Sensitivity analysis incorporating fuzzy evaluation for scaling constants of multiattribute utility functions

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Abstract: A multiattribute utility function can be represented by a function of single-attribute utility functions if the decision maker's preference satisfies additive independence or mutually utility independence. Additive independence is a preference condition stronger than mutually utility independence, and the multiattribute utility function is in the additive form if the former condition is satisfied, otherwise it is in the multiplicative form. In this paper, we propose a method for sensitivity analysis of multiattribute utility functions in multiplicative form, taking into account the imprecision of the decision maker's judgment in the procedures for determining scaling constants (attribute weights).

Keywords: Multiattribute utility function, sensitivity analysis, scaling constant (attribute weight).

1 Introduction

Multiattribute utility analysis is utilized for resolving and assessing real-world decision making problems with several alternatives. Keeney and Raiffa (1976) give comprehensive theoretical basis and deal with some applications of multiattribute utility analysis in their seminal book. Multiattribute utility analysis has been applied for evaluation of development policies of the municipal government (Bana E Costa, 1988), management of nuclear waste from power plants (Keeney and von Winterfeldt, 1994), economic analysis of South Korea (Seo *et al.*, 1999), and so forth. Evaluation of scaling constants (attribute weights) through tradeoff experiments between attributes is a crucial procedure as well as specification of single-attribute utility functions in utility analysis. In the tradeoff experiments for the evaluation of scaling constants, a decision maker gives indifferent points in a plane of two attributes and the assessment must be proceeded carefully such as to be consistent with his or her preference. Even if a certain alternative is recommended after appropriate and prudent assessments, it is desirable that the alternative is justified through sufficient sensitivity analysis.

Methodologies on sensitivity analysis for scaling constants of additive multiattribute value models are reported by Barron and Schmidt (1988) and Ringuest (1997). Barron and Schmidt present an entropy-based procedure and a least squares procedure for obtaining scaling constants of the additive multiattribute value function sufficient to equate or reverse by a certain amount the values of two alternatives. One of the two alternatives maximizes the value of the additive multiattribute value function with the scaling constants initially assessed by the decision maker, and the other is any of nondominated alternatives. They claim that the initially assessed scaling constants are sensitive if the distance between the two sets of scaling constants is short. Ringuest generalizes the Barron and Schmidt method by using the L_p metric instead of the least squares metric and gives another sensitivity criterion that the initially assessed scaling constants are insensitive if rank reversals with respect to the scaling constants are required to yield a different preferred nondominated alternative.

Wolters and Mareschal (1995) propose a method for minimal modification of scaling constants in order that a certain alternative is ranked first in a sense of sum of absolute differences between the initially assessed scaling constants and the modified scaling constants for the PROMETHEE method (Brans and Vincke, 1985; Brans *et al.*, 1986), which gives ranking of alternatives. They note that this method can be applied to the additive multiattribute models.

The method of sensitivity analysis by Butler *et al.* (1997) provides three classes of simulation methods: random weights, rank order weights, and response distribution weights. By using techniques of Monte Carlo simulation, all of scaling constants are varied simultaneously in the method. Scaling constants are generated completely at random in the procedure of the random weight while the procedure of the rank order weights preserves a rank-order of importance. In the procedure of the response distribution weights, scaling constants are randomly generated from a hypothetical response distribution. This method is implemented in the decision support system by Jiménez *et al.* (2003).

Wei *et al.* (2000), in the additive multiattribute model, give a parameter of a differential amount of value between preference information on two alternatives, and examine the structure of the scaling constants with the parameter while keeping the preference order of the decision maker on alternatives. Jiménez *et al.* (2003) claim that it is difficult for decision makers to precisely assess subjective probabilities. From this viewpoint, such probabilities can be specified as intervals in the decision support system developed by them and then the obtained utility values are represented also as intervals.

As we reviewed above, researches of sensitivity analysis on scaling constants have been done mostly for the additive multiattribute models, but few researches investigate sensitivity analysis for the multiplicative multiattribute models. Because the additive independence condition must be satisfied to employ the additive multiattribute models (Keeney and Raiffa, 1976), development of sensitivity analysis for the multiplicative multiattribute models, which require that the mutually independence condition weaker than the additive independence condition is satisfied, has been desired. In this paper, we present methods for sensitivity analysis on scaling constants of multiattribute utility functions in the multiplicative form and propose methods taking into account judgments of a decision maker in the procedures for determining scaling constants from the viewpoint that it is difficult for the decision maker to precisely assess subjective probabilities and/or indifferent points.

In section 2, we briefly review multiattribute utility analysis and its procedure, and give a motive for the proposed method of sensitivity analysis. In section 3, after showing the conventional procedures for sensitivity analysis in additive multiattribute models, we present a corresponding procedure for multiplicative multiattribute models and propose a method of sensitivity analysis incorporating fuzziness in tradeoff experiments. Section 4 is devoted to illustrating the proposed method with two numerical examples.

