant $A = A(R) > 0$ such that*

$$B_{\alpha,\varphi_p,\omega}(B(x,r); B(0,R)) \geq A \left(\int_r^R \{t^{-\alpha p}\varphi(t^{-1})\omega(B(x,t))\}^{1/(1-p)} \, dt/t \right)^{1-p}$$

whenever $B(x,r) \subset B(0,R/4)$.

Now Theorem A follows from Corollaries 2.2 and 2.4.

3. Proof of Theorem B

Let us begin with an upper estimate for the norm $C_{\alpha,\varphi_p,\omega}$-capacity of balls.

Theorem 3.1. *Suppose that $p > 1$ and that $\omega(B(0,r))$ satisfies the doubling condition. Then there exists a constant $A > 0$ such that*

$$C_{\alpha,\varphi_p,\omega}(B(0,r); B(0,R)) \leq A \left(\int_r^R \{t^{-\alpha p}\varphi(t^{-1})\omega(B(0,t))\}^{1/(1-p)} \, dt/t \right)^{(1-p)/p}$$

whenever $0 < r < R/2 < \infty$.

Proof. Set

$$\varphi^*(r) = \int_r^R \{t^{-\alpha p}\varphi(t^{-1})\omega(B(0,t))\}^{1/(1-p)} \, dt/t$$

for $r > 0$. Consider the function

$$f(y) = |y|^{-\alpha}\{|y|^{-\alpha p}\varphi(|y|^{-1})\omega(B(0,|y|))\}^{1/(1-p)}\varphi^*(r)^{-1/p}$$

for $r < |y| < R$ and $f = 0$ elsewhere. If $x \in B(0,r)$ and $y \in B(0,R) \setminus B(0,r)$, then $|x - y| < 2|y|$, so that

$$I_\alpha f(x) \geq 2^{\alpha-n}\varphi^*(r)^{-1/p} \int_{B(0,R)\setminus B(0,r)} |y|^{-n}\{|y|^{-\alpha p}\varphi(|y|^{-1})\omega(B(0,|y|))\}^{1/(1-p)} \, dy$$

$$= A_1\varphi^*(r)^{(p-1)/p}$$

with a constant $A_1 = A_1(\alpha, n) > 0$. It follows from the definition of capacity that

$$C_{\alpha,\varphi_p,\omega}(B(0,r); B(0,R)) \leq A_1^{-1}\varphi^*(r)^{(1-p)/p}\|f\|_{\varphi_p,\omega,B(0,R)}.$$

Thus it suffices to show that

$$\|f\|_{\varphi_p,\omega,B(0,R)} \leq A_2.$$

For this purpose, we first note that

$$\int_{B(0,R)} \varphi_p(f(y))\omega(y)dy$$

$$= \int_{B(0,R)\backslash B(0,r)} f(y)^p \varphi(f(y))\omega(y)dy$$

$$= \varphi^*(r)^{-1} \int_{B(0,R)\backslash B(0,r)} |y|^{-\alpha p}\{|y|^{-\alpha p}\varphi(|y|^{-1})\omega(B(0,|y|))\}^{p/(1-p)}$$

$$\times \varphi\left(|y|^{-\alpha}\{|y|^{-\alpha p}\varphi(|y|^{-1})\omega(B(0,|y|))\}^{1/(1-p)}\varphi^*(r)^{-1/p}\right)\omega(y)dy.$$

Here we see from the doubling condition of $\omega(B(0,r))$ that

$$|y|^{-\alpha}\{|y|^{-\alpha p}\varphi(|y|^{-1})\omega(B(0,|y|))\}^{1/(1-p)}\varphi^*(r)^{-1/p}$$

$$\leq A_3|y|^{-\alpha}\{|y|^{-\alpha p}\varphi(|y|^{-1})\omega(B(0,|y|))\}^{1/(1-p)}$$

$$\left(\int_r^{2R}\{t^{-\alpha p}\varphi(t^{-1})\omega(B(0,t))\}^{1/(1-p)}\ dt/t\right)^{-1/p}$$

$$\leq A_3|y|^{-\alpha}\{|y|^{-\alpha p}\varphi(|y|^{-1})\omega(B(0,|y|))\}^{1/(1-p)}$$

