Stochastic Optimal Control for Weakly Coupled Large-Scale Systems via State and Static Output Feedback

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Abstract

In this study, we investigate the infinite-horizon linear quadratic control involving state- and control-dependent noise in weakly coupled large-scale systems. In contrast to the existing results, we allow the control and state weighting matrices in the cost function to be indefinite. After establishing an asymptotic structure for the solutions of the stochastic algebraic Riccati equation (SARE), a weak coupling parameter-independent control is provided. Moreover, by solving the reduced-order linear matrix inequality (LMI), we can easily obtain the proposed control without using any numerical algorithms. As a result, although the small positive weak coupling parameter that connects the other subsystems is very small or unknown, it is possible to compute the desired controller. Finally, the extension of the result of the study to the static output feedback control problem is discussed by considering the Lagrange multiplier method.

Index Terms

weakly coupled large-scale systems, stochastic systems, linear matrix inequality (LMI), Lagrange multiplier method.

I. Introduction

The stability analysis, control, filtering, and differential games of large-scale interconnected systems that are parameterized by a small weak coupling parameter ε have been extensively investigated [8]. For example, the weakly coupled systems have been used to illustrate multi-area power systems [1], [2]. Even though weakly coupled systems have been studied in engineering and mathematics for more than forty years, weakly coupled stochastic systems that are governed by Itô's differential equation are still an interesting and challenging research area, as demonstrated in several recent papers [11], [12], [13].

Over the last decade, stochastic control problems governed by Itô's differential equation have attracted considerable research interest. Recently, the indefinite stochastic linear quadratic (LQ) control problem with state- and control-dependent noise has been investigated via linear matrix inequality (LMI) [3]. Although the results of [3] are very efficient and powerful in theory and the control is easy by solving the LMI for a normal system, the indefinite stochastic LQ control problem for a weakly coupled stochastic system, which is more complicated than a normal system, is still an issue to be considered.

In order to design the stochastic LQ control for the weakly coupled large-scale systems, the stochastic algebraic Riccati equation (SARE) that is parameterized by the positive and small coupling parameter ε should be solved. Various reliable approaches for solving the SARE have been well documented in many literatures (see e.g., [3], [7], [9], [11], [12]). If the small positive weak coupling parameter that connects the other subsystems is relatively small, these approaches are very useful. However, a limitation of these approaches is that the small parameter is assumed to be known. Thus, it is not applicable to a large class of problems where the parameters represent a small unknown perturbation whose value is not exactly known. Although Newton's method is still useful for solving the reduced-order solution of the parameter-independent SARE [11], the convergence of the algorithm becomes very sensitive when the control weighting matrix in the cost function is indefinite.

In this paper, we study the stochastic LQ control problem with the state- and control-dependent noise of weakly coupled large-scale systems. Particularly, this study is challenged by a class of stochastic LQ control problems with sign indefinite state and control weighting matrices. The main contributions of this paper can be summarized as follows. First, the asymptotic structure of the SARE is established. Second, by using the asymptotic structure, a new near-optimal controller that does not depend on the values of the small parameter is obtained. Moreover, we claim that the near-optimal controller can be computed via the LMI approach. As a result, although the small positive weak coupling parameter that connects the other subsystems is very small or unknown, it is possible to compute the desired controller under the reduced-order computation. As another important feature, it is newly shown that the resulting controller achieves $O(\varepsilon^2)$ approximation of the optimal cost. Furthermore, the static output feedback LQ control problem is solved by using the Lagrange multiplier method. A necessary condition is derived in the form of cross-coupled stochastic algebraic Riccati equations (CSARE). Finally, in order to demonstrate the efficiency and validity of the algorithm, a numerical example is included.

Notation: The notations used in this paper are fairly standard. I_n denotes an $n \times n$ identity matrix. $\mathbf{block\ diag}$ denotes a block diagonal matrix. $\| \cdot \|$ denotes the Euclidean norm of a matrix. E denotes the expectation. $\otimes$ denotes the Kronecker product.

II. Problem Formulation and Preliminary Results

Consider stochastic linear time-invariant weakly coupled large-scale systems.

$$dx(t) = [A_\varepsilon x(t) + B_\varepsilon u(t)]dt + [C_\varepsilon x(t) + D_\varepsilon u(t)]dw(t),\ x(0) = x^0, \tag{1}$$

where

$$x(t) := \begin{bmatrix} x_1(t) \\ \vdots \\ x_N(t) \end{bmatrix},\ u(t) := \begin{bmatrix} u_1(t) \\ \vdots \\ u_N(t) \end{bmatrix},$$

$$A_\varepsilon := \begin{bmatrix} A_{11} & \varepsilon A_{12} & \cdots & \varepsilon A_{1N} \\ \varepsilon A_{21} & A_{22} & \cdots & \varepsilon A_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon A_{N1} & \varepsilon A_{N2} & \cdots & A_{NN} \end{bmatrix},\ B_\varepsilon := \begin{bmatrix} B_{11} & \varepsilon B_{12} & \cdots & \varepsilon B_{1N} \\ \varepsilon B_{21} & B_{22} & \cdots & \varepsilon B_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon B_{N1} & \varepsilon B_{N2} & \cdots & B_{NN} \end{bmatrix},$$

$$C_\varepsilon := \begin{bmatrix} C_{11} & \varepsilon C_{12} & \cdots & \varepsilon C_{1N} \\ \varepsilon C_{21} & C_{22} & \cdots & \varepsilon C_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon C_{N1} & \varepsilon C_{N2} & \cdots & C_{NN} \end{bmatrix},\ D_\varepsilon := \begin{bmatrix} D_{11} & \varepsilon D_{12} & \cdots & \varepsilon D_{1N} \\ \varepsilon D_{21} & D_{22} & \cdots & \varepsilon D_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon D_{N1} & \varepsilon D_{N2} & \cdots & D_{NN} \end{bmatrix}.$$

$x_i(t) \in \Re^{n_i},\ i = 1,\ \dots,N$ represents the i-th state vectors. $u_i(t) \in \Re^{m_i},\ i = 1,\ \dots,N$ represents the i-th control inputs. $w(t) \in \Re$ is a one-dimensional standard Wiener process defined in the filtered probability space [3], [4]. Here, ε denotes a relatively small positive coupling parameter that relates the linear system with the other subsystems.

The cost function for each strategy subset is defined by

$$J(u,\ x(0)) = E \int_0^\infty \Big[x^T(t) Q_\varepsilon x(t) + u^T(t) R_0 u(t) \Big] dt, \tag{2}$$

where

$$Q_\varepsilon := \begin{bmatrix} Q_{11} & \varepsilon Q_{12} & \cdots & \varepsilon Q_{1N} \\ \varepsilon Q_{12}^T & Q_{22} & \cdots & \varepsilon Q_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon Q_{1N}^T & \varepsilon Q_{2N}^T & \cdots & Q_{NN} \end{bmatrix},\ Q_\varepsilon = Q_\varepsilon^T,$$

$$R_0 := \textbf{block diag} \begin{pmatrix} R_{11} & \cdots & R_{NN} \end{pmatrix},\ R_0 = R_0^T.$$

It should be noted that the weighting matrices Q_ε and R_0 are assumed to be sign indefinite. Moreover, suppose that R_0 is an invertible matrix.