2 Multiattribute utility analysis

To make a rational decision, it is necessary to express the preference of a decision maker quantitatively, and multiattribute utility analysis is devised so as to make it possible. Multiattribute utility analysis is effective in resolving decision making problems with multiple criteria or attributes in which the decision maker selects the most preferable alternative out of multiple discrete alternatives.

Consider n attributes: $X_1,\ldots,X_n$. Let x_i be a certain value of attribute X_i, and $\boldsymbol{x}=(x_1,\ldots,x_n)$ an n-dimensional vector of the attribute values. If the additive or the mutually independence condition is satisfied or can be assumed, the multiattribute utility function is represented as $U(x_1,\ldots,x_n)=f(u_1(x_1),\ldots,u_n(x_n))$ and can be specified by identifying f and $u_1,\ldots,u_n$.

If the mutually independence condition is satisfied, the multiattribute utility function is represented in multiplicative form as (Keeney and Raiffa, 1976):

$$1+KU(\boldsymbol{x})=\prod_{i=1}^{n}[1+Kk_iu_i(x_i)], \tag{1}$$

where k_i, $i=1,\dots,n$ are scaling constants for n attributes satisfying $0\le k_i\le 1$, $i=1,\dots,n$ and $\sum_{i=1}^{n}k_i\neq 1$. K is an additional scaling constant satisfying

$$1+K=\prod_{i=1}^{n}[1+Kk_i]. \tag{2}$$

If $\sum_{i=1}^{n}k_i<1$, then $K>0$, and if $\sum_{i=1}^{n}k_i>1$, then $-1<K<0$.

If the additive independence condition which is stranger than the mutually independence condition is satisfied, the multiattribute utility function is represented in additive form as

$$U(\boldsymbol{x})=\sum_{i=1}^{n}k_iu_i(x_i). \tag{3}$$

The additive multiattribute utility function is a special case of the multiplicative one satisfying $\sum_{i=1}^{n}k_i=1$. A procedure of multiattribute utility analysis is summarized as follows (Keeney and Raiffa, 1976):

Step 1 Enumerate all of the attributes $X_1,\dots,X_n$ which are measures quantitatively representing degrees of attainments of objectives in a decision making problem. For the effectiveness of multiattribute utility analysis, an objective hierarchy is constructed in multilevel if necessary.

Step 2 If the mutually independence condition is satisfied or can be assumed, the decision maker is asked to specify single-attribute utility functions u_i, $i=1,\dots,n$. If the preference of the decision maker conforms to one of several types of utility function arranged in advance, elicit information on the preference from the decision maker to determine parameters of the corresponding utility function.

Step 3 To obtain the multiattribute utility function in multiplicative form (1) or in additive form (3), it is necessary to determine the scaling constants k_i,

$i = 1, \ldots, n$ and K. The scaling constants k_i, $i = 1, \ldots, n$ are determined through the indifference experiments consisting of the following two questions for examining the tradeoff of the preference between attributes, and the additional scaling constant K is computed from the equation (2).

Question 1 Let x_i^0 and x_i^* be the worst value and the best value of the attribute X_i, respectively. What is the value of the probability p with which a certain event $(x_s^*, \boldsymbol{x}_{-s}^0) = (x_1^0, \ldots, x_{s-1}^0, x_s^*, x_{s+1}^0, \ldots, x_n^0)$ and a lottery $<\boldsymbol{x}^*, p, \boldsymbol{x}^0>$ come to be indifferent to each other? The certain event $(x_s^*, \boldsymbol{x}_{-s}^0)$ means that the attribute x_s takes its best value and all the other attributes $\boldsymbol{x}_{-s}^0$ take their worst values, and the superscripts 0 and $*$ indicate the worst and the best values, respectively. The lottery $<\boldsymbol{x}^*, p, \boldsymbol{x}^0>$ means that all the attributes take their best values with the probability p or their worst values with the probability $1 - p$, alternatively.

Question 2 Let the values x_i of all the attributes X_i except for the two attributes X_s and X_j be fixed to certain levels $x_i = x_i'$, $i \neq s, j$. Then what are the values x_s and x_j of the attributes X_s and X_j with which a certain event $(x_s, x_j^0; x_i', i \neq s, j)$ and another certain event $(x_s^0, x_j; x_i', i \neq s, j)$ come to be indifferent to each other? It is possible to fix the value x_j of the attribute X_j to $x_j = x_j^*$, and then answer only the value x_s of the attribute X_s.

Step 4 After the multiattribute utility function is identified through the procedures of Step 1 to Step 3, for a given set of the values x_i, $i = 1, \ldots, n$ of the attributes X_i, $i = 1, \ldots, n$, the value of the multiattribute utility function $U(\boldsymbol{x})$ and the values of the single-attribute utility functions $u_i(x_i)$, $i = 1, \ldots, n$ can be obtained.

