Stably extendible tangent bundles over lens spaces

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Abstract: The purpose of this paper is to study the stable extendibility of the tangent bundle $\tau_n(p)$ over the $(2n+1)$-dimensional standard lens space $L^n(p)$ for odd prime p. We investigate for which m the tangent bundle $\tau_n(p)$ is stably extendible to $L^m(p)$ but is not stably extendible to $L^{m+1}(p)$, where we consider $m = \infty$ if $\tau_n(p)$ is stably extendible to $L^k(p)$ for any $k \geq n$, and determine m in the case $n \geq p-3$.

1 Introduction

Let F be the real number field $\mathbb{R}$, the complex number field $\mathbb{C}$ or the quaternion number field $\mathbb{H}$. For a subspace A of a space X, a t-dimensional F-vector bundle ζ over A is defined to be *extendible* to X, if there is a t-dimensional F-vector bundle over X whose restriction to A is equivalent to ζ, that is, if ζ is equivalent to the induced bundle $i^*\eta$ of a t-dimensional F-vector bundle η over X under the inclusion map $i\colon A \to X$. We can observe the interesting studies about the extendibility of vector bundles by Schwarzenberger[12], Adams-Mahmud[1], Rees[11], Kobayashi-Maki-Yoshida[8] and so on.

In [4, p.273], we have introduced the notion of *stably extendible* vector bundle as follows: In the above situation, if $i^*\eta$ is stably equivalent to ζ, namely $i^*\eta + k$ is equivalent to $\zeta + k$ for a trivial F-vector bundle k of some dimension $k \geq 0$, ζ is defined to be stably extendible to X. Obviously, if ζ is extendible to X, then ζ is stably extendible to X. When A is an n-dimensional CW complex and the dimension t of ζ is more than or equal to $((n+2)/d) - 1$, where $d = \dim_{\mathbb{R}} F$, ζ is stably extendible to X if and only if it is extendible to X, by the stability property (cf. [3, pp.111-113]).

2000 *Mathematics Subject Classification.* Primary 55R50; Secondary 55N15.
Key Words and Phrases. stably extendible, tangent bundle, lens space, *KO*-theory.

The tangent bundle $\tau(FP^n)$ over the F–projective space FP^n is stably extendible to FP^{n+1} if and only if $n = 1, 3$ or 7 when $F = \mathbb{R}$ ([9, Theorem 4.2]), $n = 1$ when $F = \mathbb{C}$ considered $\tau(\mathbb{C}P^n)$ as a $\mathbb{C}$–vector bundle ([2, Appendix I, p.166]) and $n = 1$ when $F = \mathbb{H}$ ([5, Theorem A]). We study the stable extendibility of the tangent bundle over the standard lens space for an odd prime, and show that the difference between the stable extendibility and extendibility appears in this case.

Let $L^n(p) = S^{2n+1}/(\mathbb{Z}/p)$ be the $(2n+1)$-dimensional standard lens space mod p, and $\tau_n(p) = \tau(L^n(p))$ denote the tangent bundle over $L^n(p)$. Then, our purpose is to determine the integer $s(\tau_n(p))$ defined by

$$s(\tau_n(p)) = \max\{m \mid m \geq n \text{ and } \tau_n(p) \text{ is stably extendible to } L^m(p)\},$$

where we set $s(\tau_n(p)) = \infty$ if $\tau_n(p)$ is stably extendible to $L^m(p)$ for any $m \geq n$. We have the following result about the extendibility of $\tau_n(p)$ and $s(\tau_n(p))$.

Theorem 1.1. *Let p be an odd integer.*

(1) [8, Theorem 1.2], [9, Theorem 5.3] *$\tau_n(p)$ is extendible to $L^{n+1}(p)$ if and only if $n = 0, 1$ or 3, and $\tau_n(p)$ for $n = 0, 1$ or 3 is extendible to $L^m(p)$ for any $m \geq n$.*
(2) [10, Theorem 1, Theorem 5.3], [6, Theorem 2,Theorem 3] *When $p = 3, 5$ or 7, $s(\tau_n(p)) = \infty$ if $0 \leq n \leq p$, and $s(\tau_n(p)) = 2n + 1$ if $n \geq p + 1$.*

In this paper, we generalize the result of Theorem 1.1 (2) to the case of any odd prime p, and show the following theorems.

Theorem 1.2. *Let p be an odd prime. Then, we have*

$$s(\tau_n(p)) = 2n + 1 \text{ for } n \geq p + 1.$$

Theorem 1.3. *Let p be an odd prime. Then, we have the following.*

(1) *$s(\tau_n(p)) = \infty$ for $p - 3 \leq n \leq p$.*
(2) *$s(\tau_2(p)) = \infty$ if $p \equiv \pm 1 \mod 12$.*

Notice that $s(\tau_n(p)) = \infty$ for $n = 0, 1$ or 3 by Theorem 1.1(1), and $s(\tau_2(p)) = \infty$ for $p = 3, 5$ or 7 by Theorem 1.1(2). For the case of $p = 11$, 13 or 17, we have the following additional result.

Lemma 1.4. *$s(\tau_n(11)) = \infty$ for $n = 4$ or 5, $s(\tau_n(13)) = \infty$ for $5 \leq n \leq 7$, and $s(\tau_2(17)) = \infty$.*

These results support our following conjecture given in [6]:

Conjecture 1.5. *For any odd prime p,*

$$s(\tau_n(p)) = \infty \text{ for } 0 \leq n \leq p, \text{ and } s(\tau_n(p)) = 2n+1 \text{ for } n \geq p+1.$$

We remark that, in the case of $p-3 \leq n \leq p$ for any odd prime $p \geq 7$, a difference between the extendibility and the stable extendibility appears by Theorem 1.1(1) and Theorem 1.3, that is, $\tau_n(p)$ is not extendible to $L^{n+1}(p)$ in spite of $s(\tau_n(p)) = \infty$.

