

Stochastic Pursuit - Evasion Games by the Reachable Region Approach

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Abstract

The "Geometrical Approach to Problems of Pursuit-Evasion Games" is extended to the case that pursuer and evader are disturbed by additive white noise. It is shown how to obtain the optimal controls and a numerical method to calculate the capture probability is proposed. A simple numerical example is given to illustrate the above mentioned theory. Some remarks about the validity of the "Geometrical Approach to Problems of Pursuit-Evasion Games" are added.

1. Introduction

The study of differential games was initiated by Isaacs [1] in 1954, who used game theoretic concepts originated by von Neumann and Morgenstern [2]. His approach closely resembled the dynamic programming approach to optimization problems. Since then many papers have been published, mainly on the subject of pursuit-evasion games.

A pursuit-evasion game is a noncooperative (in general two-player) game. One player, the Pursuer, tries to capture the Evader, while the Evader tries to avoid capture. Capture means that the distance between the Pursuer's and the Evader's state becomes less than a certain prescribed positive quantity ϵ . If capture occurs before a given time T elapses, the Pursuer wins the game, otherwise the evader wins.

In most of the papers solutions were achieved, using the calculus of variations techniques or a direct application of functional analyses. In this paper we will use the topological properties of the reachable region as shown in [3] to derive solutions.

The reachable region of a system is this part of the statespace, which can be reached by the system within a given time T , using constrained controls.

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2. The Reachable Region Approach

Before formulating the actual game, we will introduce the concept of the reachable region and its application to the solution of pursuit-evasion games.

For more detailed information and proofs see [3] and [4].

Let us consider the two-player pursuit-evasion game, described by the linear differential equations :

$$\dot{x}_p(t) = Ax_p(t) + Bu(t) \quad (2.1)$$

$$\dot{x}_e(t) = Cx_e(t) + Dv(t) \quad (2.2)$$

where $x_p, x_e \in R^n$ are the state vectors and $u \in R^m, v \in R^r$ are the control vectors of Pursuer P and Evader E. A, B, C and D are $n \times n, n \times m, n \times n$ and $n \times r$ given matrices respectively.

It is assumed that u and v are restricted to the following sets of admissible strategies:

$$u(t) \in U, \quad v(t) \in V, \quad t_0 \leq t \leq T$$

with :

$$U = \{u(t), \|u\| \triangleq (\int_{t_0}^T u'(t)u(t)dt)^{1/2} \leq E_p\} \quad (2.3)$$

$$V = \{v(t), \|v\| \triangleq (\int_{t_0}^T v'(t)v(t)dt)^{1/2} \leq E_e\} \quad (2.4)$$

In other words : both players are restricted in their total energy by positive constants E_p and E_e .

The game begins at initial time t_0 . P wins, if he can satisfy the distance $|x_p(t) - x_e(t)| \leq \varepsilon$ within a given time $T, t_0 \leq T \leq \infty$; otherwise E wins.

The solutions of (2.1) and (2.2) are given by :

$$x_p(t) = \phi_p(t, t_0)x_p(t_0) + \int_{t_0}^t \phi_p(t, \tau)Bu(\tau)d\tau \quad (2.5)$$

$$x_e(t) = \phi_e(t, t_0)x_e(t_0) + \int_{t_0}^t \phi_e(t, \tau)Dv(\tau)d\tau \quad (2.6)$$

where $\phi_p(t, t_0)$ and $\phi_e(t, t_0)$ are the transition matrices of (2.1) and (2.2) and $x_p(t_0)$ and $x_e(t_0)$ are the known initial states.