It is noteworthy that the scaling constant k_s is determined by specifying the value p of Question 1 in Step 3 because one finds that $k_s u_s(x_s^*) = k_s = p$ from

$U(x_s^*, \boldsymbol{x}_{-s}^0) = pU(\boldsymbol{x}^*) + (1-p)U(\boldsymbol{x}^0)$, and the other scaling constants k_j are determined by specifying the values x_s of Question 2 because one finds that $k_j = k_s u_s(x_s)$ from $U(x_s^0, x_j^*; x_i', i \neq s, j) = U(x_s, x_j^0; x_i', i \neq s, j)$ in case $x_j = x_j^*$.

Because it is necessary to carefully and repeatedly examine the procedures of Step 1 to Step 4 and such assessment can be carried out effectively by use of computers, various computer systems for supporting multiattribute utility analysis have been developed (Sicherman, 1975; Sakawa and Seo, 1982; von Nitzsch and Weber, 1988; Seo *et al.*, 1999; Seo and Nishizaki, 1997; Jiménez *et al.*, 2003; Seo *et al.*, 2004).

It seems difficult for the decision maker to answer the questions in Step 3 precisely (Nishizaki and Seo, 1994). By performing sensitivity analysis, such difficulty may be allayed and it is expected to obtain more reliable results and some insights into the problem.

3 Sensitivity analysis of scaling constants

After reviewing the procedures by Barron and Schmidt (1988) and Ringuest (1997) for sensitivity analysis in additive multiattribute models, we present a procedure for sensitivity analysis in multiplicative multiattribute models.

3.1 Additive multiattribute models

Barron and Schmidt consider the sensitivity of a solution or a decision in an additive multiattribute model as follows. Assume that the scaling constants k_i^1, $i = 1, \ldots, n$ are assessed by the decision maker and the alternative $\boldsymbol{x}^1 = (x_1^1, \ldots, x_n^1)$ has the maximal value of the multiattribute value function. It follows that the solution to the problem is to select the alternative $\boldsymbol{x}^1$. Then, the solution is sensitive to the choice of the scaling constants if a similar set of scaling constants yields a different preferred nondominated alternative, say $\boldsymbol{x}^2 = (x_1^2, \ldots, x_n^2)$. Namely, if a small change of the scaling constants leads to a reversal of value (utility), the solution based on a set of the scaling constants $(k_1^1, \ldots, k_n^1)$ assessed by the decision maker is said to be sensitive to the choice of the scaling constants. The following linear programming problem for finding a set of the scaling constants close to the original set $(k_1^1, \ldots, k_n^1)$ of the scaling constants in a sense of the least squares metric.

$$\left.\begin{array}{ll} \text{minimize} & \sum_{i=1}^{n} (k_i - k_i^1)^2 \\ \text{subject to} & \sum_{i=1}^{n} k_i u_i(x_i^2) - \sum_{i=1}^{n} k_i u_i(x_i^1) = \Delta \\ & \sum_{i=1}^{n} k_i = 1 \\ & k_i \geq 0,\ i = 1, \ldots, n \end{array}\right\} \tag{4}$$

The objective function of problem (4) can be regarded as one of the L_p metric $L_p = \left[\sum_{i=1}^{n} (\pi_i |k_i - k_i^1|)^p\right]^{\frac{1}{p}}$ (Ringuest, 1997), where $p \in \{1, \ldots, \infty\}$ is a parameter of the metric and $\pi_i \geq 0$ is a weight for the attribute X_i.

Ringuest (1997) gives a different idea of sensitivity based on the rank reversals. Assume that a set of the scaling constants $(k_1^1, \ldots, k_n^1)$ is assessed by the decision maker, it satisfies $k_i^1 > k_j^1$, and the alternative $\boldsymbol{x}^1 = (x_1^1, \ldots, x_n^1)$ has the maximal value of the multiattribute value function. He states that the solution is insensitive to the choice of the scaling constants if the rank reversals are required to yield a different preferred nondominated solution. Namely, the insensitive solution $\boldsymbol{x}^1$ keeps having the maximal value unless the decision maker reverses the ranking of the attributes X_i and X_j.

Moreover, Ringuest formulates the following linear programming problems for $p = 1$ and $p = \infty$ which can be solved easily. In the minimization problem of the L_1 metric, the sum of the absolute deviations is minimized and then the problem can be represented by:

$$\left.\begin{array}{ll}\text{minimize} & \sum_{i=1}^{n}(k_i^+ + k_i^-) \\ \text{subject to} & \sum_{i=1}^{n} k_i u_i(x_i^2) - \sum_{i=1}^{n} k_i u_i(x_i^1) \geq 0 \\ & k_i - k_i^+ + k_i^- = k_i^1,\ i = 1,\ldots,n \\ & \sum_{i=1}^{n} k_i = 1 \\ & k_i, k_i^+, k_i^- \geq 0,\ i = 1,\ldots,n, \end{array}\right\} \tag{5}$$

where k_i^+ is the amount by which k_i exceeds k_i^1 and k_i^- is the amount by which k_i^1 exceeds k_i.

