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Characterization and Existence of Affine α-Resolvable Block Designs

(Affine α-Resolvable Block Designs の
特徴付けと存在性)

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主　論　文

Characterization and existence of affine α-resolvable block designs

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Contents

This essentially consists in a collection of four papers by Satoru Kadowaki and Sanpei Kageyama (2003, 2008a, 2008b, 2009) to make a contribution for the advancement on combinatorics in affine α-resolvable balanced incomplete block or 2-associate partially balanced incomplete block designs and to have a Ph.D. Thesis by Satoru Kadowaki. In fact, Sections 3, 4 and 5 of the thesis are based on Kadowaki and Kageyama (2003, 2008a, 2008b) respectively, which have been already published, while Sections 2, 6 and 7 are based on Kadowaki and Kageyama (2009), which has been accepted for publication.

I would like to thank all the referees of these papers for their valuable suggestions and comments which have been incorporated in this thesis.

1. Introduction

One of the earliest examples of a resolvable balanced incomplete block (BIB) design is the Kirkman (1850a) school girl problem formulated in 1850 and pursued further in another paper (Kirkman, 1850b). The problem was to find different row arrangements such that any two girls would be assigned to the same row exactly on one day. This can be seen as equivalent to finding a resolvable solution of a BIB design with parameters $v = 6t+3, b = (2t+1)(3t+1), r = 3t+1, k = 3, \lambda = 1$. In fact, we want to arrange $6t+3$ girls in $2t+1$ rows of 3 each for $3t+1$ successive days. Kirkman himself gave some solutions and many mathematicians worked on this problem in the late 19th and early 20th century. A relevant bibliography can be found in Eckenstein (1912). However, no complete solution was known until Ray-Chaudhuri and Wilson (1971) completely solved the problem. This was a celebrated open problem throughout the period 1850-1970.

Though Yates (1939, 1940) has pointed out some statistical advantages of resolvable designs and their original form had appeared earlier in the mathematical literature as described above, the interest in resolvable BIB designs was greatly enhanced by a combinatorial paper by Bose (1942), who again introduced the concept of resolvable and affine resolvable block designs (BD) clearly and also derived a fundamental inequality now called Bose's inequality. This bound plays a key role to characterize an affine resolvable BIB design. Further statistical usefulness of affine resolvable designs can be found in Bailey, Monod and Morgan (1995), and Caliński and Kageyama (2000, 2008).

The concept was generalized to α-resolvability and affine α-resolvability by Shrikhande and Raghavarao (1963). The constructions of (affine) α-resolvable BIB designs or partially balanced incomplete block (PBIB) designs with their combinatorial properties have been discussed in literature (see, for example, Bailey, Monod and Morgan, 1995, Banerjee and Kageyama, 1990, Caliński and Kageyama, 2000, 2003, Clatworthy, 1973, Furino and Mullin, 1993, Furino, Miao and Yin, 1996, Ge, 2002, Ge and Ling, 2004, Ghosh, Bhimani and Kageyama, 1989, Hanani, 1974, Jungnickel, Mullin and Vanstone, 1991, Kageyama, 1973a, 1976, Kageyama and Miao, 1998, Kageyama and Mohan, 1985, Mohan, 1980, Mohan and Kageyama, 1989, Qian, Meng and Du, 2008, Rees, 2000, Shrikhande, 1976, Shrikhande and Raghavarao, 1963, Vasiga, Furino and Ling, 2001, Zhang and Du, 2005). Furthermore, as necessary conditions for the existence of α-resolvable BIB designs, several interesting inequalities on parameters are available (see Bose, 1942, Kadowaki and Kageyama, 2003, Kageyama, 1971, 1973b). The information obtained from such bounds are useful to characterize such block designs.

A block design BD(v, b, r, k) is said to be α-resolvable if the b blocks of size k each can be grouped into t sets (called α-resolution sets) of β blocks each ($b = \beta t$) such that in each α-resolution set every treatment (or point) is replicated α times ($r = \alpha t$). An α-resolvable BD is said to be affine α-resolvable if every two distinct blocks from the same α-resolution set intersect in the same number, say, q_1, of treatments, whereas every two blocks belonging to different α-resolution sets intersect in the same number, say, q_2, of treatments. It follows (see Kageyama, 1973a, Shrikhande and Raghavarao, 1964) that for an affine α-resolvable BD$(v, b = \beta t, r = \alpha t, k)$ with block intersection numbers q_1 and q_2, the following relations $q_1 = k(\alpha - 1)/(\beta - 1)$ and $q_2 = k\alpha/\beta = k^2/v$ hold. Note that both of q_1 and q_2 must be nonnegative integers. An integral expression of q_1 without α and β in terms of design parameters only is meaningful.

When $\alpha = 1$, the definition of (affine) 1-resolvability coincides with that by Bose (1942). Hence a 1-resolvable or an affine 1-resolvable design is simply called a resolvable or an affine resolvable design, respectively. In this case $t = r$. Since an affine resolvable BIB design has the same structure as an affine geometry, by regarding treatments and blocks of a design as points and flats of a geometry respectively, a name "affine" is introduced.

In this thesis, some characterization on affine α-resolvable block designs are dealt with from a combinatorial point of view. Their topics are concerned with bounds on parameters in designs, the characterization of parameters in a closed form and existence problems with construction methods. The block designs discussed here are BIB designs and PBIB designs. The basic procedure is based on the number-theoretic and combinatorial approach. Comprehensive and useful results on combinatorics are presented. Several methods of construction are also newly shown with practical affine resolvable block designs.

2. Preliminaries

Several key definitions on technical terms used here are described in this section.

Definition 2.1. A balanced incomplete block (BIB) design with parameters v, b, r, k, λ is defined as an arrangement of v treatments into b blocks of $k(< v)$ treatments each such that

(1) each treatment occurs at most once in a block,
(2) each treatment occurs in exactly r different blocks,
(3) every pair of treatments occurs together in exactly λ blocks.

This is also denoted by BIB(v, b, r, k, λ) or BIB(v, k, λ). Though (2) can be derived from other conditions, it is traditionally mentioned. The parameter λ is called a coincidence number of the design.

It is known that the five parameters of the BIB design satisfy $vr = bk$, $\lambda(v - 1) = r(k - 1)$ and $b \geq v$ (called Fisher's inequality). In particular, when $b = v$, the BIB design is said to be symmetric. It is also known that in an α-resolvable BIB design with $b = \beta t$ and $r = \alpha t$, $b \geq v + t - 1$ (called Bose's inequality) holds, and $b = v + t - 1$ is a necessary and sufficient condition for an α-resolvable BIB design to be affine α-resolvable with the block intersection number $q_1 = k(\alpha - 1)/(\beta - 1) = k + \lambda - r$ (cf. Shrikhande and Raghavarao, 1964, Kageyama, 1973a).

In defining a 2-associate PBIB design with two distinct coincidence numbers λ_1 and λ_2 different from a BIB design, the concept of an association scheme for a set of v treatments is needed.

Given v treatments 1, 2, ..., v, a relation satisfying the following conditions is said to have an association scheme with two associate classes:

(1) Any two treatments are either 1st or 2nd associates, the relation of association being symmetric, that is, if the treatment x is ith associate of the treatment y, then y is ith associate of x for $i = 1, 2$.
(2) Each treatment x has n_i ith associates, the number n_i being independent of x for $i = 1, 2$.

(3) If any two treatments x and y are ith associates, then the number of treatments that are jth associates of x and ℓth associates of y is $p^i_{j\ell}$ and is independent of the pair of ith associates x and y for $i, j, \ell = 1, 2$.

The numbers $v, n_i, p^i_{j\ell}$ are called the parameters of an association scheme.

Definition 2.2. Given an association scheme with two associate classes for a set of v treatments, a 2-associate PBIB design is defined as an arrangement of v treatments into b blocks of size $k(< v)$ each such that

(1) each treatment occurs at most once in a block,
(2) each treatment occurs in exactly r different blocks,
(3) if two treatments are ith associates, then they occur together in exactly λ_i blocks, the number λ_i being independent of the particular pair of ith associates for $i = 1, 2$.

The numbers v, b, r, k, λ_i are called the parameters of a PBIB design. Like a BIB design, when $b = v$, the PBIB design is said to be symmetric. It holds that in a 2-associate PBIB design, $vr = bk$, $n_1 + n_2 = v - 1$, $n_1\lambda_1 + n_2\lambda_2 = r(k-1)$. Conventionally let every treatment be the 0th associate of itself and of no other treatment, and then it is seen that $n_0 = 1$ and $\lambda_0 = r$.

From Definitions 2.1 and 2.2, when $\lambda_1 = \lambda_2$, a PBIB design becomes a BIB design. In this sense, in a 2-associate PBIB design $\lambda_1 \neq \lambda_2$ in general.

Remark 2.1. Though, by a relation of Fisher's inequality and Bose's inequality, a symmetric BIB design cannot possess a property of affine α-resolvability, it is remarkable that there exists an affine α-resolvable "symmetric" PBIB design.

The known "2-associate" PBIB designs have been mainly classified into the following types depending on association schemes, i.e., group divisible, triangular, Latin-square (L_2), cyclic (see Bose and Shimamoto, 1952), all of which will be discussed here. However, the parameters $p^i_{j\ell}$ are not described in Definitions 2.3 to 2.6 (for them, see Raghavarao, 1988).

Definition 2.3. A 2-associate PBIB design is said to be group divisible (GD) if there are $v = mn$ treatments which can be divided into m groups of n treatments each, such that any two treatments of the same group are the 1st associates and any two treatments from different groups are the 2nd associates. Here $m, n \geq 2$, $n_1 = n - 1$ and $n_2 = n(m - 1)$.

The GD designs are further classified into three subclasses: Singular (S) if $r - \lambda_1 = 0$; Semi-Regular (SR) if $r - \lambda_1 > 0$ and $rk - v\lambda_2 = 0$; Regular if $r - \lambda_1 > 0$ and $rk - v\lambda_2 > 0$. By a relation $n_1\lambda_1 + n_2\lambda_2 = r(k - 1)$, it holds that $(rk - v\lambda_2) - (r - \lambda_1) = n(\lambda_1 - \lambda_2)$. The last relation shows that for an SGD design $\lambda_1 > \lambda_2$, while for an SRGD design $\lambda_2 > \lambda_1$. Note that $r - \lambda_1$ and $rk - v\lambda_2$ are eigenvalues of an information matrix of the design.

Definition 2.4. A 2-associate PBIB design is said to be triangular if there are $v = n(n - 1)/2$ treatments which are arranged into an $n \times n$ array such that

(1) the position in the principal diagonal are left blank,
(2) the $n(n - 1)/2$ positions above the principal diagonal are filled by the numbers 1, 2, ..., $n(n - 1)$ corresponding to the treatments,
(3) the $n(n - 1)/2$ positions below the principal diagonal are filled so that the array is symmetric about the principal diagonal,
(4) for any treatment x the 1st associates are exactly those that occur in the same row or in the same column as x, otherwise they are the 2nd associates.

Here $n \geq 4$, $n_1 = 2(n - 2)$ and $n_2 = (n - 2)(n - 3)/2$.

Definition 2.5. A 2-associate PBIB design is said to be L_2 (Latin-square) if there are $v = s^2$ treatments which are arranged into an $s \times s$ array such that any two treatments in the same row or in the same column of the array are the 1st associates, otherwise they are the 2nd associates. Here $s \geq 2$, $n_1 = 2(s - 1)$ and $n_2 = (s - 1)^2$.

Definition 2.6. A 2-associate PBIB design with v treatments is said to be cyclic if the set of the 1st associates of ith treatment is $(i+d_1, i+d_2, ..., i+d_{n_1})$ mod v, where the elements d_j satisfy the following conditions:

(1) The elements d_j are all different and $0 < d_j < v$ for $j = 1, 2, ..., n_1$.
(2) Among the $n_1(n_1 - 1)$ differences $d_j - d_{j'}$ each of the $d_1, d_2, ..., d_{n_1}$ occurs p_{11}^1 times and each of the $e_1, e_2, ..., e_{n_2}$ occurs p_{11}^2 times, where $d_j, e_{j'}$ are all nonzero distinct and $\{d_1, d_2, ..., d_{n_1}, e_1, e_2, ..., e_{n_2}\} \subseteq \{1, 2, ..., v\}$.

(3) For each d_i in a set $D = (d_1, d_2, ..., d_{n_1})$, there exists d_k in D such that $d_k = -d_i$.

Here $n_1 = n_2 = (v-1)/2$.

The cyclic structure is very convenient to store the information on incidence in block designs and to construct a block design effectively. However, it is shown (Ma, 1984) that all cyclic association schemes have the parameters $v = 4t+1$ being a prime and $n_1 = n_2 = 2t$ for a positive integer t Thus the cyclic design may exist only for a prime v being the number of treatments. Also it is shown that $p_{11}^1 = t-1$ and $p_{11}^2 = t$.

Definition 2.7. In a BD(v, b, r, k), the $v \times b$ incidence matrix $N = (n_{ij})$ is defined such that n_{ij} is the number of times ith treatment occurs in jth block. Hence $r = \sum_{j=1}^{b} n_{ij}$ for all i and $k = \sum_{i=1}^{v} n_{ij}$ for all j. In this thesis the usual case of $n_{ij} = 0$ or 1 for all $i = 1, 2, ..., v$ and $j = 1, 2, ..., b$ (called a binary design) is only considered, as seen, for example, from (1) of Definitions 2.1 and 2.2.

In a block design the eigenvalues of information matrices NN' and $N'N$ play a key role for the existence problem.

Two results will be needed for the present further argument.

Lemma 2.1 (cf. Shrikhande and Raghavarao, 1964, Kageyama, 2008a). In an affine α-resolvable BD$(v, b = \beta t, r = \alpha t, k)$ with the incidence matrix N, the matrix $N'N$ has eigenvalues $rk, k\{1-(\alpha-1)/(\beta-1)\}$ and 0, with multiplicities 1, $b-t$ and $t-1$, respectively.

Lemma 2.2 (cf. Lang, 1986). The matrices XY and YX have the same nonzero eigenvalues with the same multiplicities, where the matrices X and Y are of appropriate sizes.

Finally, a known equivalence result on existence of an affine α-resolvable BD is described. This can be seen from the complementation of a design.

Lemma 2.3. The existence of an affine α-resolvable BD$(v, b = \beta t, r = \alpha t, k)$ with block intersection numbers q_1 and q_2 is equivalent to the existence of an affine $(\beta-\alpha)$-resolvable BD$(v^* = v, b^* = b, r^* = (\beta-\alpha)t, k^* = v-k)$ with block intersection numbers $q_1^* = v - 2k + q_1$ and $q_2^* = v - 2k + q_2$.

The subsequent Sections 3, 4 and 5 will be devoted to combinatorial arguments about bounds on design parameters in (affine) α-resolvable BIB designs and a special discussion on nonexistence of a 2-resolvable BIB design. As for (affine) α-resolvable BIB designs, as was described in Introduction, there are much literature. Recent combinatorial developments can be found in Caliński and Kageyama (2000; Chapter 5), Caliński and Kageyama (2003; Chapter 9), and Colbourn and Dinitz (2007). Hence in this thesis we do not list up any series of affine α-resolvable BIB designs for $\alpha \geq 1$. In Sections 6 and 7 some combinatorial investigation on affine α-resolvable PBIB designs will be made comprehensively.

3. Inequality on α-resolvable BIB designs

As was described in Introduction, for a lower bound on the number of blocks, it is known (see Bose, 1942, Kageyama, 1973a, Raghavarao, 1988, Shrikhande and Raghavarao, 1964) that in an α-resolvable BIB design with parameters $v, b = \beta t, r = \alpha t, k, \lambda$,

$$b \geq v + t - 1 \tag{3.1}$$

holds for $\alpha \geq 1$. In particular, the equality $b = v + t - 1$ is a necessary and sufficient condition for an α-resolvable BIB design to be affine α-resolvable for $\alpha \geq 1$.

In this section, as an improvement of the known inequality $b \geq v + t - 1$, a new inequality $b \geq 2(v-1) + t$ on the parameters for α-resolvable BIB designs with parameters $v, b = \beta t, r = \alpha t, k, \lambda$ that are not affine α-resolvable is discussed under the condition $(\alpha, v-1) = 1$.

In general, it follows that in a BIB design with parameters v, b, r, k, λ,

$$b = \frac{r-\lambda}{k}(v-1) + r, \tag{3.2}$$

$$b - (v + r - 1) = \frac{(r-k-\lambda)(v-1)}{k}. \tag{3.3}$$

Therefore, in a 1-resolvable BIB design with parameters $v = \beta k, b = \beta r, r, k, \lambda$, a relation $(v-1, k) = 1$ holds, and hence, by (3.2), $r - \lambda$ is divisible by k, i.e., $(r-\lambda)/k$ is a positive integer. Also, as it follows from (3.3) that the Bose inequality $b \geq v + r - 1$ holds if and only if $r \geq k + \lambda$ holds, (3.2) shows that in a 1-resolvable BIB design which is not affine 1-resolvable, an improved inequality

$$b \geq 2(v-1) + r \tag{3.4}$$

holds (see Kageyama, 1971). Furthermore, by (3.2), an expression of b in terms of $v-1$ is given by $b = p(v-1) + r$ for some positive integer p. Thus the lower bound of b is improved by $v-1$ in turn starting from the value $v - 1 + r$. From now on, a discussion similar to the above one will be made when $\alpha \geq 2$.

At first, the following observation can be presented.

Lemma 3.1. In an α-resolvable BIB design with parameters $v, b = \beta t, r =$

$\alpha t, k, \lambda$, when $(\alpha, v-1) = 1$, $(r-\lambda)/k$ is a positive integer for $\alpha \geq 1$.

Proof. By (3.2), we have

$$b = \frac{m(r-\lambda)}{\alpha v}(v-1) + r.$$

As $(\alpha, v-1) = 1$ and $(v, v-1) = 1$, $m(r-\lambda)$ is divisible by αv, and hence by $\alpha v = \beta k$, $(r-\lambda)/k$ is a positive integer. ∎

In general, the complement of an α-resolvable BIB design with parameters $v, b = \beta t, r = \alpha t, k, \lambda$ is an $(\beta - \alpha)$-resolvable BIB design (see also Lemma 2.3). Hence the following characterization can be given.

Theorem 3.1. In an α-resolvable BIB design with parameters $v, b = \beta t, r = \alpha t, k, \lambda$, that is not affine α-resolvable, when $\alpha = 1$ or $\beta - \alpha = 1$, an inequality

$$b \geq 2(v-1) + t$$

holds. In particular, the lower bound of b is improved by $v-1$ in turn starting from the value $2(v-1)+t$.

Proof. When $\alpha = 1$, we have $r = t$ and hence by (3.4), $b \geq 2(v-1) + r = 2(v-1) + t$. Furthermore, as $(\alpha, v-1) = 1$, Lemma 3.1 with (3.2) shows that the lower bound of b is improved by $v-1$ in turn. On the other hand, when $\beta - \alpha = 1$, the complement of an α-resolvable BIB design is a 1-resolvable BIB design with parameters $v^* = v, b^* = b = \beta t, r^* = b - r = (\beta - \alpha)t = t, k^* = v - k, \lambda^*$. Hence we have $b = b^* \geq 2(v^* - 1) + r^* = 2(v-1) + t$. ∎

When $\alpha \geq 2$ and $\beta - \alpha \geq 2$, a certain condition will be needed to show our final target which will be described as follows.

Conjecture: In an α-resolvable BIB design with parameters $v, b = \beta t, r = \alpha t, k, \lambda$, that is not affine α-resolvable, when $(\alpha, v-1) = 1$, an inequality

$$b \geq 2(v-1) + t$$

holds.

Even if k does not divide v, the following can be given as an improvement of (3.1).

Theorem 3.2. In an α-resolvable BIB design with parameters $v, b = \beta t, r =$

$\alpha t, k, \lambda$, that is not affine α-resolvable, when $(\alpha, v-1) = 1$, an inequality

$$b \geq v - 1 + r \quad (\geq v - 1 + t)$$

holds. In particular, the lower bound of b is improved by $v-1$ in turn starting from the value $v - 1 + r$.

Proof. As $(\alpha, v-1) = 1$, Lemma 3.1 shows that $(r - \lambda)/k$ is a positive integer. Hence by (3.2) the required result is obtained. ■

Corollary 3.1. In an α-resolvable BIB design with parameters $v, b = \beta t, r = \alpha t, k, \lambda$, that is not affine α-resolvable, when $(\alpha, v-1) = 1$ and $(r-\lambda)/k \geq 2$, an inequality

$$b \geq 2(v-1) + r \quad (\geq 2(v-1) + t)$$

holds. In particular, the lower bound of b is improved by $v-1$ in turn starting from the value $2(v-1) + r$.

