

STABLE EXTENDIBILITY OF NORMAL BUNDLES OVER THE LENS SPACES

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ABSTRACT. We study the stable extendibility of \mathbb{R} -vector bundles over the $(2n+1)$ -dimensional standard lens space $L^n(p)$ with odd prime p , by focusing on the normal bundle $\nu_n^t(p)$ for an immersion of $L^n(p)$ to the Euclidean space \mathbb{R}^{2n+1+t} . We show several concrete cases in which $\nu_n^t(p)$ is stably extendible to $L^k(p)$ for any k with $k \geq n$, and also in some cases we determine exact values of m for which $\nu_n^t(p)$ is stably extendible to $L^m(p)$ but not stably extendible to $L^{m+1}(p)$.

1. INTRODUCTION

Let F be the real number field \mathbb{R} or the complex number field \mathbb{C} . Then, for a subspace A of a space X , a t -dimensional F -vector bundle α over A is said to be *extendible* to X if α is equivalent to the induced bundle $i^*\beta$ of a t -dimensional F -vector bundle β over X under the inclusion map $i: A \rightarrow X$. If $i^*\beta$ is stably equivalent to α instead of the equivalence, namely $i^*\beta + \theta$ is equivalent to $\alpha + \theta$ for a trivial vector bundle θ , α is said to be *stably extendible* to X . (Cf. [20], [6]) Obviously, if α is extendible to X , then it is stably extendible to X .

Originally, Schwarzenberger [20], [4, Appendix I] has studied extendibility of vector bundles over the real or complex projective spaces and shown an interesting characterization of infinitely extendible vector bundles. Related topological results have been obtained by Rees [3], [19], Adams–Mahmud [1], Thomas [23] and ours [6], [7]. The infinite extendibility of \mathbb{C} -vector bundles has also pulled attention from the algebraic point of view (cf., Barth–Vane de Ven [2], Sato [21]). Thus, an algebraic–topological analysis on (stable) extendibility of \mathbb{R} -vector bundles is considered to be worth studying.

Let $L^n(p) = S^{2n+1}/(\mathbb{Z}/p)$ for $n \geq 0$ denote the $(2n+1)$ -dimensional standard lens space mod p . Throughout this paper, we assume that p is an odd prime number. Then, for any \mathbb{R} -vector bundle ζ over $L^n(p)$, we set

$$(1.1) \quad s(\zeta) = \max\{m \in \mathbb{N} \mid m \geq n \text{ and } \zeta \text{ is stably extendible to } L^m(p)\}$$

if the maximum exists; and, we set $s(\zeta) = \infty$ if ζ is stably extendible to $L^m(p)$ for any $m \geq n$, namely when ζ is infinitely stably extendible.

Kobayashi-Maki-Yoshida [13], [14] and Kobayashi-Komatsu [11], [12] have studied the (stable) extendibility of vector bundles over $L^n(p)$, and shown some detailed results, in particular in the case $p = 3$. In [8] and [9], we have studied

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the stable extendibility of the tangent bundle $\tau_n(p)$ over $L^n(p)$, and obtained the following result.

Theorem 1.1. ([9, Theorems 1.2, 1.3].) *For any odd prime p , $s(\tau_n(p)) = \infty$ if $p - 3 \leq n \leq p$, and $s(\tau_n(p)) = 2n + 1$ if $n \geq p + 1$.*

Obviously, $\tau_0(p)$ and $\tau_1(p)$ are trivial vector bundles, and so infinitely extendible. In [13, Lemma 5.2], it is remarked that any orientable 2-plane bundle over the n -skeleton of a CW-complex K with $n \geq 3$ is always extendible to K . Related to such low dimensional phenomena, we can show the following.

Theorem 1.2. *Let p be an odd prime number; and, assume that $1 \leq n \leq 3$. Then, $s(\alpha) = \infty$ for any m -dimensional \mathbb{R} -vector bundle α over $L^n(p)$ with $m \geq 4$.*

Let $\nu_n^t(p)$ denote the normal bundle of an immersion $L^n(p) \rightarrow \mathbb{R}^{2n+1+t}$ for $t > 0$. If $\nu_n^t(p)$ and $\nu_n^{t'}(p)$ are two normal bundles over $L^n(p)$, they are stably equivalent in the sense that $\nu_n^t(p) + \theta$ and $\nu_n^{t'}(p) + \theta'$ are equivalent for trivial vector bundles θ and θ' of some dimensions. Thus, for a fixed prime p , the value $s(\nu_n^t(p))$ depends only on n and t if there exists an immersion $L^n(p) \rightarrow \mathbb{R}^{2n+1+t}$, and we have $s(\nu_n^t(p)) \leq s(\nu_n^{t'}(p))$ if $t \leq t'$.

Sjerve[22] has shown that $L^n(p)$ is immersible to $\mathbb{R}^{2n+2\lfloor n/2 \rfloor + 2}$ for any n , where $\lfloor r \rfloor$ denotes the maximal integer less than or equal to a rational number r . Thus, it is reasonable to investigate the value of $s(\nu_n^t(p))$ for any $t \geq 2\lfloor n/2 \rfloor + 1$. Furthermore, by the stability properties of vector bundles (cf., Husemoller [5, Chapter 9, Proposition 1.1 and Theorem 1.5]), for $t \geq 2n + 2$, we have an equivalence $\nu_n^t(p) \cong \nu_n^{2n+1} + (t - 2n - 1)$, and stable extendibility of $\nu_n^t(p)$ coincides with its extendibility. Thus, $s(\nu_n^t(p))$ for $t = 2n + 1$ or $2n + 2$ seems to be a suitable object for the first inquiry.

Concerning the infinite stable extendibility of $\nu_n^t(p)$, we have the following theorem, by which and Theorem 1.2 we could conjecture that $s(\nu_n^t(p)) = \infty$ if $n \leq p$.

Theorem 1.3. *Let $(p - 1)/2 \leq n \leq p$ for a prime $p \geq 5$, and $0 \leq n \leq 5$ for $p = 3$. Then, $\nu_n^t(p)$ for $t \geq 2n + 1$ is stably equivalent to the Whitney sum of 2-plane bundles and a trivial bundle, and thus $s(\nu_n^t(p)) = \infty$.*

We also show a corresponding result for $t \geq 2\lfloor n/2 \rfloor + 1$ in Proposition 3.4.

In the case $p = 3$ or 5 , we have some more explicit result as follows, in which the case of $p = 3$ and $t = 2n + 2$ has been shown in [15, Lemma B].

Theorem 1.4. *Let $p = 3$ or 5 ; and, assume that $t = 2n + 1$ or $t = 2n + 2$. Then, $s(\nu_n^t(p)) = \infty$ if and only if $0 \leq n \leq 5$. Furthermore, $s(\nu_n^t(p)) = \infty$ if and only if $\nu_n^t(p)$ is stably equivalent to the Whitney sum of 2-plane bundles and a trivial bundle.*

Schwarzenberger [20] has shown that any infinitely extendible \mathbb{R} -vector bundle over the real projective space $\mathbb{R}P^n$ is equivalent to the Whitney sum of 2-plane bundles and a trivial bundle. Thus, we can say that $\nu_n^t(p)$ for the values of n and t in Theorem 1.3 or Theorem 1.4 satisfy ‘‘a stable Schwarzenberger’s property.’’