For convenience let us define :

$$\alpha_p(t, t_0) = x_p(t) - \phi_p(t, t_0)x_p(t_0) \quad (2.7)$$

$$\alpha_e(t, t_0) = x_e(t) - \phi_e(t, t_0) x_e(t_0) \quad (2.8)$$

$$H_p(t, \tau) = \phi_p(t, \tau) B \quad (2.9)$$

$$H_e(t, \tau) = \phi_e(t, \tau) D \quad (2.10)$$

$$G_p(t, t_0) = \int_{t_0}^t H_p(t, \tau) H_p'(t, \tau) d\tau \quad (2.11)$$

$$G_e(t, t_0) = \int_{t_0}^t H_e(t, \tau) H_e'(t, \tau) d\tau \quad (2.12)$$

2.1 The Time-optimal Controls

Having established these preliminaries we are now faced with a classical time-optimal control problem [5]. Find the control variables, constrained in some manner, which bring the state of the controlled plant from some initial value to a desired final one in shortest time.

The optimal controls, which fulfill this requirements are, as shown in [3] and [4]:

$$u^0(t) = H_p'(T, t) G_p^{-1}(T, t_0) [x_p(T) - \phi_p(T, t_0) x_p(t_0)] \\ t_0 \leq t \leq T \quad (2.13)$$

$$v^0(t) = H_e'(T, t) G_e^{-1}(T, t_0) [x_e(T) - \phi_e(T, t_0) x_e(t_0)] \\ t_0 \leq t \leq T \quad (2.14)$$

2.2 The Reachable Region

The reachable region $R(T, t_0)$ of a player is the set of all points $\alpha(T, t_0)$ in R^n which can be reached at time T using all the strategies in the admissible set.

By inserting (2.13) into (2.3) and (2.14) into (2.4) we will get the explicit expressions of $R_p(T, t_0)$, the reachable region of the Pursuer and $R_e(T, t_0)$, the reachable region of the Evader.

$$R_p(T, t_0) = \{ x_p(T) \in R^n; \alpha_p'(T, t_0) G_p^{-1}(T, t_0) \alpha_p(T, t_0) \leq E_p^2, \\ \alpha_p = x_p(T) - \phi_p(T, t_0) x_p(t_0) \} \quad (2.15)$$

$$R_e(T, t_0) = \{ x_e(T) \in R^n; \alpha_e'(T, t_0) G_e^{-1}(T, t_0) \alpha_e(T, t_0) \leq E_e^2, \\ \alpha_e = x_e(T) - \phi_e(T, t_0) x_e(t_0) \} \quad (2.16)$$

The boundary $\delta R_p(T, t_0)$ of $R_p(T, t_0)$ is defined by (2.15) by replacing the \leq sign with the equality sign. Similarly $\delta R_e(T, t_0)$, the boundary of $R_e(T, t_0)$ can be expressed by (2.16).

In other words: the boundary of the reachable region can be reached in time T using the full energy permitted by the constrained. All points inside the boundary can be reached

either in shorter time t , $t_0 \leq t \leq T$, or by using lesser energy, or by a combination of both.

If the allowed terminal miss ϵ is not equal to zero we can take this into account by expressing an expanded reachable region of the pursuer by replacing $x_p(T)$ in (2.15) with $x_p(T) + \epsilon n_p$. Where n_p is the unit vector outward, normal to a tangent plane at any point of $\delta R_p(T, t_0)$

According to [3] game termination is only possible when the reachable region of the Pursuer includes the reachable region of the Evader and furthermore the optimal termination time T^* for both players is achieved when δR_p and δR_e have one common point x_f . This point x_f is then the optimal game termination point.

3. Pursuit-Evasion Game with White Noise Disturbance

Let us now focus on the two-player pursuit-evasion game with additive white noise, described by the following stochastic differential equations :

$$dx_p = Ax_p dt + B u dt + dw_p \tag{3.1}$$

$$dx_e = Cx_e dt + D v dt + dw_e \tag{3.2}$$

where x_p and x_e are the n-dimensional state vectors of pursuer P and evader E respectively and u and v are their control vectors. The matrices A, B, C , and D have appropriate dimensions. $\{w_p(t), -\infty \leq t \leq \infty\}$ and $\{w_e(t), -\infty \leq t \leq \infty\}$ are n-dimensional Wiener-processes with incremental covariances $R_p dt$ and $R_e dt$ respectively. It is assumed that the processes w_p and w_e are independent. They are also independent of x_p and x_e .