By Theorem 3.2 and Corollary 3.1, in an α-resolvable BIB design that is not affine α-resolvable, the conjecture is valid when $(r - \lambda)/k \geq 2$. Thus, for the general validity of the conjecture, a case of $(r - \lambda)/k = 1$ has to be taken under $(\alpha, v-1) = 1, \alpha \geq 2$ and $\beta - \alpha \geq 2$ as the remaining consideration.

Now, several necessary conditions for the existence of such designs are given.

Theorem 3.3. In an α-resolvable BIB design with parameters $v, b = \beta t, r = \alpha t, k, \lambda$, that is not affine α-resolvable, in which $(\alpha, v-1) = 1$ and $(r-\lambda)/k = 1$, it holds that

$$(i)\ (\alpha, \beta - \alpha) = 1 \text{ and } (\alpha, t) = 1; \quad (ii)\ (k, v-1) = 1 \text{ and } (\beta, v-1) = 1;$$

$$(iii)\ \alpha | \lambda\ (\Longleftrightarrow (\beta - \alpha)|(k-1)); \quad (iv)\ \alpha | k;$$

$$(v)\ (v-1)|t(k-1); \quad (vi)\ (\alpha, \beta) = 1.$$

Proof. Recall that $(r - \lambda)/k = 1$ if and only if

$$b = v - 1 + r. \tag{3.5}$$

Let $\alpha^* = \beta - \alpha$. Hence $b = \beta t = (\alpha + \alpha^*)t = r + \alpha^* t$, which, from (3.5), implies $v - 1 = \alpha^* t$. As $1 = (\alpha, v-1) = (\alpha, \alpha^* t)$, we have the condition (i). On the other hand, from $(\alpha, v-1) = 1$, we have $(\alpha v, v-1) = 1$, i.e.,

$(\beta k, v-1)=1$, which shows the condition (ii). Next, $\lambda(v-1)=r(k-1)$ implies $\lambda(v-1)/\alpha = t(k-1)$ which, from $(\alpha, v-1)=1$, shows that (iii) $\alpha|\lambda$, i.e., $\lambda = \alpha\lambda_1$ for a positive integer λ_1. [Also, $(k-1)/(\beta-\alpha) = t(k-1)/(\alpha^* t)$ $= t(k-1)/(v-1) = \lambda/\alpha$ shows that $\alpha|\lambda \Longleftrightarrow (m-\alpha)|(k-1)$.] Furthermore, from the assumption, $k = r-\lambda = \alpha(t-\lambda_1)$, which shows (iv). The above relation $\lambda = \alpha t(k-1)/(v-1)$ implies (v), because $(\alpha, v-1)=1$. Finally, let $(\alpha,\beta) = g \geq 1$, then $\alpha = \alpha_1 g$ and $\beta = \beta_1 g$ for some integers α_1 and β_1. Hence, by (i), $1 = (\alpha, \beta-\alpha) = (\alpha_1 g, (\beta_1-\alpha_1)g) \geq g$, which shows (vi). ∎

Next, in such designs some sufficient conditions for the validity of an inequality are described.

Theorem 3.4. In an α-resolvable BIB design with parameters $v, b=\beta t, r = \alpha t, k, \lambda$, that is not affine α-resolvable, when $(\alpha, v-1)=1$ and $(r-\lambda)/k=1$, any one of the following conditions is sufficient for the validity of an inequality $b \geq 2(v-1)+t$:

$$(i)\ \alpha \geq \beta-\alpha, \quad (ii)\ \alpha = k, \quad (iii)\ k|v, \quad (iv)\ (k-1, v-1)=1.$$

Proof. Case (i-1): $\alpha > \beta-\alpha$. Now $b = \beta t = \{\alpha + (\beta-\alpha)\}t$, which, from (3.5), yields $v-1 = (\beta-\alpha)t$. Hence $r = \alpha t > (\alpha-1)t \geq (\beta-\alpha)t = v-1$. Therefore it follows that $b = v-1+r = v-1+(\alpha-1)t+t \geq 2(v-1)+t$.

Case (i-2): $\alpha = \beta-\alpha$. As $(\alpha, v-1)=1$ and $v-1 = (\beta-\alpha)t$, it holds that $(\alpha, \beta-\alpha)=1$. However, this is valid only when $\alpha=1$, as $\alpha = \beta-\alpha$. Hence by Theorem 3.1, $b \geq 2(v-1)+t$.

Case (ii): $\alpha = k$. Now $\alpha v = \beta k$ implies $v = m$. Then $b = \beta t = vt = t(v-1)+t \geq 2(v-1)+t$, as $t \geq 2$.

Case (iii): $k|v$. By $v\alpha = \beta k$, the condition implies $\alpha|\beta$. Then $(\alpha, v-1) = \alpha$, as $v-1 = (\beta-\alpha)t$. By the assumption, $\alpha = 1$. Hence by Theorem 3.1, $b \geq 2(v-1)+t$.

Case (iv): $(k-1, v-1)=1$. Now, $\lambda = r(k-1)/(v-1)$ implies that $r = \ell(v-1)$ for a positive integer ℓ. When $\ell \geq 2$, by (3.5), $b = v-1+r = \ell(v-1)+v-1 = \ell(v-1)+(\beta-\alpha)t \geq 2(v-1)+t$. When $\ell=1$, we have the parameters of the BIB design as $v=2k, b=2(2k-1), r=2k-1, k, \lambda=k-1$. This means that $k|v$. Hence as in the proof of the case (iii), the proof is complete. ∎

Some sufficient conditions in terms of design parameters for the validity of the conditions given in Theorem 3.4 can be stated.

Theorem 3.5. In an α-resolvable BIB design with parameters $v, b = \beta t, r = \alpha t, k, \lambda$, that is not affine α-resolvable, when $(\alpha, v-1) = 1$ and $(r-\lambda)/k = 1$,

(1) each of $r \geq v - 1$ and $k|r$ shows (i) of Theorem 3.4;
(2) each of $(v, k) = 1, (k, v - k) = 1, v$ being a prime, $\alpha \geq 2$ and k being a prime, $k|\alpha$, and $t = 2$ shows (ii) of Theorem 3.4;
(3) $v - 1$ being a prime shows (iv) of Theorem 3.4.

Proof. Case (1). From $r = \alpha t$ and $v-1 = (\beta - \alpha)t$, it holds that $r \geq v-1$ implies $\alpha \geq \beta - \alpha$. When $k|r$, i.e., $v|b$, as $b = v - 1 + r$, we get $v|(r-1)$, i.e., $v \leq r - 1$. Hence $r > v - 1$.

Case (2). When $(v, k) = 1$, a relation $\alpha v = \beta k$ implies $k|\alpha$. Hence, by the condition (iv) of Theorem 3.3, we get $\alpha = k$. When $(k, v-k) = 1$, a relation $(\beta - \alpha)k = \alpha(v - k)$ implies $k|\alpha$. Hence, similarly we get $\alpha = k$. When v is a prime, as $v > k$, we get $(v, k) = 1$. Hence, similarly $\alpha = k$. When k is a prime and $\alpha \geq 2$, by (iv) of Theorem 3.3, $\alpha|k$ and then $\alpha = k$. When $t = 2$, as $\alpha|k$, we have $1 \leq k/\alpha < r/\alpha = 2$, which implies that $\alpha = k$.

Case (3) is trivial, because $v > k$. ∎

We cannot yet prove the conjecture entirely. Its reason can be clarified by the following Remark 3.1. This is, a number-theoretic approach may not be enough to solve the problem completely.

Remark 3.1. The following three designs satisfy all the available necessary conditions for the existence of α-resolvable BIB designs:

(i) BIB($v = 10, b = 15, r = 6, k = 4, \lambda = 2, t = 3, \alpha = 2,\ m = 5$), here $b = 15 < 2(v - 1) + t = 21$.
(ii) BIB($v = 21, b = 35, r = 15, k = 9, \lambda = 6, t = 5, \alpha = 3,\ m = 7$), here $b = 35 < 2(v - 1) + t = 45$.
(iii) BIB($v = 56, b = 77, r = 22, k = 16, \lambda = 6, t = 11,\ \alpha = 2, m = 7$), here $b = 77 < 2(v - 1) + t = 121$.

This circumstance shows that the only information on integrality on parameters of designs is not enough to show the nonexistence of α-resolvable BIB designs. However, the existence of the above α-resolvable BIB designs (i), (ii) and (iii) has not been known. Kadowaki (2001) shows the nonexistence of the

α-resolvable BIB design (i), by starting from three non-isomorphic solutions (see Mathon and Rosa, 1996) of the original BIB design with parameters $v = 10, b = 15, r = 6, k = 4, \lambda = 2$ and then by showing the impossibility of reforming three 2-resolution sets in a family of 15 blocks of 4 treatments each.

4. Nonexistence of a 2-resolvable BIB design

In the present section, a 2-resolvable BIB(10, 15, 6, 4, 2) given in Remark 3.1(i) is discussed through a combinatorial approach.

4.1. Some results

For a symmetric BIB(v, b, r, k, λ), the following result is well known.

Lemma 4.1. Any two blocks of a symmetric BIB$(v = b, r = k, \lambda)$ intersect in exactly λ points.

Let D be a symmetric BIB(v, b, r, k, λ) and B be a block of D. Remove B and all points appearing in B from the other blocks in D. Then the resulting design is a BIB$(v - k, b - 1, r, k - \lambda, \lambda)$, which is called a residual design of D.

In the residual BIB(v, b, r, k, λ), a relation $r = k + \lambda$ holds. A BIB(v, b, r, k, λ) with $r = k + \lambda$ is said to be quasi-residual. The following Lemma 4.2 is known (see Mathon and Rosa, 1996):

Lemma 4.2. When $\lambda = 1$ or 2, the existence of a quasi-residual BIB design implies the existence of the corresponding symmetric BIB design.

Lemmas 4.1 and 4.2 show the following.

Theorem 4.1. When $\lambda = 1$ or 2, any two blocks of a BIB$(v, b, r = k+\lambda, k, \lambda)$ have at most λ common points.

As an improvement of the Bose inequality $b \geq v + t - 1$ in Section 3, we conjecture the validity of an inequality $b \geq 2(v - 1) + t$ under the condition $(\alpha, v - 1) = 1$ for an α-resolvable BIB$(v,\ b = \beta t,\ r = \alpha t,\ k,\ \lambda)$ that is not affine α-resolvable. Note that when $\alpha = 1$, the conjecture has been proved (cf. Kageyama, 1971). In Section 3, we could prove the conjecture for $(r-\lambda)/k \geq 2$. Since $(r-\lambda)/k$ is a positive integer, the case of $(r-\lambda)/k = 1$ has to be considered to prove the conjecture entirely. However, as Example 4.1 shows, there are four BIB designs which may violate the conjecture. That is, such designs satisfy $b < 2(v - 1) + t$ and all available necessary conditions for the existence of the α-resolvable BIB designs.

Example 4.1. The following four designs satisfy all the available necessary conditions for the existence of α-resolvable BIB designs:

(i) BIB($v = 10, b = 15, r = 6, k = 4, \lambda = 2; t = 3, \alpha = 2,\ m = 5$), here $b = 15 < 2(v-1) + t = 21$.

(ii) BIB($v = 21, b = 35, r = 15, k = 9, \lambda = 6; t = 5, \alpha = 3,\ m = 7$), here $b = 35 < 2(v-1) + t = 45$.

(iii) BIB($v = 50, b = 70, r = 21, k = 15, \lambda = 6; t = 7,\ \alpha = 3, m = 10$), here $b = 70 < 2(v-1) + t = 105$.

(iv) BIB($v = 56, b = 77, r = 22, k = 16, \lambda = 6; t = 11,\ \alpha = 2, m = 7$), here $b = 77 < 2(v-1) + t = 121$.

However, the existence of the above α-resolvable BIB designs with (i), (ii), (iii) or (iv) has not been known. Kadowaki (2001) shows the nonexistence of the α-resolvable BIB design (i) by use of a computer only. This is not completely theoretical.

We here show the nonexistence of the 2-resolvable BIB(10, 15, 6, 4, 2) through a combinatorial approach. This approach may be useful to consider another existence problem on designs.

4.2. Nonexistence

At first, the following can be presented.

Lemma 4.3. Let q^* be the number of block intersection among blocks in a BIB(10, 15, 6, 4, 2). Then it holds that $q^* \leq 2$.

Proof. By Mathon and Rosa (1996), it is seen that the number of non-isomorphic BIB(10, 15, 6, 4, 2) is only three and their designs are all residual designs of a symmetric BIB(16, 16, 6, 6, 2). Therefore, it follows that $q^* \leq 2$, by Theorem 4.1. ■

Now, let $B_j^{(i)}$ be the jth column in the ith resolution set of an α-resolvable BIB design and let $q_{jj'}^{(i)} = |B_j^{(i)} \cap B_{j'}^{(i)}|$ for $1 \leq i \leq t$ and $1 \leq j, j' \leq m$.

Lemma 4.4. In a 2-resolvable BIB(10, 15, 6, 4, 2), $q_{jj'}^{(i)} = 1$ for all i, j, j' $(j \neq j')$.

Proof. Suppose that $q_{jj'}^{(i)} \neq 1$ for some i, j, j'. Then by Lemma 4.3, for such i, j, j' there are the following two cases: (I) $q_{jj'}^{(i)} = 2$ and (II) $q_{jj'}^{(i)} = 0$. Let N be the incidence matrix of the design.

Case (I): $q_{jj'}^{(i)} = 2$ for such i, j, j'. Now suppose that $q_{12}^{(1)} = 2$. Since $\lambda = 2$ and $\alpha = 2$ for the 2-resolvable BIB design, the upper 4×15-submatrix of the original incidence matrix N of size 10×15 can be formed without loss of generality as follows:

$$\left[\begin{array}{ccccc|ccccc|ccccc} 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & & & & & & & & & & \\ 1 & 0 & & & & & & & & & & & & & \end{array}\right] (= N_1, \text{say}).$$

Later only N_1 in N is discussed. For the 1st and 3rd rows, since $\lambda = 2$ and $\alpha = 2$, N_1 can be set as follows:

$$\left[\begin{array}{ccccc|ccccc|ccccc} 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & & & & 0 & 0 & & & 1 \\ 1 & 0 & & & & & & & & & & & & & \end{array}\right].$$

For the 2nd and 3rd rows, since $\lambda = 2$, there is only one '1' in either cell(3, 13) or cell(3, 14). Furthermore, for the 3rd row in the 3rd resolution set, since $\alpha = 2$, '1' is put in cell(3, 13) or cell(3, 14). Therefore, without loss of generality, N_1 has

$$\left[\begin{array}{ccccc|ccccc|ccccc} 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & & & & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & & & & & & & & & & & & & \end{array}\right].$$

Finally, for the 1st and 3rd rows, and for the 2nd and 3rd rows, respectively, since $\lambda = 2$ and $\alpha = 2$, N_1 becomes

$$\left[\begin{array}{ccccc|ccccc|ccccc} 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & & & & & & & & & & & & & \end{array}\right].$$

Now for this pattern, the following two cases are further considered. (I-1) cell(4, 3)$= 1$ and (I-2) cell(4, 3)$= 0$.

Case (I-1): cell(4, 3)$= 1$. For the 3rd and 4th rows, since $\lambda = 2$, by Lemma 2.1 we can get N_1 as follows:

$$\left[\begin{array}{ccccc|ccccc|ccccc} 1&1&0&0&0&1&1&0&0&0&1&1&0&0&0\\ 1&1&0&0&0&0&0&1&1&0&0&0&1&1&0\\ 1&0&1&0&0&1&0&0&0&1&0&0&1&0&1\\ 1&0&1&0&0&0&&&&0&&&0&&0 \end{array}\right]. \tag{4.1}$$

And for the 2nd and 4th rows, since $\lambda = 2$, there is only one '1' in either cell(4, 8) or cell(4, 9). Therefore N_1 has

$$\left[\begin{array}{ccccc|ccccc|ccccc} 1&1&0&0&0&1&1&0&0&0&1&1&0&0&0\\ 1&1&0&0&0&0&0&1&1&0&0&0&1&1&0\\ 1&0&1&0&0&1&0&0&0&1&0&0&1&0&1\\ 1&0&1&0&0&0&1&1&0&0&0&0&0&0&0 \end{array}\right].$$

There are no '1' at the 4th row in the 3rd resolution set. This is a contradiction to $\alpha = 2$. Note that in (4.1) if a cell (4, 14) has 1, then the 4th row in the 2nd resolution set does not satisfy $\alpha = 2$.

Case (I-2): cell(4, 3)= 0. By Lemma 2.1 it is enough to consider a case of cell(4, 6)= 0. Now this case can be further divided into the following four cases: (I-2-1) cell(4, 7)= 1 and cell(4, 8)= 1, (I-2-2) cell(4, 7)= 1 and cell(4, 8)= 0, (I-2-3) cell(4, 7)= 0 and cell(4, 8)= 1, (I-2-4) cell(4, 7)= 0 and cell(4, 8)= 0.

Case (I-2-1):cell(4, 7)= 1 and cell(4, 8)= 1. Then N_1 has

$$\left[\begin{array}{ccccc|ccccc|ccccc} 1&1&0&0&0&1&1&0&0&0&1&1&0&0&0\\ 1&1&0&0&0&0&0&1&1&0&0&0&1&1&0\\ 1&0&1&0&0&1&0&0&0&1&0&0&1&0&1\\ 1&0&0&&&0&1&1&0&0&0&0&0&0&1 \end{array}\right].$$

Hence $\alpha = 1$ at the 4th row in the 3rd resolution set. This is a contradiction. For Cases (I-2-2), (I-2-3) and (I-2-4), similarly, we get such contradiction.

Thus, when $q_{12}^{(1)} = 2$, we get a contradiction. By any permutation of columns of N, we can suppose $q_{jj'}^{(i)} = 2$ for other i, j, j'. However, similarly, a contradiction can be derived. Hence it follows that $q_{jj'}^{(i)} \neq 2$ for any i, j, j'.

Case (II): $q_{jj'}^{(i)} = 0$ for such i, j, j'. Now suppose that $q_{12}^{(1)} = 0$. Then the first four rows of the first 2-resolution set of N can be formed as follows:

$$\left[\begin{array}{ccccc} 1&0&1&0&0\\ 1&0&0&1&0\\ 1&0&0&0&1\\ 1&0&&& \end{array}\right] (= S_1, \text{say}).$$

Since $\alpha = 2$, there is only one '1' in either cell(4, 3), cell(4, 4) or cell(4, 5) in S_1. But since $q^{(i)}_{jj'} \neq 2$ by Case(I), we get a contradiction. By any permutation of columns of N, we can suppose $q^{(i)}_{jj'} = 0$ for other i, j, j' in N. However, similarly, a contradiction can be derived. Hence it follows that $q^{(i)}_{jj'} \neq 0$ for any i, j, j'.

Thus Cases (I) and (II) show that $q^{(i)}_{jj'} = 1$ for all i, j, j' $(j \neq j')$. Hence Lemma 4.4 is proved. ■

Theorem 4.2. A 2-resolvable BIB(10, 15, 6, 4, 2) does not exist.

Proof. By Lemma 4.4, without loss of generality N can be formed as follows.

$$\left[\begin{array}{ccccc|ccccc|ccccc}
1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & & & & & & & & & \\
1 & 0 & 0 & 0 & 1 & 0 & & & & & & & & & \\
0 & 1 & 1 & 0 & 0 & & & & & & & & & & \\
0 & 1 & 0 & 1 & 0 & & & & & & & & & & \\
0 & 1 & 0 & 0 & 1 & & & & & & & & & & \\
0 & 0 & 1 & 1 & 0 & & & & & & & & & & \\
0 & 0 & 1 & 0 & 1 & & & & & & & & & & \\
0 & 0 & 0 & 1 & 1 & & & & & & & & & &
\end{array}\right].$$

Now consider the 3rd row in the 2nd resolution set. Then the row vector of size 5 can be taken into the following six cases. (1-1) $(0,1,1,0,0)$, (1-2) $(0,1,0,1,0)$, (1-3) $(0,1,0,0,1)$, (1-4) $(0,0,1,1,0)$, (1-5) $(0,0,1,0,1)$, (1-6) $(0,0,0,1,1)$.

Case (1-1): the 3rd row in the 2nd resolution set is $(0,1,1,0,0)$. Since $\lambda = 2$, the 3rd row of N can be given by

$$\left(\begin{array}{ccccc|ccccc|ccccc} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & * \end{array}\right).$$

Thus $\alpha \leq 1$ at the 3rd row in the 3rd resolution set. This is a contradiction.