If $s(\nu_n^t(p)) < \infty$, it means that $s(\nu_n^t(p))$ is not stably decomposable to the Whitney sum of 2-plane bundles and a trivial bundle. From Theorem 1.4, we are tempted to conjecture that $s(\nu_n^t(p)) < \infty$ if $n \geq 2p$. Actually, we have the following theorem.

Theorem 1.5. *For any prime $p \geq 7$, $s(\nu_n^t(p)) < \infty$ if $n \geq 2p$ and $t \leq 2n + 2$.*

Next, we consider a case when the actual value of $s(\nu_n^t(p)) < \infty$ can be specified. We denote by $\lceil r \rceil$ the minimal integer greater than or equal to a rational number r , and set

$$l_a(n, p) = ap^{\lceil (n-1)/(p-1) \rceil} - (n+1)$$

for an integer $a > 0$. Let η_n be the canonical \mathbb{C} -line bundle over $L^n(p)$; and, let $r(\eta_n)$ be the underlying \mathbb{R} -plane bundle. Then, the normal bundle $\nu_n^t(p)$ is stably equivalent to $l_a(n, p)r(\eta_n)$ if $l_a(n, p) \geq 0$ (see Lemma 2.2). For an odd prime p and positive integers n and t , we consider the following condition:

$$(1.2) \quad \binom{l_a(n, p)}{\lceil (t+1)/2 \rceil} \not\equiv 0 \pmod{p} \quad \text{and} \quad p^{\lceil n/(p-1) \rceil} > \lceil (t+1)/2 \rceil,$$

where $\binom{c}{d}$ denotes the binomial coefficient. We remark that the inequality in (1.2) is satisfied when $n \geq 2p - 2$ and $t \leq 2n + 1$ (see Lemma 5.4).

Then, we have the following.

Theorem 1.6. *Assume that $t \geq n \geq p + 1$ for a prime $p \geq 5$; and assume that $t \geq n \geq 6$ for $p = 3$. Then, if the condition (1.2) is satisfied for an integer $a > 0$, we have $s(\nu_n^t(p)) = t$ (resp. $t - 1 \leq s(\nu_n^t(p)) \leq t + 1$) when t is an odd integer (resp. an even integer).*

As a special case, we have the following corollary.

Corollary 1.7. *When $p \geq 5$ and $n = 2p^k \pm 1$ for an integer $k \geq 1$, we have $s(\nu_n^{2n+1}(p)) = 2n + 1$; and, $\nu_n^n(p)$ is not stably extendible to $L^{n+1}(p)$.*

We shall show some general consequences of Theorem 1.6 in Corollary 5.5 and Corollary 5.6, which include Corollary 1.7.

From here, we organize this paper as follows. In the next section, we recall the K -groups of the lens space and prove Theorem 1.2; and, in the section 3, we consider infinitely extendible cases and prove Theorem 1.3. In the section 4, we investigate upper bounds for $s(\nu_n^t(p))$ and prove Theorem 1.5; and, in the section 5, we prove Theorem 1.6, Corollary 5.5 and Corollary 5.6. In the last section, we prove Theorem 1.4.

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2. K -GROUPS OF THE LENS SPACES

For any odd prime p , the structures of the reduced unitary K -group $K(L^n(p))$ and the reduced orthogonal K -group $KO(L^n(p))$ has been determined by Kambe [10], as in the below.

Let η_n be the canonical \mathbb{C} -line bundle over $L^n(p)$, which is the induced bundle from the canonical \mathbb{C} -line bundle over the complex projective space $\mathbb{C}P^n$ under the usual projection $L^n(p) \rightarrow \mathbb{C}P^n$. Then, we put $\sigma_n = \eta_n - 1 \in K(L^n(p))$.

Let $r : K(X) \rightarrow KO(X)$ and $c : KO(X) \rightarrow K(X)$ be the homomorphisms induced by the real restriction and the complexification of vector bundles, respectively. We put $\bar{\sigma}_n = r(\sigma_n) = r(\eta_n) - 2 \in KO(L^n(p))$. Also, let $L_0^n(p)$ be the $2n$ -skeleton of $L^n(p)$ (see [10]); and, let $j : L_0^n(p) \rightarrow L^n(p)$ be the natural inclusion map.

Then, the K -groups $K(L^n(p))$ and $KO(L^n(p))$ are represented as follows.

Theorem 2.1. ([10, Theorem 1, 2, Lemma 3.4]) *Let p be an odd prime number.*

(1) *Put $n = s(p-1) + r$ with $0 \leq r \leq p-2$. Then,*

$$K(L^n(p)) = \bigoplus_{i=1}^r \mathbb{Z}/p^{s+1}\{\sigma_n^i\} \oplus \bigoplus_{j=r+1}^{p-1} \mathbb{Z}/p^s\{\sigma_n^j\},$$

and $j^ : K(L^n(p)) \rightarrow K(L_0^n(p))$ is an isomorphism. Furthermore, we have $r(K(L_0^n(p))) = KO(L_0^n(p))$.*

(2) *The homomorphism $j^* : KO(L^n(p)) \rightarrow KO(L_0^n(p))$ induces the following isomorphism of abelian groups:*

$$KO(L^n(p)) \cong \begin{cases} KO(L_0^n(p)) & \text{if } n \not\equiv 0 \pmod{4}, \\ \mathbb{Z}/2 \oplus KO(L_0^n(p)) & \text{if } n \equiv 0 \pmod{4}. \end{cases}$$

(3) *Let $q = (p-1)/2$ and $n = s(p-1) + r$ with $0 \leq r \leq p-2$. Then,*

$$KO(L_0^n(p)) = \bigoplus_{i=1}^{\lfloor r/2 \rfloor} \mathbb{Z}/p^{s+1}\{\bar{\sigma}_n^i\} \oplus \bigoplus_{j=\lfloor r/2 \rfloor + 1}^q \mathbb{Z}/p^s\{\bar{\sigma}_n^j\}.$$

As in the previous section, $\lceil r \rceil$ (resp. $\lfloor r \rfloor$) denotes the smallest (resp. largest) integer more than (resp. less than) or equal to a rational number r . Also, we have set

$$(2.1) \quad l_a(n, p) = ap^{\lceil (n-1)/(p-1) \rceil} - (n+1)$$

for an integer $a \geq 1$. By Theorem 2.1, the order of $\bar{\sigma}_n = r(\eta_n) - 2$ in $KO(L^n(p))$ is equal to $p^{\lceil (n-1)/(p-1) \rceil}$. Thus, in $KO(L^n(p))$ we have

$$l_a(n, p)(r(\eta_n) - 2) = -(n+1)(r(\eta_n) - 2).$$

Since $\nu_n^t(p)$ is the normal bundle of an immersion $L^n(p) \rightarrow \mathbb{R}^{2n+1+t}$, we have an equivalence

$$\nu_n^t(p) + \tau_n(p) \cong 2n + 1 + t.$$

of vector bundles. Also, we have

$$\tau_n(p) + 1 \cong (n+1)r(\eta_n)$$

(cf., [17, Chapter 6, Corollary 1.6]). Then, there are equations

$$\begin{aligned} \nu_n^t(p) - t &= -(\tau_n(p) - (2n+1)) = -(n+1)(r(\eta_n) - 2) \\ &= l_a(n, p)(r(\eta_n) - 2). \end{aligned}$$

in $KO(L^n(p))$. Thus, we have the following.