The paper is organized as follows: In §2, we state some known results necessary to the proofs, and prove Theorem 1.2 in §3. The proofs of Theorem 1.3 and Lemma 1.4 are shown in §4 and §5 respectively.

2 Preliminary

Throughout the paper, p denotes an odd prime. Let η be the canonical $\mathbb{C}$–line bundle over $L^n(p)$, that is, η is the induced vector bundle from the canonical $\mathbb{C}$–line bundle over the complex projective space $\mathbb{C}P^n$ under the projection $\pi\colon L^n(p) \to \mathbb{C}P^n$, and $r(\eta)$ the underlying 2–dimensional $\mathbb{R}$–vector bundle of η. Sometimes, we denote η by η_n to make it clear that η is over $L^n(p)$.

Let $\widetilde{KO}(X)$ (resp. $\widetilde{K}(X)$) denote the reduced real (resp. complex) K–ring. Then, we have the homomorphisms $r : \widetilde{K}(X) \to \widetilde{KO}(X)$ defined by taking the underlying $\mathbb{R}$–vector bundles of given $\mathbb{C}$–vector bundles and $z : \widetilde{KO}(X) \to \widetilde{K}(X)$ defined by taking the complexifications $z(\gamma) = \gamma \otimes \mathbb{C}$ of given $\mathbb{R}$–vector bundles γ. Then, z is a ring homomorphism, and the composition zr is equal to the homomorphism $1 + t : \widetilde{K}(X) \to \widetilde{K}(X)$ where 1 is the identity map and t is the homomorphism defined by taking the conjugate vector bundles of given $\mathbb{C}$–vector bundles.

We set $\sigma = \eta - 1 \in \widetilde{K}(L^n(p))$ and $\bar{\sigma} = r(\sigma) = r(\eta) - 2 \in \widetilde{KO}(L^n(p))$. Then, the explicit structure of $\widetilde{K}(L^n(p))$ and $\widetilde{KO}(L^n(p))$ are determined by Kambe [7] as follows, where $L_0^n(p)$ is the $2n$-skeleton of $L^n(p)$ and $[x]$ for a real number x denotes the largest integer m with $m \leq x$.

Theorem 2.1. [7, Theorem 1, Theorem 2, Lemma (3.4)]

(1) *Let $n = s(p-1) + r$ with $0 \leq r < p-1$. Then,*

$$\widetilde{K}(L^n(p)) \cong (\mathbb{Z}/p^{s+1})^r \oplus (\mathbb{Z}/p^s)^{p-r-1},$$

and the direct summands are generated by $\sigma^1, \cdots, \sigma^r$ and $\sigma^{r+1}, \cdots, \sigma^{p-1}$ re-

spectively. Furthermore, the ring structure is determined by the relations

$$\sigma^p = -\sum_{i=0}^{p-1} \binom{p}{i} \sigma^i, \quad \sigma^{n+1} = 0,$$

where $\binom{a}{b}$ *denotes a binomial coefficient.*

(2) *Let* $q = (p-1)/2$ *and* $n = s(p-1) + r$ *with* $0 \leq r < p-1$. *Then,*

$$\widetilde{KO}(L_0^n(p)) \cong (\mathbb{Z}/p^{s+1})^{[\frac{r}{2}]} \oplus (\mathbb{Z}/p^s)^{q-[\frac{r}{2}]},$$

and the direct summands are generated by $\bar{\sigma}^1, \cdots, \bar{\sigma}^{[\frac{r}{2}]}$ *and* $\bar{\sigma}^{[\frac{r}{2}]+1}, \cdots, \bar{\sigma}^q$ *respectively. Also, we have*

$$\widetilde{KO}(L^n(p)) \cong \begin{cases} \widetilde{KO}(L_0^n(p)) & \text{if } n \not\equiv 0 \mod 4, \\ \mathbb{Z}/2 \oplus \widetilde{KO}(L_0^n(p)) & \text{if } n \equiv 0 \mod 4. \end{cases}$$

Furthermore, the ring structure of $\widetilde{KO}(L_0^n(p))$ *is determined by the relations*

$$\bar{\sigma}^{q+1} = -\sum_{i=1}^{q} \frac{2q+1}{2i-1} \binom{q+i-1}{2i-2} \bar{\sigma}^i, \quad \bar{\sigma}^{[n/2]+1} = 0.$$

The following property is also necessary.

Lemma 2.2. [7, Lemma (3.5)] *The homomorphism* $z : \widetilde{KO}(L_0^n(p)) \to \widetilde{K}(L_0^n(p))$ *is a monomorphism.*

About the lower bound of the stable extendibility of $\tau_n(p)$, we have shown the following proposition using the result due to Sjerve [14, Theorem A].

Proposition 2.3. [6, Proposition 3.1] *For any* $n \geq 1$, $s(\tau_n(p)) \geq 2n+1$.

Alternatively, about the upper bound of the stable extendibility of $\tau_n(p)$, the following has been shown.

Proposition 2.4. [9, Theorem 4.3] *If* $p^{[n/(p-1)]} > n+1$, *then* $\tau_n(p)$ *is not stably extendible to* $L^m(p)$ *with* $m \geq 2n+2$.

It is easy to show that $p^{[n/(p-1)]} > n+1$ holds if and only if $n \geq 2p-2$. Hence, by Propositions 2.3 and 2.4, we have the following.