Case (1-2): the 3rd row in the 2nd resolution set is $(0,1,0,1,0)$. Then by Lemma 4.3, for the 1st and 7th columns of N, since $q^* \leq 2$, it is seen that cell(4,7)= 0. Therefore, for cell(4, 8), cell(4, 9) and cell(4,10), the following three cases are possible: (1-2-1) cell(4, 8) = 1, cell(4, 9) = 1 and cell(4,10)=0, (1-2-2) cell(4, 8) = 1, cell(4, 9) = 0 and cell(4,10)=1, (1-2-3) cell(4, 8) = 0, cell(4, 9) = 1 and cell(4,10)=1. Here $N = \left[\frac{N_1}{N_2}\right]$, say, where N_1 is of size 4×15 and N_2 is of size 6×15. Later only N_1 is discussed.

Case (1-2-1): cell(4, 8) = 1, cell(4, 9) = 1 and cell(4,10)=0. Then by Lemma 4.3 with $\lambda = 2$ and $\alpha = 2$, N_1 has

$$\left[\begin{array}{ccccc|ccccc|ccccc} 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \end{array}\right].$$

Hence $\alpha = 1$ at the 4th row in the 3rd resolution set. This is a contradiction.

Case (1-2-2): cell(4, 8) = 1, cell(4, 9) = 0 and cell(4,10)=1. Similarly, N_1 has

$$\left[\begin{array}{ccccc|ccccc|ccccc} 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right].$$

Thus the 1st, 2nd, 3rd and 4th rows can be constructed.

Case (1-2-3): cell(4, 8) = 0, cell(4, 9) = 1 and cell(4,10)=1. Similarly, N_1 has

$$\left[\begin{array}{ccccc|ccccc|ccccc} 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \end{array}\right].$$

Thus, the 1st, 2nd, 3rd and 4th rows can be formed. However, by some permutation of rows and columns of N, the above matrix is isomorphic to Case (1-2-2).

Similarly, for Cases (1-3), (1-4), (1-5) and (1-6), we can construct the 1st, 2nd, 3rd and 4th rows. By some permutation of rows and columns of N, their matrices are isomorphic to Case (1-2-2) for all of cases. By Lemma 4.4 and the above discussion, it is seen that N forms

$$\left[\begin{array}{ccccc|ccccc|ccccc}
1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 & 0 & & & & & & & & & & \\
0 & 1 & 0 & 1 & 0 & & & & & & & & & & \\
0 & 1 & 0 & 0 & 1 & & & & & & & & & & \\
0 & 0 & 1 & 1 & 0 & & & & & & & & & & \\
0 & 0 & 1 & 0 & 1 & & & & & & & & & & \\
0 & 0 & 0 & 1 & 1 & & & & & & & & & &
\end{array}\right].$$

Furthermore, for the 1st and 5th rows, and the 2nd and 5th rows, since $\lambda = 2$, some argument can show that cell(5, 6)= 0. Similarly, it can be seen that cell(6, 7)= 0, cell(7, 11)= 0, cell(8, 13)= 0, cell(9, 8)= 0 and cell(10, 15)= 0. Hence N has the following form:

$$\left[\begin{array}{ccccc|ccccc|ccccc}
1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 & 0 & 0 & & & & & & & & & \\
0 & 1 & 0 & 1 & 0 & & 0 & & & & & & & & \\
0 & 1 & 0 & 0 & 1 & & & & & & 0 & & & & \\
0 & 0 & 1 & 1 & 0 & & & & & & & & 0 & & \\
0 & 0 & 1 & 0 & 1 & & & 0 & & & & & & & \\
0 & 0 & 0 & 1 & 1 & & & & & & & & & & 0
\end{array}\right].$$

Next, we consider cell(i, j) for $5 \leq i \leq 10$ and $9 \leq j \leq 10$, namely, elements of a 6×2-submatrix. Now there are six row vectors of size 2 in the 6×2-submatrix. By Lemma 4.4, the submatrix composes one (1,1), one (0,0), two (1, 0) and two (0, 1).

Now, the following two cases are taken: (I) cell(5, 9) $=$ 1 and cell(5, 10) $=$ 1, and (II) other patterns than (I).

Case (I): cell(5, 9) $=$ 1 and cell(5, 10) $=$ 1. Furthermore, each of the following five cases are considered: (I-1) cell(6, 9) $=$ 0 and cell(6, 10) $=$ 0, (I-2) cell(7, 9) $=$ 0 and cell(7, 10) $=$ 0, (I-3) cell(8, 9) $=$ 0 and cell(8, 10) $=$ 0, (I-4) cell(9, 9) $=$ 0 and cell(9, 10) $=$ 0, (I-5) cell(10, 9) $=$ 0 and cell(10, 10) $=$ 0.

Case (I-1): cell(6, 9) = 0 and cell(6, 10) = 0. Since $\lambda = 2$, N has the following:

$$\left[\begin{array}{ccccc|ccccc|ccccc}
1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & & & & & \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & & & & & & 0 & & & & \\
0 & 0 & 1 & 1 & 0 & & & & & & & & 0 & & \\
0 & 0 & 1 & 0 & 1 & & & 0 & & & & & & & \\
0 & 0 & 0 & 1 & 1 & & & & & & & & & & 0
\end{array}\right].$$

Hence $\alpha = 1$ at the 6th row in the 3rd resolution set. This is a contradiction.

Similarly, each of the other Cases (I-2), (I-3), (I-4) and (I-5) can lead a contradiction.

Case (II): not (I). As a position of (1, 1) in the 6×2-submatrix, the following five cases are taken: (II-1) cell(6, 9) = 1 and cell(6, 10) = 1, (II-2) cell(7, 9) = 1 and cell(7, 10) = 1, (II-3) cell(8, 9) = 1 and cell(8, 10) = 1, (II-4) cell(9, 9) = 1 and cell(9, 10) = 1, (II-5) cell(10, 9) = 1 and cell(10, 10) = 1.

Case (II-1): cell(6, 9) = 1 and cell(6, 10) = 1. Since $\lambda = 2$, N has

$$\left[\begin{array}{ccccc|ccccc|ccccc}
1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 & 0 & 0 & & & & & & & & & \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & & & 1 & 1 & \\
0 & 1 & 0 & 0 & 1 & & & & & & 0 & & & & \\
0 & 0 & 1 & 1 & 0 & & & & & & & & 0 & & \\
0 & 0 & 1 & 0 & 1 & & & 0 & & & & & & & \\
0 & 0 & 0 & 1 & 1 & & & & & & & & & & 0
\end{array}\right].$$

Then the 13th and 14th columns show a contradiction to $q_{jj'}^{(i)} = 1$ in Lemma 4.4.

Similarly, each of the other Cases (II-2), (II-3), (II-4) and (II-5) leads a contradiction.

Hence we get a contradiction for all of the cases. Thus, it can be shown that a 2-resolvable BIB(10, 15, 6, 4, 2) does not exist. ■

It should be noted that the present combinatorial approach will take much time to deal with other BIB designs (ii)-(iv) as in Example 4.1. This task has not been completed.

5. Bounds in α-resolvable BIB designs

Based on the argument made in Section 3, in an α-resolvable BIB design, three lower bounds, $b \geq v-1+t, b \geq v-1+2t, b \geq 2(v-1)+t$, on the number of blocks will be discussed under some conditions. An interesting characteristic will be described with the possibility of having $b < 2(v-1)+t$.

It is also known (Bose, 1942, Raghavarao, 1988) that in an α-resolvable BIB design $b \geq v+t-1$ holds and that $b = v+t-1$ is a necessary and sufficient condition for an α-resolvable BIB design to be affine α-resolvable. In this sense, for a class of α-resolvable BIB designs that are not affine α-resolvable, necessarily an inequality $b \geq v+t$ holds. But it seems that this bound is further improved.

Though there is another direction of the improvement (Kageyama, 1973b), the present problem on an improvement of a bound will be considered for α-resolvable BIB designs that are not affine α-resolvable under $(v-1, \alpha) = 1$ which always holds when $\alpha = 1$. This assumption may be reasonable. Since $(v-1, \alpha) = 1$ implies that $(r-\lambda)/k$ is a positive integer (see Section 5.1), in this section the present problem will be dealt with by separating into two cases of $(r-\lambda)/k \geq 2$ and $(r-\lambda)/k = 1$.

Under this set-up, it is well-known (cf. Kageyama, 1971) that $b \geq 2(v-1)+r$ in general for $\alpha = 1$. When $\alpha \geq 2$, it is clear that $b \geq v+t$ in general. However, when $(r-\lambda)/k \geq 2$, it is known (see Theorem 5.1.2) that $b \geq 2(v-1)+t$ holds under $(v-1, \alpha) = 1$. The remaining case is now when $(r-\lambda)/k = 1$ and $\alpha \geq 2$.

In this case, unfortunately there are parameters' combinations of such BIB designs for each of $b < 2(v-1)+t, b = 2(v-1)+t$ and $b > 2(v-1)+t$. Some cases are shown to be existent. In fact, it can be seen that there exist α-resolvable BIB designs having $b \geq 2(v-1)+t$. So far the existence of designs for the case $b < 2(v-1)+t$ is not known, even if the admissible parameters are available. There remains some possibility of showing the nonexistence of all BIB designs with $b < 2(v-1)+t$. However, it seems (cf. Kadowaki and Kageyama, 2008a) that it is a tough problem. Thus, a different status on a bound (in terms of v and r) of b may be seen between two cases of $(r-\lambda)/k \geq 2$ and $(r-\lambda)/k = 1$.

5.1. Basic results

In an α-resolvable BIB$(v, b = \beta t, r = \alpha t, k, \lambda)$, it follows that

$$
\begin{aligned}
b &= \frac{r-\lambda}{k}(v-1)+r \\
&= \frac{\beta(r-\lambda)}{\alpha v}(v-1)+r
\end{aligned}
\tag{5.1.1}
$$

which implies that when $(v-1,\alpha)=1$, αv divides $\beta(r-\lambda)$, i.e., $(r-\lambda)/k$ is a positive integer. In this case, a conjecture is proposed in Section 3 such that in an α-resolvable BIB$(v, b=\beta t, r=\alpha t, k, \lambda)$, which is not affine α-resolvable, when $(v-1,\alpha)=1$, an inequality $b \geq 2(v-1)+t$ holds. Note that the lower limit is expressed through v and t only.

Now, when $\alpha = 1$, the following is known.

Theorem 5.1.1 (Kageyama, 1971). In a resolvable BIB$(v, b=\beta r, r, k, \lambda)$, that is not affine resolvable, an inequality $b \geq 2(v-1)+r$ holds.

Next, when $\alpha \geq 2$, the following is shown.

Theorem 5.1.2 (Kadowaki and Kageyama, 2003). In an α-resolvable BIB$(v, b=\beta t, r=\alpha t, k, \lambda)$ with $(v-1,\alpha)=1$ that is not affine α-resolvable, if $(r-\lambda)/k \geq 2$, an inequality $b \geq 2(v-1)+t$ holds.

By Theorems 5.1.1 and 5.1.2, a case of $(r-\lambda)/k=1$ only has to be taken under $(v-1,\alpha)=1$ and $\alpha \geq 2$ as the remaining exhaustive investigation for the present problem.

5.2. Several bounds

Through an argument as in the previous subsection, consider here a class of α-resolvable BIB$(v, b=\beta t, r=\alpha t, k, \lambda)$, that is not affine α-resolvable, under $(v-1,\alpha)=1, \alpha \geq 2$ and $(r-\lambda)/k=1$. This class is denoted by α-BIBD* throughout this section.

At first the following relation is derived.

Lemma 5.2.1. In an α-BIBD*, a relation $v \geq t+1$ holds.

Proof. By (5.1.1), the α-BIBD* shows that $b=v+r-1$. Then $v-1=b-r=(\beta-\alpha)t \geq t$, since $\beta \geq \alpha+1$. Hence $v \geq t+1$. ■

As in the above proof, note that in an α-BIBD*, $(t,\alpha)=1$ and $(\beta,\alpha)=1$.

Furthermore, Lemma 5.2.1 yields the following.

Corollary 5.2.1. In an α-BIBD*, a relation $2(v-1)+t \geq v-1+2t$ holds.

Now the following bound is given.

Theorem 5.2.1. In an α-BIBD*, an inequality $b \geq v-1+2t$ holds. In particular, the bound is attained if and only if $\alpha = 2$.

Proof. By $(r-\lambda)/k = 1$ with (5.1.1), $b = v-1+r = v-1+\alpha t \geq v-1+2t$, since $\alpha \geq 2$. ∎

Note that $v-1+2t = 2(v-1)+t$ if and only if $v = t+1$. Hence we have the following.

Corollary 5.2.2. In an α-BIBD* with $v = t+1$, an inequality $b \geq 2(v-1)+t$ holds.

Example 5.2.1. Consider a 2-BIBD*(12, 33, 22, 8, 14) with $t = 11$ (cf. Kageyama and Mohan, 1983). Then $(v-1, \alpha) = 1, v = t+1$ and $b = v-1+2t = 2(v-1)+t$.

From a point of view of investigating the present conjecture, there is a problem on the existence of an α-BIBD* with b satisfying $2(v-1)+t > b > v-1+2t$ in general. By Lemma 3.1 and Corollary 3.1, when $v = t+1$, this problem is solved, while when $v > t+1$ the problem is still open.

Theorem 5.2.2. There does not exist an α-BIBD* with $\lambda = 1$.

Proof. A relation $\lambda(v-1) = r(k-1)$ implies that $v-1 = \alpha t(k-1)$. Hence $(v-1, \alpha) \neq 1$ since $\alpha \geq 2$. ∎

Lemma 5.2.2. In an α-BIBD* with $\lambda = 2^n, n \geq 1$, an inequality $b < 2(v-1)+t$ holds.

Proof. It follows from $r = k+\lambda$ that $v = 1+r(k-1)/\lambda = k+k(k-1)/2^n$, which implies that $2^n|k$ or $2^n|(k-1)$.

Case 1: $k-1 = 2^n\ell, \ell \geq 1$. Now it holds that $v = (\ell+1)(2^n\ell+1), r = (\ell+1)2^n+1, b = (\ell+1)[(\ell+1)2^n+1]$. Hence $b-2(v-1) = -(\ell+1)[(\ell-1)2^n+1]+2 \leq 0$, i.e., $b < 2(v-1)+t$.

Case 2: $k = 2^n\ell, \ell \geq 1$. Similarly, $v = \ell[(\ell+1)2^n-1], r = (\ell+1)2^n, b = (\ell+1)[(\ell+1)2^n-1]$. Hence $b-2(v-1) = -(\ell-1)[(\ell+1)2^n-1]+2 < 0$ if $\ell \geq 2$, i.e., $b < 2(v-1)+t$. Then $\ell = 1$ yields that $v = 2^{n+1}-1, b = 2(2^{n+1}-1), r =$

$2^{n+1}, k = 2^n, \lambda = 2^n$. Since $b = \beta t, r = \alpha t$ and $t \geq 2$, it necessarily holds that $\alpha = 2^n, \beta = 2^{n+1} - 1$ and $t = 2$. Hence $(v-1, \alpha) = 2 \neq 1$. ■

Remark 5.2.1. Any existing example of the design satisfying Lemma 3.2 is not found. In particular, when $n = 1$ in Lemma 3.2, Kadowaki and Kageyama (2008a) disprove the existence of a 2-BIBD*(10, 15, 6, 4, 2) with $t = 3$ and $b = 15 < 2(v-1) + t = 21$.

Lemma 5.2.3. In an α-BIBD* with $\lambda = p^n, p$ being a prime ≥ 3 for a positive integer n, $\lambda|(k-1)$ or $\lambda|k$. Furthermore,

(i) when $\lambda|(k-1)$, an inequality $b < 2(v-1) + t$ holds;
(ii) when $k = \lambda$, parameters $v = 2\lambda - 1, b = 2(2\lambda - 1), r = 2\lambda, k = \lambda, \alpha = \lambda, \beta = 2\lambda - 1, t = 2$ hold, i.e., $b = 2(v-1) + t$;
(iii) when $k = \ell\lambda, \ell \geq 2$, an inequality $b < 2(v-1) + t$ holds.

Proof. Since $r = k + \lambda$, it follows that $v = k + k(k-1)/p^n$, which implies that $p^n|k$ or $p^n|(k-1)$.

Case 1: $k-1 = \ell p^n, \ell \geq 1$. It holds that $v = (\ell+1)(\ell p^n+1), r = (\ell+1)p^n + 1, b = (\ell+1)[(\ell+1)p^n+1]$. Hence $b - 2(v-1) = -(\ell+1)[(\ell-1)p^n+1]+2 \leq 0$, i.e., $b < 2(v-1) + t$.

Case 2: $k = \ell p^n, \ell \geq 1$. Similarly, $v = \ell[(\ell+1)p^n - 1], r = (\ell+1)p^n, b = (\ell+1)[(\ell+1)p^n - 1]$ and hence $b - 2(v-1) = -(\ell+1)[(\ell-1)p^n - 1] < 0$ if $\ell \geq 2$, i.e., $b < 2(v-1) + t$ when $k = \ell\lambda$ and $\ell \geq 2$. So next $\ell = 1$ yields $v = 2p^n - 1, b = 2(2p^n - 1), r = 2p^n, k = p^n, \lambda = p^n$. Since $b = \beta t, r = \alpha t$ and $t \geq 2$, it follows that $\alpha = p^n, \beta = 2p^n - 1$ and $t = 2$. In this case $(v-1, \alpha) = 1$ since $p \geq 3$. ■

Example 5.2.2. Any existing example of the designs for cases (i) and (iii) in Lemma 3.3 is unknown. For the case (ii), there exists a 3-BIBD*(5, 10, 6, 3, 3) with $t = 2$ and $\beta = 5$. Here $b = 2(v-1) + t$. The blocks and two resolution sets are given by [(3, 4, 5), (2, 4, 5), (1, 3, 5), (1, 2, 4), (1, 2, 3)], [(2, 3, 5), (2, 3, 4), (1, 3, 4), (1, 4, 5), (1, 2, 5)]. Also when $\lambda = 5$, there exists a 5-BIBD*(9, 18, 10, 5, 5) with $t = 2$ and $\beta = 9$ (cf. Kageyama and Mohan, 1983).

Remark 5.2.2. A general result on existence of the design (ii) in Lemma 3.3 is presented as follows. Since $v < 2k$, its complement can be considered, i.e., an α^*-BIBD*$(v^* = 2\lambda - 1, b^* = 2(2\lambda - 1), r^* = 2(\lambda - 1), k^* = \lambda - 1, \lambda^* =$

$\lambda - 2, \alpha^* = \lambda - 1, \beta^* = 2\lambda - 1, t^* = 2)$, whose existence is seen by Theorem 4.4.15 of Hedayat and Kageyama (1980) when λ and $2\lambda - 1$ is a prime or a prime power, since one initial block constructed there through GF$(2\lambda - 1)$ produces a resolution set.

Next, we consider an α-BIBD* with a composite number on λ whose smallest case (being not a power) is $\lambda = 6 = 2 \times 3$.

Lemma 5.2.4. In an α-BIBD* with $\lambda = 6$, an inequality $b < 2(v-1) + t$ holds, except for an existing 2-BIBD*(6, 15, 10, 4, 6) with $t = 5$ having $b = 2(v-1) + t = v - 1 + 2t$.

Proof. Since $r = k + \lambda$, it follows that $v = k + k(k-1)/6$, which implies that $2|k, 3|k, 6|k, 2|(k-1), 3|(k-1)$, or $6|(k-1)$.

Case 1: $k - 1 = 2\ell, \ell \geq 1$. It holds that $v = (2\ell+1)(\ell+3)/3, r = 2\ell+7, b = (\ell+3)(2\ell+7)/3$. Hence $b - 2(v-1) = -(\ell+3)(2\ell-5)/3 + 2 \leq 0$ if $\ell \geq 3$, i.e., $b < 2(v-1) + t$. When $\ell = 1, 2$, there is no design for an α-BIBD*.

Case 2: $k - 1 = 3\ell, \ell \geq 1$. Similarly, $v = (3\ell+1)(\ell+2)/2, r = 3\ell+7, b = (\ell+2)(3\ell+7)/2$. Hence $b - 2(v-1) = -(\ell+2)(3\ell-5)/2 + 2 \leq 0$ if $\ell \geq 2$, i.e., $b < 2(v-1) + t$. Next, $\ell = 1$ yields $v = 6, b = 15, r = 10, k = 4, \lambda = 6$. Since $b = \beta t, r = \alpha t$ and $t \geq 2$, it follows that $\alpha = 2, \beta = 3$ and $t = 5$. This 3-resolvable BIB design exists as a complement of the design of No. 2 in Kageyama (1972).

Case 3: $k - 1 = 6\ell, \ell \geq 1$. Similarly, $v = (\ell+1)(6\ell+1), r = 6\ell+7, b = (\ell+1)(6\ell+7)$. Hence $b - 2(v-1) = -(\ell+1)(6\ell-5) + 2 \leq 0$ if $\ell \geq 1$, i.e., $b < 2(v-1) + t$.