The initial states $x_p(t_0)$ and $x_e(t_0)$ are normal with mean m_{0p} and m_{0e} and covariance matrices R_{0p} and R_{0e} .

The controls are restricted to the following sets of strategies :

$$u(t) \in U, \quad v(t) \in V, \quad t_0 \leq t \leq T$$

where

$$U = \{u(t), \|u\| \triangleq \left(\int_{t_0}^T u'(t) u(t) dt \right)^{1/2} \leq E_p\} \tag{3.3}$$

$$V = \{v(t), \|v\| \triangleq \left(\int_{t_0}^T v'(t) v(t) dt \right)^{1/2} \leq E_e\} \tag{3.4}$$

E_p and E_e are positive constants.

According to [6] the stochastic processes $x_p(t)$ and $x_e(t)$ are normal processes since the values of x_p and x_e at particular times are linear combinations of normal variables. The stochastic processes $x_p(t)$ and $x_e(t)$ can thus be completely characterized by their mean value functions and covariance functions.

The mean value functions for P and E are :

$$\frac{dmx_p}{dt} = Amx_p + Bu, \quad mx_p(t_0) = m_{0p} \quad (3.5)$$

$$\frac{dmx_e}{dt} = Cmx_e + Du, \quad mx_e(t_0) = m_{0e} \quad (3.6)$$

where $mx_p(t) = E(x_p(t))$ and $mx_e(t) = E(x_e(t))$.

$P_p(t) = \text{cov}[x_p(t), x_p(t)]$, the covariance of $x_p(t)$ and $P_e(t) = \text{cov}[x_e(t), x_e(t)]$, the covariance of $x_e(t)$ are determined as follows :

$$\frac{dP_p}{dt} = AP_p + P_pA' + R_p, \quad P_p(t_0) = R_{0p} \quad (3.7)$$

$$\frac{dP_e}{dt} = CP_e + P_eC' + R_e, \quad P_e(t_0) = R_{0e} \quad (3.8)$$

Using the reachable region approach we can now calculate the optimal game termination point and the optimal game termination time for the mean value functions of pursuer $mx_p(t)$ and of the evader $mx_e(t)$, and with these results find the optimal open loop controls for the mean value functions. We will not use the capture distance ϵ to determine a slightly expanded reachable region for the pursuer, but use ϵ as a parameter to determine the probability of capture.

3.1 Capture Probability

Having obtained the optimal open loop controls, using the reachable region approach for the mean value functions, it is now left to examine how good these controls suit the actual noise disturbed game. We can do this by calculating the probability that capture will occur, if we apply these controls. We have to do this calculation for each state separately because the noise characteristics as well as the capture distances $\epsilon = (\epsilon_1, \epsilon_2, \dots, \epsilon_n)$ may be different.

As the integration of the density function of a normal distributed random variable has no closed analytical solution, we have to do this calculation numerically. In the following $i = (1, 2, \dots, n)$ indicates the state.

We will consider the situation at time $t_p, t_0 \leq t_p \leq T$. For this time we will calculate the mean values of the i^{th} state of pursuer $mx_{pi}(t_p)$ and evader $mx_{ei}(t_p)$ and their covariances $P_{pi}(t_p)$ and $P_{ei}(t_p)$. With this information we can give the probability of every value $x_{pi}(t_p)$ and $x_{ei}(t_p)$. We will however consider only values of $x_{pi}(t_p)$ and $x_{ei}(t_p)$ out of the following intervals :

$$mx_{pi}(t_p) + 4s_{pi}(t_p) \leq x_{pi}(t_p) \leq mx_{pi}(t_p) - 4s_{pi}(t_p) \quad (3.9)$$