Corollary 2.5. [6, Theorem 1] $s(\tau_n(p)) = 2n+1$ *if* $n \geq 2p-2$.

Thus, Theorem 1.2 extends Corollary 2.5 in the case $p+1 \leq n \leq 2p-2$, and we shall prove it in the next section.

3 Proof of Theorem 1.2

As mentioned in the previous sections, $s(\tau_n(p)) \geq 2n+1$ for any $n \geq 1$ by Proposition 2.3, and $s(\tau_n(p)) = 2n+1$ for $n \geq 2p-2$ by Corollary 2.5. Also, $s(\tau_n(3)) = 2n+1$ for $n \geq 4$ by Theorem 1.1(2). Thus, the rest of this section is devoted to prove the following proposition, which establishes Theorem 1.2.

Proposition 3.1. *Assume that $p \geq 5$ and $n = p+m$ with $1 \leq m \leq p-3$. Then, $s(\tau_n(p)) \leq 2n+1$.*

Now, under the assumption on p and n in Proposition 3.1, we suppose that $s(\tau_n(p)) \geq 2n+2$, and derive a contradiction. Thus, it is supposed that there is a $(2n+1)$-dimensional vector bundle α over $L^{2n+2}(p)$ satisfying that $i^*\alpha$ is stably equivalent to $\tau_n(p)$ for the inclusion map $i\colon\ L^n(p) \to L^{2n+2}(p)$.

By Theorem 2.1, we have

$$\widetilde{KO}(L_0^n(p)) \cong \mathbb{Z}/p^2\{\bar{\sigma}, \cdots, \bar{\sigma}^{\left[\frac{m+1}{2}\right]}\} \oplus \mathbb{Z}/p\{\bar{\sigma}^{\left[\frac{m+1}{2}\right]+1}, \cdots, \bar{\sigma}^q\} \tag{3.1}$$

since $n = (p-1)+(m+1)$, where $q = (p-1)/2$, and

$$\widetilde{KO}(L_0^{2n+2}(p)) \cong \mathbb{Z}/p^{s+1}\{\bar{\sigma}, \cdots, \bar{\sigma}^{\left[\frac{r}{2}\right]}\} \oplus \mathbb{Z}/p^s\{\bar{\sigma}^{\left[\frac{r}{2}\right]+1}, \cdots, \bar{\sigma}^q\} \tag{3.2}$$

when $2n+2 = s(p-1)+r$ with $s \geq 0$ and $0 \leq r \leq p-2$, where $L_0^{2n+2}(p)$ is the $(4n+4)$–skeleton of $L^{2n+2}(p)$.

We set $[\alpha] = j^*\alpha - (2n+1) \in \widetilde{KO}(L_0^{2n+2}(p))$ for the inclusion map $j : L_0^{2n+2}(p) \to L^{2n+2}(p)$. By (3.2), we can represent $[\alpha]$ as

$$[\alpha] = \sum_{i=1}^{q} a_i\bar{\sigma}^i \in \widetilde{KO}(L_0^{2n+2}(p)),$$

where a_i for $1 \leq i \leq q$ are some integers. Then, we have $i^*[\alpha] = \sum\limits_{i=1}^{q} a_i\bar{\sigma}^i \in \widetilde{KO}(L_0^n(p))$ for the inclusion map $i\colon\ L_0^n(p) \to L_0^{2n+2}(p)$. On the other hand, we recall that the tangent bundle $\tau_n(p)$ satisfies $\tau_n(p)+1 = (n+1)r(\eta_n)$. Since $i^*[\alpha] = \tau_n(p) - (2n+1)$, it follows $i^*[\alpha] = (n+1)\bar{\sigma}$. Thus, by (3.1), we have

$$a_1 \equiv n+1 = p+m+1 \mod p^2 \quad \text{and} \quad a_i \equiv 0 \mod p \quad \text{for} \quad 2 \leq i \leq q.$$

Hence, we can put a_i as follows using some integers b_i:

$$a_i = \begin{cases} b_1p+m+1 & \text{if} \ \ i=1 \\ b_ip & \text{if} \ \ 2 \leq i \leq q. \end{cases}$$

Here, the integer b_1 satisfies

$$b_1 \equiv 1 \mod p. \tag{3.3}$$

Hence, we have

$$[\alpha] = (m+1)\bar{\sigma} + \sum_{i=1}^{q} b_i p \bar{\sigma}^i. \tag{3.4}$$

Let $z : \widetilde{KO}(L_0^{2n+2}(p)) \to \widetilde{K}(L_0^{2n+2}(p))$ and $r : \widetilde{K}(L_0^{2n+2}(p)) \to \widetilde{KO}(L_0^{2n+2}(p))$ be the homomorphisms mentioned in §2. Then, since $zr(\eta - 1) = \eta + \bar{\eta} - 2$, where $\bar{\eta}$ denotes the conjugate vector bundle of η, we have

$$z(\bar{\sigma})^i = (\eta + \bar{\eta} - 2)^i = \sum_{j=0}^{i} \binom{i}{j} (-2)^{i-j}((\eta+\bar{\eta})^j - 2^j). \tag{3.5}$$

Since $\eta\bar{\eta} = 1$,

$$(\eta+\bar{\eta})^j - 2^j = \sum_{k=0}^{[j/2]} \binom{j}{k} (\eta^{j-2k} + \bar{\eta}^{j-2k} - 2). \tag{3.6}$$