In the minimization problem of the L_∞ metric, the maximal absolute deviation is minimized and then the problem can be represented by:

$$\left.\begin{array}{ll}\text{minimize} & d \\ \text{subject to} & \sum_{i=1}^{n} k_i u_i(x_i^2) - \sum_{i=1}^{n} k_i u_i(x_i^1) \geq 0 \\ & k_i - k_i^+ + k_i^- = k_i^1,\ i = 1,\ldots,n \\ & k_i^+ + k_i^- \leq d,\ i = 1,\ldots,n \\ & \sum_{i=1}^{n} k_i = 1 \\ & k_i, k_i^+, k_i^- \geq 0,\ i = 1,\ldots,n. \end{array}\right\} \tag{6}$$

3.2 Multiplicative multiattribute models

Following the formulation by Ringuest (1997), the minimization problem of the L_p metric is represented by:

$$\left.\begin{array}{ll}\text{minimize} & L_p \\ \text{subject to} & \frac{1}{K}\left\{\prod_{i=1}^{n}[Kk_i u_i(x_i^2)+1] - \prod_{i=1}^{n}[Kk_i u_i(x_i^1)+1]\right\} \geq 0 \\ & K+1 = \prod_{i=1}^{n}[Kk_i+1] \\ & 0 \leq k_i \leq 1,\ i = 1,\ldots,n. \end{array}\right\} \tag{7}$$

When $p = 1$, it follows that the objective function is represented by $L_1 = \sum_{i=1}^{n}(k_i^+ + k_i^-)$ and the conditions $k_i - k_i^+ + k_i^- = k_i^1$, $k_i^+, k_i^- \geq 0$, $i = 1,\ldots,n$ are added into the constraints of problem (7). When $p=2$, it follows that the objective function is represented by $L_2=\sum_{i=1}^{n}(k_i-k_i^1)^2$. Moreover, when $p=\infty$, it follows that the objective function is represented by $L_\infty=d$ and the conditions $k_i-k_i^+ + k_i^- = k_i^1$, $k_i^+ + k_i^- \leq d$, $k_i^+, k_i^- \geq 0$, $i=1,\ldots,n$ are added into the constraints.

Problem (7) is a nonconvex programming problem, and the degree of the problem depends on the number of attributes. Then, it is difficult to develop a unified method for obtaining the global optimal solution to nonconvex problem (7) for any of the possible numbers of attributes. To overcome this difficulty, we employ genetic algorithms which recently attract a great deal of considerable attention as methods for optimization, adaptation and learning. Especially, it has been shown that they effectively work in nonconvex programming problems (Goldberg, 1989; Michalewicz, 1996; Sakawa, 2001).

3.3 A computational method based on genetic algorithms

To employ genetic algorithms, we formulate the following problem by taking the left hand side of the first constraint of problem (7) into the objective function:

$$\left.\begin{array}{ll} \text{minimize} & f=\alpha L_p+\beta\max\left\{\dfrac{1}{K}\left\{\displaystyle\prod_{i=1}^{n}[Kk_iu_i(x_i^1)+1]-\prod_{i=1}^{n}[Kk_iu_i(x_i^2)+1]\right\},0\right\} \\ \text{subject to} & K+1=\displaystyle\prod_{i=1}^{n}[Kk_i+1] \\ & 0\leq k_i\leq 1,\ i=1,\ldots,n. \end{array}\right\} \tag{8}$$

We solve problem (8) instead of problem (7). In our artificial genetic system, each of individuals represents a set of the scaling constants k_i, $i=1,\ldots,n$ and we employ a representation of an individual by the floating point implementation (Michalewicz, 1996). Each individual $\mathbf{s}=(k_1,\ldots,k_n)$ in the initial population is generated so as to satisfy the condition $0\leq k_i\leq 1$, $i=1,\ldots,n$, and the genetic operations are devised such that an offspring generated satisfies the condition. The additional scaling constant K can be obtained from the equation $1+K=\prod_{i=1}^{n}[1+Kk_i]$. Through the genetic operations in many generations, individuals evolve into those with smaller objective function values and an approximate optimal solution to problem (8) can be obtained. The parameters α and β in problem (8) are adjusted so that individuals effectively evolve in the artificial genetic system.

For the fitness function, because the objective function of problem (8) is minimized, we employ the following fitness function:

$$F(f) = 1/(1+f). \tag{9}$$

As a reproduction operator, the elitist roulette wheel selection is adopted in our artificial genetic systems. The elitist roulette wheel selection is a combination of the roulette wheel selection and the elitism. The roulette wheel selection is the most popular way of the selection. This reproduction allocates offsprings using a roulette wheel with slots sized according to fitness values. If the fitness of an individual in the past populations is larger than that of any individual in the current population, this individual is preserved into the current generation. Because the elitism is employed, the best fitness among the population does not decrease as the generation advances.