The stochastic LQ problem for weakly coupled large-scale systems is given below.

"Find a matrix K_ε such that the control $u(t) = K_\varepsilon x(t)$ minimizes the cost function (2) along the trajectories of the system (1) corresponding to all admissible controls."

According to [3], [16] (see also [17] chapter 5), the class of admissible controls consists of the stochastic processes; $u = \{u(t)\}_{t \ge 0}$ adapted to the filtration generated by the Wiener process $w(t)$, having the additional properties: $E \int_0^\infty \|u(t)\|^2 dt < +\infty$ and $\lim_{t\to\infty} \|x_u(t,\ x_0)\|^2 = 0$, for all $x_0 \in \Re^n$, where $x_u(t,\ x_0)$ is the solution of (1) determined by the input u and starting from x_0 at $t = 0$.

By using the existing result [3], an optimal feedback control will be as given in

$$u^*(t) = -\tilde{R}_\varepsilon^{-1} (B_\varepsilon^T P_\varepsilon + D_\varepsilon^T P_\varepsilon C_\varepsilon) x(t), \tag{3}$$

where $\tilde{R}_\varepsilon := R_0 + D_\varepsilon^T P_\varepsilon D_\varepsilon > 0$ and

$$\boldsymbol{F}(P_\varepsilon) := P_\varepsilon A_\varepsilon + A_\varepsilon^T P_\varepsilon + C_\varepsilon^T P_\varepsilon C_\varepsilon - (B_\varepsilon^T P_\varepsilon + D_\varepsilon^T P_\varepsilon C_\varepsilon)^T \tilde{R}_\varepsilon^{-1} (B_\varepsilon^T P_\varepsilon + D_\varepsilon^T P_\varepsilon C_\varepsilon) + Q_\varepsilon = 0. \tag{4}$$

Furthermore, the following result was proved [3]. The feedback gains K_ε can be obtained by solving the following semidefinite programming (SDP). Moreover, P_ε is a maximal solution of P_ε^*, which is the unique optimal solution.

$$\text{maximize} \quad \textbf{Tr}\ [P_\varepsilon], \tag{5a}$$

$$\text{subject to} \quad \begin{bmatrix} P_\varepsilon A_\varepsilon + A_\varepsilon^T P_\varepsilon + C_\varepsilon^T P_\varepsilon C_\varepsilon + Q_\varepsilon & (B_\varepsilon^T P_\varepsilon + D_\varepsilon^T P_\varepsilon C_\varepsilon)^T \\ B_\varepsilon^T P_\varepsilon + D_\varepsilon^T P_\varepsilon C_\varepsilon & R_0 + D_\varepsilon^T P_\varepsilon D_\varepsilon \end{bmatrix} \ge 0, \tag{5b}$$

$$R_0 + D_\varepsilon^T P_\varepsilon D_\varepsilon > 0. \tag{5c}$$

If the positive weak coupling parameter is sufficiently small, the Riccati direct method or LMI approach that is based on the SDP is very useful. However, a limitation of these approaches is that the small parameter ε is assumed to be known. Thus, we propose the design method of the parameter-independent controller by means of the reduced-order computation.

Without loss of generality, the stochastic LQ control problem is investigated under the following basic assumptions [4].

Assumption 1: For each $i \in \{1,\ 2,\ \dots,N\}$ the subsystems in

$$dx_i(t) = [A_{ii}x_i(t) + B_{ii}u_i(t)]dt + [C_{ii}x_i(t) + D_{ii}u_i(t)]dw(t) \tag{6}$$

is stochastic stabilizable.

For precise definition, as well as necessary and sufficient conditions for stochastic stabilizability, we refer to [3], [17].

Remark 1: If $K_{ii} \in \Re^{m_i \times n_i}$ is a stabilizing feedback gain for the subsystems (6), we set $K_\varepsilon = \bar{K} = \mathbf{block\ diag}(K_{11},\ \dots,K_{NN})$. On the basis of the fact that the exponential stability in the mean square is preserved under small perturbations of the coefficients of the system one deduces that there exists ε^* such that the control $u(t) = \bar{K}x(t)$ stabilizes the system (1) for any $\varepsilon \in (0,\ \varepsilon^*]$. Therefore, if Assumption 1 is fulfilled, then the system in (1) is stochastic stabilizable for $\varepsilon > 0$, which is sufficiently small.

A. Asymptotic Structure of SARE

Firstly, in order to obtain the controller, the asymptotic structure of SARE (4) is established. Since A_ε, B_ε, C_ε and D_ε include the term of the small parameter ε, the solution P_ε of SARE (4) with the following structure is considered [11], [12], [13].

$$P_\varepsilon := \begin{bmatrix} P_{11} & \varepsilon P_{12} & \cdots & \varepsilon P_{1N} \\ \varepsilon P_{12}^T & P_{22} & \cdots & \varepsilon P_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon P_{1N}^T & \varepsilon P_{2N}^T & \cdots & P_{NN} \end{bmatrix}. \tag{7}$$

By substituting the coefficient matrices and $P_{i\varepsilon}$ into SARE (4), setting $\varepsilon = 0$, and partitioning SARE (4), the following reduced-order algebraic Riccati equations (AREs) are obtained, where $\bar{P}_{ii},\ i = 1,\ \dots,N$ is the zeroth-order solutions of SARE (4) as $\varepsilon = 0$.

$$\boldsymbol{F}_i(\bar{P}_{ii}) := \bar{P}_{ii}A_{ii} + A_{ii}^T\bar{P}_{ii} + C_{ii}^T\bar{P}_{ii}C_{ii} - (B_{ii}^T\bar{P}_{ii} + D_{ii}^T\bar{P}_{ii}C_{ii})^T\tilde{R}_{ii}^{-1}(B_{ii}^T\bar{P}_{ii} + D_{ii}^T\bar{P}_{ii}C_{ii}) + Q_{ii} = 0,\ i = 1,\ \dots,N, \tag{8}$$

where $\tilde{R}_{ii} := R_{ii} + D_{ii}^T\bar{P}_{ii}D_{ii} > 0$.

In order to guarantee the existence of a stabilizing solution of SARE (4), the following conditions are assumed.

Assumption 2: There exists a solution $\bar{P}_{ii}^0$ for all

$$\begin{bmatrix} \bar{P}_{ii}^0A_{ii} + A_{ii}^T\bar{P}_{ii}^0 + C_{ii}^T\bar{P}_{ii}^0C_{ii} + Q_{ii} & (B_{ii}^T\bar{P}_{ii}^0 + D_{ii}^T\bar{P}_{ii}^0C_{ii})^T \\ B_{ii}^T\bar{P}_{ii}^0 + D_{ii}^T\bar{P}_{ii}^0C_{ii} & R_{ii} + D_{ii}^T\bar{P}_{ii}^0D_{ii} \end{bmatrix} \geq 0 \tag{9}$$

and $R_{ii} + D_{ii}^T\bar{P}_{ii}^0D_{ii} > 0$.