Hence, substituting (3.6) into (3.5), we have

$$z(\bar{\sigma}^i) = z(\bar{\sigma})^i = \sum_{j=0}^{i} \sum_{k=0}^{[j/2]} (-2)^{i-j} \binom{i}{j} \binom{j}{k} (\eta^{j-2k} + \bar{\eta}^{j-2k} - 2), \tag{3.7}$$

Thus, by (3.4) and (3.7), we have

$$\begin{aligned} z[\alpha] = {} & (m+1)(\eta+\bar{\eta}-2) \\ & + \sum_{i=1}^{q} \sum_{j=0}^{i} \sum_{k=0}^{[j/2]} b_i p (-2)^{i-j} \binom{i}{j} \binom{j}{k} (\eta^{j-2k} + \bar{\eta}^{j-2k} - 2). \end{aligned} \tag{3.8}$$

Let $c(\beta) = \sum_{j \geq 0} c_j(\beta)$ be the total Chern class of a $\mathbb{C}$–vector bundle β over a space X, where $c_j(\beta) \in H^{2j}(X;\mathbb{Z})$ denotes the j–th Chern class of β and $c_0(\beta) = 1$. As is known, the multiplicative property $c(\beta + \gamma) = c(\beta)c(\gamma)$ holds for any $\mathbb{C}$–vector bundles β and γ. Also, since $c(k) = 1$ for a trivial $\mathbb{C}$–vector bundle k, $c(\beta + k) = c(\beta)$, and the Chern class $c_j(\beta - b)$ of an element $\beta - b \in \widetilde{K}(X)$, where b is the dimension of β, is also defined to be $c_j(\beta)$. We denote the mod p reductions of $c(\beta)$ and $c_j(\beta)$ by the same letters. Then, the following lemma holds.

Lemma 3.2. *For the Chern class of $z[\alpha] \in \widetilde{K}(L_0^{2n+2}(p))$, we have*

$$c(z[\alpha]) = (1-x^2)^{m+1} \left(1 - \sum_{i=1}^{q} b_i \left(\sum_{j=0}^{i} \sum_{k=0}^{[j/2]} (-2)^{i-j} \binom{i}{j} \binom{j}{k} (j-2k)^2 \right) x^{2p} \right).$$

Proof. From (3.8), we have

$$c(z[\alpha]) = c(\eta+\bar{\eta})^{m+1}\prod_{i=1}^{q}\prod_{j=0}^{i}\prod_{k=0}^{[j/2]} c(\eta^{j-2k}+\bar{\eta}^{j-2k})^{b_i p(-2)^{i-j}\binom{i}{j}\binom{j}{k}}. \qquad (3.9)$$

We recall that $\bigoplus_{i\geq 0} H^i(L_0^{2n+2}(p);\mathbb{Z}/p) = \mathbb{Z}/p[x]/(x^{2n+3})$,where $x = c_1(\eta)$, and we have $c(\eta^{j-2k}+\bar{\eta}^{j-2k}) = 1-(j-2k)^2x^2$. Since $h^p \equiv h \mod p$ for any integer h, it follows from (3.9)

$$c(z[\alpha]) = (1-x^2)^{m+1}\prod_{i=1}^{q}\prod_{j=0}^{i}\prod_{k=0}^{[j/2]}(1-(j-2k)^2x^{2p})^{b_i(-2)^{i-j}\binom{i}{j}\binom{j}{k}}.$$

Remark that $(x^{2p})^2 = 0$, because $n \leq 2p-3$ by the assumption and thus $(x^{2p})^2 \in H^{8p}(L_0^{2n+2}(p);\mathbb{Z}/p) = 0$. Therefore, we have

$$\begin{aligned} c(z[\alpha]) &= (1-x^2)^{m+1}\prod_{i=1}^{q}\prod_{j=0}^{i}\prod_{k=0}^{[j/2]}\left(1-b_i(-2)^{i-j}\binom{i}{j}\binom{j}{k}(j-2k)^2x^{2p}\right) \\ &= (1-x^2)^{m+1}\left(1-\sum_{i=1}^{q}\sum_{j=0}^{i}\sum_{k=0}^{[j/2]} b_i(-2)^{i-j}\binom{i}{j}\binom{j}{k}(j-2k)^2x^{2p}\right) \\ &= (1-x^2)^{m+1}\left(1-\sum_{i=1}^{q} b_i\left(\sum_{j=0}^{i}\sum_{k=0}^{[j/2]}(-2)^{i-j}\binom{i}{j}\binom{j}{k}(j-2k)^2\right)x^{2p}\right), \end{aligned}$$

as is required. □

For each $1 \leq i \leq q$ and $0 \leq j \leq i$, we put

$$I_j = \sum_{k=0}^{[j/2]}\binom{j}{k}(j-2k)^2 \quad \text{and} \quad K_i = \sum_{j=0}^{i}(-2)^{i-j}\binom{i}{j}I_j.$$

Then, the equation in Lemma 3.2 is represented as

$$c(z[\alpha]) = (1-x^2)^{m+1}\left(1-\left(\sum_{i=1}^{q} b_iK_i\right)x^{2p}\right). \qquad (3.10)$$

We shall show the following.