For the crossover, we employ the single arithmetical crossover and the whole arithmetical crossover (Michalewicz, 1996). In the single arithmetical crossover, only a single element is crossed. If $\boldsymbol{k}^1$ and $\boldsymbol{k}^2$ are selected and the ith elements k_i^1 and k_i^2 are to be crossed, the ith elements $k_i^{1'}$ and $k_i^{2'}$ of the resulting offsprings are represented as

$$\left.\begin{array}{l} {k_i}^{1'} = (1-a){k_i}^1 + a{k_i}^2 \\ {k_i}^{2'} = (1-a){k_i}^2 + a{k_i}^1, \end{array}\right\} \tag{10}$$

where a is a random number in the interval such that the resulting offsprings satisfy the condition $0 \leq k_i \leq 1$. The whole arithmetical crossover is defined as the following linear combination of the two vectors $\boldsymbol{k}^1$ and $\boldsymbol{k}^2$ corresponding to the selected individuals:

$$\left.\begin{array}{l} \boldsymbol{k}^{1'} = (1-a)\boldsymbol{k}^1 + a\boldsymbol{k}^2 \\ \boldsymbol{k}^{2'} = (1-a)\boldsymbol{k}^2 + a\boldsymbol{k}^1, \end{array}\right\} \tag{11}$$

where a is also a random number in the interval such that the resulting offsprings satisfy the condition $0 \leq k_i \leq 1$, $i = 1, \ldots, n$.

For the mutation, we employ the uniform mutation (Michalewicz, 1996). After a certain scaling constant k_i is randomly selected, it is replaced with a number generated in the same way as the generation method of the initial population.

3.4 Sensitivity analysis incorporating fuzziness in tradeoff experiments

In this subsection, we present a method of sensitivity analysis incorporating fuzziness in tradeoff experiments. In order to obtain multiplicative multiattribute utility functions (1) or additive multiattribute utility functions (3), we must identify single-attribute utility functions and evaluate the scaling constants k_i, $i = 1, \ldots, n$, K.

As shown in section 2, the scaling constants can be derived through the two questions. By answering the probability p in Question 1, one finds that

$$U(x_s^*, \boldsymbol{x}_{-s}^0) = pU(\boldsymbol{x}^*) + (1-p)U(\boldsymbol{x}^0), \tag{12}$$

and then it follows that $k_s = p$. By answering the attribute value x_s in Question 2, one finds that

$$U(x_s, x_j^0; x_i', i \neq s, j) = U(x_s^0, x_j^*; x_i', i \neq s, j), \tag{13}$$

and therefore it follows that $k_j = k_s u_s(x_s) = p u_s(x_s)$. For all pairs of the attribute X_s and the other attributes X_j, $j \neq s$, the decision maker is asked Question 2 and then the remaining scaling constants k_j, $j \neq s$ are obtained.

Because it may be difficult for the decision maker to exactly evaluate the subjective probability p or the attribute values x_s, the importance of sensitivity analysis is recognized. After the multiattribute utility function is identified and all of the alternatives are evaluated, we conduct sensitivity analysis. To take into account fuzziness of the decision maker's judgments in identifying the multiattribute utility function, we ask the decision maker intervals of the probability $[\underline{p}, \overline{p}]$ and the attribute values $[\underline{x}_s, \overline{x}_s]$ which represent the intervals of the minimal and the maximal possible values estimated by the decision maker's self. We propose a new procedure of sensitivity analysis by using this kind of information about fuzzy preference of the decision maker.

Let $\hat{p}$ be an initial subjective probability assessed by the decision maker in Question 1 and $\hat{x}_s$ an initial attribute value in Question 2. Let k_i^1, $i = 1, \ldots, n$ be a corresponding set of the scaling constants. Assume that $\boldsymbol{x}^1 = (x_1^1, \ldots, x_n^1)$ is the most preferred alternative, i.e., the alternative with the maximal utility value.

In the proposed method of sensitivity analysis, we first elicit the interval $[\underline{p}, \overline{p}]$ of the probability and the intervals $[\underline{x}_s, \overline{x}_s]$ of the attribute values from the decision maker, and then we can compute intervals $[\underline{k}_i^1, \overline{k}_i^1]$, $i = 1, \ldots, n$ of the corresponding scaling constants.

If, for any $k_i \in [\underline{k}_i^1, \overline{k}_i^1]$, the alternative $\boldsymbol{x}^1 = (x_1^1, \ldots, x_n^1)$ keeps having the largest value of the multiattribute utility function, we can admit that the obtained solution is not sensitive to choice of the scaling constants. On the other hand, if there exists a certain set of the scaling constants $k_i^2 \in [\underline{k}_i^1, \overline{k}_i^1]$ and a different alternative $\boldsymbol{x}^2 = (x_1^2, \ldots, x_n^2)$ such that the utility value of the alternative $\boldsymbol{x}^1 = (x_1^1, \ldots, x_n^1)$ is smaller than that of the alternative $\boldsymbol{x}^2 = (x_1^2, \ldots, x_n^2)$, the obtained solution is sensitive to choice of the scaling constants.