$$\left(\int_{|y|}^{2|y|}\{t^{-\alpha p}\varphi(t^{-1})\omega(B(0,t))\}^{1/(1-p)}\ dt/t\right)^{-1/p}$$

$$\leq A_4\varphi(|y|^{-1})^{-1/p}\omega(B(0,|y|))^{-1/p}$$

for $y \in B(0,R) \backslash B(0,r)$. Further we can find constants $A_0 > 0$ and $\gamma > 0$ such that

$$\omega(B(0,t)) \geq A_0 t^\gamma$$

for all $t > 0$. Hence, as in Theorem 2.1, we obtain by $(\varphi 6)$:

$$\int_{B(0,R)} \varphi_p(f(y))\omega(y)dy$$

$$\leq A_5\varphi^*(r)^{-1}\int_{B(0,R)\backslash B(0,r)} |y|^{-\alpha p}\{|y|^{-\alpha p}\varphi(|y|^{-1})\omega(B(0,|y|))\}^{p/(1-p)}\varphi(|y|^{-1})\omega(y)dy$$

$$\leq A_6\varphi^*(r)^{-1}\int_r^R\{t^{-\alpha p}\varphi(t^{-1})\omega(B(0,t))\}^{1/(1-p)}dt/t$$

$$= A_6,$$

as required. $\square$

Corollary 3.2. *Suppose that $p > 1$ and ω is doubling. Then there exists a constant $A > 0$ such that*

$$C_{\alpha,\varphi_p,\omega}(B(x,r); B(0,R)) \leq A\left(\int_r^R\{t^{-\alpha p}\varphi(t^{-1})\omega(B(x,t))\}^{1/(1-p)}\ dt/t\right)^{(1-p)/p}$$

whenever $B(x,r) \subset B(0,R/4)$.

Next we give a lower estimate for the norm $C_{\alpha,\varphi_p,\omega}$-capacity of balls.

Theorem 3.3. *Suppose $p > 1$ and $\omega \in A_p$. For $R > 0$, there exists a constant $A = A(R) > 0$ such that*

$$C_{\alpha,\varphi_p,\omega}(B(0,r); B(0,R)) \geq A \left(\int_r^R \{t^{-\alpha p} \varphi(t^{-1}) \omega(B(0,t))\}^{1/(1-p)} \ dt/t \right)^{(1-p)/p}$$

whenever $0 < r < R/2 < \infty$.

Proof. For $0 < r < R/2$ take a nonnegative measurable function f on $B(0,R)$ such that

$$I_\alpha f(x) \geq 1 \qquad \text{for } x \in B(0,r).$$

Then we have by Fubini's theorem

$$
\begin{aligned}
\int_{B(0,r)} dx \quad &\leq \quad \int_{B(0,r)} I_\alpha f(x) \ dx \\
&\leq \quad \int_{B(0,R)} \left(\int_{B(0,r)} |x-y|^{\alpha-n} \ dx \right) f(y) \ dy \\
&\leq \quad A_1 r^n \int_{B(0,R)} (r+|y|)^{\alpha-n} f(y) \ dy,
\end{aligned}
$$

so that

$$1 \leq A_1 \int_{B(0,R)} (r+|y|)^{\alpha-n} f(y) \ dy.$$

We show that

$$\int_{B(0,R)} (r+|y|)^{\alpha-n} f(y) \ dy$$

$$\leq A_2 \left(\int_r^R \{t^{-\alpha p} \varphi(t^{-1}) \omega(B(0,t))\}^{1/(1-p)} \ dt/t \right)^{(p-1)/p} \|f\|_{\varphi_p,\omega,B(0,R)}.$$