Lemma 3.3. *$K_1 = 1$ and $K_i = 0$ for $2 \leq i \leq q$.*

Proof. First, we assume that j is odd. Then, we have the following equalities:

$$\sum_{k=0}^{[j/2]}\binom{j}{k} = \frac{1}{2}2^j = 2^{j-1};$$

$$\sum_{k=0}^{[j/2]} k(j-k)\binom{j}{k} = \sum_{k=1}^{[j/2]} \frac{j!}{(k-1)!(j-k-1)!}$$
$$= j(j-1)\sum_{k=1}^{[j/2]}\binom{j-2}{k-1} = j(j-1)\sum_{k'=0}^{[(j-2)/2]}\binom{j-2}{k'}$$
$$= j(j-1)\frac{2^{j-2}}{2} = j(j-1)2^{j-3}.$$

Hence,

$$I_j = j^2\sum_{k=0}^{[j/2]}\binom{j}{k} - 4\sum_{k=0}^{[j/2]} k(j-k)\binom{j}{k} = j^2 2^{j-1} - 4j(j-1)2^{j-3} = j2^{j-1}.$$

Next, we assume that j is even. Then, we have

$$\sum_{k=0}^{[j/2]}\binom{j}{k} = \frac{1}{2}\left(2^j - \binom{j}{j/2}\right) + \binom{j}{j/2} = 2^{j-1} + \frac{1}{2}\binom{j}{j/2}, \quad \text{and}$$

$$\sum_{k=0}^{[j/2]} k(j-k)\binom{j}{k} = j(j-1)\sum_{k=1}^{[j/2]}\binom{j-2}{k-1} = j(j-1)\left(2^{j-3} + \frac{1}{2}\binom{j-2}{(j/2)-1}\right).$$

Hence, I_j is transformed as follows:

$$\sum_{k=0}^{[j/2]}(j^2 - 4kj + 4k^2)\binom{j}{k} = j^2\left(2^{j-1} + \frac{1}{2}\binom{j}{j/2}\right) - 4j(j-1)\left(2^{j-3} + \frac{1}{2}\binom{j-2}{(j/2)-1}\right)$$
$$= j2^{j-1} + \frac{1}{2}j^2\binom{j}{j/2} - 2j(j-1)\binom{j-2}{(j/2)-1}.$$

Here,

$$2j(j-1)\binom{j-2}{(j/2)-1} = \frac{1}{2}j^2\binom{j}{j/2},$$

and thus we have the same conclusion $I_j = j2^{j-1}$ as in the case of odd j.

Therefore, we have

$$K_i = \sum_{j=0}^{i}(-2)^{i-j}\binom{i}{j}j2^{j-1} = (-1)^i 2^{i-1}\sum_{j=0}^{i}(-1)^j j\binom{i}{j}$$
$$= (-1)^i 2^{i-1}\sum_{j=1}^{i}(-1)^j i\binom{i-1}{j-1} = (-1)^{i+1}2^{i-1} i\sum_{j'=0}^{i-1}(-1)^{j'}\binom{i-1}{j'}.$$

Since

$$\sum_{j'=0}^{i-1}(-1)^{j'}\binom{i-1}{j'} = (1-1)^{i-1} = 0 \ \text{ for } \ i \geq 2 \quad \text{and} \quad = 1 \ \text{ for } \ i = 1,$$

we have the required result. ☐

Now, we can complete the proof of Proposition 3.1 as follows. By (3.10) and lemma 3.3, we have

$$c(z[\alpha]) = (1-x^2)^{m+1}(1-b_1x^{2p}) = 1 + \cdots + (-1)^m b_1 x^{2p+2m+2}.$$

Recall that $n = p+m$ and $b_1 \equiv 1 \mod p$ by (3.3). Since $H^{4n+4}(L_0^{2n+2}; \mathbb{Z}/p) = \mathbb{Z}/p$ generated by x^{2n+2}, $c_{2n+2}(z[\alpha]) \neq 0$. On the other hand, since α is of dimension $2n+1$, we have $c_{2n+2}(z[\alpha]) = 0$, which contradicts the above. Thus, we have completed the proof of Proposition 3.1, and obtained Theorem 1.2.

4 Proof of Theorem 1.3

In the remainder of the article, we denote the stable equivalence of two $\mathbb{R}$–vector bundles (resp. $\mathbb{C}$–vector bundles) ζ and γ with the same dimensions over a space X simply by $\zeta = \gamma$ considering them as elements of the K–ring $KO(X)$ (resp. $K(X)$) if there is no confusion. In order to prove Theorem 1.3, we first have the following combinatorial congruence.

Lemma 4.1. *Let p be an odd prime, and k an integer with $0 \leq k \leq p-2$. Then, the following holds.*

$$\sum_{j=k}^{p-1} \binom{j}{k} \equiv 0 \mod p.$$

Proof. Let S be the value of the left hand side in the required congruence. Then, S appears as the coefficient of x^k in the expansion of the polynomial $f(x) = \sum_{j=k}^{p-1}(1+x)^j$ on the variable x. But, since

$$\begin{aligned} f(x) &= (1+x)^k(1+(1+x)+\cdots+(1+x)^{p-k-1}) \\ &= (1+x)^k\frac{(1+x)^{p-k}-1}{x} = \frac{1}{x}((1+x)^p-(1+x)^k), \end{aligned}$$

S is equal to the coefficient of x^{k+1} in the expansion of $(1+x)^p - (1+x)^k$. Hence, we have

$$S = \binom{p}{k+1} \equiv 0 \mod p$$

since $0 \leq k \leq p-2$, which shows the required result. ☐

Using Lemma 4.1, the next lemma follows.