It should be noted that under Assumptions 1 and 2, for all i, the reduced-order SAREs in (8) have a maximal and stabilizing solution $\bar{P}_{ii}$, which verifies $R_{ii} + D_{ii}^T\bar{P}_{ii}^*D_{ii} > 0$. Moreover, since the weight matrices are sign indefinite, it is not expected that the stabilizing solution P_ε of (4) and the stabilizing solution $\bar{P}_{ii}$ of (8) are positive semidefinite.

The asymptotic expansion of SARE (4) at $\varepsilon = 0$ is described by the following lemma.

Lemma 1: Under Assumptions 1 and 2, there exists the small constant σ^* such that for all $\varepsilon \in (0,\ \sigma^*)$, SARE (4) admits a unique stabilizing solution P_ε^*, which verifies (5c). Moreover, this solution can be written as given in

$$P_\varepsilon := P_\varepsilon^* = \bar{P} + O(\varepsilon) = \textbf{block diag}\begin{pmatrix} \bar{P}_{11} & \cdots & \bar{P}_{NN} \end{pmatrix} + O(\varepsilon). \tag{10}$$

Proof: This can be proved by applying the implicit function theorem on SARE (4). In order to do this, it is sufficient to show that the corresponding Jacobian is nonsingular at $\varepsilon = 0$. The Jacobian $\boldsymbol{J}$ is given by

$$\boldsymbol{J} := \left. \frac{\partial \mathrm{vec} \boldsymbol{F}(P_\varepsilon)}{\partial (\mathrm{vec} P_\varepsilon)^T} \right|_{\varepsilon=0} = \textbf{block diag}\begin{pmatrix} \nabla_1 & \cdots & \nabla_N \end{pmatrix}. \tag{11}$$

For each $i \in \{1,\ \dots\ , N\}$, the matrix ∇_i is invertible due to the fact that $\bar{P}_{ii}$ is the stabilizing solution of (8). Thus, $\det \boldsymbol{J} \neq 0$, i.e., $\boldsymbol{J}$ is non-singular for $\varepsilon = 0$. As a consequence of the implicit function theorem, this implies that there exists a unique continuous mapping $P_\varepsilon := \boldsymbol{G}(\varepsilon)$ that possesses the Taylor series expansion at $\varepsilon = 0$; in other words, $P_\varepsilon := \boldsymbol{G}(0) + \sum_{l=1}^{\infty} \frac{\varepsilon^l}{l!} \boldsymbol{G}^{(l)}(0)$. Thus, we have an equation in the form given in (10). On the other hand, taking into account the fact that $\bar{P}_{ii}$ verifies $R_{ii} + D_{ii}^T \bar{P}_{ii}^* D_{ii} > 0$, then for a sufficiently small parameter ε, P_ε will verify (5c). Thus, the proof is complete. ■

Remark 2: In Lemma 1, the existence of a bound on σ^* is only guaranteed. Since it is well-known that it is very hard to compute the exact bound of σ^* [8], this issue is still an open problem.

III. Parameter Independent Controller

Since there exist many cases such that the parameters represent small unknown perturbations whose values are not exactly known, it is desirable to have the parameter-independent controller. Therefore, a parameter-independent stochastic LQ controller is considered. Using the result in (10), the approximate stochastic controller is given below.

$$\bar{u}(t) := \bar{K} x(t) = -\tilde{R}^{-1} (\bar{B}^T \bar{P} + \bar{D}^T \bar{P} \bar{C}) x(t), \tag{12}$$

where $\bar{R} = R_0 + \bar{D}^T \bar{P} \bar{D}$ and

$$\bar{B} := \textbf{block diag}\begin{pmatrix} B_{11} & \cdots & B_{NN} \end{pmatrix},$$
$$\bar{D} := \textbf{block diag}\begin{pmatrix} D_{11} & \cdots & D_{NN} \end{pmatrix},$$
$$\bar{C} := \textbf{block diag}\begin{pmatrix} C_{11} & \cdots & C_{NN} \end{pmatrix}.$$

In order to obtain the parameter-independent controller (12), we have to solve the reduced-order SARE (8). Various reliable approaches for solving SARE have been well documented in many literatures (see e.g., [3], [7], [9], [11], [12]). However, there are some difficulties with SARE (8). First, R_0 has the sign indefinite and the $\tilde{R}_{ii}^{-1}$ involves the unknown $\bar{P}_{ii}$. Second, there is an additional strictly positive definiteness constraint. Thus, we try to examine the LMI design method. Let us consider the following reduced-order SDP.

$$\text{maximize} \quad \textbf{Tr}\ [P_{ii}], \tag{13a}$$

$$\text{subject to} \quad \begin{bmatrix} P_{ii} A_{ii} + A_{ii}^T P_{ii} + C_{ii}^T P_{ii} C_{ii} + Q_{ii} & L_{ii}^T \\ L_{ii} & R_{ii} + D_{ii}^T P_{ii} D_{ii} \end{bmatrix} \geq 0, \ \text{where } L_{ii} := B_{ii}^T P_{ii} + D_{ii}^T P_{ii} C_{ii},$$

$$R_{ii} + D_{ii}^T P_{ii} D_{ii} > 0, \ i = 1,\ \dots\ , N. \tag{13b}$$

By using the existing result [3], we obtain the following result.

Lemma 2: Under Assumptions 1 and 2, SARE (8) has a maximal and stabilizing solution $\bar{P}_{ii}$ verifying $R_{ii}+D_{ii}^T\bar{P}_{ii}^*D_{ii} > 0$, which is the unique optimal solution to the above-mentioned SDP problem.

A. Degradation of Cost

The main result for the degradation of the value of the cost function (2) via the proposed approximate controller (12) is presented as follows.

Theorem 1: Under Assumptions 1 and 2, the use of the approximate controller (12) results in (14).

$$J(u^*,\ x(0)) = J(\bar{u},\ x(0)) + O(\varepsilon^2), \tag{14}$$

where

$$J(u^*,\ x(0)) = x^T(0)P_\varepsilon x(0), \tag{15a}$$

$$J(\bar{u},\ x(0)) = x^T(0)X_\varepsilon x(0), \tag{15b}$$

$$X_\varepsilon(A_\varepsilon + B_\varepsilon\bar{K}) + (A_\varepsilon + B_\varepsilon\bar{K})^T X_\varepsilon + (C_\varepsilon + D_\varepsilon\bar{K})^T X_\varepsilon(C_\varepsilon + D_\varepsilon\bar{K}) + \bar{K}^T R_0\bar{K} + Q_\varepsilon = 0, \tag{15c}$$

$$\bar{K} = -\bar{R}^{-1}(\bar{B}^T\bar{P} + \bar{D}^T\bar{P}\bar{C}). \tag{15d}$$

Proof: Let us consider Newton's method.

$$\begin{aligned} &P_\varepsilon^{(n+1)}[A_\varepsilon - B_\varepsilon(\tilde{R}_\varepsilon^{(n)})^{-1}L_\varepsilon^{(n)}] + [A_\varepsilon - B_\varepsilon(\tilde{R}_\varepsilon^{(n)})^{-1}L_\varepsilon^{(n)}]^T P_\varepsilon^{(n+1)} \\ &\quad +[C_\varepsilon - D_\varepsilon(\tilde{R}_\varepsilon^{(n)})^{-1}L_\varepsilon^{(n)}]^T P_\varepsilon^{(n+1)}[C_\varepsilon - D_\varepsilon(\tilde{R}_\varepsilon^{(n)})^{-1}L_\varepsilon^{(n)}] + L_\varepsilon^{(n)T}(\tilde{R}_\varepsilon^{(n)})^{-1}R_0(\tilde{R}_\varepsilon^{(n)})^{-1}L_\varepsilon^{(n)} + Q_\varepsilon = 0, \end{aligned} \tag{16}$$

where $P_\varepsilon^{(0)} = \bar{P}$, $\tilde{R}_\varepsilon^{(n)} := R_0 + D_\varepsilon^T P_\varepsilon^{(n)} D_\varepsilon$ and $L_\varepsilon^{(n)} := B_\varepsilon^T P_\varepsilon^{(n)} + D_\varepsilon^T P_\varepsilon^{(n)} C_\varepsilon$.