Lemma 2.2. *Assume that $l_a(n, p) \geq 0$ for a positive integer a . Then, $\nu_n^t(p)$ and $l_a(n, p)r(\eta_n)$ is stably equivalent in the sense that $\nu_n^t(p) + \theta \cong l_a(n, p)r(\eta_n) + \theta'$ for trivial vector bundles θ and θ' of some dimensions.*

Now, we begin the proof of Theorem 1.2. By Theorem 2.1, we have $KO(L^1(p)) = 0$. Thus, any \mathbb{R} -vector bundle α over $L^1(p)$ is stably trivial, and we have $s(\alpha) = \infty$. Although this is enough for Theorem 1.2 in the case of $L^1(p)$, we remark the following unstable property.

Lemma 2.3. *Any \mathbb{R} -vector bundle α over $L^1(q)$ with odd integer q is trivial.*

Proof. Since $H^1(L^1(q); \mathbb{Z}/2) = 0$, the first Whitney class $w_1(\alpha)$ is 0. Thus, α is orientable, and also α is trivial when α is an \mathbb{R} -line bundle. Thus, we assume that the dimension m of α satisfies $m \geq 2$. Then, α is classified by an element of the base point free homotopy set $[L^1(q), BSO(m)]$, where $BSO(m)$ is a connected classifying space of the rotation group $SO(m)$. Then, it is sufficient to show that the base point preserving homotopy set $[L^1(q), BSO(m)]_*$ consists of 1 element, since $[L^1(q), BSO(m)]_* \rightarrow [L^1(q), BSO(m)]$ is surjective. From a cell structure of $L^1(p) = (S^1 \cup_q e^2) \cup e^3$, we have an exact sequence

$$[S^3, BSO(m)]_* \rightarrow [L^1(q), BSO(m)]_* \rightarrow [S^1 \cup_q e^2, BSO(m)]_*$$

of sets. But, $[S^3, BSO(m)]_* = \pi_3(BSO(m)) = \pi_2(SO(m)) = 0$, where $\pi_i(X)$ denotes the i -dimensional homotopy group of a space X . Since there is an exact sequence

$$\pi_2(BSO(m)) = \mathbb{Z}/2 \xrightarrow{\times q} \pi_2(BSO(m)) \rightarrow [S^1 \cup_q e^2, BSO(m)]_* \rightarrow \pi_1(BSO(m)) = 0,$$

$[S^1 \cup_q e^2, BSO(m)]_*$ has only 1 element. Thus, $[L^1(q), BSO(m)]_*$ consists of 1 element, and we have completed the proof. \square

We set $M = L^2(p)$ or $M = L^3(p)$. Then, by Theorem 2.1,

$$(2.2) \quad KO(M) = \mathbb{Z}/p\{\bar{\sigma}\}.$$

Using this fact, we have the following.

Lemma 2.4. *Let α be any \mathbb{R} -vector bundle of dimension $m \geq 4$ over M . Then, α is stably equivalent to an m -dimensional vector bundle which is the sum of two 2-plane bundles and a trivial vector bundle, and thus $s(\alpha) = \infty$.*

Proof. Let $p_1(\beta)$ (resp. $c_1(\gamma)$) denote the first Pontrjagin class (resp. the first Chern class) of an \mathbb{R} -vector bundle β (resp. a \mathbb{C} -vector bundle) (cf. [18]). We refer the necessary properties about $p_1(\beta)$ and $c_1(\gamma)$ here to the first part of the section 4. Then, $x = c_1(\eta_n) \in H^2(M; \mathbb{Z}) = \mathbb{Z}/p$ and $p_1(r(\eta_n)) = x^2 \in H^4(M; \mathbb{Z}) = \mathbb{Z}/p$ are the respective generators, where $n = 2$ and 3 according as $M = L^2(p)$ and $L^3(p)$. We also have $p_1(r(\eta_n^k)) = k^2 x^2$ for any positive integer k , where η_n^k is the tensor product over \mathbb{C} of k numbers of η_n .

By (2.2), $\alpha - m = a(r(\eta_n) - 2)$ and $r(\eta_n^k) - 2 = b_k(r(\eta_n) - 2)$ in $KO(M) = \mathbb{Z}/p$ for some integers a and b_k , respectively. But, comparing the first Pontrjagin classes on both sides of the latter equation, we have $b_k = k^2$ and thus $r(\eta_n^k) - 2 = k^2(r(\eta_n) - 2)$.

It is known that there are positive integers i and j satisfying

$$i^2 + j^2 \equiv a \pmod{p}$$

(cf., [16, II, Proposition 3.4]). Hence, $\alpha - m = r(\eta_n^i) + r(\eta_n^j) - 4$ in $KO(M)$. Since $m \geq 4$, α is stably equivalent to the m -dimensional vector bundle $r(\eta^i) + r(\eta^j) + (m - 4)$. Since the 2-plane bundle $r(\eta_n^k)$ for any $k \geq 0$ is infinitely extendible, we obtain the required result. \square

We have established the proof of Theorem 1.2 by Lemma 2.3 and Lemma 2.4. Moreover, by the above deduction, we can remark that Theorem 1.2 is still valid for the standard lens space of mod $q = p^k$ for any $k \geq 1$.

3. INFINITE EXTENDIBILITY

Recall that $\nu_n^t(p)$ is the normal bundle of an immersion $L^n(p) \rightarrow \mathbb{R}^{2n+1+t}$ and $s(\nu_n^t(p))$ is the value defined in (1.1). Also, let $l_a(n, p)$ be the integer given in (2.1). We remark again that p denotes an odd prime number.

By Lemma 2.2, if $0 \leq 2l_a(n, p) \leq t$ holds, then $\nu_n^t(p)$ is stably equivalent to the t -dimensional vector bundle $l_a(n, p)r(\eta_n) + (t - 2l_a(n, p))$. Thus, we have the following lemma, which implies that $s(\nu_n^t(p)) = \infty$ if t is sufficiently large for any fixed p and n .

Lemma 3.1. *$s(\nu_n^t(p)) = \infty$ if $0 \leq 2l_a(n, p) \leq t$ for some integer $a \geq 1$. Furthermore, in this case, $\nu_n^t(p)$ is stably equivalent to the Whitney sum of 2-plane bundles and a trivial bundle.*

Now, we examine the cases when the condition in Lemma 3.1 is satisfied. First, we consider the condition $0 \leq 2l_a(n, p)$.

Lemma 3.2. *For an odd prime p and an integer $n \geq 0$, we have the following.*

- (1) $l_1(n, p) \geq 0$ if and only if $n \neq 1$ and $n \neq p$.
- (2) $l_a(n, p) \geq 0$ for any $a \geq 2$.