Lemma 4.2. *Let p be an odd prime and $0 \leq n \leq p-2$. Then, over $L^n(p)$, we have the following stable equivalences:*

$$\eta_n^{p-1} + \eta_n^{p-2} + \cdots + \eta_n + 1 = p;$$

$$r(\eta_n^{[\frac{p}{2}]}) + r(\eta_n^{[\frac{p}{2}]-1}) + \cdots + r(\eta_n) + 1 = p.$$

Proof. Since $0 \leq n \leq p-2$, $\widetilde{K}(L^n(p)) \cong \bigoplus_{i=1}^n \mathbb{Z}/p\{\sigma^i\}$ and $\sigma^{n+1} = 0$ by Theorem 2.1(1). Hence, $\sigma^{p-1} = 0$. Then, using Lemma 4.1, we have

$$\begin{aligned}
\sum_{j=0}^{p-1} \eta_n^j &= \sum_{j=0}^{p-1} (\sigma+1)^j = \sum_{j=0}^{p-1} \left(\sum_{k=0}^{j} \binom{j}{k} \sigma^k \right) \\
&= \sum_{k=0}^{p-1} \left(\sum_{j=k}^{p-1} \binom{j}{k} \right) \sigma^k = \sum_{k=0}^{p-2} \left(\sum_{j=k}^{p-1} \binom{j}{k} \right) \sigma^k \\
&= p + \sum_{k=1}^{p-2} \left(\sum_{j=k}^{p-1} \binom{j}{k} \right) \sigma^k = p.
\end{aligned}$$

Thus, we obtain the first required stable equivalence. About the second stable equivalence, since $\eta_n^p = 1$ and $\eta_n \bar{\eta}_n = 1$, $r(\eta_n^i) = r(\bar{\eta}_n^{p-i}) = r(\eta_n^{p-i})$ for $0 \leq i \leq p$. Therefore, from the first equivalence, we have

$$2r(\eta_n^{[\frac{p}{2}]}) + 2r(\eta_n^{[\frac{p}{2}]-1}) + \cdots + 2r(\eta_n) + 2 = 2p.$$

Since $\widetilde{KO}(L^{p-2}(p))$ is a torsion group without 2–torsion by Theorem 2.1(2), dividing both sides of the above equivalence by 2, we have the required equivalence for $n = p-2$. Then, taking the induced vector bundles $r(\eta_n^j) = i^* r(\eta_{p-2}^j)$ for $0 \leq n \leq p-3$ and $1 \leq j \leq p-1$, we have the required result. □

Proof of Theorem 1.3(1). Recall that $\tau_n(p) = (n+1)r(\eta_n) - 1$, and notice that $p(r(\eta_n) - 2) = 0$ by Theorem 2.1(2) when $0 \leq n \leq p$. Thus, we have $\tau_p(p) = (p+1)r(\eta_p) - 1 = r(\eta_p) + 2p - 1$. Since the vector bundle $r(\eta_p) + 2p - 1$ over $L^p(p)$ is extendible to $L^m(p)$ for every $m \geq p$, we have $s(\tau_p(p)) = \infty$. Similarly, since $\tau_{p-1}(p) = pr(\eta_{p-1}) - 1 = 2p - 1$, we have $s(\tau_{p-1}(p)) = \infty$. As for the case of $n = p-2$ or $p-3$, using Lemma 4.2, we have

$$\tau_{p-2}(p) = (p-1)r(\eta_{p-2}) - 1 = 2 \sum_{i=0}^{[p/2]-2} r(\eta_{p-2}^{[\frac{p}{2}]-i}) + r(\eta_{p-2}) + 1,$$

$$\tau_{p-3}(p) = (p-2)r(\eta_{p-3}) - 1 = 2 \sum_{i=0}^{[p/2]-2} r(\eta_{p-3}^{[\frac{p}{2}]-i}) + 1.$$

Since $r(\eta_n), r(\eta_n^2), \cdots, r(\eta_n^{[\frac{p}{2}]-1})$ are extendible to $L^m(p)$ for every $m \geq n$, we have $s(\tau_{p-2}(p)) = \infty$ and $s(\tau_{p-3}(p)) = \infty$, as is required. □

In order to show Theorem 1.3(2), we must develop some properties of vector bundles over orbit spaces. For an $\mathbb{R}$–vector bundle ζ over a space X, we denote the i-th Pontrjagin class of ζ by $p_i(\zeta) \in H^{4i}(X;\mathbb{Z})$, which also denotes its mod p reduction $p_i(\zeta) \in H^{4i}(X;\mathbb{Z}/p)$. Let M be an orbit manifold by a free action of a finite group G on a sphere S^m. Then, Sjerve [15] has shown the following theorem.

Theorem 4.3. [15, Theorem (1.8)] *Let ζ be a $(2r+1)$–dimensional $\mathbb{R}$–vector bundle over $M = S^m/G$. If an integer s satisfies the Condition 4.4 below, there exists a $2s$–dimensional $\mathbb{R}$–vector bundle γ which satisfies $\gamma+(2r+1-2s) = \zeta$.*

Condition 4.4. [15, (1.4) and (1.8)]

(1) For any prime divisor r of $|G|$, $r > \max([m/2] - s + 1, s + 1)$ holds.
(2) $p_{s+1}(\zeta) \equiv p_{s+2}(\zeta) \equiv \cdots \equiv p_r(\zeta) \equiv 0$ mod 2–torsions.
(3) There exists $u \in H^{2s}(M;\mathbb{Z})$ satisfying $u^2 = p_s(\zeta)$.
(4) There exists a $2s$–dimensional vector bundle β over S^m which satisfies $\beta + (2r + 1 - 2s) = \pi^*\zeta$ for the projection $\pi : S^m \to M$.

Now,we prove Theorem 1.3(2).