There exists a small $\tilde{\delta}$ such that for all $\varepsilon \in (0,\ \tilde{\delta}),\ 0 < \tilde{\delta} \le \bar{\delta}$, the iterative algorithm represented by equation (16) converges to a maximal solution P_ε with a rate equal to that of quadratic convergence. In other words, the following condition is satisfied [11].

$$\|P_\varepsilon^{(n)} - P_\varepsilon\| = O(\varepsilon^{2^n}),\ n = 0,\ 1,\ \dots\ . \tag{17}$$

On the other hand, setting $n = 0$ for Newton's method (16), we obtain

$$P_\varepsilon^{(1)}(A_\varepsilon + B_\varepsilon\bar{K}) + (A_\varepsilon + B_\varepsilon\bar{K})^T P_\varepsilon^{(1)} + (C_\varepsilon + D_\varepsilon\bar{K})^T P_\varepsilon^{(1)}(C_\varepsilon + D_\varepsilon\bar{K}) + \bar{K}^T R_0\bar{K} + Q_\varepsilon = 0. \tag{18}$$

Subtracting (15d) from (18), $V_\varepsilon = X_\varepsilon - P_\varepsilon^{(1)}$ satisfies the following stochastic algebraic Lyapunov equation (SALE)

$$V_\varepsilon^{(1)}(A_\varepsilon + B_\varepsilon\bar{K}) + (A_\varepsilon + B_\varepsilon\bar{K})^T V_\varepsilon^{(1)} + (C_\varepsilon + D_\varepsilon\bar{K})^T V_\varepsilon^{(1)}(C_\varepsilon + D_\varepsilon\bar{K}) = 0. \tag{19}$$

Under Assumption 1, since the above SALE (19) has the unique solution $V_\varepsilon = 0$, the following equation holds by using the result given in (17).

$$X_\varepsilon = P_\varepsilon^{(1)} = P_\varepsilon + O(\varepsilon^2). \tag{20}$$

This is the desired result. ■

B. Fixed Point Iteration

When the reduced-order SARE (8) or LMI (13b) is solved, it is well known that the norm of the matrices $\|C_{ii}\|$ and $\|D_{ii}\|$ for the practical plant are small [12], [13]. Thus, using this feature, the fixed point algorithm for solving SARE (8) is established. By taking into account the fact that $C_{ii} = \mu C$ and $D_{ii} = \mu D$, we consider SALE (21) in its general form.

$$\boldsymbol{H}(\mu,\ P) := PA + A^T P + \mu^2 C^T PC - (B^T P + \mu^2 D^T PC)^T \hat{R}^{-1}(B^T P + \mu^2 D^T PC) + Q = 0, \tag{21}$$

where $\hat{R} := R + \mu^2 D^T PD$ and $R = R^T$ is assumed to be sign indefinite and invertible.

It should be noted that if the parameter μ is sufficiently small, the fixed point algorithm is also useful in the sense that only the required workspace of $\Re^{n_i \times n_i}$ is required. Moreover, the solution can be obtained directly by using the `are` function in MATLAB.

By setting $\mu = 0$ for the previous SARE (21), the following ordinary algebraic Riccati equation (ARE) holds.

$$\boldsymbol{H}(0,\ \hat{P}) := \hat{P}A + A^T \hat{P} - \hat{P}BR^{-1}B^T \hat{P} + Q = 0, \tag{22}$$

where $\hat{P}$ is the zeroth-order solutions of SARE (21).

The asymptotic structure of the solutions $P = P(\mu)$ is given.

Lemma 3: There exists a small $\bar{\mu} > 0$ such that for all $\mu \in (0,\ \bar{\mu})$, SARE (21) permits a unique solution P in the neighbourhood of $\mu = 0$, which can be written as given in

$$P(\mu) = \hat{P} + O(\mu^2). \tag{23}$$

Proof: This can be done by applying an implicit function theorem to SARE (21). In order to do so, it is sufficient to show that the corresponding Jacobian is non-singular at $\mu = 0$. Obtaining the partial derivative of the function $\boldsymbol{H}(\mu,\ P)$, with respect to P, and setting $\mu = 0$ yields (24).

$$\left.\frac{\partial \mathrm{vec}\boldsymbol{H}(\mu,\ P)}{\partial(\mathrm{vec}P)^T}\right|_{\mu=0} = (A - BR^{-1}B^T\hat{P})^T \otimes I + I \otimes (A - BR^{-1}B^T\hat{P})^T. \tag{24}$$

Obviously, $A - BR^{-1}B^T\hat{P}$ is nonsingular because the ARE (22) has stabilizing solutions under stabilizable and detectable conditions. Thus, the corresponding Jacobian is non-singular at $\mu = 0$. The conclusion of Lemma 3 is obtained directly by using the implicit function theorem. ■

In order to obtain solutions for SARE (21), the following algorithm that is based on the fixed point algorithm is considered.

$$\begin{aligned} &P^{(i+1)}[A - \mu^2 B(\hat{R}^{(i)})^{-1}D^T P^{(i)}C] + [A - \mu^2 B(\hat{R}^{(i)})^{-1}D^T P^{(i)}C]^T P^{(i+1)} \\ &\quad -P^{(i+1)}B(\hat{R}^{(i)})^{-1}B^T P^{(i+1)} + \mu^2 C^T P^{(i)}C - \mu^4 C^T P^{(i)}D(\hat{R}^{(i)})^{-1}D^T P^{(i)}C + Q = 0,\ i = 0,\ 1,\ \ldots\ . \end{aligned} \tag{25}$$

where $P^{(0)} = \hat{P}$ and $\hat{R}^{(i)} := R + D^T P^{(i)} D$.