Applying the above mentioned method of sensitivity analysis to the multiattribute multiplicative utility function can be realized through a genetic algorithm similar to the method presented in the previous subsection. Because it is important to check whether or not the scaling constants k_i, $i = 1, \ldots, n$ are in the intervals $[\underline{k}_i^1, \overline{k}_i^1]$, $i = 1, \ldots, n$, we employ the L_∞ metric, and formulate the following problem:

$$\left.\begin{array}{ll}\text{minimize} & \alpha d+\beta \max\left\{\dfrac{1}{K}\left\{\prod_{i=1}^{n}[Kk_iu_i(x_i^1)+1]-\prod_{i=1}^{n}[Kk_iu_i(x_i^2)+1]\right\},0\right\} \\ \text{subject to} & k_i-\overline{k}_i^{+}+\overline{k}_i^{-}=\overline{k}_i^{1},\ i=1,\ldots,n \\ & k_i-\underline{k}_i^{+}+\underline{k}_i^{-}=\underline{k}_i^{1},\ i=1,\ldots,n \\ & \max\{\overline{k}_i^{-},\underline{k}_i^{+}\}\le d,\ i=1,\ldots,n \\ & \underline{k}_i^{-},\underline{k}_i^{+},\overline{k}_i^{-},\overline{k}_i^{+}\ge 0,\ i=1,\ldots,n \\ & K+1=\prod_{i=1}^{n}[Kk_i+1] \\ & 0\le k_i\le 1,\ i=1,\ldots,n. \end{array}\right\} \tag{14}$$

Although problem (14) is a formulation based on the L_∞ metric, for the L_1 or the L_2 metric, a similar problem can be formulated. We solve problem (14) by using the genetic algorithm, and each of individuals represents a set of the scaling constants k_i, $i=1,\ldots,n$. An individual in the initial population is generated so as to satisfy the condition $0\le k_i\le 1$, $i=1,\ldots,n$, and the genetic operations are devised such that an offspring generated satisfies the condition. The additional scaling constant K can be obtained from the equation $1+K=\prod_{i=1}^{n}[1+Kk_i]$. By directly computing d of problem (14) for each individual, the constraints of problem (14) can be reduced to those of problem (8). Thus, for solving problem (14), we can use a computational method based on the genetic algorithm which is the same as that for problem (8) except for the fitness function.

We formally give the definition of sensitivity of a solution in sensitivity analysis incorporating fuzziness in tradeoff experiments as follows. Let $\boldsymbol{k}^1=(k_1^1,\ldots,k_n^1)$ be a set of the scaling constants determined by Step 3 in the procedure of multiattribute utility analysis, and let $\boldsymbol{x}^1$ be a preferable alternative maximizing the multiattribute utility $U(\boldsymbol{x};\boldsymbol{k}^1)$. Moreover, let $[\underline{k}_i^1,\overline{k}_i^1]$, $i=1,\ldots,n$ be the permissible interval of k_i^1 derived by the fuzzy assessment, and let $\boldsymbol{k}^2=(k_1^2,\ldots,k_n^2)$ be a set of the scaling constants obtained by solving problem (14) with the L_∞ metric.

Definition 1 *If $k_i^2\in[\underline{k}_i^1,\overline{k}_i^1]$, $i=1,\ldots,n$, then, in the sense of the L_∞ metric, $\boldsymbol{x}^1$ is said to be sensitive to the choice of the scaling constants. If there exists an index*

$i \in \{1,\ldots,n\}$ such that $k_i^2 \notin [\underline{k}_i^1, \overline{k}_i^1]$, then, in the sense of the L_∞ metric, $\boldsymbol{x}^1$ is said to be not sensitive to the choice of the scaling constants.

A similar definition for the L_1 or the L_2 metric can be also given. In particular, for the later part of the above definition, because the proposed method is an approximate computational method, we cannot rule out the possibility that even though there exists a set of the scaling constants $\boldsymbol{k}^2$ satisfying $k_i^2 \in [\underline{k}_i^1, \overline{k}_i^1]$, $i = 1,\ldots,n$, the proposed solution method cannot find it. However, if a sufficient amount of computation of searching through the proposed solution method cannot find such a set of the scaling constants in the permissible interval, from the viewpoint of the approximate computation, it is not irrational to claim that $\boldsymbol{x}^1$ is not sensitive.

4 Numerical examples

We illustrate the proposed method for sensitivity analysis for scaling constants of multiplicative multiattribute utility functions with two multiattribute decision problems. The first example is a case where the obtained solution is insensitive to choice of the scaling constants, and the second one is a sensitive case.