Proof of Theorem 1.3(2). We apply Theorem 4.3 in the case $M = S^5/(\mathbb{Z}/p) = L^2(p)$ and $\zeta = 3r(\eta_2) + 1$, under the assumption that p is an odd prime with $p \equiv \pm 1 \mod 12$. Thus, taking $m = 5$ and $r = 3$, we shall prove that

$$\text{Condition 4.4 is satisfied for } s = 1 \text{ in our case.} \tag{4.1}$$

Then, by Theorem 4.3, there exists a 2–dimensional $\mathbb{R}$–vector bundle γ which satisfies $\gamma + 5 = 3r(\eta_2) + 1$. Since $\tau_2(p) + 1 = 3r(\eta_2)$, it turns out that $\tau_2(p)$ is stably equivalent to $\gamma+3$. Since any 2–dimensional vector bundle over $L^n(p)$ is extendible to $L^m(p)$ for every $m \geq n$ in general (cf. [8]), we have $s(\tau_2(p)) = \infty$ if $p \equiv \pm 1 \mod 12$, as is required.

Now, we prove (4.1). First, the condition (1) is satisfied obviously, and also the condition (2) since $p_i(3r(\eta_2)+1) \in H^{4i}(L^2(p);\mathbb{Z}) = 0$ for $i \geq 2$. The condition (4) holds because $\pi^*(3r(\eta_2) + 1)$ is a trivial vector bundle over S^5 for the projection $\pi : S^5 \to L^2(p)$. Thus, it remains to ascertain the condition (3). Since $p_1(3r(\eta_2) + 1) = 3x^2 \in H^4(L^2(p);\mathbb{Z}) \cong \mathbb{Z}/p$, where $x = p_1(r(\eta_2))$, it is required to show that there exists an element $u \in H^2(L^2(p);\mathbb{Z})$ which satisfies $u^2 \equiv 3x^2 \mod p$. Since $H^2(L^2(p);\mathbb{Z}) \cong \mathbb{Z}/p$ generated by x, we put $u = ax$ using an integer a. Then, $u^2 \equiv 3x^2 \mod p$ holds if and only if $a^2 \equiv 3 \mod p$. Using the quadratic residue, we see that there exists an integer a satisfying $a^2 \equiv 3 \mod p$ if and only if $p = \pm 1 \mod 12$ (cf. [16, p.80]). In fact, using the Legendre notation $\left(\dfrac{q}{p}\right)$, $a^2 \equiv 3 \mod p$ holds for some integer a if and only if

$\left(\frac{3}{p}\right) = +1$. By the law of reciprocity, we have $\left(\frac{3}{p}\right) = (-1)^{(p-1)/2}\left(\frac{p}{3}\right)$. Also, using the first complementary law, $\left(\frac{p}{3}\right) = +1$ or -1 according as $p \equiv 1$ or -1 mod 3. Hence, we have $\left(\frac{3}{p}\right) = +1$ if and only if $p \equiv \pm 1 \mod 12$. Thus, we have the required result. □

We remark that the condition $p \equiv \pm 1 \mod 12$ in Theorem 1.3(2) is satisfied by an infinite number of primes p by the following theorem due to Dirichlet (cf. [13, p.25]), and thus $s(\tau_2(p)) = \infty$ holds for infinitely many primes p.

Theorem 4.5 (Dirichlet). *If integers m and k are prime each other, that is, the greatest common divisor $(m, k) = 1$, then there is an infinite number of primes p which satisfy $p \equiv k \mod m$.*

5 Proof of Lemma 1.4

First, we remark the following.

Lemma 5.1. *For any $j \geq 1$, there is the following stable equivalence over $L^n(p)$:*

$$r(\eta_n)^j = \sum_{k=0}^{[(j-1)/2]} \binom{j}{k} r(\eta_n^{j-2k}).$$

Proof. We recall that the complexification homomorphism $z : \widetilde{KO}(L_0^{n+1}(p)) \to \widetilde{K}(L_0^{n+1}(p))$ is a monomorphism by Lemma 2.2. Then, as an element of $\widetilde{K}(L_0^{n+1}(p))$,

$$\begin{aligned} z(r(\eta_{n+1})^j - 2^j) &= (\eta_{n+1} + \bar{\eta}_{n+1})^j - 2^j = \sum_{k=0}^{[(j-1)/2]} \binom{j}{k} (\eta_{n+1}^{j-2k} + \bar{\eta}_{n+1}^{j-2k} - 2) \\ &= z\left(\sum_{k=0}^{[(j-1)/2]} \binom{j}{k} (r(\eta_{n+1}^{j-2k}) - 2)\right) \\ &= z\left(\sum_{k=0}^{[(j-1)/2]} \binom{j}{k} r(\eta_{n+1}^{j-2k}) - \left(2^j + \epsilon_j \binom{j}{[j/2]}\right)\right), \end{aligned}$$

where $\epsilon_j = 1$ or 0 according as j is an even or odd integer. Thus, the required stable equivalence holds over $L_0^{n+1}(p)$. Then, applying the homomorphism $j^* : \widetilde{KO}(L_0^{n+1}(p)) \to \widetilde{KO}(L^n(p))$ induced by the inclusion $j : L^n(p) \to L_0^{n+1}(p)$, we have the required stable equivalence over $L^n(p)$. □

Using Lemma 5.1, the element $\bar{\sigma}^i \in \widetilde{KO}(L^n(p))$ as in Theorem 2.1 is represented using $r(\eta_n^j)$ for $0 \le j \le i$ as follows:

$$\bar{\sigma}^i = (r(\eta_n)-2)^i = \sum_{j=0}^{i} \binom{i}{j}(-2)^{i-j} r(\eta_n)^j = \sum_{j=0}^{i} \binom{i}{j}(-2)^{i-j} \sum_{k=0}^{[(j-1)/2]} \binom{j}{k} r(\eta_n^{j-2k}).$$

Thus, we have the following.