Proof. We shall prove (1) and (2) simultaneously. Since $l_a(0, p) = a - 1$ and $l_a(1, p) = a - 2$, the conclusions hold for $n = 0$ and $n = 1$ obviously. Thus, we assume $n \geq 2$; and, put $n = (s - 1)(p - 1) + k + 1$ using integers $s \geq 1$ and $1 \leq k \leq p - 1$. Then, $\lceil (n - 1)/(p - 1) \rceil = s$ and $l_a(n, p) = ap^s - ((s - 1)(p - 1) + k + 2)$. Now, consider the real variable function

$$f_{a,k}(x) = ap^x - ((x - 1)(p - 1) + k + 2)$$

for fixed p , a and k . Then, the inequality $l_a(n, p) \geq 0$ holds if and only if $f_{a,k}(s) \geq 0$. We have $f'_{a,k}(x) = a(\log p)p^x - (p - 1)$ and $f''_{a,k}(x) = a(\log p)^2 p^x > 0$ for any $x > 0$. Since $f'_{a,k}(1) = (a \log p - 1)p + 1 > 0$, $f_{a,k}(x)$ is monotonously increasing for $x \geq 1$. Now, $f_{a,k}(1) = ap - (k + 2)$. Thus, when $a = 1$ and $1 \leq k \leq p - 2$ or when $a \geq 2$ and $1 \leq k \leq p - 1$, we have $f_{a,k}(1) \geq 0$ and hence $f_{a,k}(s) \geq 0$ for any $s \geq 1$. When $a = 1$ and $k = p - 1$, we have $f_{1,p-1}(1) = l_1(p, p) = -1$. Also, for any $s \geq 2$, $a \geq 1$ and $1 \leq k \leq p - 1$, we have $f_{a,k}(s) \geq p^2 - 2p > 0$. Hence, $l_a(n, p) > 0$ for any $a \geq 1$ and $n \geq p + 1$. Thus, we have obtained the required result. \square

Next, we consider the case when the condition $2l_1(n, p) \leq t$ in Lemma 3.1 is satisfied for $t = 2n + 1$ or $t = 2\lfloor n/2 \rfloor + 1$.

Lemma 3.3. *Let $n \geq 2$. Then, we have the following.*

- (1) *For $p \geq 5$, $2l_1(n, p) \leq 2n + 1$ if and only if $(p-1)/2 \leq n \leq p$. For $p = 3$, $2l_1(n, 3) \leq 2n + 1$ if and only if $2 \leq n \leq 5$.*
- (2) *$2l_1(n, p) \leq 2\lfloor n/2 \rfloor + 1$ if and only if $(2p - \delta)/3 \leq n \leq p$, where $\delta = 1$ or 2 according as n is odd or even.*

Proof. We shall use the similar method as in the previous lemma. Thus, we put $n = (s-1)(p-1) + k + 1$ using integers $s \geq 1$ and $1 \leq k \leq p-1$. Then, $l_1(n, p) = p^s - ((s-1)(p-1) + k + 2)$.

Now, we prove (1). Consider the real variable function

$$g_k(x) = p^x - 2((x-1)(p-1) + k) - 3$$

for fixed p and k . Then, $2l_1(n, p) \leq 2n + 1$ if and only if $g_k(s) \leq 0$. We have $g'_k(x) = (\log p)p^x - 2(p-1)$ and $g''_k(x) = (\log p)^2 p^x > 0$. Then, since $g'_k(2) = (p \log p - 2)p + 2 > 0$, $g_k(x)$ is monotonously increasing for $x \geq 2$. We have $g_k(1) \leq 0$ if and only if $(p-1)/2 \leq k+1$, since $g_k(1) = p - 2k - 3$. Also, since $g_k(2) = p^2 - 2p - 2k - 1$ and $k \leq p-1$, $g_k(2) \leq 0$ if and only if $(p^2 - 2p - 1)/2 \leq k \leq p-1$, which is possible when and only when $p = 3$ and $k = 1$ or 2 . Thus, for $p \geq 5$, $2l_1(n, p) \leq 2n + 1$ if and only if $s = 1$ and $(p-1)/2 \leq k+1$, that is, if and only if $(p-1)/2 \leq n = k+1 \leq p$ as required. When $p = 3$, $g_k(1) \leq 0$ if and only if $n = 2$ or 3 , and $g_k(2) \leq 0$ if and only if $n = 4$ or 5 . Moreover, when $p = 3$, $g_k(3) = 16 - 2k > 0$ and thus $2l_1(n, 3) > 2n + 1$ for any $n \geq 6$. Hence, $2l_1(n, 3) \leq 2n + 1$ if and only if $2 \leq n \leq 5$, as required. In this way, we have proved (1).

Since we can prove (2) just similarly by considering the real variable function

$$h_k(x) = p^x - ((x-1)(p-1) + k + 2) - ((x-1)(p-1)/2 + \lfloor (k+1)/2 \rfloor)$$

for fixed p and k , we omit the details. □

Using these lemmas, we can complete the proof of Theorem 1.3.

Proof of Theorem 1.3. Since $s(\nu_n^t(p)) \leq s(\nu_n^{t'}(p))$ if $t \leq t'$ as remarked in the first section, we have only to prove the result when $t = 2n + 1$. Thus, for any $n \geq 0$ with $n \neq p$, the required result follows from Lemma 3.1, Lemma 3.2(1) and Lemma 3.3(1). When $n = p$, since $0 < 2l_2(p, p) = 2p - 2 < 2n + 1$, we have also the required result by Lemma 3.1. □

Similarly, using Lemmas 3.1, Lemma 3.2(1) and Lemma 3.3(2), we have a corresponding result in the case $t \geq 2\lfloor n/2 \rfloor + 1$, as follows.

Proposition 3.4. *For $t \geq 2\lfloor n/2 \rfloor + 1$, $s(\nu_n^t(p)) = \infty$ if $(2p-1)/3 \leq n < p$.*

4. UPPER BOUND FOR THE EXTENDIBILITY

Let $p_i(\alpha) \in H^{4i}(L^n(p); \mathbb{Z})$ be the i -th Pontrjagin class of an \mathbb{R} -vector bundle α over $L^n(p)$ (cf., [18]); and, let $P(\alpha) = 1 + p_1(\alpha) + \cdots + p_i(\alpha) + \cdots$ denote the total Pontrjagin class, where we use the capital letter P instead of p to avoid the confusion with the prime p . Then, since $H^*(L^n(p); \mathbb{Z})$ has no 2-torsion, the multiplicative property $P(\alpha + \beta) = P(\alpha)P(\beta)$ holds. Also, since $P(\theta) = 1$ for a trivial bundle θ , we have $P(\alpha) = P(\beta)$ if α and β are stably equivalent, and thus we can consider the Pontrjagin class $p_i(u)$ of any element $u \in KO(L^n(p))$. Later on, we shall use the fundamental property that $p_i(\zeta) = 0$ if $2i > m$ for any m -dimensional \mathbb{R} -vector bundle ζ .

Let $x \in H^2(L^n(p); \mathbb{Z})$ be the Euler class of the \mathbb{C} -line bundle η_n . Then, $H^{2i}(L^n(p); \mathbb{Z}) \cong \mathbb{Z}/p$ is generated by x^i for $1 \leq i \leq n$ (cf., [22]). Let η_n^k be the tensor product over \mathbb{C} of k (resp. $-k$) numbers of η_n (resp. the conjugate bundle $\bar{\eta}_n$ of η_n) if k is a positive (resp. negative) integer. Then, $P(r(\eta_n^k)) = 1 + k^2 x^2$ since the Euler class of η_n^k is equal to kx (cf., [4]), which we have already used in the proof of Lemma 2.4.

To investigate an upper bound of $s(\nu_n^t(p))$, we use the following proposition, which is shown essentially in the proof of Theorem 1.1 in [13], but we give a proof for completeness.