4.1 Problem 1: an example with an insensitive solution

Consider a five-attribute decision problem (Problem 1) with two alternatives whose attribute values are shown in Table 1. Moreover, assume that the decision maker identifies the following single-attribute utility functions for the five attributes:

$$u_1(x_1) = \frac{1}{1-e^{-1}}(1-e^{-0.1x_1}) \qquad u_2(x_2) = \frac{1}{2.67}(1-e^{0.13x_2})+1$$
$$u_3(x_3) = \frac{1}{0.492}(1-e^{0.04x_3})+1 \qquad u_4(x_4) = \frac{1}{1-e^{-2}}(1-e^{-0.2x_4})$$
$$u_5(x_5) = \frac{1}{1-e^{-0.16}}(1-e^{-0.016x_5}),$$

where u_1, u_4 and u_5 are monotone increasing functions, and u_2 and u_3 are monotone decreasing functions.

Table 1: Attribute values of the two alternatives in Problem 1

	x_1	x_2	x_3	x_4	x_5
Alternative 1: $\boldsymbol{x}^1$	7.0	7.0	5.0	6.5	5.5
Alternative 2: $\boldsymbol{x}^2$	6.0	6.0	6.0	6.0	6.0

Consider the decision maker to have specified the probability $p = 0.5$ such that a certain event $(x_1^*, x_2^0, x_3^0, x_4^0, x_5^0)$ is indifferent to a lottery $<\boldsymbol{x}^*, p, \boldsymbol{x}^0>$ in Question 1, and for the two attribute X_1 and X_2, the attribute value $x_1 = 7.0$ such that a certain event $(x_1, x_2^0, x_3^0, x_4^0, x_5^0)$ is indifferent to another certain event $(x_1^0, x_2^*, x_3^0, x_4^0, x_5^0)$ in Question 2. Similarly, assume that the decision maker has specified the attribute values $x_1 = 5.0$ for the two attribute X_1 and X_3, $x_1 = 7.8$ for the two attribute X_1 and X_4, and $x_1 = 4.0$ for the two attribute X_1 and X_5. From the above assessment, we can compute the scaling constants $k_1^1 = 0.50$, $k_2^1 = 0.398195$, $k_3^1 = 0.311230$, $k_4^1 = 0.428395$, $k_5^1 = 0.260773$, and $K^1 = -0.869032$. Then, the multiattribute utility function in multiplicative form is represented by

$$
\begin{aligned}
U(\boldsymbol{x};\boldsymbol{k}^1) = & \frac{1}{K^1}\Big\{(1+K^1k_1^1\frac{1}{1-e^{-1}}(1-e^{-0.1x_1})) \\
& (1+K^1k_2^1(\frac{1}{2.67}(1-e^{0.13x_2})+1)) \\
& (1+K^1k_3^1(\frac{1}{0.492}(1-e^{0.04x_3})+1)) \\
& (1+K^1k_4^1\frac{1}{1-e^{-2}}(1-e^{-0.2x_4})) \\
& (1+K^1k_5^1\frac{1}{1-e^{-0.16}}(1-e^{-0.016x_5}))-1\Big\},
\end{aligned}
$$

and one finds that

$$
\begin{aligned}
U(\boldsymbol{x}^1;\boldsymbol{k}^1) &= 0.826450, \\
U(\boldsymbol{x}^2;\boldsymbol{k}^1) &= 0.812364.
\end{aligned}
$$

Thus, it follows that the alternative 1, $\boldsymbol{x}^1$, is favorable to the decision maker.

Next, we proceed to the procedure of sensitivity analysis. When the probability p in Question 1 is estimated to be in the interval $[0.48, 0.53]$, it follows that the scaling constant k_1 is also in the interval $[0.48, 0.53]$. Moreover, in the indifference experiment between the two attributes X_1 and X_2 in Question 2, when the attribute value x_1 is estimated to be in the interval $[6.95, 7.20]$, it follows that the scaling constant k_2 is in the interval $[0.380337, 0.430331]$. Similarly, assuming that on the second thought the decision maker still evaluates the attribute value x_1 for the two attribute X_1 and X_3 certainly at $x_1 = 5.0$, which is the same value as the first assessment, it follows that the scaling constant k_3 is in the interval $[0.298780, 0.329903]$. Assuming that the decision maker estimates the attribute value x_1 for the two attribute X_1 and X_4 to be in the interval $[7.75, 7.95]$, it follows that the scaling constant k_4 is in the interval $[0.409514, 0.459820]$. Assuming that on the second thought the decision maker still evaluates the attribute value x_1 for the two attribute X_1 and X_5 certainly at $x_1 = 4.0$, which is the same value as the first assessment, it follows that the scaling constant k_5 is in the interval $[0.250342, 0.276419]$. The first assessment and the fuzzy assessment for sensitivity analysis are given in Table 2.