Case 4: $k = 6\ell, \ell \geq 1$. Similarly, $v = \ell(6\ell+5), r = 6(\ell+1), b = (\ell+1)(6\ell+5)$. Hence $b - 2(v-1) = -(\ell-1)(6\ell+5) + 2 < 0$ if $\ell \geq 2$, i.e., $b < 2(v-1) + t$. Next, when $\ell = 1$, there is no design.

Case 5: $k = 3\ell, \ell \geq 1$. Similarly, $v = \ell(3\ell+5)/2, r = 3(\ell+2), b = (\ell+2)(3\ell+5)/2$. Hence $b - 2(v-1) = -(\ell-2)(3\ell+5)/2 + 2 < 0$ if $\ell \geq 3$, i.e., $b < 2(v-1) + t$. Next, when $\ell = 1, 2$, there is no design.

Case 6: $k = 2\ell, \ell \geq 1$. Similarly, $v = \ell(2\ell+5)/3, r = 2(\ell+3), b = (\ell+3)(2\ell+5)/3$. Hence $b - 2(v-1) = -(\ell-3)(2\ell+5)/3 + 2 < 0$ if $\ell \geq 4$, i.e., $b < 2(v-1) + t$. Next, when $\ell = 1, 3$, there is no design, while $\ell = 2$ yields $v = 6, b = 15, r = 10, k = 4, \lambda = 6$ which is the same as the design in

Case 2. ■

Remark 5.2.3. Any existing example of the design satisfying Lemma 3.4 is not found.

Similarly, another case of a composite number $\lambda = 10 = 2 \times 5$ can be investigated as follows. The proof is routine and similar to that of Lemma 5.2.4 and hence it is omitted.

Lemma 5.2.5. In an α-BIBD* with $\lambda = 10$, an inequality $b < 2(v-1)+t$ holds, except for an existing 5-BIBD*(7, 21, 15, 5, 10) with $t = 3$ having $b > 2(v-1)+t$.

Example 5.2.3. The existing 5-BIBD*(7, 21, 15, 5, 10) in Lemma 5.2.5 has 21 blocks and 3 resolution sets as follows: [(3, 4, 5, 6, 7), (2, 4, 5, 6, 7), (1, 3, 5, 6, 7), (1, 2, 4, 6, 7), (1, 2, 3, 5, 7), (1, 2, 3, 4, 6), (1, 2, 3, 4, 5)], [(2, 3, 5, 6, 7), (2, 3, 4, 6, 7), (1, 3, 4, 5, 7), (1, 2, 4, 5, 6), (1, 4, 5, 6, 7), (1, 2, 3, 5, 6), (1, 2, 3, 4, 7)], [(2, 3, 4, 5, 7), (2, 3, 4, 5, 6), (1, 3, 4, 6, 7), (1, 3, 4, 5, 6), (1, 2, 4, 5, 7), (1, 2, 3, 6, 7), (1, 2, 5, 6, 7)]. Here $b = 21 > 2(v-1)+t = 15$. In fact, $b = 3(v-1)+t$.

Remark 5.2.4. Among α-BIBD* satisfying $b \geq 2(v-1)+t$, most of designs have $b = 2(v-1)+t$. As in Example 5.2.3, a design with $b > 2(v-1)+t$ can be constructed in α-BIBD*. Furthermore, we can show the existence of such designs, for example, α-BIBD* of Nos. 44 ($x = 3$), 65 ($x = 3$), 75 ($x = 4$), 80 ($x = 5$), 84 ($x = 3$), 89 ($x = 4$), 98 ($x = 5$) in Kageyama and Mohan (1983), where $b = x(v-1)+t$ that is an interesting relation. Also note that the design of Example 5.2.3 is the smallest example with $x = 3$ among designs with $b > 2(v-1)+t$.

Finally, to investigate further the real existence of an α-BIBD* with $b < 2(v-1)+t$, this problem can be also considered in terms of a GD design. It is to utilize the fact (Theorem 8.5.1 of Raghavarao, 1988) that the existence of an α-BIBD*$(v, b = \beta t, r = \alpha t, k, \lambda)$ is equivalent to the existence of an α-resolvable singular GD design with parameters $v^* = nv, b^* = b = \beta t, r^* = r = \alpha t, k^* = nk, \lambda_1^* = r, \lambda_2^* = \lambda; m^* = v, n^* = n$ for a positive integer $n \geq 2$. The α-resolvability is obvious. In this case, $b < 2(v-1)+t$ is equivalent to $b^* < 2(v^*/n - 1) + t$. In Table IV of Clatworthy (1973), unfortunately any example of such GD designs is not available within the scope of parameters.

In fact, the existing GD designs in the table shows $b^* \geq 2(v^*/n - 1) + t$.

5.3. Remarks

An improvement of bounds on the number of blocks has been discussed in this section. The basic bound is $b \geq v + t - 1$ for an α-resolvable BIB design with $\alpha \geq 1$. Our problem is for a class of α-resolvable BIB designs that are not affine α-resolvable. Usually in this case, the basic bound is improved by one, i.e., $b \geq v + t$. Though the present problem is considered under a condition $(v - 1, \alpha) = 1$, the basic bound could be improved for both cases of (i) $\alpha = 1$ and (ii) $\alpha \geq 2$ and $(r - \lambda)/k \geq 2$. However, the remaining case of $\alpha \geq 2$ and $(r - \lambda)/k = 1$ (quasi-residual design) shows us a completely different status. For this case the improved bound $b \geq 2(v-1)+t$ cannot be derived similarly to the cases (i) and (ii). Finally, a relationship on parameters is described.

Proposition 5.3.1. In an α-BIBD*, it holds that

(1) $b < 2(v-1)+t \iff \beta > 2\alpha - 1 \iff t < (v-1)/(\alpha - 1)$,
(2) $b = 2(v-1)+t \iff \beta = 2\alpha - 1 \iff t = (v-1)/(\alpha - 1)$,
(3) $b > 2(v-1)+t \iff \beta < 2\alpha - 1 \iff t > (v-1)/(\alpha - 1)$,

where t is the number of resolution sets and β is the number of blocks in each resolution set.

Proof. Note that $b = v + r - 1$ is equivalent to $v - 1 = (\beta - \alpha)t$. Hence, if $b = v + r - 1 < 2(v-1) + t$, then $(\alpha - 1)t < (\beta - \alpha)t$, i.e., $\beta > 2\alpha - 1$, and conversely. Also, if $b\ (= v + r - 1) = v + \alpha t - 1 < 2(v-1) + t$, then $(\alpha - 1)t < v - 1$, and conversely. The other cases are similar. ■

Thus, in an α-BIBD*, $b = v + \alpha t - 1$, which is in terms of v, t and α, whose lower limit cannot be expressed in terms of v and t only. As Proposition 5.3.1 reveals, several relations on parameters may occur in an α-BIBD*. Without having the parameter α, an inequality $b \geq v+2t-1$ may be the best in general as in Theorem 5.2.1. As in Lemmas 5.2.2 – 5.2.5, the bound $b < 2(v-1)+t$ is derived as a parametric characterization. As far as the authors are aware of, any example of the design for such cases cannot be constructed.

Remark 5.3.1. Note that $\beta \geq 2\alpha \iff b \geq 2r \iff v \geq 2k$. The relations (2) and (3) in Proposition 5.3.1 make an improvement to (i) of Theorem 2.4 of Kadowaki and Kageyama (2003). Also, in (2), $(\beta, \alpha) = 1$.

6. Affine α-resolvable PBIB designs

The present section is devoted to the combinatorial investigation on a property of affine α-resolvability in a 2-associate PBIB design.

In literature there are much combinatorial discussions on α-resolvable PBIB designs (see, for example, Bose, 1977, Kageyama, 1977, 2007, 2008a, 2008b, Kageyama and Mohan, 1985). However, there are not many papers on "affine" α-resolvable PBIB designs. As was mentioned in Section 2, there are several types of 2-associate PBIB designs. Among them, two types are at first considered here.

Let us take a class of cyclic PBIB designs (see Definition 2.6). In this case the following can be seen.

Theorem 6.1. There does not exist an affine α-resolvable cyclic 2-associate PBIB design for any $\alpha \geq 1$.

Proof. In this design, it is known that the number of treatments is $v = 4t + 1$ being a prime. On the other hand, the affine α-resolvability requires that $q_2 = k^2/v$ is an integer. Now since v is a prime and $v > k$, q_2 is not an integer. Hence the proof is complete. ■

Next consider a class of triangular PBIB designs with $v = n(n-1)/2$ (see Definition 2.4). No example has been found for an affine α-resolvable triangular design for $\alpha \geq 1$ in literature. Recently the following has been shown.

Theorem 6.2 (Kageyama, 2007, 2008b). There does not exist an affine α-resolvable triangular design for $1 \leq \alpha \leq 10$.

Then Kageyama has conjectured that there does not exist an affine α-resolvable triangular design for any $\alpha \geq 1$. Since the attractive result on existence could not be further obtained, the existence problem of affine α-resolvable triangular designs will not be discussed in this thesis.

As of today, a cyclic design forms the only class of 2-associate PBIB designs which do not possess entirely a property of affine α-resolvability in design theory. A class of triangular designs may be the next such candidate.

For further argument, the following lemma is useful. This can be derived by use of Lemmas 2.1 and 2.2.

Lemma 6.1 (cf. Kageyama, 2007). In a 2-associate PBIB design, having the incidence matrix N, with parameters v, b, r, k, λ_i, θ_i, ρ_i, $i = 0, 1, 2$, where $\lambda_0 = r, \theta_0 = rk$, $\rho_0 = 1$, θ_1 and θ_2 are the nonnegative eigenvalues (other than rk) of NN' with respective multiplicities ρ_1 and ρ_2, when $\theta_1 > 0$ and $\theta_2 > 0$, the design does not possess a property of affine α-resolvability.

Remark 6.1. Similarly to λ_i as in Definition 2.2(3), the eigenvalues θ_i are corresponding to ith associates of an association scheme, $i = 0, 1, 2$ (Caliński and Kageyama, 2003, Raghavarao, 1988). Since in a cyclic 2-associate PBIB design all the eigenvalues of NN' are positive (see, pp. 126 and 129 in Raghavarao, 1988), Lemma 6.1 can yield the same result as in Theorem 6.1.

The following result plays a crucial role to characterize affine α-resolvable 2-associate PBIB designs in this thesis.

Theorem 6.3. Let N be the $v \times b$ incidence matrix of an affine α-resolvable 2-associate PBIB design with parameters $v, b = \beta t, r = \alpha t, k, \lambda_1$, $\lambda_2, q_1 = k(\alpha - 1)/(\beta - 1)$ and $q_2 = k^2/v$, and further let θ_i be eigenvalues of NN' with multiplicities ρ_i, $i = 0, 1, 2$, where $\theta_0 = rk$ and $\rho_0 = 1$. Then, when $\theta_i > 0$ and $\theta_{i'} = 0, i \neq i' \in \{1, 2\}$, $q_1 = k - \theta_i$ and $b = t + \rho_i$ hold.

Proof. By Lemma 2.1, $N'N$ has the only nonzero eigenvalue (other than rk) $k\{1 - (\alpha - 1)/(\beta - 1)\}$, which is equal to $k - q_1$, with multiplicity $b - t$. Then (i) when $\theta_1 > 0$ and $\theta_2 = 0$, Lemma 2.2 implies that $k - q_1 = \theta_1$ and $b - t = \rho_1$, while (ii) when $\theta_1 = 0$ and $\theta_2 > 0$, Lemma 2.2 implies that $k - q_1 = \theta_2$ and $b - t = \rho_2$. On account of Lemma 6.1 note that a case of $\theta_1 > 0$ and $\theta_2 > 0$ does not occur in this design. ∎

Remark 6.2. In Theorem 6.3, when $\theta_1 = \theta_2 = 0$, i.e., NN' has the only one nonzero eigenvalue rk, the design is orthogonal and then $N = \mathbf{1}_v \mathbf{1}_b'$, which is not incomplete (cf. Caliński and Kageyama, 2003, Chapters 6 and 7), where $\mathbf{1}_s$ is an $s \times 1$ column vector all of whose elements are 1. Hence, the orthogonal design is not a PBIB design, but a randomized block design.

In a 2-associate PBIB design, when $\lambda_1 = \lambda_2$, the PBIB design becomes a

BIB design and hence, as eigenvalues of NN', $\theta_1 = r - \lambda$ only other than rk. Therefore, by Theorem 6.3, in an affine α-resolvable BIB design $q_1 = k+\lambda-r$ holds (see the statements after Definition 2.1).

The largest, simplest and perhaps most important class of 2-associate PBIB designs is known as GD (group divisible). In a GD design the eigenvalues of NN' have $\theta_1 = rk - v\lambda_2$ and $\theta_2 = r - \lambda_1$ (other than rk) with respective multiplicities $\rho_1 = m - 1$ and $\rho_2 = m(n-1)$. Hence by Definition 2.3 and Lemma 6.1 the following has been provided.

Theorem 6.4 (cf. Kageyama, 2008a). There does not exist an affine α-resolvable regular GD design for any $\alpha \geq 1$.

By Remark 6.2, other two subclasses (i.e., SGD and SRGD) of GD designs will be discussed in subsequent Sections 6.1 to 6.4 below.

6.1. Affine α-resolvable SGD designs

By Definition 2.3, the present section is devoted to a GD design with $r = \lambda_1$, i.e., of singular type. Note that $\lambda_1 > \lambda_2$ in an SGD design. Shah and Kabe (1981) discussed the affine α-resolvability of an SGD design, but their argument does not help the present consideration.

It is known (Bose and Connor, 1952) that the existence of an SGD($v = mn, b, r = \lambda_1, k, \lambda_1, \lambda_2$) is equivalent to the existence of a BIB($v^*, b^*, r^*, k^*, \lambda^*$), where $v = nv^*, b = b^*, r = r^*, k = nk^*, \lambda_1 = r^*, \lambda_2 = \lambda^*, m = v^*, n = n$. This result can be obtained from replacing each treatment of the BIB design by a group of n treatments for $n \geq 2$. It is obvious that the present replacement procedure preserves a property of affine α-resolvability between a BIB design and an SGD design. Hence the following result has been established.

Theorem 6.1.1. The existence of an affine α-resolvable SGD($v = nv^*, b = b^* = \beta t, r = r^* = \alpha t, k = nk^*, \lambda_1 = r^*, \lambda_2 = \lambda^*; m = v^*, n = n$) with $q_1 = nk^*(\alpha-1)/(\beta-1)$ and $q_2 = n(k^*)^2/v^*$ is equivalent to the existence of an affine α-resolvable BIB($v^*, b^* = \beta t, r^* = \alpha t, k^*, \lambda^*$) with $q_1^* = k(\alpha-1)/(\beta-1)$ and $q_2^* = k^2/v$.

Now an integral expression of q_1 is derived like $q_1 = k + \lambda - r$ in an affine

α-resolvable BIB design.

Corollary 6.1.1. In an affine α-resolvable SGD design, $q_1 = k(\alpha - 1)/(\beta - 1) = k - \lambda_1 k + v\lambda_2$ holds.

Proof. Since $\theta_1 = rk - v\lambda_2$ and $\theta_2 = r - \lambda_1 = 0$, by Theorem 6.3 we have $q_1 = k - \theta_1 = k - rk + v\lambda_2$. ∎

Now the parameters of an affine α-resolvable SGD design with parameters $v = mn, b = \beta t, r = \alpha t, k, \lambda_1, \lambda_2$, $q_1 = k(\alpha - 1)/(\beta - 1)$ and $q_2 = k^2/v$ are characterized. The following can be shown.

Theorem 6.1.2. The parameters of an affine α-resolvable SGD design are given by

$$v = mn, b = \frac{\beta(m-1)}{\beta - 1}, r = \frac{\alpha(m-1)}{\beta - 1}, k = \frac{\alpha mn}{\beta}, \lambda_1 = \frac{\alpha(m-1)}{\beta - 1},$$

$$\lambda_2 = \frac{\alpha(\alpha m - \beta)}{\beta(\beta - 1)};\ t = \frac{m-1}{\beta - 1}, q_2 = \frac{\alpha^2 mn}{\beta^2},$$

where $\alpha m/\beta$ is an integer.

Proof. Since eigenvalues of NN' are $rk - v\lambda_2$ and $r - \lambda_1 = 0$ with respective multiplicities $m - 1$ and $m(n-1)$, it follows from Theorem 6.3 that $b - t = m - 1$, i.e., $t = (m-1)/(\beta-1)$ which also implies that $m > \beta$. Then we obtain the expression of parameters as $v = mn, b = \beta t = \beta(m-1)/(\beta-1), r = \alpha t = \alpha(m-1)/(\beta-1), k = vr/b = \alpha mn/\beta, \lambda_1 = r = \alpha(m-1)/\beta$. Furthermore, by a relation $r(k-1) = n_1\lambda_1 + n_2\lambda_2$, we get $\lambda_2 = \alpha(\alpha m - \beta)/[\beta(\beta - 1)]$. Also by Theorem 6.1.1, $k/n = \alpha m/\beta$ must be an integer. ∎

Thus, all parameters of an affine α-resolvable SGD design can be expressed in terms of m, n, α and β. It is clear that these parameters satisfy Corollary 6.1.1.

6.2. Table of affine resolvable SGD designs with $v \leq 100$ and $r, k \leq 20$

There are a number of affine α-resolvable SGD designs with parameters $v = mn, b = \beta t, r = \alpha t, k, \lambda_1, \lambda_2, q_1, q_2$. In this section, we restrict ourselves to the case of $\alpha = 1$. Even so, by Lemma 2.3, some of other affine α-resolvable SGD designs can be constructed for some $\alpha \geq 2$. Now, since $q_2 = k^2/v$, by Theorem 6.1.2 we have the expression of parameters as

$$v = mn, b = \frac{\beta(m-1)}{\beta - 1}, r = \frac{m-1}{\beta - 1}, k = \frac{mn}{\beta},$$

$$\lambda_1 = \frac{m-1}{\beta - 1}, \lambda_2 = \frac{m-\beta}{\beta(\beta - 1)}, q_2 = \frac{mn}{\beta^2},$$

where m/β is an integer. Since $m > \beta$, according to the value being a positive integer (≥ 2) of m/β, we now systematically search the designs with admissible parameters (i.e., of satisfying necessary conditions for the existence) within the scope of $v \leq 100$ and $r, k \leq 20$. All the design parameters should be integers. In fact, there are 41 parameters' combinations, all of which have explicit information on the existence of affine α-resolvability. By Theorem 6.1.1 the existence problem completely depends on the existence status of the corresponding affine resolvable BIB($v^* = v/n, b^* = b, r^* = r = \lambda_1, k^* = k/n, \lambda^* = \lambda_2$) whose combinatorics has been discussed widely in literature (cf. Kageyama, 1972, Shrikhande, 1976, Furino, Miao and Yin, 1996). For example, the existence of a "self-complementary" (i.e., $v = 2k$) affine resolvable SGD design with parameters $v = mn, b = 2(m-1), r = m-1, k = mn/2, \lambda_1 = m-1, \lambda_2 = (m-2)/2, q_1 = 0, q_2 = mn/4$ is equivalent to the existence of an affine resolvable BIB($v^* = m, b^* = 2(m-1), r^* = m-1, k^* = m/2, \lambda^* = (m-2)/2$) for even m.

In Table 6.2, the admissible parameters of affine resolvable SGD designs are listed along with existence information. The designs are numbered in the ascending order of m and for the same m in the order of n. Since $q_1 = 0$, the parameter is not listed. "Non-E" means the nonexistence of the design, Kx+$\{y\}$ in Source 1 means that the design is constructed through an affine resolvable BIB design of No. x in Kageyama (1972) in which each treatment is replaced by a group of y new treatments. In Source 2, when an affine resolvable SGD design does not exist, the status on existence of the corresponding BIB design, i.e., an SGD design, which is not affine resolvable, is described.