Theorem 2: Let us assume that the conditions of Lemma 1 hold. Then, there exists a small σ^* such that for all $\mu \in (0,\ \sigma^*)$, the fixed point algorithm (25) converges to the exact solution of P^* at a linear convergence rate. In other words, the following relations are satisfied.

$$\|P^{(i)} - P^*\| = O(\mu^{2(i+1)}),\ i = 0,\ 1,\ \ldots\ . \tag{26}$$

Proof: The proof of Theorem 2 can be obtained by mathematical induction. It is easy to verify that the first order approximation P corresponding to the small parameter μ is $\hat{P}$. It follows from this equation that

$$\|P^{(0)} - P\| = \|\hat{P} - P\| = O(\mu^2). \tag{27}$$

When $i = h$, $h \geq 1$, it is assumed that

$$\|P^{(h)} - P\| = O(\mu^{2(h+1)}). \tag{28}$$

Subtracting (21) from (25) and setting $i = h$, the following equation holds under the above assumption.

$$\begin{aligned}&(P^{(h+1)} - P)(A - B\hat{R}^{-1}B^TP - \mu^2 B\hat{R}^{-1}D^TPC) + (A - B\hat{R}^{-1}B^TP - \mu^2 B\hat{R}^{-1}D^TPC)^T(P^{(h+1)} - P)\\ &\quad -(P^{(h+1)} - P)BR^{-1}B^T(P^{(h+1)} - P) + \mu^2 C^T(P^{(h)} - P)C + O(\mu^{2(h+2)}) = 0.\end{aligned} \tag{29}$$

Therefore, under the stabilizable and detectable conditions, the following relations hold.

$$\|P^{(h+1)} - P\| = O(\mu^{2(h+2)}). \tag{30}$$

Consequently, the error equations (26) hold for all $i \in \mathbf{N}$. This completes the proof of Theorem 2. ■

C. Numerical Algorithms for Solving Reduced-order SARE (6)

For computing the maximal and stabilizing solution of (8) in the case of weight matrices with sign indefinite, we propose two iterative procedures.

1) An Algorithm on Basis of Stochastic Lyapunov Iterations:

Step 1. *Choose K_{ii}^0 as a stabilizing feedback gain for the subsystems in* (6). *This can be obtained by $K_{ii}^0 = W_{ii}X_{ii}^{-1}$, where for each i, the pair $(X_{ii},\ W_{ii})$ is a solution of the following LMI:*

$$\begin{bmatrix} \Gamma_i(X_{ii},\ W_{ii}) & C_{ii}X_{ii} + D_{ii}W_{ii} \\ X_{ii}C_{ii}^T + W_{ii}^TD_{ii}^T & -X_{ii} \end{bmatrix} < 0 \tag{31}$$

where $\Gamma_i(X_{ii},\ W_{ii}) = A_{ii}X_{ii} + X_{ii}A_{ii}^T + W_{ii}^TB_{ii}^T + B_{ii}W_{ii}$.

Construct $P_{ii}^{(1)}$ as a solution of the following LMI

$$\begin{aligned}&(A_{ii} + B_{ii}K_{ii}^0)^TP_{ii}^{(1)} + P_{ii}^{(1)}(A_{ii} + B_{ii}K_{ii}^0) + (C_{ii} + D_{ii}K_{ii}^0)^TP_{ii}^{(1)}(C_{ii} + D_{ii}K_{ii}^0)\\ &\quad +Q_{ii} + K_{ii}^{0T}R_{ii}K_{ii}^0 + I_{n_i} \leq 0,\ 1 \leq i \leq N,\end{aligned} \tag{32a}$$

$$K_{ii}^{(1)} = -(R_{ii} + D_{ii}^TP_{ii}^{(1)}D_{ii})^{-1}(B_{ii}P_{ii}^{(1)} + D_{ii}^TP_{ii}^{(1)}C_{ii}). \tag{32b}$$

Step k, $k \geq 2$. *Construct $P_{ii}^{(k)}$, $K_{ii}^{(k)}$ from*

$$\begin{aligned}&(A_{ii} + B_{ii}K_{ii}^{(k-1)})^TP_{ii}^{(k)} + P_{ii}^{(k)}(A_{ii} + B_{ii}K_{ii}^{(k-1)}) + (C_{ii} + D_{ii}K_{ii}^{(k-1)})^TP_{ii}^{(k)}(C_{ii} + D_{ii}K_{ii}^{(k-1)})\\ &\quad +Q_{ii} + K_{ii}^{(k-1)T}R_{ii}K_{ii}^{(k-1)} + \frac{1}{k}I_{n_i} = 0,\end{aligned} \tag{33a}$$

$$K_{ii}^{(k)} = -(R_{ii} + D_{ii}^TP_{ii}^{(k)}D_{ii})^{-1}(B_{ii}^TP_{ii}^{(k)} + D_{ii}^TP_{ii}^{(k)}C_{ii}). \tag{33b}$$

It can be seen (see for example [14]) that under Assumptions 1 and 2, the sequence $\{P_{ii}^{(k)}\}_{k\geq 1}$ is convergent and its limit is the maximal and stabilizing solution of (7), which verifies $R_{ii} + D_{ii}^T \bar{P}_{ii}^0 D_{ii} > 0$.

2) An Algorithm on Basis of Standard Lyapunov Iterations:

Step 1. *The same as in the previous algorithm.*

Step k**,** $k \geq 2$**.** *Construct* $P_{ii}^{(k)}, K_{ii}^{(k)}$ *by*

$$(A_{ii} + B_{ii}K_{ii}^{(k-1)})^T P_{ii}^{(k)} + P_{ii}^{(k)}(A_{ii} + B_{ii}K_{ii}^{(k-1)}) + (C_{ii} + D_{ii}K_{ii}^{(k-1)})^T P_{ii}^{(k-1)}(C_{ii} + D_{ii}K_{ii}^{(k-1)}) + Q_{ii} + K_{ii}^{(k-1)T} R_{ii} K_{ii}^{(k-1)} + \frac{1}{k} I_{n_i} = 0, \tag{34a}$$

$$K_{ii}^{(k)} = -(R_{ii} + D_{ii}^T P_{ii}^{(k-1)} D_{ii})^{-1}(B_{ii}^T P_{ii}^{(k)} + D_{ii}^T P_{ii}^{(k-1)} C_{ii}). \tag{34b}$$

In [15], one shows that under Assumptions 1 and 2, $P_{ii}^{(k)}$ converges to $\bar{P}_{ii}$ when $k \to \infty$.

Remark 3: The Newton-type algorithm is converges faster than the one based on Lyapunov iterations. However, it requires the solution of a more complicated linear equation as given in (34a). The algorithm based on Lyapunov iterations requires solutions of standard Lyapunov equations for each $k \geq 2$.

IV. Extension to Static Output Feedback

The static output feedback problem is one of the most important problems. The implementation of LQ control using the static output feedback was investigated by several researchers [10], [12]. Despite the reliable result obtained in [10], there still remains an important problem that should be solved analytically-the static output feedback case; this case has not been investigated. Moreover, in [12], the control-dependent noise has not been considered. Another difficulty that is faced when solving the LQ control using the static output feedback is the nonconvexity of the solution set. In order to obtain a feasible solution set, the Lagrange multiplier method is considered for the optimization of the cost.

In this section, the control $u(t)$ is restricted to the static output feedback.

$$u(t) := F_\varepsilon y(t), \quad y(t) := E_\varepsilon x(t), \tag{35}$$

where

$$F_\varepsilon := \begin{bmatrix} F_{11} & \varepsilon F_{12} & \cdots & \varepsilon F_{1N} \\ \varepsilon F_{21} & F_{22} & \cdots & \varepsilon F_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon F_{N1} & \varepsilon F_{N2} & \cdots & F_{NN} \end{bmatrix}, \ E_\varepsilon := \begin{bmatrix} E_{11} & \varepsilon E_{12} & \cdots & \varepsilon E_{1N} \\ \varepsilon E_{21} & E_{22} & \cdots & \varepsilon E_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon E_{N1} & \varepsilon E_{N2} & \cdots & E_{NN} \end{bmatrix}, \ y(t) := \begin{bmatrix} y_1(t) \\ \vdots \\ y_N(t) \end{bmatrix}$$

and $y_i(t) \in \Re^{r_i}$, $i = 1, \ \ldots, N$ represents the i-th output.