Proposition 4.1. *Let $k > n > 0$ and $h \geq 0$ be integers; and, let α be a vector bundle over $L^k(p)$. We assume that the following two conditions are satisfied.*

- (i) *For the inclusion map $i : L^n(p) \rightarrow L^k(p)$, $i^*\alpha$ is stably equivalent to $hr(\eta_n)$. That is, $i^*\alpha + \theta \cong hr(\eta_n) + \theta'$ for trivial vector bundles θ and θ' of some dimensions.*
- (ii) $2p^{\lfloor n/(p-1) \rfloor} > k$.

Then, we have $P(\alpha) = (1 + x^2)^h$.

Proof. First, by Theorem 2.1(1), the K -group $K(L^m(p))$ for any $m > 0$ is generated additively by $\sigma_m = \eta_m - 1, \sigma_m^2, \dots, \sigma_m^{p-1}$. Since we have

$$(4.1) \quad \sigma_m^i = \sum_{j=0}^i \binom{i}{j} (-1)^{i-j} (\eta_m^j - 1) \quad \text{and} \quad \eta_m^i - 1 = \sum_{j=1}^i \binom{i}{j} \sigma_m^j,$$

$K(L^m(p))$ is also generated additively by $\eta_m - 1, \eta_m^2 - 1, \dots, \eta_m^{p-1} - 1$. Let t be the dimension of α ; and, let $j : L_0^k(p) \rightarrow L^k(p)$ be the inclusion map. Then, since $r(K(L_0^k(p))) = KO(L_0^k(p))$ by Theorem 2.1(1), we have

$$j^*(\alpha - t) = r \left(\sum_{i=1}^{p-1} b_i (\eta_k^i - 1) \right) \in KO(L_0^k(p))$$

for some integers b_i . Since $i^*(\alpha - t) = hr(\eta_n - 1) \in KO(L^n(p))$ by the assumption, it follows

$$(4.2) \quad r((b_1 - h)(\eta_n - 1) + \sum_{i=2}^{p-1} b_i (\eta_n^i - 1)) = 0.$$

We notice that $cr(\eta_n^i) = \eta_n^i + \eta_n^{p-i}$ for the complexification homomorphism $c : KO(L^n(p)) \rightarrow K(L^n(p))$, and we apply c on both sides of (4.2). Then, we have

$$(4.3) \quad \sum_{i=1}^{p-1} (b_i + b_{p-i} - d_i)(\eta_n^i - 1) = 0,$$

where we put

$$d_i = \begin{cases} h & \text{if } i = 1 \text{ or } i = p - 1, \\ 0 & \text{otherwise.} \end{cases}$$

Using the latter relation in (4.1), (4.3) is written as

$$(4.4) \quad \sum_{j=1}^{p-1} \left(\sum_{i=j}^{p-1} \binom{i}{j} (b_i + b_{p-i} - d_i) \right) \sigma_n^j = 0.$$

We put $s = \lfloor n/(p-1) \rfloor$. Then, by Theorem 2.1(1), the order of σ_n^j is equal to p^s or p^{s+1} , and thus, from (4.4), it follows

$$\sum_{i=j}^{p-1} \binom{i}{j} (b_i + b_{p-i} - d_i) \equiv 0 \pmod{p^s}$$

for any j with $1 \leq j \leq p-1$. Thus, we have

$$b_j + b_{p-j} = d_j + u_j p^s$$

for $1 \leq j \leq p-1$, where u_j are some integers. Then, we can calculate the total Pontrjagin class of $j^* \alpha$ as follows.

$$\begin{aligned} j^* P(\alpha) &= P \left(\sum_{i=1}^{p-1} b_i r(\eta_k^i) \right) = \prod_{i=1}^{p-1} P(r(\eta_k^i))^{b_i} = \prod_{i=1}^{p-1} (1 + i^2 x^2)^{b_i} \\ &= \prod_{i=1}^{(p-1)/2} (1 + i^2 x^2)^{b_i + b_{p-i}} = \prod_{i=1}^{(p-1)/2} (1 + i^2 x^2)^{d_i + u_i p^s} \\ &= (1 + x^2)^h \prod_{i=1}^{(p-1)/2} (1 + i^{2p^s} x^{2p^s})^{u_i}. \end{aligned}$$

But, since $2p^s > k$ by the assumption, $x^{2p^s} \in H^{4p^s}(L_0^k(p); \mathbb{Z}) = 0$. Hence, we have $j^* P(\alpha) = (1 + x^2)^h$, and obtain the required result, since $j^* : H^{4i}(L^k(p); \mathbb{Z}) \rightarrow H^{4i}(L_0^k(p); \mathbb{Z})$ is an isomorphism for any i . \square

Now, we put

$$(4.5) \quad k_0 = p^{\lfloor (n-1)/(p-1) \rfloor - 1} - (n+1) \quad \text{for } n \geq 2.$$

Then, we have the following lemma.

Lemma 4.2. *Let $n \geq 2$. Then, the inequality $k_0 > n + 1$ holds if and only if $n \geq 2p$ when $p \geq 7$, $n = 10, 11$ or $n \geq 14$ when $p = 5$ and $n \geq 8$ when $p = 3$.*

Proof. We use the similar method as in the proof of Lemma 3.2. Thus, we put $n = (s - 1)(p - 1) + u + 1$ for $s \geq 1$ and $1 \leq u \leq p - 1$. Then, $k_0 - (n + 1) = p^{s-1} - 2((s - 1)(p - 1) + u + 2)$. Consider the real variable function

$$f_u(x) = p^{x-1} - 2((x - 1)(p - 1) + u + 2)$$

for fixed p and u . Then, the inequality $k_0 > n + 1$ holds if and only if $f_u(s) > 0$. We have $f'_u(x) = (\log p)p^{x-1} - 2(p - 1)$ and $f''_u(x) = (\log p)^2 p^{x-1} > 0$. Since $f'_u(3) = (\log p)p^2 - 2p + 2 = p((\log p)p - 2) + 2 > 0$, $f_u(x)$ is monotonously increasing for $x \geq 3$. Now, $f_u(1) = -2u - 3 < 0$, $f_u(2) = -(p + 2u + 2) < 0$, and $f_u(3) = p^2 - 4p - 2u = p(p - 6) + 2(p - u) > 0$ if $p \geq 7$. Thus, we have the required result for $p \geq 7$. Also, we have $f_u(3) = 5 - 2u$ if $p = 5$, $f_u(3) = -2u - 3 < 0$ if $p = 3$, and $f_u(4) = p^3 - 6p - 2u + 2 \geq p(p^2 - 8) + 4 > 0$. Hence, for $p = 5$ (resp. $p = 3$), the inequality $k_0 > n + 1$ holds if and only if $n = 10$, $n = 11$ or $n \geq 14$ (resp. $n \geq 8$), as required. \square

Now, we can complete the proof of Theorem 1.5, as follows.