Table 6.2. Affine resolvable SGD designs

No.	m	n	v	b	r	k	λ_1	λ_2	q_2	Source 1	Source 2	Remark
1	4	2	8	6	3	4	3	1	2	K1+{2}		S6
2	4	3	12	6	3	6	3	1	3	K1+{3}		S27
3	4	4	16	6	3	8	3	1	4	K1+{4}		S61
4	4	5	20	6	3	10	3	1	5	K1+{5}		S106
5	4	6	24	6	3	12	3	1	6	K1+{6}		
6	4	7	28	6	3	14	3	1	7	K1+{7}		
7	4	8	32	6	3	16	3	1	8	K1+{8}		
8	4	9	36	6	3	18	3	1	9	K1+{9}		
9	4	10	40	6	3	20	3	1	10	K1+{10}		
10	6	2	12	10	5	6	5	2	3	Non-E	BIB(6,3,2) + {2}	※ 1
11	6	4	24	10	5	12	5	2	6	Non-E	BIB(6,3,2) + {4}	※ 1
12	6	6	36	10	5	18	5	2	9	Non-E	BIB(6,3,2) + {6}	※ 1
13	8	2	16	14	7	8	7	3	4	K5+{2}		S63
14	8	3	24	14	7	12	7	3	6	K5+{3}		
15	8	4	32	14	7	16	7	3	8	K5+{4}		
16	8	5	40	14	7	20	7	3	10	K5+{5}		
17	9	2	18	12	4	6	4	1	2	K6+{2}		S37
18	9	3	27	12	4	9	4	1	3	K6+{3}		S91
19	9	4	36	12	4	12	4	1	4	K6+{4}		
20	9	5	45	12	4	15	4	1	5	K6+{5}		
21	9	6	54	12	4	18	4	1	6	K6+{6}		
22	10	2	20	18	9	10	9	4	5	Non-E	BIB(10,5,4) + {2}	※ 1
23	10	4	40	18	9	20	9	4	10	Non-E	BIB(10,5,4) + {4}	※ 1
24	12	2	24	22	11	12	11	5	6	K12+{2}		
25	12	3	36	22	11	18	11	5	9	K12+{3}		
26	14	2	28	26	13	14	13	6	7	Non-E	BIB(14,7,6) + {2}	※ 1
27	15	3	45	21	7	15	7	2	5	Non-E	Non-E	※ 2
28	16	2	32	20	5	8	5	1	2	K17+{2}		S74
29	16	2	32	30	15	16	15	7	8	K18+{2}		
30	16	3	48	20	5	12	5	1	3	K17+{3}		
31	16	4	64	20	5	16	5	1	4	K17+{4}		
32	16	5	80	20	5	20	5	1	5	K17+{5}		
33	18	2	36	34	17	18	17	8	9	Non-E	BIB(18,9,8) + {2}	※ 1
34	20	2	40	38	19	20	19	9	10	K25+{2}		
35	25	2	50	30	6	10	6	1	2	K28+{2}		S121
36	25	3	75	30	6	15	6	1	3	K28+{3}		
37	25	4	100	30	6	20	6	1	4	K28+{4}		
38	27	2	54	39	13	18	13	4	6	K30+{2}		
39	36	2	72	42	7	12	7	1	2	Non-E	Non-E	※ 3
40	40	2	80	52	13	20	13	3	5	Non-E	BIB(40,10,3) ?	※ 4
41	49	2	98	56	8	14	8	1	2	K40+{2}		

The column of Remark shows some information:

For example, S6 denotes an SGD design number from Table IV of Clatworthy (1973). An actual affine resolvable solution is also given there.

※1: Though a BIB($v^* = v/n, b^* = b, r^* = r, k^* = k/n, \lambda^* = \lambda_2$) exists, $(k^*)^2/v^*$ is not an integer. Hence the corresponding affine resolvable solution does not exist.

※2: A BIB($v = 15, b = 21, r = 7, k = 5, \lambda = 2$) does not exist (Takeuchi, 1962). Hence an affine resolvable solution does not exist.

※3: A BIB($v = 36, b = 42, r = 7, k = 6, \lambda = 1$) does not exist (Takeuchi, 1962). Hence an affine resolvable solution does not exist.

※4: In a BIB($v = 40, b = 52, r = 13, k = 10, \lambda = 3$), k^2/v is not an integer and hence such an affine resolvable solution of a design of No. 40 does not exist, but the existence as a BIB design (or an SGD design) is in doubt.

6.3. Affine α-resolvable SRGD designs

In this section an affine α-resolvable SRGD design with parameters $v = mn, b = \beta t, r = \alpha t, k, \lambda_1, \lambda_2$, $q_1 = k(\alpha - 1)/(\beta - 1)$ and $q_2 = k^2/v$, in which $rk - v\lambda_2 = 0$, is considered. Note that $\lambda_2 > \lambda_1$ in an SRGD design.

Now an integral expression of q_1 is derived like $q_1 = k + \lambda - r$ in an affine α-resolvable BIB design and as in Corollary 6.1.1.

Corollary 6.3.1. In an affine α-resolvable SRGD design, $q_1 = k(\alpha - 1)/(\beta - 1) = k + \lambda_1 - r$ holds.

Proof. Since $\theta_1 = rk - v\lambda_2 = 0$ and $\theta_2 = r - \lambda_1$, Theorem 6.3 implies that $q_1 = k + \lambda_1 - r$. ■

Furthermore, a typical result is remarked.

Lemma 6.3.1 (Bose and Connor, 1952). In an SRGD design, k is divisible by m.

Next the following characterization of parameters is obtained.

Theorem 6.3.1. The parameters of an affine α-resolvable SRGD design are given by

$$v = mn, b = \frac{\beta m(n-1)}{\beta - 1}, r = \frac{\alpha m(n-1)}{\beta - 1}, k = \frac{\alpha mn}{\beta}, \lambda_1 = \frac{\alpha m(\alpha n - \beta)}{\beta(\beta - 1)},$$

$$\lambda_2 = \frac{\alpha^2 m(n-1)}{\beta(\beta-1)};\ t = \frac{m(n-1)}{\beta - 1}, q_2 = \frac{\alpha^2 mn}{\beta^2},$$

where $\alpha n/\beta$ is an integer.

Proof. Since eigenvalues of NN' are $rk - v\lambda_2 = 0$ and $r - \lambda_1$ with respective multiplicities $m-1$ and $m(n-1)$, by Theorem 6.3 it holds that $b - t = m(n-1)$, i.e., $b = v + t - m$ which also implies that $t = m(n-1)/(\beta-1)$. Then it follows that $v = mn, b = \beta t = \beta m(n-1)/(\beta-1), r = \alpha t = \alpha m(n-1)/(\beta-1), k = vr/b = \alpha mn/\beta, \lambda_2 = rk/v = \alpha^2 m(n-1)/[\beta(\beta-1)]$. Furthermore, from a relation $r(k-1) = n_1\lambda_1 + n_2\lambda_2$, we get $\lambda_1 = \alpha m(\alpha n - \beta)/[\beta(\beta-1)]$. Also by Lemma 6.3.1, $k/m = \alpha n/\beta$ must be an integer. ∎

Thus, all parameters of an affine α-resolvable SRGD design can be expressed in terms of m, n, α and β. It is clear that these parameters satisfy Corollary 6.3.1.

There are 14 affine resolvable SRGD designs listed by Clatworthy (1973), among of which 12 designs are symmetric. That is, only two affine resolvable "nonsymmetric" SRGD designs are available within the scope of parameters in Clatworthy (1973).

When the SRGD design is symmetric, we have $t = m$ and $n = \beta$. Hence Theorem 6.3.1 yields the following.

Corollary 6.3.2. The parameters of an affine α-resolvable symmetric SRGD design are given by

$$v = b = mn, r = k = \alpha m, \lambda_1 = \frac{\alpha m(\alpha - 1)}{n - 1}, \lambda_2 = \frac{\alpha^2 m}{n};$$

$$t = m, \beta = n.$$

All the existing affine α-resolvable symmetric SRGD designs satisfy $m = n$. In this case Corollary 6.3.2 yields the following since $n = \beta$.

Corollary 6.3.3. The parameters of an affine α-resolvable symmetric SRGD design with $m = n$ are given by

$$v = b = m^2, r = k = \alpha m, \lambda_1 = \frac{\alpha m(\alpha - 1)}{m - 1}, \lambda_2 = \alpha^2;\ t = \beta = m.$$

Note that in Corollary 6.3.3

$$\lambda_1 = \alpha(\alpha - 1) + \frac{\alpha(\alpha - 1)}{m - 1}$$

which causes some restriction on the values of $\alpha(\geq 2)$ for given m in $v = mn$.

As a method of construction of an SRGD design belonging to Corollary 6.3.3, Kageyama and Mohan (1985, Corollary 2.1) show that when v^* is a prime, the existence of a symmetric BIB($v^* = b^*, r^* = k^*, \lambda^*$) implies the existence of an affine α-resolvable symmetric SRGD design with parameters $v = b = (v^*)^2, r = k = v^*k^*, \lambda_1 = \lambda^*v^*, \lambda_2 = (k^*)^2, q_1 = \lambda^*v^*, q_2 = (k^*)^2, \alpha = r^*, t = \beta = v^*$ for $m = n = v^*$. By use of this result, for example, the following can be given. (i) Since a symmetric BIB(3, 3, 2, 2, 1) exists, we get a design of No. 6 of Table 6.4, i.e., SR23. (ii) Since a symmetric BIB(5, 5, 4, 4, 3) exists, we get an affine 4-resolvable SRGD design with parameters $v = b = 25, r = k = 20, \lambda_1 = 15, \lambda_2 = 16, t = \beta = 5; m = n = 5$, whose complement is, by Lemma 2.3, an affine resolvable SRGD design with parameters $v = b = 25, r = k = 5, \lambda_1 = 0, \lambda_2 = 1; m = n = 5$, i.e., a design of No. 13, which may be different from SR60. (iii) Since a symmetric BIB(7, 7, 3, 3, 1) exists (cf. Takeuchi, 1962), we get an affine 3-resolvable SRGD design with parameters $v = b = 49, r = k = 21, \lambda_1 = 7, \lambda_2 = 9, t = \beta = 7$ for $m = n = 7$.

For the next section the case of $\alpha = 1$ will be investigated in detail. For an affine resolvable SRGD design, $t = r$ and then Theorem 6.3.1 with $q_2 = k^2/v$ shows the expression of design parameters as

$$v = mn, b = \frac{\beta m(n - 1)}{\beta - 1}, r = \frac{m(n - 1)}{\beta - 1}, k = \frac{mn}{\beta}, \lambda_1 = \frac{m(n - \beta)}{\beta(\beta - 1)},$$

$$\lambda_2 = \frac{m(n - 1)}{\beta(\beta - 1)}, q_1 = 0, q_2 = \frac{mn}{\beta^2}, \frac{k}{m} = \frac{n}{\beta}.$$

Then it holds that $\lambda_2 - \lambda_1 = m/\beta$. Therefore, there exist positive integers x and y such that

$$m = x\beta \text{ and } n = y\beta.$$

These x and y can be used to express the required parameters as

$$v = xy\beta^2, b = \frac{x\beta^2(\beta y - 1)}{\beta - 1}, r = \frac{x\beta(y\beta - 1)}{\beta - 1}, k = xy\beta, \tag{6.3.1}$$

$$\lambda_1 = \frac{x\beta(y - 1)}{\beta - 1}, \lambda_2 = \frac{x(y\beta - 1)}{\beta - 1}, q_1 = 0, q_2 = xy, \frac{k}{m} = y. \tag{6.3.2}$$

In this case $\lambda_2 - \lambda_1 = x$ and $\lambda_1 = \beta(\lambda_2 - xy)$ (≥ 0). Note that $\lambda_1 = 0$ if and only if $y = 1$, i.e., the design is symmetric.

Now a way of presentation of the design parameters is made according to four patterns on the values of positive integers x and y.

Case 1: $x = y = 1$, i.e., $m = n = \beta$. In this case we have the design parameters as

$$v = b = \beta^2, r = k = \beta, \lambda_1 = 0, \lambda_2 = 1, q_2 = 1, \frac{k}{m} = 1,$$

which is symmetric. In fact, the existing SR1, SR23, SR44, SR60, SR87, SR97 and SR105 in Table VI of Clatworthy (1973) belong to this class. By Lemma 2.3, note that the complement of the design of Case 1 is an affine $(\beta-1)$-resolvable symmetric SRGD design with parameters $v^* = b^* = \beta^2, r^* = k^* = \beta(\beta-1), \lambda_1^* = \beta(\beta-2), \lambda_2^* = \beta(\beta-2)+1, q_1^* = \beta(\beta-2), q_2^* = \beta(\beta-2)+1$, and vice versa. For the present case a construction result can be provided.

Theorem 6.3.2. When β is a prime or a prime power, there exists an affine resolvable symmetric SRGD design with parameters

$$v = b = \beta^2, r = k = \beta, \lambda_1 = 0, \lambda_2 = 1, q_1 = 0, q_2 = 1; m = n = \beta.$$

Proof. It is well known (cf. Caliński and Kageyama, 2003; Chapter 6) that when β is a prime or a prime power, an affine resolvable BIB$(v^* = \beta^2, b^* = \beta(\beta + 1), r^* = \beta + 1, k^* = \beta, \lambda^* = 1)$ can be constructed by use of an affine plane. The dual of this design can yield an SRGD design with parameters $v = \beta(\beta + 1), b = \beta^2, r = \beta, k = \beta + 1, \lambda_1 = 0, \lambda_2 = 1$. In this design by deleting a group of β treatments corresponding to a partition for the affine resolvability of the original BIB design, we can obtain an SRGD design with parameters $v = b = \beta^2, r = k = \beta, \lambda_1 = 0, \lambda_2 = 1$. The remaining problem is to introduce the affine resolvability for the present design. It can be shown that this affine resolvability is naturally given when the incidence structure corresponding to β treatments of a group deleted in the dual design is

$$I_\beta \otimes \mathbf{1}'_\beta,$$

where $A \otimes B$ denotes the Kronecker product of matrices A and B, and I_β is the identity matrix of order β. ■

Remark 6.3.1. From the combinatorial structure on incidence in the construction process given in the proof of Theorem 6.3.2, it is obvious that the existence of an "affine resolvable" SRGD design as in Theorem 6.3.2 is equivalent to the existence of an affine plane of order β.

Case 2: $y = 1$, i.e., $n = \beta$. In this case we have the design parameters as

$$v = b = x\beta^2, r = k = x\beta, \lambda_1 = 0, \lambda_2 = x, q_2 = x, \frac{k}{m} = 1,$$

which is symmetric. When $x = 1$, this case coincides with Case 1 and then $x > 1$ is mainly considered. In fact, the existing SR36, SR72, SR92, SR95 and SR102 in Table VI of Clatworthy (1973) belong to this class for $x = 2, 2, 4, 2$ and 3, respectively. By Lemma 2.3 note that the complement of the design of Case 2 is an affine $(\beta - 1)$-resolvable symmetric SRGD design with parameters $v^* = b^* = x\beta^2, r^* = k^* = x\beta(\beta - 1), \lambda_1^* = x\beta(\beta - 2), \lambda_2^* = x[\beta(\beta - 2) + 1], q_1^* = x\beta(\beta - 2), q_2^* = x[\beta(\beta - 2) + 1]$.

As a method of construction of a design belonging to Case 2, Bose, Shrikhande and Bhattacharya (1953) show that when s is a prime or a prime power, there exists an affine resolvable symmetric SRGD design with parameters $v = b = s^3, r = k = s^2, \lambda_1 = 0, \lambda_2 = s, q_2 = s; m = s^2, n = s$. Here $x = s$ and $y = 1$. When $s = 2$ and 3, we have designs of Nos. 8 and 23 in Table 6.4, respectively. When $s = 4$, we can obtain a solution of an affine resolvable SRGD design of No. 37 with parameters $v = b = 64, r = k = 16, \lambda_1 = 0, \lambda_2 = 4, q_2 = 4; m = 16, n = 4$.

Furthermore, to construct affine resolvable symmetric SRGD design of Case 2, a special type of a difference scheme (cf. Hedayat, Sloane and Stufken, 1999) will be utilized.

An $m \times m$ matrix A with entries from a set $S = \{0, 1, ..., s-1\}$ for $s \geq 2$ is here called a difference scheme, denoted by $DS(m, s; x)$, if on a vector difference in any two columns of A every entry of S occurs x times.

Remark 6.3.2. The same concept as the difference scheme has been discussed under other names of a difference matrix $D(m, m, s)$ or a generalized Hadamard matrix $GH(s, x)$ by interchanging roles of rows and columns (see Beth, Jungnickel and Lenz, 1999, Colbourn and Dinitz, 2007).

It is easily seen that (i) all entries in the first row and first column of a $DS(m, s; x)$ can be set 0, and (ii) in each of columns except for the first, every entry of S occurs x times. The property (ii) implies that $m = xs$ in a $DS(m, s; x)$.

Furthermore, the following properties can be derived (see Beth, Jungnickel and Lenz, 1999, pp. 532 – 534, especially, Remark 3.9(a), or Hedayat, Sloane and Stufken, 1999, p. 115).

(iii) In each of rows except for the first one of the $DS(m, s; x)$, every entry of S occurs x times.

(iv) On a vector difference in any two rows of a $DS(m, s; x)$, every entry of S occurs x times.

Now the following construction result can be shown.

Theorem 6.3.3. The existence of a $DS(m, s; x)$ implies the existence of an affine resolvable symmetric SRGD design with parameters

$$v = b = xs^2, r = k = xs, \lambda_1 = 0, \lambda_2 = x, q_1 = 0, q_2 = x; m = xs, n = s$$

for $s \geq 2$.

Proof. Replace the entries $0, 1, ..., s-1$ in an $m \times m$ matrix as a $DS(m, s; x)$ by $s \times s$ matrices $\pi^i I_s, i = 0, 1, ..., s-1$, respectively, where π is a row permutation such that $\pi R_\ell = R_{\ell+1}$ and R_ℓ is the ℓth row of I_s. Then from $m = xs$ such replacement can show the required design with a GD association scheme on an $xs \times s$ array. In fact, under the property (ii), parameters $v = b = xs^2, k = xs, \lambda_1 = 0, m = xs$ and $n = s$ are obvious. The property (iii) with $m = xs$ implies $r = xs$. It is also clear that the replacement of $s \times s$ $(0, 1)$-matrices shows the resolvability consisting of m resolution sets of

s blocks each, and then $q_1 = 0$. Furthermore, the properties (i) and (ii) of the $DS(m, s; x)$ with properties (iii) and (iv) can yield $\lambda_2 = x$ and $q_2 = x$ (affine resolvability). ∎

When $s = 5$ and $x = 2$ in Theorem 6.3.3, it is illustrated by use of a $DS(10, 5; 2)$ given as follows (see Table 6.35 in Hedayat, Sloane and Stufken, 1999).

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 3 & 1 & 2 & 1 & 0 & 4 & 2 & 3 \\ 0 & 3 & 1 & 2 & 4 & 4 & 2 & 0 & 1 & 3 \\ 0 & 1 & 2 & 4 & 3 & 1 & 2 & 3 & 0 & 4 \\ 0 & 2 & 4 & 3 & 1 & 4 & 1 & 3 & 2 & 0 \\ 0 & 2 & 3 & 2 & 3 & 0 & 4 & 1 & 4 & 1 \\ 0 & 1 & 1 & 3 & 0 & 2 & 4 & 4 & 3 & 2 \\ 0 & 0 & 4 & 4 & 2 & 3 & 3 & 1 & 1 & 2 \\ 0 & 3 & 0 & 1 & 1 & 2 & 3 & 2 & 4 & 4 \\ 0 & 4 & 2 & 0 & 4 & 3 & 1 & 2 & 3 & 1 \end{bmatrix}$$

which obviously satisfies the above properties (i) to (iv).