For this purpose, suppose that $\|f\|_{\varphi_p,\omega,B(0,R)} \leq 1$. For $0 < \delta < \alpha$, we see from Hölder's inequality that

$$\int_{B(0,R)} (r+|y|)^{\alpha-n} f(y) \ dy$$

$$= \int_{\{y \in B(0,R): f(y) > (r+|y|)^{-\delta}\}} (r+|y|)^{\alpha-n} f(y) \ dy$$

$$+ \int_{\{y \in B(0,R): f(y) \leq (r+|y|)^{-\delta}\}} (r+|y|)^{\alpha-n} f(y) \ dy$$

$$\leq \left(\int_{B(0,R)} [(r+|y|)^{\alpha-n} \{\varphi((r+|y|)^{-\delta}) \omega(y)\}^{-1/p}]^{p'} dy \right)^{1/p'}$$

$$\times \left(\int_{B(0,R)} \varphi_p(f(y)) \omega(y) \ dy \right)^{1/p} + \int_{B(0,R)} (r+|y|)^{\alpha-n-\delta} dy.$$

Here note from $\omega \in A_p$, $(\varphi 2)$ and $(\varphi 3)$ that

$$\int_{B(0,R)} [(r+|y|)^{\alpha-n} \{\varphi((r+|y|)^{-\delta})\omega(y)\}^{-1/p}]^{p'} dy$$

$$\leq A_3 \int_0^R (r+t)^{(\alpha-n)p/(p-1)} \varphi((r+t)^{-1})^{1/(1-p)} \left(\int_{B(0,t)} \omega(y)^{1/(1-p)} dy\right) \frac{dt}{t}$$

$$\leq A_4 \int_0^R (r+t)^{\alpha p/(p-1)} \varphi((r+t)^{-1})^{1/(1-p)} \left(\int_{B(0,r+t)} \omega(y) dy\right)^{1/(1-p)} \frac{dt}{t}$$

$$\leq A_4 \int_r^{2R} \{t^{-\alpha p}\varphi(t^{-1})\omega(B(0,t))\}^{1/(1-p)} dt/t$$

$$\leq A_5 \int_r^R \{t^{-\alpha p}\varphi(t^{-1})\omega(B(0,t))\}^{1/(1-p)} dt/t,$$

so that

$$\int_{B(0,R)} (r+|y|)^{\alpha-n} f(y) dy$$

$$\leq A_5 \left(\int_r^R \{t^{-\alpha p}\varphi(t^{-1})\omega(B(0,t))\}^{1/(1-p)} dt/t\right)^{1/p'}$$

$$\times \left(\int_{B(0,R)} \varphi_p(f(y))\omega(y) dy\right)^{1/p} + A_6$$

$$\leq A_7 \left(\int_r^R \{t^{-\alpha p}\varphi(t^{-1})\omega(B(0,t))\}^{1/(1-p)} dt/t\right)^{1/p'}.$$

Hence we establish

$$C_{\alpha,\varphi_p,\omega}(B(0,r); B(0,R)) \geq A \left(\int_r^R \{t^{-\alpha p}\varphi(t^{-1})\omega(B(0,t))\}^{1/(1-p)} dt/t\right)^{(1-p)/p},$$

as required. □

Corollary 3.4. *Suppose $p > 1$ and $\omega \in A_p$. For $R > 0$, there exists a constant $A = A(R) > 0$ such that*

$$C_{\alpha,\varphi_p,\omega}(B(x,r); B(0,R)) \geq A \left(\int_r^R \{t^{-\alpha p}\varphi(t^{-1})\omega(B(x,t))\}^{1/(1-p)} dt/t\right)^{(1-p)/p}$$

whenever $B(x,r) \subset B(0,R/4)$.

As in the proof of Theorem A, Theorem B follows readily from Corollaries 3.2 and 3.4.

References

[1] D. R. Adams, Weighted capacity and the Choquet integral, Proc. Amer. Math. Soc. **102** (1988), 879-887. MR934860 (89h:31007)

[2] D. R. Adams and L. I. Hedberg, Function spaces and potential theory, Springer-Verlag, Berlin-Heidelberg-New York, 1996. MR1411441 (97j:46024)

[3] D. R. Adams and R. Hurri-Syrjänen, Vanishing exponential integrability for functions whose gradients belong to $L^n(\log(e + L))^\alpha$, J. Funct. Anal. **197** (2003), 162-178. MR1957679 (2004b:46035)

[4] D. R. Adams and R. Hurri-Syrjänen, Capacity estimates, Proc. Amer. Math. Soc. **131** (2003), 1159-1167. MR1948107 (2003k:46034)

[5] N. Aissaoui and A. Benkirane, Capacités dans les espaces d'Orlicz, Ann. Sci. Math. Québec **18** (1994), 1-23. MR1273865 (95h:31009)

[6] C. Bennett and R. Sharpley, Interpolation of Operators, Academic Press, Inc., New York, 1988. MR928802 (89e:46001)

[7] D. E. Edmunds and W. D. Evans, Hardy operators, function spaces and embeddings, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2004. MR2091115 (2005g:46068)

[8] T. Futamura, Y. Mizuta, T. Ohno and T. Shimomura, Orlicz-Sobolev capacity of balls, to appear in Illinois J. Math.