Corollary 5.2. *As elements of* $\widetilde{KO}(L^n(p))$,

$$\begin{aligned}
\bar{\sigma}^6 &= r(\eta_n^6) - 12r(\eta_n^5) + 66r(\eta_n^4) - 220r(\eta_n^3) + 495r(\eta_n^2) - 792r(\eta_n) + 924, \\
\bar{\sigma}^5 &= r(\eta_n^5) - 10r(\eta_n^4) + 45r(\eta_n^3) - 120r(\eta_n^2) + 210r(\eta_n) - 252, \\
\bar{\sigma}^4 &= r(\eta_n^4) - 8r(\eta_n^3) + 28r(\eta_n^2) - 56r(\eta_n) + 70, \\
\bar{\sigma}^3 &= r(\eta_n^3) - 6r(\eta_n^2) + 15r(\eta_n) - 20, \\
\bar{\sigma}^2 &= r(\eta_n^2) - 4r(\eta_n) + 6.
\end{aligned}$$

Proof of Lemma 1.4. First, we consider the case of $p = 11$, and let $n = 4, 5$. Then, by Theorem 2.1,

$$\widetilde{KO}(L_0^n(11)) = \mathbb{Z}/11\{\bar{\sigma}\} \oplus \mathbb{Z}/11\{\bar{\sigma}^2\}$$

with the relation $\bar{\sigma}^3 = 0$. Thus, we have $11\bar{\sigma} = 11\bar{\sigma}^2 = 0$ and $\bar{\sigma}^i = 0$ for $i \ge 3$. Thus, substituting the relations in Corollary 5.2 into the equation $\bar{\sigma}^4 + 10\bar{\sigma}^3 + 3(11\bar{\sigma}^2) + 3(11\bar{\sigma}) = 0$, we have

$$r(\eta_n^4) + 2r(\eta_n^3) + r(\eta_n^2) - 5r(\eta_n) + 2 = 0.$$

Hence, it follows

$$\begin{aligned}
\tau_4(11) &= 5r(\eta_4) - 1 = r(\eta_4^4) + 2r(\eta_4^3) + r(\eta_4^2) + 1, \\
\tau_5(11) &= 6r(\eta_5) - 1 = r(\eta_5^4) + 2r(\eta_5^3) + r(\eta_5^2) + r(\eta_5) + 1.
\end{aligned}$$

Since $r(\eta_n^i)$ for any $i \ge 0$ over $L^n(11)$ is stably extendible to $L^m(11)$ for any $m \ge n$, we have the required result $s(\tau_n(11)) = \infty$ for $n = 4, 5$.

We can proceed similarly to prove the remaining statements. In the case of $p = 13$ and $n = 5$, doing the same way as above, by substituting the relations in Corollary 5.2 into the equation $\bar{\sigma}^5 + 11\bar{\sigma}^4 + 44\bar{\sigma}^3 + 6(13\bar{\sigma}^2) + 4(13\bar{\sigma}^2) = 0$, we have

$$r(\eta_5^5) + r(\eta_5^4) + r(\eta_5^3) + 2r(\eta_5^2) - 6r(\eta_5) + 2 = 0.$$

Thus, we have

$$\tau_5(13) = 6r(\eta_5) - 1 = r(\eta_5^5) + r(\eta_5^4) + r(\eta_5^3) + 2r(\eta_5^2) + 1,$$

and obtain the required result.

Let $p = 13$ and $n = 6$ or 7. Then, by Theorem 2.1,

$$\widetilde{KO}(L^n(13)) = \mathbb{Z}/13\{\bar{\sigma}\} \oplus \mathbb{Z}/13\{\bar{\sigma}^2\} \oplus \mathbb{Z}/13\{\bar{\sigma}^3\}$$

with the relation $\bar{\sigma}^4 = 0$. Thus, substituting the relations in Corollary 5.2 to the equation $\bar{\sigma}^6 + 14\bar{\sigma}^5 + 74\bar{\sigma}^4 + 14(13\bar{\sigma}^3) + 16(13\bar{\sigma}^2) + 7(13\bar{\sigma}) = 0$, we have

$$r(\eta_n^6) + 2r(\eta_n^5) + 3r(\eta_n^2) - 7r(\eta_n) + 2 = 0.$$

Hence, it follows

$$\begin{aligned}\tau_6(13) &= 7r(\eta_6) - 1 = r(\eta_6^6) + 2r(\eta_6^5) + 3r(\eta_6^2) + 1,\\ \tau_7(13) &= 8r(\eta_7) - 1 = r(\eta_7^6) + 2r(\eta_7^5) + 3r(\eta_7^2) + r(\eta_7) + 1,\end{aligned}$$

and thus we obtain the required result $s(\tau_n(13)) = \infty$ for $n = 6, 7$.

As for the case of $p = 17$ and $n = 2$,

$$\widetilde{KO}(L_2(17)) = \mathbb{Z}/17\{\bar{\sigma}\}$$

with the relation $\bar{\sigma}^2 = 0$. Then, substituting the relations in Corollary 5.2 to the equation $\bar{\sigma}^4 + 8\bar{\sigma}^3 + 21\bar{\sigma}^2 + 17\bar{\sigma} = 0$, we obtain

$$r(\eta_2^4) + r(\eta_2^2) - 3r(\eta_2) + 2 = 0.$$

Hence, it follows

$$\tau_2(17) = 3r(\eta_2) - 1 = r(\eta_2^4) + r(\eta_2^2) + 1,$$

and thus we have $s(\tau_2(17)) = \infty$, and we have completed the proof. □

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