Example 6.3.1. There exists an affine resolvable symmetric SRGD design with parameters $v = b = 50, r = k = 10, \lambda_1 = 0, \lambda_2 = 2, q_2 = 2; m = 10, n = 5$, whose GD association scheme of 50 treatments is

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 15 \\ 16 & 17 & 18 & 19 & 20 \\ 21 & 22 & 23 & 24 & 25 \\ 26 & 27 & 28 & 29 & 30 \\ 31 & 32 & 33 & 34 & 35 \\ 36 & 37 & 38 & 39 & 40 \\ 41 & 42 & 43 & 44 & 45 \\ 46 & 47 & 48 & 49 & 50 \end{bmatrix}.$$

Now, replace 0, 1, 2, 3, 4 in the above $DS(10, 5; 2)$ by the following five matrices of order 5:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix},$$

respectively. Then the 50 blocks of 10 resolution sets (i.e., a resolution set showing a bracket [ ] below) of 5 blocks each are given by

[(1, 6, 11, 16, 21, 26, 31, 36, 41, 46), (2, 7, 12, 17, 22, 27, 32, 37, 42, 47), (3, 8, 13, 18, 23, 28, 33, 38, 43, 48), (4, 9, 14, 19, 24, 29, 34, 39, 44, 49), (5, 10, 15, 20, 25, 30, 35, 40, 45, 50)],

[(1, 10, 14, 17, 23, 28, 32, 36, 44, 50), (2, 6, 15, 18, 24, 29, 33, 37, 45, 46), (3, 7, 11, 19, 25, 30, 34, 38, 41, 47), (4, 8, 12, 20, 21, 26, 35, 39, 42, 48), (5, 9, 13, 16, 22, 27, 31, 40, 43, 49)],

[(1, 9, 12, 18, 25, 29, 32, 40, 41, 48), (2, 10, 13, 19, 21, 30, 33, 36, 42, 49), (3, 6, 14, 20, 22, 26, 34, 37, 43, 50), (4, 7, 15, 16, 23, 27, 35, 38, 44, 46), (5, 8, 11, 17, 24, 28, 31, 39, 45, 47)],

[(1, 7, 13, 20, 24, 28, 34, 40, 42, 46), (2, 8, 14, 16, 25, 29, 35, 36, 43, 47), (3, 9, 15, 17, 21, 30, 31, 37, 44, 48), (4, 10, 11, 18, 22, 26, 32, 38, 45, 49), (5, 6, 12, 19, 23, 27, 33, 39, 41, 50)],

[(1, 8, 15, 19, 22, 29, 31, 38, 42, 50), (2, 9, 11, 20, 23, 30, 32, 39, 43, 46), (3, 10, 12, 16, 24, 26, 33, 40, 44, 47), (4, 6, 13, 17, 25, 27, 34, 36, 45, 48), (5, 7, 14, 18, 21, 28, 35, 37, 41, 49)],

[(1, 7, 15, 17, 25, 26, 33, 39, 43, 49), (2, 8, 11, 18, 21, 27, 34, 40, 44, 50), (3, 9, 12, 19, 22, 28, 35, 36, 45, 46), (4, 10, 13, 20, 23, 29, 31, 37, 41, 47), (5, 6, 14, 16, 24, 30, 32, 38, 42, 48)],

[(1, 6, 13, 18, 22, 30, 35, 39, 44, 47), (2, 7, 14, 19, 23, 26, 31, 40, 45, 48), (3, 8, 15, 20, 24, 27, 32, 36, 41, 49), (4, 9, 11, 16, 25, 28, 33, 37, 42, 50), (5, 10, 12, 17, 21, 29, 34, 38, 43, 46)],

[(1, 10, 11, 19, 24, 27, 35, 37, 43, 48), (2, 6, 12, 20, 25, 28, 31, 38, 44, 49), (3, 7, 13, 16, 21, 29, 32, 39, 45, 50), (4, 8, 14, 17, 22, 30, 33, 40, 41, 46), (5, 9, 15, 18, 23, 26, 34, 36, 42, 47)],

[(1, 8, 12, 16, 23, 30, 34, 37, 45, 49), (2, 9, 13, 17, 24, 26, 35, 38, 41, 50), (3, 10, 14, 18, 25, 27, 31, 39, 42, 46), (4, 6, 15, 19, 21, 28, 32, 40, 43, 47), (5, 7, 11, 20, 22, 29, 33, 36, 44, 48)],

[(1, 9, 14, 20, 21, 27, 33, 38, 45, 47), (2, 10, 15, 16, 22, 28, 34, 39, 41, 48), (3, 6, 11, 17, 23, 29, 35, 40, 42, 49), (4, 7, 12, 18, 24, 30, 31, 36, 43, 50), (5, 8, 13, 19, 25, 26, 32, 37, 44, 46)].

Six designs of Nos. 23, 29, 30, 33, 39 and 42 in Table 6.4 are also constructed by use of Theorem 6.3.3 with $DS(9,3;3), DS(12,3;4), DS(12,2^2;3)$, $DS(14,7;2)$, $DS(18,3;6)$ and $DS(20,5;4)$, respectively. Many useful information on the existence of a difference scheme can be found in Beth, Jungnickel and Lenz (1999), Hedayat, Sloane and Stufken (1999; Chapter 6) or Colbourn and Dinitz (2007).

Another characterization for Case 2 is provided. It is clear (see, for example, Hedayat, Sloane and Stufken, 1999, Theorem 7.6) that a $DS(2x, 2; x)$ exists iff a Hadamard matrix of order $2x$ exists. Here Theorem 6.3.3 with $s = 2$ can be especially expressed as an equivalence existence.

Theorem 6.3.4. The existence of a Hadamard matrix of order $2x$ is equivalent to the existence of an affine resolvable symmetric SRGD design with parameters

$$v = b = 4x, r = k = 2x, \lambda_1 = 0, \lambda_2 = x, q_1 = 0, q_2 = x; m = 2x, n = 2.$$

Proof. (Necessity) In a Hadamard matrix H of order $2x$, replace $+1$ and -1 by I_2 and $\mathbf{1}_2\mathbf{1}_2' - I_2$ respectively. Then the relation $HH' = 2xI_{2x} = H'H$ can yield that $\lambda_1 = 0$ and $\lambda_2 = x$ with the affine resolvability. Thus the required design can be obtained. Or apply Theorem 6.3.3.

(Sufficiency) Since $v = 2k$, from the properties of the GD association scheme on a $2x \times 2$ array, the resolvability and $\lambda_1 = 0$, it follows that the $4x \times 4x$ incidence matrix is partitioned into $(2x)^2$ submatrices of order 2, whose pattern is either I_2 or $\mathbf{1}_2\mathbf{1}_2' - I_2$. Now replace I_2 and $\mathbf{1}_2\mathbf{1}_2' - I_2$ by $+1$ and -1 respectively. Then we get a $2x \times 2x$ matrix H whose elements are $+1$ or -1. In the original incidence matrix of the design, each of four rows (consisting of two columns each) corresponding to the replacement (which follows the above partition of the incidence matrix) has one of four patterns as $(I_2,\ I_2)'$, $(I_2,\ 1_21_2' - I_2)'$, $(1_21_2' - I_2,\ 1_21_2' - I_2)'$, $(1_21_2' - I_2,\ I_2)'$. Hence, on account of $\lambda_2 = x$, it can be shown that $HH' = 2xI_{2x}$. ∎

In Theorem 6.3.4, when $x = 6$, by use of a Hadamard matrix H_{12} of order 12 as

$$\begin{bmatrix} 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 \end{bmatrix},$$

we can obtain an affine resolvable symmetric SRGD design of No. 28 in Table 6.4. This will be given in Example 6.3.2.

Example 6.3.2. There exists an affine resolvable symmetric SRGD design with parameters $v = b = 24, r = k = 12, \lambda_1 = 0, \lambda_2 = 6, q_2 = 6; m = 12, n = 2$, whose GD association scheme of 24 treatments is given by the usual 12×2 array. If the entries $+1$ and -1 in H_{12} are replaced by

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

respectively, then the 24 blocks of 12 resolution sets of 2 blocks each are given by

[(1,3,5,7,9,11,13,15,17,19,21,23), (2,4,6,8,10,12,14,16,18,20,22,24)],
[(2,3,5,8,9,11,13,16,18,20,21,24), (1,4,6,7,10,12,14,15,17,19,22,23)],
[(2,4,5,7,10,11,13,15,18,20,22,23), (1,3,6,8,9,12,14,16,17,19,21,24)],
[(2,3,6,7,9,12,13,15,17,20,22,24), (1,4,5,8,10,11,14,16,18,19,21,23)],
[(2,4,5,8,9,11,14,15,17,19,22,24), (1,3,6,7,10,12,13,16,18,20,21,23)],
[(2,4,6,7,10,11,13,16,17,19,21,24), (1,3,5,8,9,12,14,15,18,20,22,23)],
[(2,4,6,8,9,12,13,15,18,19,21,23), (1,3,5,7,10,11,14,16,17,20,22,24)],

[(2,3,6,8,10,11,14,15,17,20,21,23), (1,4,5,7,9,12,13,16,18,19,22,24)],
[(2,3,5,8,10,12,13,16,17,19,22,23), (1,4,6,7,9,11,14,15,18,20,21,24)],
[(2,3,5,7,10,12,14,15,18,19,21,24), (1,4,6,8,9,11,13,16,17,20,22,23)],
[(2,4,5,7,9,12,14,16,17,20,21,23), (1,3,6,8,10,11,13,15,18,19,22,24)],
[(2,3,6,7,9,11,14,16,18,19,22,23), (1,4,5,8,10,12,13,15,17,20,21,24)].

It is well known that a necessary condition for the existence of a Hadamard matrix is that the order is either 2 or a multiple of 4. Then Theorem 6.3.4 can produce the following.

Corollary 6.3.4. When x is odd (≥ 3), there does not exist an affine resolvable symmetric SRGD design with parameters $v = b = 4x, r = k = 2x, \lambda_1 = 0, \lambda_2 = x, q_1 = 0, q_2 = x; m = 2x, n = 2$.

Remark 6.3.3. The existence of a Hadamard matrix of order $2x$ is known for all $2x \leq 664$ (i.e., the smallest order in which a Hadamard matrix is undecided is 668) (Kharaghani and Tayfeh-Rezaie, 2005). Hence an affine resolvable symmetric SRGD design of Theorem 6.3.4 exists for all even $x \leq 332$. In fact, it is conjectured that a Hadamard matrix always exists for any order ($\equiv 0 \bmod 4$) (see Hall, 1986).

Remark 6.3.4. By Theorem 6.3.3, Theorem 6.3.4 and Corollary 6.3.4, the nonexistence information on designs of Nos. 14, 17, 25, 27, 32, 34, 35 and 38 in Source 1 of Table 6.4 for $y = 1$ implies the nonexistence of difference schemes (difference matrices, generalized Hadamard matrices) $DS(m, s; x)$ (or $GH(s, x)$) in $DS(6, 2; 3), DS(6, 6; 1), DS(10, 2; 5), DS(10, 10; 1)$, $DS(14, 2; 7)$, $DS(15, 3; 5)$, $DS(15, 5; 3)$ and $DS(18, 2; 9)$, respectively. Since the existence of $DS(12, 6; 2)$ and $DS(20, 2^2; 5)$ is unknown, designs of Nos. 31 and 41 may not be constructed through Theorem 6.3.3. In general, it also follows from Theorem 6.3.4 and Corollary 6.3.4 that there does not exist a difference scheme $DS(2x, 2; x)$ for any odd $x \geq 3$.

Case 3: $x = 1$, i.e., $m = \beta$. In this case we have the design parameters as

$$v = y\beta^2, b = \frac{\beta^2(y\beta - 1)}{\beta - 1}, r = \frac{\beta(y\beta - 1)}{\beta - 1}, k = y\beta,$$

$$\lambda_1 = \frac{\beta(y - 1)}{\beta - 1}, \lambda_2 = \frac{y\beta - 1}{\beta - 1}, q_2 = y, \frac{k}{m} = y.$$

When $y = 1$, this case coincides with Case 1 and then $y > 1$ is mainly considered. In fact, the existing SR38 and SR71 in Table VI of Clatworthy (1973) belong to this class for $y = 2$ and 3, respectively. In this case, all the existing designs satisfy $v = 2k$ (self-complementary). However, note that the parameters of an unknown design of No. 12 do not satisfy $v = 2k$.

As a method of construction of a design belonging to Case 3, Kageyama, Banerjee and Verma (1989) show that the existence of an affine resolvable BIB($v^* = 2k^*, b^* = 2r^*, r^* = 2k^* - 1, k^*, \lambda^* = k^* - 1$) implies the existence of an affine resolvable SRGD design with parameters $v = 4k^*, b = 4(2k^* - 1), r = 2(2k^* - 1), k = 2k^*, \lambda_1 = 2(k^* - 1), \lambda_2 = 2k^* - 1; m = 2, n = 2k^*$. Here $x = 1$ and $y = k^*$. Note that this design has only possibility of existence when k^* is even. When $k^* = 2$ we have a design of No. 2 in Table 6.4, i.e., SR38. When $k^* = 4$, a design of No. 4 in Table 6.4 is newly constructed as will be constructed in Example 6.3.3, because there exists an affine resolvable BIB(8, 14, 7, 4, 3) (cf. Kageyama, 1972).

Example 6.3.3. There exists an affine resolvable SRGD design with parameters $v = 16, b = 28, r = 14, k = 8, \lambda_1 = 6, \lambda_2 = 7, q_2 = 4; m = 2, n = 8$ whose GD association scheme of 16 treatments is

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \end{bmatrix}.$$

The 28 blocks of 14 resolution sets of 2 blocks each are given by

[(1,2,3,5,9,10,11,13), (4,6,7,8,12,14,15,16)],
[(4,6,7,8,9,10,11,13), (1,2,3,5,12,14,15,16)],
[(2,3,4,6,10,11,12,14), (1,5,7,8,9,13,15,16)],
[(1,5,7,8,10,11,12,14), (2,3,4,6,9,13,15,16)],
[(3,4,5,7,11,12,13,15), (1,2,6,8,9,10,14,16)],
[(1,2,6,8,11,12,13,15), (3,4,5,7,9,10,14,16)],
[(1,4,5,6,9,12,13,14), (2,3,7,8,10,11,15,16)],
[(2,3,7,8,9,12,13,14), (1,4,5,6,10,11,15,16)],
[(2,5,6,7,10,13,14,15), (1,3,4,8,9,11,12,16)],
[(1,3,4,8,10,13,14,15), (2,5,6,7,9,11,12,16)],
[(1,3,6,7,9,11,14,15), (2,4,5,8,10,12,13,16)],
[(2,4,5,8,9,11,14,15), (1,3,6,7,10,12,13,16)],
[(1,2,4,7,9,10,12,15), (3,5,6,8,11,13,14,16)],
[(3,5,6,8,9,10,12,15), (1,2,4,7,11,13,14,16)].

This is constructed by use of Theorem 1 and Corollary 2 of Kageyama, Banerjee and Verma (1989) with an affine resolvable solution, [(0, 1, 2, 4), (3, 5, 6, ∞)] mod 7, of a BIB(8, 14, 7, 4, 3) having the incidence matrix N, i.e., the constructed design has

$$N \otimes \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + (\mathbf{1}_8 \mathbf{1}'_{14} - N) \otimes \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

with some renumbering of 16 new treatments to suit the present GD association scheme from the original scheme

$$\begin{bmatrix} 1 & 3 & 5 & 7 & 9 & 11 & 13 & 15 \\ 2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 \end{bmatrix}.$$

Case 4: $x > 1$ and $y > 1$. In this case we have the design parameters as in (6.3.1) and (6.3.2). In general, since

$$\lambda_1 = x(y-1) + \frac{x(y-1)}{\beta - 1},$$

for given x and y there are a finite number of values of β since λ_1 is an integer. Thus all parameters of an affine resolvable SRGD design are systematically expressed in terms of parameters x, y and β.

Some special cases are taken below.

Case 4.1: $x = 2$ and $y = 2$. In this case $\lambda_1 = 2 + 2/(\beta - 1)$ which implies $\beta = 2, 3$. When $\beta = 2$, we have the parameters $v = 16, b = 24, r = 12, k = 8, \lambda_1 = 4, \lambda_2 = 6; m = n = 4$. This is a design of No. 10 in Table 6.4 and will be constructed as in Example 6.3.4. This is the only existing affine resolvable SRGD design for $x > 1$ and $y > 1$ as far as the author is aware of. When $\beta = 3$, we have the parameters $v = 36, b = 45, r = 15, k = 12, \lambda_1 = 3, \lambda_2 = 5; m = n = 6$ a design of which is shown to be nonexistent by Theorem 12.6.2 in Raghavarao (1988).

Example 6.3.4. There exists an affine resolvable SRGD design with parameters $v = 16, b = 24, r = 12, k = 8, \lambda_1 = 4, \lambda_2 = 6, q_2 = 4; m = n = 4$ whose GD association scheme of 16 treatments is

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}.$$

The 24 blocks of 12 resolution sets of 2 blocks each are given by

[(1,2,5,6,9,10,13,14), (3,4,7,8,11,12,15,16)],
[(1,2,5,6,11,12,15,16), (3,4,7,8,9,10,13,14)],
[(1,2,7,8,9,10,15,16), (3,4,5,6,11,12,13,14)],
[(1,2,7,8,11,12,13,14), (3,4,5,6,9,10,15,16)],
[(1,3,5,7,9,11,13,15), (2,4,6,8,10,12,14,16)],
[(1,3,5,7,10,12,14,16), (2,4,6,8,9,11,13,15)],
[(1,3,6,8,9,11,14,16), (2,4,5,7,10,12,13,15)],
[(1,3,6,8,10,12,13,15), (2,4,5,7,9,11,14,16)],
[(1,4,5,8,9,12,13,16), (2,3,6,7,10,11,14,15)],
[(1,4,5,8,10,11;14,15), (2,3,6,7,9,12,13,16)],
[(1,4,6,7,9,12,14,15), (2,3,5,8,10,11,13,16)],
[(1,4,6,7,10,11,13,16), (2,3,5,8,9,12,14,15)].

This is constructed by trial and error under some manner.

Case 4.2: $x = 2$ and $y = 3$. In this case $\lambda_1 = 4 + 4/(\beta - 1)$ which implies $\beta = 2, 3, 5$. When $\beta = 2$, we have the parameters $v = 24, b = 40, r = 20, k = 12, \lambda_1 = 8, \lambda_2 = 10; m = 4, n = 6$ whose affine resolvable solution as a design of No. 11 in Table 6.4 is unknown. However, a 5-resolvable solution under the usual 4×6 GD association scheme of 24 treatments can be given by trial and error as follows.

[(1,2,3,7,8,9,13,14,15,19,20,21), (1,2,4,7,8,10,13,14,16,19,20,22),
(1,3,5,7,9,11,13,15,17,19,21,23), (1,4,6,7,10,12,13,16,18,19,22,24),
(1,5,6,7,11,12,13,17,18,19,23,24), (2,3,6,8,9,12,14,15,18,20,21,24),
(2,4,5,8,10,11,14,16,17,20,22,23), (2,5,6,8,11,12,14,17,18,20,23,24),
(3,4,5,9,10,11,15,16,17,21,22,23), (3,4,6,9,10,12,15,16,18,21,22,24)];
[(1,2,3,7,8,9,16,17,18,22,23,24), (1,2,4,7,8,10,15,17,18,21,23,24),
(1,3,5,7,9,11,14,16,18,20,22,24), (1,4,6,7,10,12,14,15,17,20,21,23),
(1,5,6,7,11,12,14,15,16,20,21,22), (2,3,6,8,9,12,13,16,17,19,22,23),
(2,4,5,8,10,11,13,15,18,19,21,24), (2,5,6,8,11,12,13,15,16,19,21,22),
(3,4,5,9,10,11,13,14,18,19,20,24), (3,4,6,9,10,12,13,14,17,19,20,23)];
[(1,2,3,10,11,12,13,14,15,22,23,24), (1,2,4,9,11,12,13,14,16,21,23,24),
(1,3,5,8,10,12,13,15,17,20,22,24), (1,4,6,8,9,11,13,16,18,20,21,23),
(1,5,6,8,9,10,13,17,18,20,21,22), (2,3,6,7,10,11,14,15,18,19,22,23),
(2,4,5,7,9,12,14,16,17,19,21,24), (2,5,6,7,9,10,14,17,18,19,21,22),
(3,4,5,7,8,12,15,16,17,19,20,24), (3,4,6,7,8,11,15,16,18,19,20,23)];
[(1,2,3,10,11,12,16,17,18,19,20,21), (1,2,4,9,11,12,15,17,18,19,20,22),

(1,3,5,8,10,12,14,16,18,19,21,23), (1,4,6,8,9,11,14,15,17,19,22,24),
(1,5,6,8,9,10,14,15,16,19,23,24), (2,3,6,7,10,11,13,16,17,20,21,24),
(2,4,5,7,9,12,13,15,18,20,22,23), (2,5,6,7,9,10,13,15,16,20,23,24),
(3,4,5,7,8,12,13,14,18,21,22,23), (3,4,6,7,8,11,13,14,17,21,22,24)],

where 10 blocks in each square bracket shows a 5-resolution set. Note that this is not affine 5-resolvable.

When $\beta = 3$, we get the parameters $v = 54, b = 72, r = 24(> 20), k = 18, \lambda_1 = 6, \lambda_2 = 8; m = 6, n = 9$ whose solution as a design is unknown. When $\beta = 5$, we obtain the parameters $v = 150, b = 175, r = 35, k = 30, \lambda_1 = 5, \lambda_2 = 7; m = 10, n = 15$ a design of which is shown to be nonexistent by Theorem 12.6.2 in Raghavarao (1988).

Case 4.3: $x = 3$ and $y = 2$. In this case $\lambda_1 = 3 + 3/(\beta - 1)$ which implies $\beta = 2, 4$. When $\beta = 2$, we have the parameters $v = 24, b = 36, r = 18, k = 12, \lambda_1 = 6, \lambda_2 = 9; m = 6, n = 4$ a design of which is shown to be nonexistent by Theorem 12.6.2 in Raghavarao (1988). When $\beta = 4$, we get the parameters $v = 96, b = 112, r = 28(> 20), k = 24(> 20), \lambda_1 = 4, \lambda_2 = 7; m = 12, n = 8$ whose solution as a design is unknown.