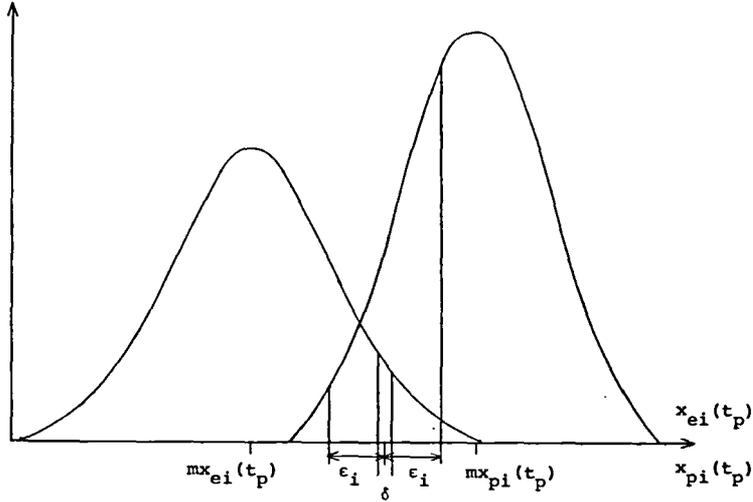


Fig. 1. Density functions of $x_{ei}(t_p)$ and $x_{pi}(t_p)$.

$$mx_{ei}(t_p) + 4s_{ei}(t_p) \leq x_{ei}(t_p) \leq mx_{ei}(t_p) - 4s_{ei}(t_p) \quad (3.10)$$

where $s_{pi}(t_p)$ and $s_{ei}(t_p)$ are the standard deviations of the i^{th} state of pursuer and evader at time t_p .

The interval :

$$[mx_{ei}(t_p) + 4s_{ei}(t_p), mx_{ei}(t_p) - 4s_{ei}(t_p)]$$

has now to be divided into m subintervals of length δ . We have to choose m that $\delta \ll \epsilon_i$. The capture probability $PC_i(t_p)$ of the i^{th} state at time t_p can now be calculated :

$$\begin{aligned} PC_i &= \sum_{k=0}^{m-1} P [x_{ei} : mx_{ei} + 4s_{ei} - k\delta \leq x_{ei} \\ &\geq mx_{ei} + 4s_{ei} - (k+1)\delta] \\ &\cdot P[x_{pi} : mx_{ei} + 4s_{ei} - k\delta - \delta/2 + \epsilon_i \leq x_{pi} \\ &\geq mx_{ei} + 4s_{ei} - k\delta - \delta/2 - \epsilon_i] \end{aligned} \quad (3.11)$$

The dependence upon t_p of the variables in (3.11) was omitted for simplicity.

If we calculate PC_i for each state variable at sufficient enough points of time $t_p, t_0 \leq t_p \leq T, (p = 1, 2, \dots)$ we will get numerically the functions $PC_i(t), t_0 \leq t \leq T$. The total capture probability PCT : the joint probability of all states, is expressed by :

$$PCT(t_p) = \prod_{i=1}^n PC_i(t_p) \quad (3.12)$$

With these results, we can now judge if the open loop controls will give sufficient results.

4. Stochastic Difference Equation

If we want to simulate the beforehand mentioned "Pursuit-Evasion Game with White Noise Disturbance" with a digital computer, we have to change the stochastic differential equations into stochastic difference equations.

Let us consider the following stochastic differential equation :

$$dx = Axdt + Budt + dv \quad (4.1)$$

where x is an n -dimensional state vector, u the control vector and $\{v(t), -\infty \leq t \leq \infty\}$ is an n -dimensional Wienerprocess with incremental covariance Rdt .