Proof of Theorem 1.5. Let $p \geq 7$, $n \geq 2p$ and $t \leq 2n + 2$. We suppose that $\nu_n^t(p)$ is stably extendible to $L^{2k_0}(p)$ for k_0 in (4.5), and we shall deduce a contradiction. Thus, there is a vector bundle α of dimension t over $L^{2k_0}(p)$ satisfying that $i^*(\alpha)$ is stably equivalent to $\nu_n^t(p)$ for the inclusion map $i : L^n(p) \rightarrow L^{2k_0}(p)$. Then, we have $i^*\alpha + \theta \cong \nu_n^t(p) + \theta \cong l_2(n, p)r(\eta_n) + \theta'$ for trivial vector bundles θ and θ' of some dimensions, where $l_2(n, p) \geq 0$ by Lemma 2.2(2). Since $\lfloor n/(p - 1) \rfloor \geq \lceil (n - 1)/(p - 1) \rceil - 1$, we have $2p^{\lfloor n/(p-1) \rfloor} > 2k_0$. Also, $2k_0 > n$ by Lemma 4.2. Hence, we can apply Proposition 4.1 to α for $k = 2k_0$ and $h = l_2(n, p)$. Thus, we have $p(\alpha) = (1 + x^2)^{l_2(n, p)}$. Then,

$$p_{k_0}(\alpha) = \binom{l_2(n, p)}{k_0} x^{2k_0} \neq 0 \text{ in } H^{4k_0}(L^{2k_0}(p); \mathbb{Z}) \cong \mathbb{Z}/p,$$

because it holds

$$\binom{l_2(n, p)}{k_0} = \binom{(2p - 1)p^{\lfloor (n-1)/(p-1) \rfloor - 1} + p^{\lfloor (n-1)/(p-1) \rfloor - 1} - (n + 1)}{p^{\lfloor (n-1)/(p-1) \rfloor - 1} - (n + 1)} \not\equiv 0 \pmod{p}.$$

However, since $2k_0 > 2n + 2 \geq t$ by Lemma 4.2 and the assumption, we have $p_{k_0}(\alpha) = 0$ on the other hand, which is a contradiction. Thus, we have established the required result. \square

Concerning the cases $p = 3$ and $p = 5$, we have the following lemma which will be used in the proof of Theorem 1.4, the proof of which is just the same as the above using Lemma 4.2.

Lemma 4.3. *For $t \leq 2n + 2$, $s(\nu_n^t(3)) < \infty$ if $n \geq 8$, and $s(\nu_n^t(5)) < \infty$ if $n \geq 10$ with $n \neq 12, 13$.*

5. PROOF OF THEOREM 1.6

Let $\text{g. dim } \alpha$ denote the geometric dimension of a vector bundle α . Then, α is stably equivalent to a $(\text{g. dim } \alpha)$ -dimensional vector bundle. About the geometric

dimensions of vector bundles over the lens space $L^m(p)$, Sjerve [22] has shown the following result, where $\pi_m: S^{2m+1} \rightarrow L^m(p)$ is the canonical projection.

Theorem 5.1 ([22], Theorem A). *Let ζ be a k -dimensional \mathbb{R} -vector bundle over $L^m(p)$. Then, if $\zeta - k \in KO(L^m(p)) \cap \ker \pi_m^*$, it follows $\text{g. dim } \zeta \leq 2\lfloor m/2 \rfloor + 1$.*

Using Theorem 5.1, we have the following.

Proposition 5.2. *For any normal bundle $\nu_n^t(p)$, we have $s(\nu_n^t(p)) \geq 2\lfloor t/2 \rfloor - 1$.*

Proof. We put $m = 2\lfloor t/2 \rfloor - 1$. By Lemma 2.2, $\nu_n^t(p)$ is stably equivalent to $l_a(n, p)r(\eta_m)$ for some $a \geq 1$. Since $\pi_m^*(r(\eta_m) - 2) = 0$ in $KO(S^{2m+1})$, $l_a(n, p)r(\eta_m)$ is stably equivalent to a $(2\lfloor m/2 \rfloor + 1)$ -dimensional vector bundle β over $L^m(p)$ by Theorem 5.1. But, we have $2\lfloor m/2 \rfloor + 1 \leq t$ since $m = 2\lfloor t/2 \rfloor - 1$. Then, the t -dimensional vector bundle $\gamma = \beta + (t - 2\lfloor m/2 \rfloor - 1)$ over $L^m(p)$ satisfies that $i^*\gamma$ is stably equivalent to $\nu_n^t(p)$, where $i: L^n(p) \rightarrow L^m(p)$ is the inclusion map. Thus, $\nu_n^t(p)$ is stably extendible to $L^m(p)$, and we have the required result. \square

Using Proposition 4.1, we also have the following.

Lemma 5.3. *Let α be a t -dimensional \mathbb{R} -vector bundle over $L^n(p)$; and, assume that α is stably equivalent to $lr(\eta_n)$ for some integer $l \geq 0$. Then, if the incongruence*

$$\binom{l}{k} \not\equiv 0 \pmod{p}$$

is satisfied for an integer k with $(t+1)/2 \leq k < p^{\lfloor n/(p-1) \rfloor}$, α is not stably extendible to $L^{2k}(p)$.

Proof. Suppose that α is stably extendible to $L^{2k}(p)$. Then, there exists a t -dimensional vector bundle β over $L^{2k}(p)$ satisfying that $i^*\beta$ is stably equivalent to α , where $i: L^n(p) \rightarrow L^{2k}(p)$ is the inclusion map. Then, $i^*\beta$ is stably equivalent to $lr(\eta_n)$; and, $2p^{\lfloor n/(p-1) \rfloor} > 2k$ by the assumption. Hence, we can apply Proposition 4.1 to β , and thus we have $P(\beta) = (1+x^2)^l$. Since $\binom{l}{k} \not\equiv 0 \pmod{p}$ by the assumption, $p_k(\beta) = \binom{l}{k}x^{2k} \neq 0$ in $H^{4k}(L^{2k}(p); \mathbb{Z}) \cong \mathbb{Z}/p$. On the other hand, since the dimension of β is t and since $2k \geq t+1$ by the assumption, we have $p_k(\beta) = 0$, which contradicts the above. Thus, we have the required result. \square

Now, we can prove Theorem 1.6 as follows.

Proof of Theorem 1.6. First, we remark that $l_a(n, p) \geq 0$ for any $a \geq 1$ by the assumption $n \geq p+1$ and Lemma 3.2. Since the conditions in Lemma 5.3 for $\alpha = \nu_n^t(p)$, $l = l_a(n, p)$ and $k = \lfloor (t+1)/2 \rfloor$ are satisfied by the assumptions in Theorem 1.6, $\nu_n^t(p)$ is not stably extendible to $L^{2\lfloor (t+1)/2 \rfloor}(p)$. On the other hand, by Proposition 5.2, $\nu_n^t(p)$ is stably extendible to $L^{2\lfloor t/2 \rfloor - 1}(p)$. When t is odd, $(2\lfloor t/2 \rfloor - 1, 2\lfloor (t+1)/2 \rfloor) = (t, t+1)$, and hence we have $s(\nu_n^t(p)) = t$ as required. When t is even, $(2\lfloor t/2 \rfloor - 1, 2\lfloor (t+1)/2 \rfloor) = (t-1, t+2)$, and hence $s(\nu_n^t(p)) = t-1$, t or $t+1$ as required. Thus, we have completed the proof. \square

Concerning the latter condition $p^{\lfloor n/(p-1) \rfloor} > \lceil (t+1)/2 \rceil$ in (1.2), we have the following expression when $t = 2n + 1$ and $t = 2\lfloor n/2 \rfloor + 1$.