Case 4.4: $x = 3$ and $y = 3$. In this case $\lambda_1 = 6 + 6/(\beta - 1)$ which implies $\beta = 2, 3, 4, 7$. When $\beta = 2$, we have the parameters $v = 36, b = 60, r = 30, k = 18, \lambda_1 = 12, \lambda_2 = 15; m = n = 6$ a design of which is shown to be nonexistent by Theorem 12.6.2 in Raghavarao (1988). When $\beta = 3$, we get the parameters $v = 81, b = 108, r = 36, k = 27, \lambda_1 = 9, \lambda_2 = 12; m = n = 9$ whose solution is unknown as a design. When $\beta = 4$, we obtain the parameters $v = 144, b = 176, r = 44, k = 36, \lambda_1 = 8, \lambda_2 = 11; m = n = 12$ whose solution is unknown as a design. When $\beta = 7$, we obtain the parameters $v = 441, b = 490, r = 70, k = 63, \lambda_1 = 7, \lambda_2 = 10; m = n = 21$ a design of which is shown to be nonexistent by Theorem 12.6.2 in Raghavarao (1988). All designs of Case 4.4 have r or $k > 20$ which are beyond the scope in Table 6.4.

Other cases may have r and/or $k > 20$.

The above-mentioned information will be summarized in Table 6.4 of the next section.

6.4. Table of affine resolvable SRGD designs with $v \leq 100$ and $r, k \leq 20$

According to the values of positive integers x and y as expressed in (6.3.1) and (6.3.2), we now systematically search affine resolvable SRGD designs with admissible parameters within the scope of $v \leq 100$ and $r, k \leq 20$. In fact, there are 42 parameters' combinations, among of which 26 designs are existent, 11 designs do not exist, while other 5 cases are unknown for the existence.

In Table 6.4, the admissible parameters of the affine resolvable SRGD designs are listed along with existence information. The designs are numbered in the ascending order of m and for the same m in the order of n. Since $q_1 = 0$, the parameter is not listed. "Non-E" means the nonexistence of the design. Source 1 has some information on the existence of the corresponding affine resolvable SRGD design, while Source 2 shows some information on the existence of the corresponding SRGD design when the affine resolvable solution does not exist or is unknown. The symbol ? means that the existence or nonexistence of the corresponding design is unknown. Half of the existence is confirmed in Table VI of Clatworthy (1973), for example, SR1, etc. By Theorem 12.6.2 of Raghavarao (1988), it can be seen that affine resolvable designs of Nos. 7, 14, 16, 17, 18, 32, 34 and 35 do not exist. The nonexistence of designs of Nos. 17 and 27 also follows from Remark 6.3.1 since an affine plane of order 6 or 10 does not exist (cf. Lam, Thiel and Swiercz, 1989). The nonexistence of designs of Nos. 25 and 38 follows from Corollary 6.3.4.

Table 6.4. Affine resolvable SRGD designs

No.	m	n	v	b	r	k	λ_1	λ_2	q_2	Source 1	Source 2	x	y
1	2	2	4	4	2	2	0	1	1	SR1		1	1
2	2	4	8	12	6	4	2	3	2	SR38		1	2
3	2	6	12	20	10	6	4	5	3	SR71		1	3
4	2	8	16	28	14	8	6	7	4	Exist		1	4
5	2	10	20	36	18	10	8	9	5	?	?	1	5
6	3	3	9	9	3	3	0	1	1	SR23		1	1
7	3	9	27	36	12	9	3	4	3	Non-E	?	1	3
8	4	2	8	8	4	4	0	2	2	SR36		2	1
9	4	4	16	16	4	4	0	1	1	SR44		1	1
10	4	4	16	24	12	8	4	6	4	Exist		2	2
11	4	6	24	40	20	12	8	10	6	?	Exist	2	3
12	4	16	64	80	20	16	4	5	4	?	?	1	4
13	5	5	25	25	5	5	0	1	1	SR60		1	1
14	6	2	12	12	6	6	0	3	3	Non-E	SR67	3	1
15	6	3	18	18	6	6	0	2	2	SR72		2	1
16	6	4	24	36	18	12	6	9	6	Non-E	?	3	2
17	6	6	36	36	6	6	0	1	1	Non-E	?	1	1
18	6	6	36	45	15	12	3	5	4	Non-E	?	2	2
19	7	7	49	49	7	7	0	1	1	SR87		1	1
20	8	2	16	16	8	8	0	4	4	SR92		4	1
21	8	4	32	32	8	8	0	2	2	SR95		2	1
22	8	8	64	64	8	8	0	1	1	SR97		1	1
23	9	3	27	27	9	9	0	3	3	SR102		3	1
24	9	9	81	81	9	9	0	1	1	SR105		1	1
25	10	2	20	20	10	10	0	5	5	Non-E	SR108	5	1
26	10	5	50	50	10	10	0	2	2	Exist		2	1
27	10	10	100	100	10	10	0	1	1	Non-E	?	1	1
28	12	2	24	24	12	12	0	6	6	Exist		6	1
29	12	3	36	36	12	12	0	4	4	Exist		4	1
30	12	4	48	48	12	12	0	3	3	Exist		3	1
31	12	6	72	72	12	12	0	2	2	?	?	2	1
32	14	2	28	28	14	14	0	7	7	Non-E	?	7	1
33	14	7	98	98	14	14	0	2	2	Exist		2	1
34	15	3	45	45	15	15	0	5	5	Non-E	?	5	1
35	15	5	75	75	15	15	0	3	3	Non-E	?	3	1
36	16	2	32	32	16	16	0	8	8	Exist		8	1
37	16	4	64	64	16	16	0	4	4	Exist		4	1
38	18	2	36	36	18	18	0	9	9	Non-E	?	9	1
39	18	3	54	54	18	18	0	6	6	Exist		6	1
40	20	2	40	40	20	20	0	10	10	Exist		10	1
41	20	4	80	80	20	20	0	5	5	?	?	5	1
42	20	5	100	100	20	20	0	4	4	Exist		4	1

6.5. Affine α-resolvable L_2 designs

For the description of an L_2 design with $v = s^2$ treatments, having the incidence matrix N, see Definition 2.5. Note (cf. Raghavarao, 1988) that

NN' has eigenvalues $r + (s-2)\lambda_1 - (s-1)\lambda_2$ ($= \theta_1$, say) and $r - 2\lambda_1 + \lambda_2$ ($= \theta_2$, say) other than simple rk with respective multiplicities $2(s-1)$ and $(s-1)^2$.

Now we consider an affine α-resolvable L_2 design with parameters $v = s^2, b = \beta t, r = \alpha t, k, \lambda_1, \lambda_2, q_1 = k(\alpha - 1)/(\beta - 1)$ and $q_2 = k^2/v$.

By Lemma 6.1, we have the following.

Theorem 6.5.1. If $r + (s-2)\lambda_1 - (s-1)\lambda_2 > 0$ and $r - 2\lambda_1 + \lambda_2 > 0$, then there does not exist an affine α-resolvable L_2 design for any $\alpha \geq 1$.

Therefore, by Remark 6.2, other two cases are considered to investigate L_2 designs with the affine α-resolvability.

Case 6.5.1: $\theta_1 = r + (s-2)\lambda_1 - (s-1)\lambda_2 = 0$ and $\theta_2 = r - 2\lambda_1 + \lambda_2 > 0$.

In this case, it is clear that $\lambda_2 > \lambda_1$.

At first, an integral expression of q_1 is derived like $q_1 = k + \lambda - r$ in an affine α-resolvable BIB design and as in Corollaries 6.1.1 and 6.3.1 for affine α-resolvable GD designs.

Corollary 6.5.1. In an affine α-resolvable L_2 design of Case 6.5.1, $q_1 = k - r + 2\lambda_1 - \lambda_2$ holds.

Proof. Since $\theta_1 = r + (s-2)\lambda_1 - (s-1)\lambda_2 = 0$ and $\theta_2 = r - 2\lambda_1 + \lambda_2 > 0$, Theorem 6.3 implies that $q_1 = k(\alpha - 1)/(\beta - 1) = k - r + 2\lambda_1 - \lambda_2$. ■

Furthermore, a useful result is remarked.

Lemma 6.5.1 (Kageyama and Tsuji, 1977). In an L_2 design of Case 6.5.1, k is divisible by s.

In the present case the following can be shown.

Theorem 6.5.2. The parameters of an affine α-resolvable L_2 design of Case 6.5.1 are given by

$$v = s^2, b = \frac{\beta(s-1)^2}{\beta - 1}, r = \frac{\alpha(s-1)^2}{\beta - 1}, k = \frac{\alpha s^2}{\beta},$$

$$\lambda_1 = \frac{\alpha(s-1)(\alpha s-\beta)}{\beta(\beta-1)}, \lambda_2 = \frac{\alpha(\alpha s^2+\beta-2\alpha s)}{\beta(\beta-1)}; \; t = \frac{(s-1)^2}{\beta-1},$$

where $\alpha s/\beta$ is an integer.

Proof. Since eigenvalues of NN' are $r+(s-2)\lambda_1-(s-1)\lambda_2 = 0$ and $r-2\lambda_1+\lambda_2 > 0$ with respective multiplicities $2(s-1)$ and $(s-1)^2$, by Theorem 6.3 it holds that $b-t = (s-1)^2$, i.e., $b = v+t-2s+1$ which also implies that $t = (s-1)^2/(\beta-1)$. Then it follows that $v = s^2, b = \beta t = \beta(s-1)^2/(\beta-1), r = \alpha t = \alpha(s-1)^2/(\beta-1), k = vr/b = \alpha s^2/\beta$. Furthermore, from relations $r(k-1) = n_1\lambda_1+n_2\lambda_2$ and $r+(s-2)\lambda_1-(s-1)\lambda_2 = 0$, we get $\lambda_1 = \alpha(s-1)(s\alpha-\beta)/[\beta(\beta-1)]$ and $\lambda_2 = \alpha(s^2\alpha+\beta-2s\alpha)/[\beta(\beta-1)]$. Also by Lemma 6.5.1, $k/s = \alpha s/\beta$ must be an integer. ■

Thus, all parameters of an affine α-resolvable L_2 design of Case 6.5.1 can be expressed in terms of s, α and β. It is clear that these parameters satisfy Corollary 6.5.1.

Note that

$$q_1 = \frac{s^2\alpha(\alpha-1)}{\beta(\beta-1)} \text{ and } q_2 = \left(\frac{s\alpha}{\beta}\right)^2.$$

For the next section the case of $\alpha = 1$ will be investigated in detail. For an affine resolvable L_2 design of Case 6.5.1, $t = r$ and then Theorem 6.5.2 shows the expression of design parameters as

$$v = s^2, b = \frac{\beta(s-1)^2}{\beta-1}, r = \frac{(s-1)^2}{\beta-1}, k = \frac{s^2}{\beta}, \lambda_1 = \frac{(s-1)(s-\beta)}{\beta(\beta-1)},$$

$$\lambda_2 = \frac{s^2+\beta-2s}{\beta(\beta-1)}, q_1 = 0, q_2 = \frac{s^2}{\beta^2}, \frac{k}{s} = \frac{s}{\beta}.$$

Then there exists a positive integer ℓ such that

$$s = \ell\beta$$

which implies that

$$v = (\ell\beta)^2, b = \frac{\beta(\ell\beta-1)^2}{\beta-1}, r = \frac{(\ell\beta-1)^2}{\beta-1}, k = \ell^2\beta, \tag{6.5.1}$$

$$\lambda_1 = \ell(\ell-1) + \frac{(\ell-1)^2}{\beta-1}, \lambda_2 = \frac{\ell^2\beta+1-2\ell}{\beta-1}, q_2 = \ell^2. \tag{6.5.2}$$

Thus all parameters of an affine resolvable L_2 design of Case 6.5.1 are expressed in terms of ℓ and β. In particular, the above expression of λ_1 means that for given ℓ, we have a finite number of β since λ_1 in (6.5.2) is an integer. For example, some ℓ are investigated.

(i) $\ell = 1$: $\lambda_1 = 0$ and then we have the design parameters as $v = \beta^2, b = \beta(\beta-1), r = \beta - 1, k = \beta, \lambda_1 = 0, \lambda_2 = 1$. The existing LS36 and LS61 in Table XII of Clatworthy (1973) belong to this case. Note (Raghavarao, 1988; Theorem 8.10.1) that there exists an L_2 design, whose solution may not be affine resolvable, with the above parameters for any β of a prime or a prime power. However, the following can be further obtained.

Theorem 6.5.3. The existence of an affine resolvable symmetric SRGD design with parameters

$$v = b = n^2, r = k = n, \lambda_1 = 0, \lambda_2 = 1, q_1 = 0, q_2 = 1; m = n$$

is equivalent to the existence of an affine resolvable L_2 design of Case 6.5.1 with parameters

$$v^* = n^2, b^* = n(n-1), r^* = n-1, k^* = n, \lambda_1^* = 0, \lambda_2^* = 1, q_1^* = 0, q_2^* = 1.$$

Proof. In the first resolution set of the given affine resolvable SRGD design, without loss of generality, we can put the incidence structure, by suitable permutations on rows for each of n groups of n treatments, as follows:

$$\mathbf{1}_n \otimes I_n,$$

where the GD association scheme is

$$\begin{bmatrix} 1 & 2 & \cdots & n \\ n+1 & n+2 & \cdots & 2n \\ \vdots & \vdots & \cdots & \vdots \\ (n-1)n+1 & (n-1)n+2 & \cdots & n^2 \end{bmatrix}. \qquad (6.5.3)$$

Now, by deleting the first resolution set $\mathbf{1}_n \otimes I_n$ of n blocks from the original affine resolvable SRGD design, it can be seen that the remaining structure forms an affine resolvable L_2 design of Case 6.5.1 with parameters $v^* = v = n^2, b^* = b - n = n(n-1), r^* = r - 1 = n - 1, k^* = k = n, \lambda_1^* = \lambda_1$ or $\lambda_2 - 1$,

$\lambda_2^* = \lambda_2$, whose association scheme is the same as in (6.5.3) by following Definition 2.5. The converse process is obvious. ∎

We should know the existence of the SRGD design in Theorem 6.5.3 as described in Theorem 6.3.2 and Remark 6.3.1. Four designs of Nos. 1, 6, 7 and 8 in Table 6.6.1 are provided by Theorem 6.5.3 with $n = 3, 7, 8$ and 9, respectively. When $n = 4$ and 5, the designs are available as LS36 and LS61.

(ii) $\ell = 2$: $\lambda_1 = 2 + 1/(\beta - 1)$ which yields $\beta = 2$. Hence we have the parameters as $v = 16, b = 18, r = 9, k = 8, \lambda_1 = 3, \lambda_2 = 5$ whose solution is known as LS100 in Table XII of Clatworthy (1973).

(iii) $\ell = 3$: $\lambda_1 = 6 + 4/(\beta - 1)$ which yields $\beta = 2, 3, 5$. When $\beta = 2$, we have $v = 36, b = 50, r = 25, k = 18, \lambda_1 = 10, \lambda_2 = 13$. When $\beta = 3$, we have $v = 81, b = 96, r = 32, k = 27, \lambda_1 = 8, \lambda_2 = 22$. When $\beta = 5$, we have $v = 225, b = 245, r = 49, k = 45, \lambda_1 = 7, \lambda_2 = 10$. All have r and/or $k > 20$ which are beyond the scope in Table 6.6.1.

(iv) $\ell \geq 4$: Since $r, k > 30$, the parameters are not described here.

Case 6.5.2: $\theta_1 = r + (s-2)\lambda_1 - (s-1)\lambda_2 > 0$ and $\theta_2 = r - 2\lambda_1 + \lambda_2 = 0$.

In this case, it is clear that $\lambda_1 > \lambda_2$.
At first, an integral expression of q_1 is derived like $q_1 = k+\lambda-r$ in an affine α-resolvable BIB design and as in Corollary 6.5.1 for an affine α-resolvable L_2 design of Case 6.5.1.

Corollary 6.5.2. In an affine α-resolvable L_2 design of Case 6.5.2, $q_1 = k - r - (s-2)\lambda_1 + (s-1)\lambda_2$ holds.

Proof. Since $\theta_1 = r + (s-2)\lambda_1 - (s-1)\lambda_2 > 0$ and $\theta_2 = r - 2\lambda_1 + \lambda_2 = 0$, Theorem 6.3 implies the required expression. ∎

Furthermore, the same result as in Corollary 6.5.1 is remarked in this case as follows.

Lemma 6.5.2 (Kageyama and Tsuji, 1977). In an L_2 design of Case 6.5.2, k is divisible by s.

In this case the following is also seen.

Theorem 6.5.4. The parameters of an affine α-resolvable L_2 design of Case 6.5.2 are given by

$$v = s^2, b = \frac{2\beta(s-1)}{\beta - 1}, r = \frac{2\alpha(s-1)}{\beta - 1}, k = \frac{\alpha s^2}{\beta},$$

$$\lambda_1 = \frac{\alpha(\alpha s + \beta s - 2\beta)}{\beta(\beta - 1)}, \lambda_2 = \frac{2\alpha(\alpha s - \beta)}{\beta(\beta-1)};\ t = \frac{2(s-1)}{\beta - 1},$$

where $\alpha s/\beta$ is an integer.

Proof. Since eigenvalues of NN' are $r + (s-2)\lambda_1 - (s-1)\lambda_2 > 0$ and $r - 2\lambda_1 + \lambda_2 = 0$ with respective multiplicities $2(s-1)$ and $(s-1)^2$, by Theorem 6.3 it holds that $b - t = 2(s-1)$, i.e., $t = 2(s-1)/(\beta-1)$. Then it follows that $v = s^2, b = \beta t = 2\beta(s-1)/(\beta-1), r = \alpha t = 2\alpha(s-1)/(\beta - 1), k = vr/b = \alpha s^2/\beta$. Furthermore, from relations $r(k-1) = n_1\lambda_1 + n_2\lambda_2$ and $r - 2\lambda_1 + \lambda_2 = 0$, we obtain $\lambda_1 = \alpha(s\alpha + s\beta - 2\beta)/[\beta(\beta-1)]$ and $\lambda_2 = 2\alpha(s\alpha - \beta)/[\beta(\beta-1)]$. Also by Lemma 6.5.2, $k/s = \alpha s/\beta$ must be an integer. ∎

Thus, all parameters of an affine α-resolvable L_2 design of Case 6.5.2 can be expressed in terms of s, α and β. It is clear that these parameters satisfy Corollary 6.5.2.

Note that

$$q_1 = \frac{s^2\alpha(\alpha-1)}{\beta(\beta-1)} \text{ and } q_2 = \left(\frac{s\alpha}{\beta}\right)^2.$$

For the next section the case of $\alpha = 1$ will be investigated in detail. For an affine resolvable L_2 design of Case 6.5.2, $t = r$ and then Theorem 6.5.4 shows the design parameters as

$$v = s^2, b = \frac{2\beta(s-1)}{\beta-1}, r = \frac{2(s-1)}{\beta-1}, k = \frac{s^2}{\beta}, \lambda_1 = \frac{(s-2)\beta + s}{\beta(\beta-1)},$$

$$\lambda_2 = \frac{2(s-\beta)}{\beta(\beta-1)}, q_1 = 0, q_2 = \frac{s^2}{\beta^2}, \frac{k}{s} = \frac{s}{\beta}.$$

Then there exists a positive integer ℓ such that

$$s = \ell\beta$$

which implies that

$$v = (\ell\beta)^2, b = \frac{2\beta(\ell\beta - 1)}{\beta - 1}, r = \frac{2(\ell\beta - 1)}{\beta - 1}, k = \ell^2\beta, \tag{6.5.4}$$

$$\lambda_1 = \ell + \frac{2(\ell - 1)}{\beta - 1}, \lambda_2 = \frac{2(\ell - 1)}{\beta - 1}, q_2 = \ell^2. \tag{6.5.5}$$

Thus all parameters of affine resolvable L_2 design of Case 6.5.2 are expressed in terms of ℓ and β. In particular, the above expression of λ_1 or λ_2 in (6.5.5) means that for given ℓ, we have a finite number of β. For example, some ℓ are investigated.

(i) $\ell = 1$: The design of this case always exists for any β as the following shows.