Multiplying (4.1) with e^{-At} , we get :

$$\frac{d}{dt} (e^{-At} x) dt = e^{-At} Budt + e^{-At} dv \quad (4.2)$$

Integration of (4.2) gives :

$$\begin{aligned} x(t_{i+1}) &= \phi(t_{i+1} - t_i) x(t_i) + \int_{t_i}^{t_{i+1}} \phi(t_{i+1} - t) Bu(t) dt \\ &+ \int_{t_i}^{t_{i+1}} \phi(t_{i+1} - t) dv(t) \end{aligned} \quad (4.3)$$

where the matrix ϕ is defined by :

$$\begin{aligned} \frac{d\phi(t - t_i)}{dt} &= A(t) \phi(t - t_i) \quad , \quad t_i \leq t \leq t_{i+1} \\ \phi(0) &= I \end{aligned} \quad (4.4)$$

Of particular interest in (4.3) is the term :

$$\bar{v}(t_i) = \int_{t_i}^{t_{i+1}} \phi(t_{i+1} - t) dv(t) \quad (4.5)$$

We will find that :

$$E\bar{v}(t_i) = E \int_{t_i}^{t_{i+1}} \phi(t_{i+1} - t) dv(t) = 0 \quad (4.6)$$

and :

$$\begin{aligned} E\bar{v}(t_i) \bar{v}^T(t_i) &= E \int_{t_i}^{t_{i+1}} \phi(t_{i+1} - t) dv(t) dv^T(s) \phi^T(t_{i+1} - s)] \\ &= \int_{t_i}^{t_{i+1}} \phi(t_{i+1} - t) R(t) \phi^T(t_{i+1} - t) dt \end{aligned} \quad (4.7)$$

Therefore $\{\bar{v}(t_i), i = 1, 2, \dots\}$ is a sequence of independent normal random variables with zero mean value and covariance given by (4.7). For more detailed information see [6].

5. Example

To illustrate the application of the method shown in the previous chapters, we will consider a pursuit-evasion game, described by the following system of equations :

$$dx_p = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} x_p dt + \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} u dt + dw_p, \quad (5.1)$$

$$dx_e = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} x_e dt + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} v dt + dw_e, \quad (5.2)$$

with covariance matrices :

$$R_{0p} = \begin{bmatrix} 0.0025 & 0 \\ 0 & 0.0025 \end{bmatrix}, \quad R_p = \begin{bmatrix} 0.05 & 0 \\ 0 & 0.1 \end{bmatrix} \quad (5.3)$$

$$R_{0e} = \begin{bmatrix} 0.0025 & 0 \\ 0 & 0.0025 \end{bmatrix}, \quad R_e = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.01 \end{bmatrix} \quad (5.4)$$

and energy constraints :

$$E_p = 2, \quad E_e = 1.3918$$

The play begins at initial time $t_0 = 0$ and the mean values of the initial positions are :

$$m_{0p} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}, \quad m_{0e} = \begin{bmatrix} 2 \\ 6.1773 \end{bmatrix} \quad (5.5)$$

We will obtain the game termination time T^0 and the game termination point x_r for the mean value functions :

$$T^0 = 2, \quad x_f = \begin{bmatrix} -0.7 \\ 0.9281 \end{bmatrix}$$

The results are shown in Figs. 2~11.