Lemma 5.4. *Let $s = \lfloor n/(p-1) \rfloor$. Then, we have the following.*

- (1) $p^s > n + 1$ if and only if $n \geq 2(p-1)$.
- (2) $p^s > \lfloor n/2 \rfloor + 1$ if and only if $n \geq p-1$.

Proof. We set $n = s(p-1) + r$ for $s \geq 0$ and $0 \leq r \leq p-2$. Then, we have $s = \lfloor n/(p-1) \rfloor$. For the proof of (1), we consider the real variable function $f(x) = p^x - (x(p-1) + r + 1)$ for fixed p and r . Then, it is sufficient to show that $f(s) > 0$ if and only if $s \geq 2$. Since $f'(1) = (\log p - 1)p + 1 > 0$ and $f''(s) = (\log p)^2 p^s > 0$, $f(x)$ is monotonously increasing for $x \geq 1$. But, $f(0) = f(1) = -r \leq 0$ and $f(2) = p^2 - 2p - r \geq p^2 - 3p + 2 > 0$, and thus we have the required result.

We can prove (2) just the same way using the function $g(x) = p^x - x(p-1)/2 - \lfloor r/2 \rfloor - 1$ instead of $f(x)$, and we omit the details. \square

As a corollary of Theorem 1.6, we have the following.

Corollary 5.5. *Let p be a prime number with $p \geq 5$; and, assume that $n \geq 2(p-1)$. Then, if the p -adic expansion of $n+1$ satisfies the below condition (5.1), we have $s(\nu_n^{2n+1}(p)) = 2n+1$.*

$$(5.1) \quad n+1 = \sum_{i=1}^m a_i p^{t_i} \quad \text{with} \quad 1 \leq a_i \leq \frac{p-1}{2} \quad (1 \leq i \leq m-1) \quad \text{and} \quad 1 \leq a_m \leq \frac{p}{2},$$

where $m \geq 1$ and $t_1 > t_2 > \dots > t_m \geq 0$.

Proof. Since we are considering the case that $t = 2n + 1$ and $n \geq 2(p-1)$, the condition $t \geq n \geq p+1$ in Theorem 1.6 is cleared. Also, we have $p^{\lfloor n/(p-1) \rfloor} \geq n+1$ by Lemma 5.4(1). Thus, the inequality in (1.2) is satisfied, since $\lceil (t+1)/2 \rceil = n+1$ in this case. Also, $l_a(n, p) \geq 0$ for any $a \geq 1$ by Lemma 3.2. From (2.1) and (5.1), the p -adic expansion of $l_a(n, p)$ is represented as

$$\begin{aligned} l_a(n, p) &= (a-1)p^{\lceil (n-1)(p-1) \rceil} + (p-1)p^{\lceil (n-1)(p-1) \rceil - 1} + \dots \\ &\quad + (p-a_1-1)p^{t_1} + \dots + (p-a_i-1)p^{t_i} + \dots + (p-a_m)p^{t_m}. \end{aligned}$$

Then, we have

$$\binom{l_a(n, p)}{n+1} \equiv \binom{p-a_1-1}{a_1} \dots \binom{p-a_{m-1}-1}{a_{m-1}} \binom{p-a_m}{a_m} \not\equiv 0 \pmod{p},$$

because it holds

$$\binom{p-a_i-1}{a_i} \not\equiv 0 \pmod{p} \quad \text{for} \quad 1 \leq i \leq m-1 \quad \text{and} \quad \binom{p-a_m}{a_m} \not\equiv 0 \pmod{p}$$

by the conditions on a_i in (5.1). Hence, the incongruence in (1.2) is also satisfied in this case. Thus, we have the required result by Theorem 1.6. \square

We have also the following corollary of Theorem 1.6 using Lemma 5.4(2) instead of Lemma 5.4(1). Since the proof is quite similar with the above, we omit the description of the proof.

Corollary 5.6. *Let p be a prime number with $p \geq 5$; and assume that n is an odd integer with $n \geq p+1$. Then, if the p -adic expansion of $(n+1)/2$ satisfies the below condition (5.2), we have $s(\nu_n^n(p)) = n$, that is, $\nu_n^n(p)$ is not stably extendible to $L^{n+1}(p)$.*

$$(5.2) \quad \frac{n+1}{2} = \sum_{i=1}^m a_i p^{t_i} \text{ with } 1 \leq a_i \leq \frac{p-1}{3} \ (1 \leq i \leq m-1) \text{ and } 1 \leq a_m \leq \frac{p}{3},$$

where $m \geq 1$ and $t_1 > t_2 > \cdots > t_m \geq 0$.

Corollary 1.7 is a special case of Corollary 5.5 and Corollary 5.6.

6. PROOF OF THEOREM 1.4

In this section, we shall complete the proof of Theorem 1.4. First, we consider the case $p = 3$. By Theorem 1.3, $s(\nu_n^t(3)) = \infty$ for $0 \leq n \leq 5$ and $t \geq 2n+1$. Also, by Lemma 4.3, $s(\nu_n^t(3)) < \infty$ for $n \geq 8$ and $t \leq 2n+2$. Thus, it is sufficient to prove the following lemma, since $s(\nu_6^{13}(3)) \leq s(\nu_6^{14}(3))$ and $s(\nu_7^{15}(3)) \leq s(\nu_7^{16}(3))$.

Lemma 6.1. *We have $s(\nu_6^{14}(3)) < 40$ and $s(\nu_7^{16}(3)) < 38$.*

Proof. Let $n = 6$ or 7 . Then, since $l_1(6, 3) = 20$ and $l_1(7, 3) = 19$, the required inequality is represented as $s(\nu_n^{2n+2}(3)) < 2l_1(n, 3)$. We suppose that $\nu_n^{2n+2}(3)$ is stably extendible to $L^{2l_1(n,3)}(3)$. Then, there exists a $(2n+2)$ -dimensional \mathbb{R} -vector bundle α over $L^{2l_1(n,3)}(3)$ whose restriction to $L^n(3)$ is stably equivalent to $\nu_n^{2n+2}(3)$. Since $n = 6$ or $n = 7$, $\nu_n^{2n+2}(3)$ is stably equivalent to $l_1(n, 3)r(\eta_n)$ by Lemma 2.2 and Lemma 3.2, and $2 \cdot 3^{\lfloor n/2 \rfloor} > 2l_1(n, 3)$ holds. Thus, by Proposition 4.1 we have $P(\alpha) = (1+x^2)^{l_1(n,3)}$, where $x = c_1(\eta_n) \in H^2(L^m(p); \mathbb{Z}) = \mathbb{Z}/p$ is a generator for $m = 2l_1(n, 3)$. Therefore, $p_{l_1(n,3)}(\alpha) = x^{2l_1(n,3)} \neq 0$, which contradicts that the dimension of α is $2n+2$ and $2l_1(n, 3) > 2n+2$. Thus, we have completed the proof. \square

Next, we consider the case $p = 5$. In this case, $s(\nu_n^t(5)) = \infty$ for $2 \leq n \leq 5$ and $t \geq 2n+1$ by Theorem 1.3, and $s(\nu_n^t(5)) < \infty$ for $n \geq 10$ with $n \neq 12, 13$ and $t \leq 2n+2$ by Lemma 4.3. Thus, it is sufficient to show that $s(\nu_n^{2n+2}(5)) < \infty$ for $6 \leq n \leq 9$ and $n = 12, 13$, since $s(\nu_n^{2n+1}(5)) \leq s(\nu_n^{2n+2}(5))$.