Theorem 6.5.5. There exists an affine resolvable L_2 design of Case 6.5.2 with parameters

$$v = \beta^2, b = 2\beta, r = 2, k = \beta, \lambda_1 = 1, \lambda_2 = 0, q_1 = 0, q_2 = 1.$$

Proof. It follows that the present design can be provided by the incidence matrix as

$$[I_\beta \otimes \mathbf{1}_\beta : \mathbf{1}_\beta \otimes I_\beta].$$

Here the association scheme is given by the $\beta \times \beta$ array as

$$\begin{bmatrix} 1 & 2 & \cdots & \beta \\ \beta + 1 & \beta + 2 & \cdots & 2\beta \\ \vdots & \vdots & \cdots & \vdots \\ (\beta - 1)\beta + 1 & (\beta - 1)\beta + 2 & \cdots & \beta^2 \end{bmatrix}$$

which is the same structure as in (6.5.3). ∎

When $\beta = 2$, a design of No. 10 in Table 6.6.2 is provided. The existing LS7, LS28, LS51, LS74, LS84, LS102, LS119 and LS137 in Table XII of Clatworthy (1973) belong to this case.

(ii) $\ell = 2$: $\lambda_1 = 2 + 2/(\beta - 1)$ which yields $\beta = 2, 3$. When $\beta = 2$, we have $v = 16, b = 12, r = 6, k = 8, \lambda_1 = 4, \lambda_2 = 2$ a design of which exists as LS98 in Table XII of Clatworthy (1973). When $\beta = 3$, we have $v = 36, b = 15, r = 5, k = 12, \lambda_1 = 3, \lambda_2 = 1$ a design of which does not exist by Theorem 12.6.6 of Raghavarao (1988).

(iii) $\ell = 3$: $\lambda_1 = 3 + 4/(\beta - 1)$ which yields $\beta = 2, 3, 5$. When $\beta = 2$, we have $v = 36, b = 20, r = 10, k = 18, \lambda_1 = 7, \lambda_2 = 4$ whose solution is unknown. When $\beta = 3$, we have $v = 81, b = 24, r = 8, k = 27, \lambda_1 = 5, \lambda_2 = 2$. When $\beta = 5$, we have $v = 225, b = 35, r = 7, k = 45, \lambda_1 = 4, \lambda_2 = 1$. The last two designs have $k > 20$ which are beyond the scope of Table 6.6.2.

(iv) $\ell \geq 4$: Since r and/or $k > 30$, the parameters are not described here.

6.6. Tables of affine resolvable L_2 designs with $v \leq 100$ and $r, k \leq 20$

According to the values of positive integers ℓ in (6.5.1), (6.5.2), (6.5.4) and (6.5.5), we now systematically search affine resolvable L_2 designs, of two cases, with admissible parameters within the scope of $v \leq 100$ and $r, k \leq 20$. In fact, there are 21 parameters' combinations, among of which 17 designs are existent, 3 designs do not exist, while only one case is unknown for the existence.

In Tables 6.6.1 and 6.6.2, the admissible parameters of the affine resolvable L_2 designs are listed along with existence information. The designs are numbered in the ascending order of v and for the same v in the order of b. Since $q_1 = 0$, the parameter is not listed. "Non-E" means the nonexistence of the design. Most of the existence is confirmed in Table VII of Clatworthy (1973), for example, LS36, etc. Source has some information on the existence of the corresponding affine resolvable L_2 design (cf. Clatworthy, 1973). Comment shows theorems on the construction. The existence of designs of Nos. 1, 6, 7 and 8 is newly shown by Theorem 6.5.3. It is also shown by Theorems 12.6.5 and 12.6.6 of Raghavarao (1988) that two designs of Nos. 5 and 16 do not exist. The nonexistence of a design of No. 9 is shown by Theorem 6.5.3 with Remark 6.3.1. A design of No. 10 is newly listed by Theorem 6.5.5.

Table 6.6.1. Affine resolvable L_2 designs with $r + (s-2)\lambda_1 - (s-1)\lambda_2 = 0$

No.	v	b	r	k	λ_1	λ_2	q_2	Source	Comment
1	9	6	2	3	0	1	1	Exist	Theorem 6.5.3
2	16	12	3	4	0	1	1	LS36	Theorem 6.5.3
3	16	18	9	8	3	5	4	LS100	
4	25	20	4	5	0	1	1	LS61	Theorem 6.5.3
5	36	30	5	6	0	1	1	Non-E	
6	49	42	6	7	0	1	1	Exist	Theorem 6.5.3
7	64	56	7	8	0	1	1	Exist	Theorem 6.5.3
8	81	72	8	9	0	1	1	Exist	Theorem 6.5.3
9	100	90	9	10	0	1	1	Non-E	

Table 6.6.2. Affine resolvable L_2 designs with $r - 2\lambda_1 + \lambda_2 = 0$

No.	v	b	r	k	λ_1	λ_2	q_2	Source	Comment
10	4	4	2	2	1	0	1	Exist	Theorem 6.5.5
11	9	6	2	3	1	0	1	LS7	Theorem 6.5.5
12	16	8	2	4	1	0	1	LS28	Theorem 6.5.5
13	16	12	6	8	4	2	4	LS98	
14	25	10	2	5	1	0	1	LS51	Theorem 6.5.5
15	36	12	2	6	1	0	1	LS74	Theorem 6.5.5
16	36	15	5	12	3	1	4	Non-E	
17	36	20	10	18	7	4	9	?	
18	49	14	2	7	1	0	1	LS84	Theorem 6.5.5
19	64	16	2	8	1	0	1	LS102	Theorem 6.5.5
20	81	18	2	9	1	0	1	LS119	Theorem 6.5.5
21	100	20	2	10	1	0	1	LS137	Theorem 6.5.5

7. Bounds in affine resolvable PBIB designs

A simple comparison between the number of treatments v and the number of blocks b will be made. As mentioned in Section 2, Fisher's inequality $b \geq v$ holds for a BIB design, but it is not always valid in a PBIB design.

The following results are well known (cf. Raghavarao, 1988): (i) In a regular GD design $b \geq v$ holds. (ii) In an SGD design with $v = mn$, $b \geq m$ holds. (iii) In an SRGD design with $v = mn$, $b \geq v - (m - 1)$ holds. (iv) In an L_2 design with $v = s^2$, $\theta_1 = r + (s-2)\lambda_1 - (s-1)\lambda_2$ and $\theta_2 = r - 2\lambda_1 + \lambda_2$, (iv-1) when $\theta_1 > 0$ and $\theta_2 > 0$, $b \geq v$ holds, (iv-2) when $\theta_1 > 0$ and $\theta_2 = 0$, $b \geq v-(s-1)^2$ holds, and (iv-3) when $\theta_1 = 0$ and $\theta_2 > 0$, $b \geq v-2(s-1)$ holds. Thus, for the incidence matrix N of a block design, if one of eigenvalues of NN' is zero, then an inequality $b \geq v$ may not hold in general. Through the property of affine resolvability, this inequality will be examined as in Theorem 7.1.

By Theorem 6.3, we can see some relations on v and b through other parameters in an affine α-resolvable 2-associate PBIB design. Even so, a property of the affine resolvability shows the following as a simple comparison between v and b only.

Theorem 7.1. In affine resolvable PBIB designs, it holds that

(1) for an SGD design, $b < v$;

(2) for an SRGD design, $b \geq v$;

(3) for an L_2 design with $\theta_1 > 0$ and $\theta_2 = 0$, $b < v$.

Proof. (1) It follows that $b = r + m - 1 = (m-1)/(\beta - 1) + m - 1 = [1 + 1/(\beta-1)](m-1) < 2(m-1) \leq n(m-1) < nm = v$. (2) Since $\lambda_1 = r - k \geq 0$, $r \geq k$ and hence $b = \beta r \geq \beta k = v$. (3) Since $b = v - 1 - (s-1)^2 + r$, $v - b = s^2 - 2s + 2 - 2(s-1)/(\beta - 1) = s^2 - 2(s-1)[1 + 1/(\beta - 1)] > s^2 - 4(s-1) = (s-2)^2 \geq 0$. ∎

Note that (2) in Theorem 7.1 is interesting in the sense that one of eigenvalues is zero and further the Fisher inequality holds. Also note that in an affine resolvable L_2 design with $\theta_1 = 0$ and $\theta_2 > 0$, two cases $b < v$ or $b > v$ hold. Both such examples exist. For example, see the existing LS51, LS61 and LS100 in Table XII of Clatworthy (1973).

The argument made in this section is motivated by the discussion given in Sections 3, 4 and 5.

Conclusions

We show the usefulness of number-theoretic approach to investigate combinatorial structure of affine α-resolvable BIB or PBIB designs. In particular, much fruitful contribution is made for 2-associate PBIB designs. Usually, this kind of approach may not yield much results in design theory. However, as far as a property of the affine α-resolvability is concerned, the approach is powerful. Of course, this does not solve the problem completely. We may require other combinatorial consideration. For example, some new direct constructions of affine α-resolvable block designs should be devised. Finding a recursive method of construction may not be easy for the affine α-resolvable block designs, especially for PBIB designs.

If we restrict ourselves to $\alpha = 1$, i.e., affine resolvability, then we could get more concise results on existence. Within the practical range of parameters in 2-associate PBIB designs, it reveals that there are not many such designs as in tables given in Sections 6.2, 6.4 and 6.6. In fact, we cannot find many new series of such PBIB designs other than ones in Theorems 6.3.2, 6.3.3, 6.3.4 and 6.5.5, except for designs constructed by use of the result that the complement of an affine resolvable block design is an affine α-resolvable block design for some α. Theorems 6.3.4 and 6.5.3 have some potential to produce many affine resolvable designs.

As a practical investigation (i.e., $v \leq 100$ and $r, k \leq 20$) of affine resolvable SGD, SRGD, L_2, triangular and cyclic designs, only six designs are left unknown (i.e., five SRGD designs, one L_2 design).

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References

[1] R. A. Bailey, H. Monod and J. P. Morgan, Construction and optimality of affine-resolvable designs, *Biometrika* **82** (1995), 187-200.

[2] S. Banerjee and S. Kageyama, Existence of α-resolvable nested incomplete block designs, *Utilitas Math.* **38** (1990), 237-243.

[3] T. Beth, D. Jungnickel and H. Lenz, *Design Theory* Volume 1, 2nd Edition, Cambridge Univ. Press, UK, 1999.

[4] Bhagwandas and S. Kageyama, Patterned constructions of partially balanced incomplete block designs, *Commun. Statist. -Theor. Meth.* **14** (1985), 1259-1267.

[5] R. C. Bose, A note on the resolvability of balanced incomplete block designs, *Sankhyā A* **6** (1942), 105-110.

[6] R. C. Bose, Symmetric group divisible designs with the dual property, *J. Statist. Plann. Inference* **1** (1977), 87-101.

[7] R. C. Bose and W. S. Connor, Combinatorial properties of group divisible incomplete block designs, *Ann. Math. Stat.* **23** (1952), 367-383.

[8] R. C. Bose and T. Shimamoto, Classification and analysis of partially balanced incomplete block designs with two associate classes, *J. Am. Stat. Assn.* **47** (1952), 151-184.

[9] R. C. Bose, S. S. Shrikhande and K. N. Bhattacharya, On the construction of group divisible incomplete block designs, *Ann. Math. Stat.* **24** (1953), 167-195.

[10] T. Caliński and S. Kageyama, *Block Designs: A Randomization Approach, Vol. I: Analysis*, Lecture Notes in Statistics 150, Springer, New York, 2000.

[11] T. Caliński and S. Kageyama, *Block Designs: A Randomization Approach, Vol. II: Design*, Lecture Notes in Statistics 170, Springer, New York, 2003.

[12] T. Caliński and S. Kageyama, On the analysis of experiments in affine resolvable designs, *J. Statist. Plann. Inference* **138** (2008), 3350-3356.

[13] W. H. Clatworthy, *Tables of Two-Associate-Class Partially Balanced Designs*, NBS Applied Mathematics Series 63, U.S. Department of Commerce, National Bureau of Standards, Washington, D.C., 1973.

[14] C. J. Colbourn and J. H. Dinitz (editors), *The CRC Handbook of Combinatorial Designs*, CRC Press, New York, 1996.

[15] C. J. Colbourn and J. H. Dinitz (editors), *Handbook of Combinatorial Designs*, Second Edition, Chapman & Hall/CRC, New York, 2007.

[16] O. Eckenstein, Bibliography of Kirkman's school girl problem, *Messenger of Math.* **41-42** (1912), 33-36.

[17] S. Furino and R. C. Mullin, Block designs and large holes and α-resolvable BIBDs, *J. Combin. Designs* **1** (1993), 101-112.

[18] S. Furino, Y. Miao and J. Yin, *Frames and Resolvable Designs*, CRC Press, Boca Raton FL, 1996.

[19] G. Ge, Resolvable block designs with block size four, *Discrete Math.* **243** (2002), 109-119.

[20] G. Ge and A. C. H. Ling, A survey on resolvable group divisible designs with block size four, *Discrete Math.* **279** (2004), 225-245.

[21] D. K. Ghosh, G. C. Bhimani and S. Kageyama, Resolvable semi-regular group divisible designs, *J. Japan Statist. Soc.* **19** (1989), 163-165.

[22] M. Hall, Jr., *Combinatorial Theory*, 2nd ed., John Wiley, 1986.

[23] H. Hanani, On resolvable balanced incomplete block designs, *J. Combin. Theory(A)* **17** (1974), 275-289.

[24] A. S. Hedayat and S. Kageyama, The family of t-designs – Part I. *J. Statist. Plann. Inference* **4** (1980), 173-212.

[25] A. S. Hedayat, N. J. A. Sloane and J. Stufken, *Orthogonal Arrays: Theory and Applications*, Springer, New York.

[26] D. Jungnickel, R. C. Mullin and S. A. Vanstone, The spectrum of α-resolvable block designs with block size 3, *Discrete Math.* **97** (1991), 269-277.

[27] S. Kadowaki, Combinatorial properties of α-resolvable BIB designs (in Japanese), *Master thesis*, Graduate School of School Education, Hiroshima University, 2001.

[28] S. Kadowaki and S. Kageyama, An inequality on α-resolvable balanced incomplete block designs, *Bulletin of the ICA.* **37** (2003), 51-57.

[29] S. Kadowaki and S. Kageyama, A 2-resolvable BIBD(10, 15, 6, 4, 2) does not exist, *Bulletin of the ICA.* **53** (2008a), 87-98.

[30] S. Kadowaki and S. Kageyama, On bounds of the number of blocks in α-resolvable BIB designs, *J. Stat. & Appl.* **3** (2008b), 23-31.

[31] S. Kadowaki and S. Kageyama, Existence of α-resolvable PBIB designs with some constructions, *Hiroshima Math. J.* (2009), to appear.

[32] S. Kageyama, An improved inequality for balanced incomplete block designs, *Ann. Math. Statist.* **42** (1971), 1448-1449.

[33] S. Kageyama, A survey of resolvable solutions of balanced incomplete block designs, *Int. Stat. Rev.* **40** (1972), 269-273.

[34] S. Kageyama, On μ-resolvable and affine μ-resolvable balanced incomplete block designs, *Ann. Statist.* **1** (1973a), 195-203.

[35] S. Kageyama, On the inequality for BIBDs with special parameters, *Ann. Statist.* **1** (1973b), 204-207.

[36] S. Kageyama, Resolvability of block designs, *Ann. Statist.* **4** (1976), 655-661.

[37] S. Kageyama, Conditions for α-resolvability and affine α-resolvability of incomplete block designs, *J. Japan Statist. Soc.* **7** (1977), 19-25.

[38] S. Kageyama, On non-existence of affine α-resolvable triangular designs, *J. Statist. Theor. Practice* **1** (2007), 291-298.

[39] S. Kageyama, Non-validity of affine α-resolvability in regular group divisible designs, *J. Statist. Plann. Inference* **138** (2008a), 3295-3298.

[40] S. Kageyama, Non-existence of affine α-resolvable triangular designs under $1 \leq \alpha \leq 10$, *J. Indian Soc. Agri. Statist.* **62** (2008b), 132-137.

[41] S. Kageyama, S. Banerjee and A. Verma, A construction of self complementary semi-regular group divisible designs, *Sankhyā B* **51** (1989), 335-338.

[42] S. Kageyama and Y. Miao, A construction for resolvable designs and its generalizations, *Graph and Combin.* **14** (1998), 11-24.

[43] S. Kageyama and R. N. Mohan, On μ-resolvable BIB designs, *Discrete Math.* **45** (1983), 113-122.

[44] S. Kageyama and R. N. Mohan, Construction of α-resolvable PBIB designs, *Calcutta Statist. Assoc. Bull.* **34** (1985), 221-224.

[45] S. Kageyama and T. Tanaka, Some families of group divisible designs, *J. Statist. Plann. Inference* **5** (1981), 231-241.

[46] S. Kageyama and T. Tsuji, Characterization of certain incomplete block designs, *J. Statist. Plann. Inference* **1** (1977), 151-161.

[47] H. Kharaghani and B. Tayfeh-Rezaie, A Hadamard matrix of order 428, *J. Combin. Designs* **13** (2005), 435-440.

[48] T. P. Kirkman, Query, *Ladies and Gentleman's Diary,* (1950a), 48.

[49] T. P. Kirkman, Note on an unanswered prize question, *Cambridge Dublin Math. J.* **5** (1950b), 191-204.

[50] C. W. H. Lam, L. Thiel and S. Swiercz, The non-existence of finite projective planes of order 10, *Can. J. Math.* **XLI** (1989), 1117-1123.

[51] S. Lang, *Introduction to Linear Algebra*, 2nd Edition, Springer, New York, 1986.

[52] S. L. Ma, Partial difference sets, *Discrete Math.* **52** (1984), 75-89.

[53] R. Mathon and A. Rosa, 2-(v, k, λ) Designs of Small Order, *The CRC Handbook of Combinatorial Designs* (ed. C. J. Colbourn and J. H. Dinitz), CRC Press, New York, 1996, 3-41.

[54] R. N. Mohan, A note on the construction of certain BIB designs, *Discrete Math.* **29** (1980), 209-211.

[55] R. N. Mohan and S. Kageyama, Two constructions of α-resolvable PBIB designs, *Utilitas Math.* **36** (1989), 115-119.

[56] Y. Qian, Z. Meng and B. Du, α-Resolvable group divisible designs with block size four and group size three, Utilitas Math. **77** (2008), 201-224.

[57] D. Raghavarao, *Constructions and Combinatorial Problems in Design of Experiments*, Dover, New York, 1988.

[58] D. K. Ray-Chaudhuri and R. M. Wilson, Solution of Kirkman's school girl problem, *Proc. Symp. in Pure Mathematics, Combinatorics. Amer. Math. Soc.* **19** (1971), 187-203.

[59] R. S. Rees, Group-divisble Designs with Block Size k having $k+1$ Groups, for $k = 4, 5$, *J Combin Designs* **8** (2000), 363-386.

[60] S. M. Shah and D. G. Kabe, On affine α-resolvability of a singular group divisible design, *Biom. J.* **23** (1981), 109-112.

[61] S. S. Shrikhande, Affine resolvable balanced incomplete block designs: a survey, *Aequationae Math.* **14** (1976), 251-269.

[62] S. S. Shrikhande and D. Raghavarao, A method of construction of incomplete block designs, *Sankhyā A* **25** (1963), 399-402.

[63] S. S. Shrikhande and D. Raghavarao, Affine α-resolvable incomplete block designs, *Contributions to Statistics, Volume presented to Professor P. C. Mahalanobis on his 75th birthday*, Pergamon Press, Oxford and Statistical Publishing Society, Calcutta, 1964, 471-480.

[64] K. Takeuchi, A table of difference sets generating balanced incomplete block designs, *Rev. Inst. Internat. Statist.* **30:3** (1962), 361-366.

[65] T. M. J. Vasiga, S. Furino and A. C. H. Ling, The spectrum of α-resolvable designs with block size four, *J. Combin. Designs* **9** (2001), 1-16.

[66] F. Yates, The recovery of inter-block information in variety trials arranged in three-dimensional lattices, *Ann. Eugen.* **9** (1939), 136-156.

[67] F. Yates, The recovery of inter-block information in balanced incomplete block designs, *Ann. Eugen.* **10** (1940), 317-325.

[68] Y. Zhang and B. Du, α-resolvable group divisible designs with block size three, *J. Combin. Designs* **13** (2005), 139-151.

公表論文

(1) An Inequality on α-Resolvable Balanced Incomplete Block Designs, S. Kadowaki and S. Kageyama, *Bulletin of the ICA*, **37**(2003), 51-57.

(2) A 2-resolvable BIBD(10, 15, 6, 4, 2) does not exist, S. Kadowaki and S. Kageyama, *Bulletin of the ICA*, **53**(2008), 87-98.

(3) On Bounds of the Number of Blocks in α-resolvable BIB Designs, S. Kadowaki and S. Kageyama, *J. Stat. & Appl*, **3**(2008), 23-31.

(4) Existence of α-resolvable PBIB designs with some constructions, S. Kadowaki and S. Kageyama, *Hiroshima Math. J.* (2009), to appear.