Using the static output feedback control of (35) and the assumption that $E[x(0)x^T(0)] = I_{\bar{n}}$, $\bar{n} := \sum_{i=1}^N n_i$, it is immediately determined that the closed-loop stochastic system is exponentially mean square stable (EMSS) [4]; further, the integral portion of $J(u, \ x(0))$ satisfies the relation in

$$J(u, \ x(0)) = \mathbf{Tr}[X_\varepsilon], \tag{36}$$

if there exists a unique solution for the following stochastic algebraic Lyapunov equality (SALE) of X_ε.

$$\boldsymbol{G}_1(X_\varepsilon,\ F_\varepsilon) := X_\varepsilon \hat{A}_\varepsilon + \hat{A}_\varepsilon^T X_\varepsilon + \hat{C}_\varepsilon^T X_\varepsilon \hat{C}_\varepsilon + E_\varepsilon^T F_\varepsilon^T R_0 F_\varepsilon E_\varepsilon + Q_\varepsilon = 0, \tag{37}$$

where $\hat{A}_\varepsilon := A_\varepsilon + B_\varepsilon F_\varepsilon E_\varepsilon$, $\hat{C}_\varepsilon := C_\varepsilon + D_\varepsilon F_\varepsilon E_\varepsilon$.

The solution of SALE (37) is assumed to have the following structure [11].

$$X_\varepsilon := \begin{bmatrix} X_{11} & \varepsilon X_{12} & \cdots & \varepsilon X_{1N} \\ \varepsilon X_{12}^T & X_{22} & \cdots & \varepsilon X_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon X_{1N}^T & \varepsilon X_{2N}^T & \cdots & X_{NN} \end{bmatrix}. \tag{38}$$

Substituting these matrices into SALE (37), setting $\varepsilon = 0$, and partitioning SALE (37), the following reduced-order in SALE (39) is obtained, where $\bar{X}_{ii}$ and $\bar{F}_{ii}$, $i = 1,\ \ldots\ , N$ are the zeroth-order solutions of SALE (37).

$$\bar{X}_{ii}\hat{A}_{ii} + \hat{A}_{ii}^T \bar{X}_{ii} + \hat{C}_{ii}^T \bar{X}_{ii} \hat{C}_{ii} + E_{ii}^T \bar{F}_{ii}^T R_{ii} \bar{F}_{ii} C_{ii} + Q_{ii} = 0, \tag{39}$$

where $\hat{A}_{ii} := A_{ii} + B_{ii}\bar{F}_{ii}E_{ii}$ and $\hat{C}_{ii} := C_{ii} + D_{ii}\bar{F}_{ii}E_{ii}$.

In order to develop the necessary conditions for this problem, $\bar{F}_{ii}$, $i = 1,\ \ldots\ , N$ must be restricted to the following set $\mathbf{F}_i := \{F_{ii} \in \Re^{m_i \times l_i}\ |$ there exists a unique symmetric matrix X_{ii} that satisfies SALE (39).$\}$

The asymptotic expansion of SALE (37) for $\varepsilon = 0$ is described by the following lemma.

Lemma 4: Let us suppose that $\bar{F}_{ii} \in \mathbf{F}_i$. There exists a small constant σ_1^* such that for all $\varepsilon \in (0,\ \sigma_1^*)$, SALE (37) admits a unique solution X_ε^* that can be expressed as given in

$$X_\varepsilon^* = \bar{X} + O(\varepsilon), \tag{40}$$

where $\bar{X} = \textbf{block diag}\begin{pmatrix} \bar{X}_{11} & \cdots & \bar{X}_{NN} \end{pmatrix}$.

Proof: This can be proved by applying the implicit function theorem to SALE (37). In order to do so, it is sufficient to show that the corresponding Jacobian is nonsingular at $\varepsilon = 0$. It should be noted that $\bar{F}_{ii} \in \mathbf{F}_i$ if and only if $I_{n_i} \otimes \hat{A}_{ii}^T + \hat{A}_{ii}^T \otimes I_{n_i} + \hat{C}_{ii}^T \otimes \hat{C}_{ii}^T$ is non-singular. Since the abovementioned relation is similar to that mentioned in [11], it is omitted. ■

The necessary conditions for the optimality will be obtained.

Theorem 3: Let us assume that $F_{ii} \in \mathbf{F}_i$ solves the static output feedback control problem. Then, it is necessary that there exist symmetric solutions X_ε and S_ε that satisfy SALE (41a) and SALE (41b), respectively; $F_{i\varepsilon}$ is obtained using (41c).

$$\boldsymbol{G}_1(X_\varepsilon,\ F_\varepsilon) = 0 \tag{41a}$$

$$\boldsymbol{G}_2(S_\varepsilon,\ F_\varepsilon) = S_\varepsilon \hat{A}_\varepsilon^T + \hat{A}_\varepsilon S_\varepsilon + \hat{C}_\varepsilon S_\varepsilon \hat{C}_\varepsilon^T + I_{\bar{n}} = 0, \tag{41b}$$

$$\boldsymbol{G}_3(X_\varepsilon,\ S_\varepsilon,\ F_\varepsilon) = \underline{R}_\varepsilon F_\varepsilon E_\varepsilon S_\varepsilon E_\varepsilon^T + (B_\varepsilon^T X_\varepsilon + D_\varepsilon^T X_\varepsilon C_\varepsilon) S_\varepsilon E_\varepsilon^T = 0, \tag{41c}$$

where $\underline{R}_\varepsilon := R_0 + D_\varepsilon^T X_\varepsilon D_\varepsilon > 0$.

Proof: The result can be proved by using the Lagrange multiplier method. First, the closed-loop cost with the static output feedback controller $u(t) = F_\varepsilon y(t) = F_\varepsilon E_\varepsilon x(t)$ is obtained by using the relation $J = \mathbf{Tr}[X_\varepsilon]$, where X_ε is the solution of SALE (37). Let us consider the Lagrange function $\mathcal{L}$

$$\mathcal{L}(X_\varepsilon,\ S_\varepsilon,\ F_\varepsilon) = \mathbf{Tr}\ [X_\varepsilon] + \mathbf{Tr}\ [\boldsymbol{G}_1(X_\varepsilon,\ F_\varepsilon)S_\varepsilon], \tag{42}$$

where S_ε is a symmetric matrix of Lagrange multipliers.

Using the Lagrange multiplier method, the necessary conditions for $F_{i\varepsilon}$ to be optimal can be determined by setting $\partial\mathcal{L}/X_\varepsilon$ and $\partial\mathcal{L}/F_{i\varepsilon}$ to zero and solving the resulting equations given in (41c) simultaneously for $F_{i\varepsilon}$. ■

It should be noted that Theorem 3 only provides the necessary conditions for a controller to be optimal.