First, consider the cases $n = 6$ and $n = 7$.

Lemma 6.2. *We have $s(\nu_6^{14}(5)) < 16$ and $s(\nu_7^{16}(5)) < 30$.*

Proof. First, we prove $s(\nu_6^{14}(5)) < 16$. Suppose that $\nu_6^{14}(5)$ is stably extendible to $L^{16}(5)$. Then, there is a 14-dimensional vector bundle α over $L_0^{16}(5)$ satisfying that $i^*\alpha$ is stably equivalent to $\nu_6^{14}(5)$, where $L_0^{16}(5)$ is the 32-skeleton of $L^{16}(5)$ and $i : L^6(5) \rightarrow L_0^{16}(5)$ is the inclusion map. By Theorem 2.1, $KO(L^6(5)) = \mathbb{Z}/25\{\bar{\sigma}_6\} \oplus \mathbb{Z}/5\{\bar{\sigma}_6^2\}$ and $KO(L_0^{16}(5)) = \mathbb{Z}/5^4\{\bar{\sigma}_{16}\} \oplus \mathbb{Z}/5^4\{\bar{\sigma}_{16}^2\}$, where $\bar{\sigma}_n = r(\eta_n) - 2$. Since $i^*(\alpha - 14) = \nu_6^{14}(5) - 14 = l_1(6, 5)\bar{\sigma}_6 = 18\bar{\sigma}_6$ and since $i^*(\bar{\sigma}_{16}^i) = \bar{\sigma}_6^i$, we have

$$(6.1) \quad \alpha - 14 = (18 + 25s)\bar{\sigma}_{16} + 5t\bar{\sigma}_{16}^2$$

in $KO(L_0^{16}(5))$ for some integers s and t . As mentioned about the total Pontrjagin class in the section 4, we have $P(\bar{\sigma}_{16}) = P(r(\eta_{16}) - 2) = 1 + x^2$. Also, since $\bar{\sigma}_{16}^2 = r(\eta_{16}^2) - 4r(\eta_{16}) + 6$, we have $P(\bar{\sigma}_{16}^2) = P(r(\eta_{16}^2))P(r(\eta_{16}))^{-4} = (1 + 4x^2)(1 + x^2)^{-4}$. Thus, it follows from (6.1) that

$$\begin{aligned} P(\alpha) &= (1 + x^2)^{18+25s-20t}(1 + 4x^2)^{5t} \\ &= (1 + x^2)^3(1 + x^{10})^{3+5s-4t}(1 - x^{10})^t \\ &= (1 + x^2)^3(1 + (3 - 4t)x^{10})(1 - tx^{10}) = (1 + x^2)^3(1 + 3x^{10}) \\ &= 1 + 3x^2 + \cdots - x^{14} + 3x^{16}, \end{aligned}$$

where we use the property that $(1 + y)^5 = 1 + y^5$ and $x^{20} = 0$ in $H^{2*}(L_0^{16}(5); \mathbb{Z})$. Hence, we have $p_8(\alpha) = 3x^{16} \neq 0$, which contradicts that the dimension of α is 14.

We can proceed similarly for $s(\nu_7^{16}(5))$. Suppose that $\nu_7^{16}(5)$ is stably extendible to $L^{30}(5)$. Then, there is a 16-dimensional vector bundle β over $L^{30}(5)$ satisfying that $i^*\beta$ is stably equivalent to $\nu_7^{16}(5)$, where $i : L^7(5) \rightarrow L^{30}(5)$ is the inclusion map. By Theorem 2.1, $KO(L^7(5)) = \mathbb{Z}/25\{\bar{\sigma}_7\} \oplus \mathbb{Z}/5\{\bar{\sigma}_7^2\}$ and $KO(L^{30}(5)) = \mathbb{Z}/5^8\{\bar{\sigma}_{30}\} \oplus \mathbb{Z}/5^7\{\bar{\sigma}_{30}^2\}$. Since $i^*(\beta - 16) = \nu_7^{16}(5) - 16 = l_1(7, 5)\bar{\sigma}_7 = 17\bar{\sigma}_7$ in $KO(L^7(5))$, we have

$$\beta - 16 = (17 + 25s)\bar{\sigma}_{30} + 5t\bar{\sigma}_{30}^2$$

in $KO(L^{30}(5))$ for some integers s and t . Then, it follows that

$$\begin{aligned} P(\beta) &= (1 + x^2)^{17+25s-20t}(1 + 4x^2)^{5t} \\ &= (1 + x^2)^2(1 + x^{10})^{3+5s-4t}(1 - x^{10})^t \\ &= (1 + x^2)^2(1 + (3 - 4t)x^{10} + \binom{3 + 5s - 4t}{2}x^{20} + \binom{3 + 5s - 4t}{3}x^{30}) \\ &\quad (1 - tx^{10} + \binom{t}{2}x^{20} - \binom{t}{3}x^{30}). \end{aligned}$$

Put $p_{10}(\beta) = ux^{20}$ using some integer u . Then, calculating the coefficient of x^{20} in the above expression, we have $u \equiv 3, 2, 1, 0$ or $4 \pmod{5}$ according to $t \equiv 0, 1, 2, 3$ or $4 \pmod{5}$. Thus, $p_{10}(\beta) \neq 0$ unless $t \equiv 3 \pmod{5}$. When $t \equiv 3 \pmod{5}$, we have $p_{15}(\beta) = 2x^{30} \neq 0$ from the coefficient of x^{30} in the above expression. Since the dimension of β is 16, we have a contradiction that $p_{10}(\beta) = 0$ and $p_{15}(\beta) = 0$. Thus, we have the required result. \square

To obtain $s(\nu_n^t(5)) < \infty$ for $n = 8, 9, 12$ or 13 , we apply the following theorem due to Kobayashi–Maki–Yoshida [15].

Theorem 6.3. ([15, Theorem 3.6]) *Let p be an odd prime number, and assume that $n \equiv 0, 1 \pmod{p-1}$. Then, if*

$$(6.2) \quad \lfloor t/2 \rfloor < p^{\lfloor (n-2)/(p-1) \rfloor + 1} - (n+1),$$

we have $s(\nu_n^t(p)) < 2(p^{\lfloor (n-2)/(p-1) \rfloor + 1} - (n+1))$.

About the condition (6.2), we have $p^{\lfloor (n-2)/(p-1) \rfloor + 1} - (n+1) = l_1(n, p)$ since $n \equiv 0, 1 \pmod{p-1}$. Furthermore, by the method used in the proof of Lemma

3.2, we can easily prove that $l_1(n, p) > n + 1$ if $p \geq 5$ and $n \geq p + 1$. Thus, we have the following corollary.

Corollary 6.4. *Let $p \geq 5$, $n \geq p + 1$ and $t \leq 2n + 2$. Then, if $n \equiv 0, 1 \pmod{p - 1}$, we have $s(\nu_n^t(p)) < 2l_1(n, p)$.*

Hence, we have $s(\nu_n^{2n+2}(5)) < \infty$ for $n = 8, 9, 12$ and 13 , as required. Thus, we have completed the proof of Theorem 1.4.

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