Table 2: First assessment and fuzzy assessment for sensitivity analysis

First assessment			Fuzzy assessment		
	$p = 0.5$	$k_1^1 = 0.50$		$p \in [0.48, 0.53]$	$k_1 \in [0.48, 0.53]$
$X_1 : X_2$	$x_1 = 7.0$	$k_2^1 = 0.398195$	$X_1 : X_2$	$x_1 \in [6.95, 7.20]$	$k_2 \in [0.380337, 0.430331]$
$X_1 : X_3$	$x_1 = 5.0$	$k_3^1 = 0.311230$	$X_1 : X_3$	$x_1 = 5.0$	$k_3 \in [0.298780, 0.329903]$
$X_1 : X_4$	$x_1 = 7.8$	$k_4^1 = 0.428395$	$X_1 : X_4$	$x_1 \in [7.75, 7.95]$	$k_4 \in [0.409514, 0.459820]$
$X_1 : X_5$	$x_1 = 4.0$	$k_5^1 = 0.260773$	$X_1 : X_5$	$x_1 = 4.0$	$k_5 \in [0.250342, 0.276419]$

We apply the proposed method to the example. By solving problem (14), we find the scaling constants $k_1^2 = 0.406905$, $k_2^2 = 0.475965$, $k_3^2 = 0.236084$, $k_4^2 = 0.348620$, $k_5^2 = 0.349488$, and $K^2 = -0.843109$ which lead to a reversal of utilities, i.e., $U(\boldsymbol{x}^1; \boldsymbol{k}^2) = 0.790946 < U(\boldsymbol{x}^2; \boldsymbol{k}^2) = 0.791380$. Because we have $k_i^2 \notin [\underline{k}_i^1, \overline{k}_i^1]$, $i = 1, \ldots, 5$, it follows that the obtained solution $(\boldsymbol{x}^1; \boldsymbol{k}^1)$ is not sensitive to choice of the scaling constants.

4.2 Problem 2: an example with a sensitive solution

Consider the same five-attribute decision problem with different two alternatives (Problem 2) and the attribute values are shown in Table 3.

Table 3: Attribute values of the two alternatives in Problem 2

	x_1	x_2	x_3	x_4	x_5
Alternative 1: $\boldsymbol{x}^1$	5.1	5.3	5.1	6.8	4.6
Alternative 2: $\boldsymbol{x}^2$	6.0	6.0	6.0	6.0	6.0

Assuming that the decision maker has identified the same single-attribute utility functions as those of Problem 1 and evaluated the same probability p and the same attribute values x_1 as those of Problem 1, we have the same scaling constants $k_1^1 = 0.50$, $k_2^1 = 0.398195$, $k_3^1 = 0.311230$, $k_4^1 = 0.428395$, $k_5^1 = 0.260773$, and $K = -0.869032$. Thus, the utilities of the two alternatives are computed as follows:

$$U(\boldsymbol{x}^1;\boldsymbol{k}^1) = 0.812713,$$
$$U(\boldsymbol{x}^2;\boldsymbol{k}^1) = 0.812364,$$

and then the alternative 1, $\boldsymbol{x}^1$, is favorable to the decision maker. Moreover, assuming that the decision maker has evaluated the same intervals of the probability p and the attribute values x_1 as those of Problem 1, we have the same intervals of the scaling constants $0.48 \leq k_1 \leq 0.53$, $0.380377 \leq k_2 \leq 0.430331$, $0.298780 \leq k_3 \leq 0.329903$, $0.409514 \leq k_4 \leq 0.459820$, and $0.250342 \leq k_5 \leq 0.276419$.

We apply the proposed method to the example and then obtain the scaling constants $k_1^2 = 0.506511$, $k_2^2 = 0.399170$, $k_3^2 = 0.314339$, $k_4^2 = 0.427068$, $k_5^2 = 0.266703$ and $K^2 = -0.823072$ which lead to a reversal of utilities, i.e., $U(\boldsymbol{x}^1;\boldsymbol{k}^2) = 0.814017 < U(\boldsymbol{x}^2;\boldsymbol{k}^2) = 0.814199$. Because we have $k_i^2 \in [\underline{k}_i^1, \overline{k}_i^1]$, $i = 1, \ldots, 5$, it follows that the obtained solution is sensitive to choice of the scaling constants.

5 Conclusions

We have developed the methods for sensitivity analysis of multiattribute utility functions in the multiplicative form. In the previous methods, although closeness of the distance between two sets of the scaling constants plays an important role for defining sensitivity of solutions to choice of the scaling constants, it supposed that it is difficult to judge the closeness. We have defined the sensitivity to choice of the scaling constants by assessing the possible intervals of the subjective probability and the attribute values in the indifference experiments and calculating the corresponding intervals of the scaling constants instead of by specifying a certain distance between two sets of the scaling constants.

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