[9] J. Heinonen, T. Kilpeläinen and O. Martio, Nonlinear potential theory of degenerate elliptic equations, Clarendon Press, 1993. MR1207810 (94e:31003)

[10] J. Joensuu, On null sets of Sobolev-Orlicz capacities, Illinois J. Math. **52** (2008), 1195-1211. MR2595762

[11] J. Joensuu, Orlicz-Sobolev capacities and their null sets, Rev. Mat. Complut. **23** (2010), 217-232. MR2578579

[12] J. Joensuu, Estimates for certain Orlicz-Sobolev capacities of an Euclidean ball, Nonlinear Anal. **72** (2010), 4316-4330.

[13] T. Kilpeläinen, Weighted Sobolev spaces and capacity, Ann. Acad. Sci. Fenn. Ser. A. I. Math. **19** (1994), 95-113. MR1246890 (95h:46054)

[14] S. E. Kuznetsov, Removable singularities for $Lu = \Psi(u)$ and Orlicz capacities, J. Funct. Anal. **170** (2000), 428-449. MR1740658 (2001g:35078)

[15] N. G. Meyers, A theory of capacities for potentials in Lebesgue classes, Math. Scand. **8** (1970), 255-292. MR0277741 (43:3474)

[16] Y. Mizuta, Continuity properties of potentials and Beppo-Levi-Deny functions, Hiroshima Math. J. **23**, 79-153 (1993). MR1211771 (94d:31005)

[17] Y. Mizuta, Potential theory in Euclidean spaces, Gakkōtosyo, Tokyo, 1996. MR1428685 (98e:31001)

[18] Y. Mizuta, E. Nakai, T. Ohno and T. Shimomura, Sobolev embeddings for Riesz potentials of functions in Morrey spaces $L^{(1,\varphi)}(G)$, to appear in J. Math. Soc. Japan.

[19] Y. Mizuta and T. Shimomura, Differentiability and Hölder continuity of Riesz potentials of functions in Orlicz classes, Analysis **20** (2000), 201-223. MR1778254 (2001h:46052)

[20] Y. Mizuta and T. Shimomura, Vanishing exponential integrability for Riesz potentials of functions in Orlicz classes, Illinois J. Math. **51** (2007), 1039-1060. MR2417414 (2009d:31014)

[21] Y. Mizuta and T. Shimomura, Continuity properties of Riesz potentials of Orlicz functions, Tohoku Math. J. **61** (2009), 225-240. MR2541407

[22] W. P. Ziemer, Weakly differentiable functions, Springer-Verlag, New York, 1989. MR1014685 (91e:46046)

DEPARTMENT OF MATHEMATICS, GRADUATE SCHOOL OF SCIENCE, HIROSHIMA UNIVERSITY, HIGASHI-HIROSHIMA 739-8521, JAPAN
  E-mail address: yomizuta@hiroshima-u.ac.jp

GENERAL ARTS, HIROSHIMA NATIONAL COLLEGE OF MARITIME TECHNOLOGY, HIGASHINO OOSAKIKAMIJIMA TOYOTAGUN 725-0231, JAPAN
  E-mail address: ohno@hiroshima-cmt.ac.jp
  Current address: Department of Mathematics, Faculty of Education, Oita University, Dannohara Oita 870-1192, Japan

DEPARTMENT OF MATHEMATICS, GRADUATE SCHOOL OF EDUCATION, HIROSHIMA UNIVERSITY, HIGASHI-HIROSHIMA 739-8524, JAPAN
  E-mail address: tshimo@hiroshima-u.ac.jp