If $E_\varepsilon S_\varepsilon E_\varepsilon^T$ is nonsingular, then (41c) may be solved for $F_{i\varepsilon}$ to obtain

$$F_\varepsilon = -\underline{R}_\varepsilon^{-1}(B_\varepsilon^T X_\varepsilon + D_\varepsilon^T X_\varepsilon C_\varepsilon)S_\varepsilon E_\varepsilon^T (E_\varepsilon S_\varepsilon E_\varepsilon^T)^{-1}. \tag{43}$$

In the remaining part of this section, we shall discuss the asymptotic structure of S_ε and F_ε.

Lemma 5: If $\bar{F}_{ii} \in \mathbf{F}_i$, there exists a small constant σ_2^* such that for all $\varepsilon \in (0,\ \sigma_2^*)$, SALEs (41a) and (41b), and the linear equation (41c) admit the positive definite solution S_ε^* and feedback gain F_ε^* that can be expressed as

$$S_\varepsilon^* = \bar{S} + O(\varepsilon),\ F_\varepsilon^* = \bar{F} + O(\varepsilon), \tag{44}$$

where $\bar{S} = \mathbf{block\ diag}\begin{pmatrix} \bar{S}_{11} & \cdots & \bar{S}_{NN} \end{pmatrix}$, $\bar{F} = \mathbf{block\ diag}\begin{pmatrix} \bar{F}_{11} & \cdots & \bar{F}_{NN} \end{pmatrix}$, $\underline{R}_{ii} = R_{ii} + D_{ii}^T \bar{X}_{ii} D_{ii}$,

$$\bar{S}_{ii}\hat{A}_{ii}^T + \hat{A}_{ii}\bar{S}_{ii} + \hat{C}_{ii}\bar{S}_{ii}\hat{C}_{ii}^T + I_{n_i} = 0, \tag{45a}$$

$$\underline{R}_{ii}\bar{F}_{ii}E_{ii}\bar{S}_{ii}E_{ii}^T + (B_{ii}^T\bar{X}_{ii} + D_{ii}^T\bar{X}_{ii}C_{ii})\bar{S}_{ii}E_{ii}^T. \tag{45b}$$

Without loss of generality, as an additional technical assumption, we suppose that F_{ii} is confined to the following set. $\mathbf{L}_i := \{F_{ii} \in \mathbf{F}_i \mid\ E_{ii}\bar{S}_{ii}E_{ii}^T > 0$, where $\bar{S}_{ii}$ satisfies (45a). $\}$.

The positive definiteness condition holds, for example, when $\bar{S}_{ii}$ is positive definite, and when C_{ii} has a full row rank. In this case, $\bar{F}_{ii}$ can be written as given in

$$\bar{F}_{ii} = -\underline{R}_{ii}^{-1}(B_{ii}^T\bar{X}_{ii} + D_{ii}^T\bar{X}_{ii}C_{ii})\bar{S}_{ii}E_{ii}^T(E_{ii}\bar{S}_{ii}E_{ii}^T)^{-1}. \tag{46}$$

Let us consider the following new iterative algorithm.

$$X_{ii}^{(n+1)}\hat{A}_{ii}^{(n)} + \hat{A}_{ii}^{(n)T}X_{ii}^{(n+1)} + \hat{C}_{ii}^T X_{ii}^{(n+1)}\hat{C}_{ii} + E_{ii}^T F_{ii}^{(n)T} R_{ii} F_{ii}^{(n)} E_{ii} + Q_{ii} = 0, \tag{47a}$$

$$S_{ii}^{(n+1)}\hat{A}_{ii}^{(n)T} + \hat{A}_{ii}^{(n)}S_{ii}^{(n+1)} + \hat{C}_{ii}S_{ii}^{(n+1)}\hat{C}_{ii}^T + I_{\bar{n}} = 0, \tag{47b}$$

$$F_{ii}^{(n+1)} = F_{ii}^{(n)} - \alpha[\underline{R}_{ii}^{-1}(B_{ii}^T X_{ii}^{(n+1)} + D_{ii}^{(n)T} X_{ii}^{(n+1)} C_{ii})S_{ii}^{(n+1)}E_{ii}^T(E_{ii}S_{ii}^{(n+1)}E_{ii}^T)^{-1} + F_{ii}^{(n)}], \tag{47c}$$

where where $\hat{A}_{ii}^{(n)} := A_{ii} + B_{ii}F_{ii}^{(n)}E_{ii}$, $\hat{C}_{ii}^{(n)} := C_{ii} + D_{ii}F_{ii}^{(n)}E_{ii}$ and $\underline{R}_{ii}^{(n)} := R_{ii} + D_{ii}^T\bar{X}_{ii}^{(n)}D_{ii}$, $n = 0,\ 1,\ \ldots$ and $\alpha \in (0,\ 1]$ is chosen so as to ensure the minimum is not overshot, that is, $J^{(n+1)} = \mathbf{Tr}[X_{ii}^{(n+1)}] < J^{(n)} = \mathbf{Tr}[X_{ii}^{(n)}]$. Moreover, $F_{ii}^{(0)}$, $i = 1,\ \ldots\ , N$ is chosen as the initial condition such that the reduced-order closed-loop system $dx_i(t) = [A_{ii} + B_{ii}F_{ii}^{(0)}E_{ii}]x_i(t) + [C_{ii} + D_{ii}F_{ii}^{(0)}E_{ii}]x_i(t)dw(t)$ is EMSS.

$$K_{\mathrm{lmi}} = \begin{bmatrix} -1.0179 & -2.0191e-1 & -1.0508 & -2.0382 & -4.2781e-2 & -1.5471e-1 & -1.1503e-5 & -6.9335e-2 & -6.5577e-2 \\ -9.1683e-2 & -1.0177e-1 & 5.8259e-2 & -1.6340e-1 & -1.0531e-1 & 1.4590e-2 & -4.8600e-3 & 3.1346e-2 & -4.9614e-1 \end{bmatrix}.$$

Theorem 4: The sequence $F_{ii}^{(n)}$, $n = 0,\ 1,\ \ldots$ in (47c) converges to a stationary point in $\mathbf{F}_i$.

Before proving the theorem, we define the following set.

$\mathbf{G}_i := \{\bar{F}_{ii} \in \Re^{m_i \times r_i} \ |\ \ \hat{A}_{ii}$ is Hurwitz. $\}$.

Proof: From (45b), the gradient of the Lagrangian with respect to $\bar{F}_{ii}$ is given by $L := \underline{R}_{ii}\bar{F}_{ii}E_{ii}\bar{S}_{ii}E_{ii}^T + (B_{ii}^T\bar{X}_{ii} + D_{ii}^T\bar{X}_{ii}C_{ii})\bar{S}_{ii}E_{ii}^T$. The inner product of the search direction $\Delta\bar{F}_{ii}$ with the gradient L is $\beta(\bar{F}_{ii}) := \mathbf{Tr}[L\Delta\bar{F}_{ii}^T]$, where $\Delta\bar{F}_{ii} := -\underline{R}_{ii}^{-1}(B_{ii}^T\bar{X}_{ii} + D_{ii}^T\bar{X}_{ii}C_{ii})\bar{S}_{ii}E_{ii}^T(E_{ii}\bar{S}_{ii}E_{ii}^T)^{-1} - \bar{F}_{ii}$. We have $\beta(\bar{F}_{ii}) := -\mathbf{Tr}[\Lambda_i^T\Lambda_i] < 0$, where $\Lambda_i := (E_{ii}\bar{S}_{ii}E_{ii}^T)^{-1/2}[E_{ii}\bar{S}_{ii}E_{ii}^T\bar{F}_{ii}^T\underline{R}_{ii}^{1/2} + E_{ii}\bar{S}_{ii}(B_{ii}^T\bar{X}_{ii} + D_{ii}^T\bar{X}_{ii}C_{ii})^T\underline{R}_{ii}^{-1/2}]$ if $\bar{F}_{ii} \in \mathbf{G}_i$, $i = 1,\ \ldots, N$ and $L \neq 0$. The continuity of the gradient implies that for each iteration, there exists some α^* that is sufficiently small such that (47c) is satisfied for $0 < \alpha \leq \alpha^*$. Under these conditions, the sequence $J^{(n)}$, $n = 0,\ 1,\ \ldots$ with $F_{ii}^{(n)}$ is convergent because it is monotonic and bounded. Finally, the continuity of J implies that the sequence $F_{ii}^{(n)}$, $n = 0,\ 1,\ \ldots$ is also convergent. This completes the proof of Theorem 4. ■

V. Numerical Example

In order to demonstrate the efficiency of the proposed algorithm, a numerical example is provided. The system matrices are given as follows.

$$A_{11} = \begin{bmatrix} 0 & 0.315 & 0 & -0.315 \\ 0 & 0 & 1 & 0 \\ 0 & -1.888 & -0.0498 & 1.888 \\ 1 & 0 & 0 & 1 \end{bmatrix},\ A_{12} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A_{21} = \begin{bmatrix} 0 & 1.888 & 0 & -1.888 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0.41666 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},\ A_{22} = \begin{bmatrix} -0.0498 & 0 & 0 & 0 & 6 \\ 0 & -3.333 & 0 & 3.333 & 0 \\ 0 & 0 & -3.333 & 0 & 3.333 \\ 0 & 0 & 0 & -1 & 0 \\ -0.41666 & 0 & 0 & 0 & -1 \end{bmatrix},$$

$$B_{11}^T = \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix},\ B_{22}^T = \begin{bmatrix} 0.5 & 0 & 0 & 0 & 0.5 \end{bmatrix},$$

$$C_{11} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0.249 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},\ C_{22} = \begin{bmatrix} 0.249 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$D_{11}^T = \begin{bmatrix} 0 & 0 & -0.6 & 0 \end{bmatrix},\ D_{22}^T = \begin{bmatrix} -0.6 & 0 & 0 & 0 & 0 \end{bmatrix},\ E_{11} = \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix},\ E_{22} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \end{bmatrix},$$

$$B_{12}=0,\ B_{21}=0,\ C_{12}=0,\ C_{21}=0,\ D_{12}=0,\ D_{21}=0,\ E_{12}=0,\ E_{21}=0,$$
$$Q=\textbf{block diag}\begin{pmatrix} I_4 & 0.25I_5 \end{pmatrix},\ R=\textbf{block diag}\begin{pmatrix} 1 & 2 \end{pmatrix}.$$

A. State Feedback Case

Referring to the design procedure on the basis of SDP (13), the parameter-independent control (15d) is given in

$$\bar{K}=-\bar{R}^{-1}(\bar{B}^T\bar{P}+\bar{D}^T\bar{P}\bar{C})=\begin{bmatrix} \bar{K}_{\text{lmi11}} & 0 \\ 0 & \bar{K}_{\text{lmi22}} \end{bmatrix}, \tag{48}$$

where

$$\bar{K}_{\text{lmi11}}=\begin{bmatrix} -1.0385 & -2.1923e-1 & -1.0525 & -2.0446 \end{bmatrix},$$
$$\bar{K}_{\text{lmi22}}=\begin{bmatrix} -9.5168e-2 & 0 & -4.9027e-3 & 0 & -4.5492e-1 \end{bmatrix}.$$

Now, setting $\varepsilon=0.1$, the optimal feedback control (3) $u^*(t):=K_{\text{lmi}}x(t)$ is given at the top of the previous page. It should be noted that the optimal one can be computed via the fixed point algorithm (25).

We evaluate the costs using the near-optimal controller (48). We assume that the initial conditions are zero mean independent random vector with covariance matrix $E[x(0)x(0)^T]=I_9$. The average values of the performance index are $E[J_{\text{app}}]=2.5965e+1$, $E[J_{\text{opt}}]=2.5609e+1$. Hence, the loss of performance J_{app} is less than 1.3919% when compared with J_{opt}. The values of the cost functional for various ε are listed in Table 1, where $\phi=(E[J_{\text{app}}]-E[J_{\text{opt}})/\varepsilon^2$.

Table 1. Degradation of cost.

ε	$E[J_{\text{app}}]$	$E[J_{\text{opt}}]$	ϕ
$1.0e-1$	$2.5965e+1$	$2.5609e+1$	$3.5646e+1$
$1.0e-2$	$2.5577e+1$	$2.5573e+1$	$3.6560e+1$
$1.0e-3$	$2.5573e+1$	$2.5573e+1$	$3.6570e+1$
$1.0e-4$	$2.5573e+1$	$2.5573e+1$	$3.6570e+1$
$1.0e-5$	$2.5573e+1$	$2.5573e+1$	$3.6570e+1$

It is easy to verify that $J_{\text{app}}=J_{\text{opt}}+O(\varepsilon^2)$ because of $\phi<\infty$. Thus, formula (14) has been verified.

B. Static Output Feedback Case

The small parameter is chosen as $\varepsilon=0.001$. It should be noted that we cannot apply the technique proposed in [3] to this system since the static output feedback case is considered. By using the proposed algorithm (47), an exact solution of SALE (41) and the parameter independent static output feedback gains (46) are given below.

$$F_\varepsilon=\begin{bmatrix} -2.2302 & -1.6537e-4 \\ -2.6681e-3 & -8.8309e-2 \end{bmatrix}, \tag{49a}$$
$$\bar{F}_{11}=-2.2302,\ \bar{F}_{22}=-8.8310e-2. \tag{49b}$$

We find that the algorithm (47) under $\alpha=0.5$ converges to the exact solution with an accuracy of $\|G_1(X_\varepsilon^{(i)},\ F_\varepsilon)\|+\|G_2(S_\varepsilon^{(i)},\ F_\varepsilon)\|+\|G_3(X_\varepsilon^{(i)},\ S_\varepsilon^{(i)},\ F_\varepsilon)\|<10^{-12}$ after 103 iterations. Therefore, it can be seen that the algorithm (47) works well and it is reliable.

VI. Conclusion

In this paper, the indefinite linear quadratic control involving state- and control-dependent noise in weakly coupled large-scale stochastic systems has been investigated. Since the sign indefinite of the control and state weighting matrices in the cost function is allowed, we can apply the proposed method to solve a wider class of problems as compared with the existing methods. Moreover, by solving the reduced-order LMI, the proposed controller can be obtained without using any numerical algorithms. As a result, although the positive weak coupling parameter is very small or unknown, it is possible to design the controller effectively. As another important implication, the static output feedback case has also been investigated. Finally, the numerical example demonstrates the reliability of the proposed method.

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