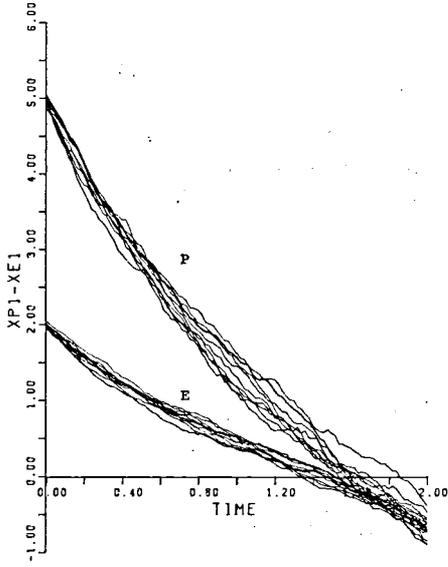


Fig. 2. Time-trajectories of x_{p1} of P and x_{e1} of E

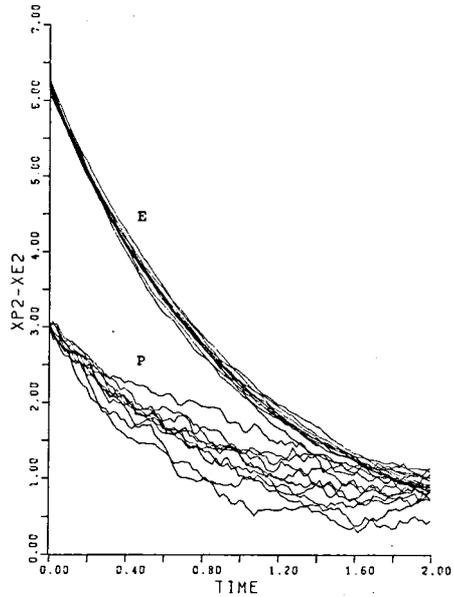


Fig. 3. Time-trajectories of x_{p2} of P and x_{e2} of E

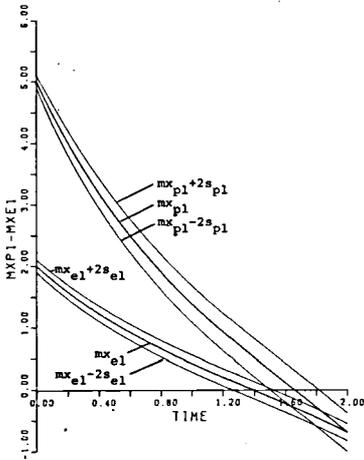


Fig. 4. Time-trajectories of mx_{p1} of P and mx_{e1} of E

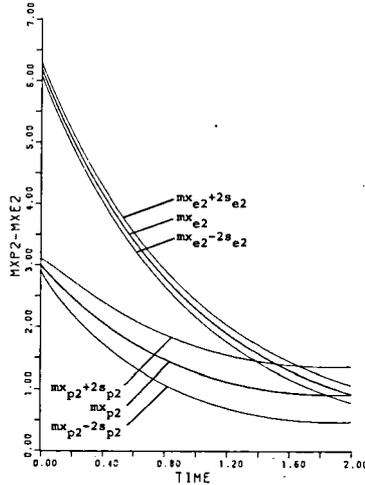


Fig. 5. Time-trajectories of mx_{p2} of P and mx_{e2} of E

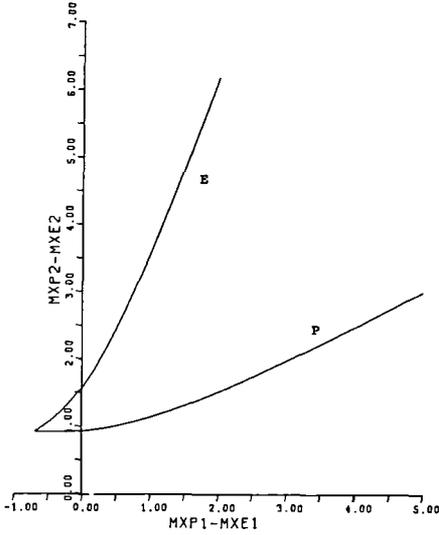


Fig. 6. State trajectories of mx , of P and mx , of E

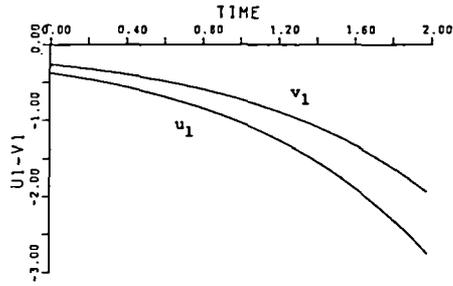


Fig. 7. Optimal strategies of P and E

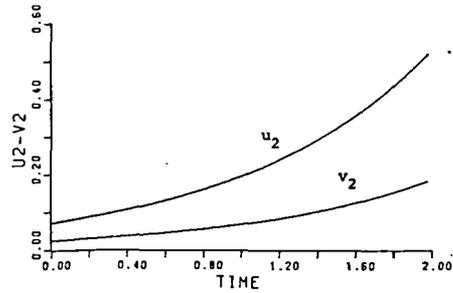


Fig. 8. Optimal strategies of P and E

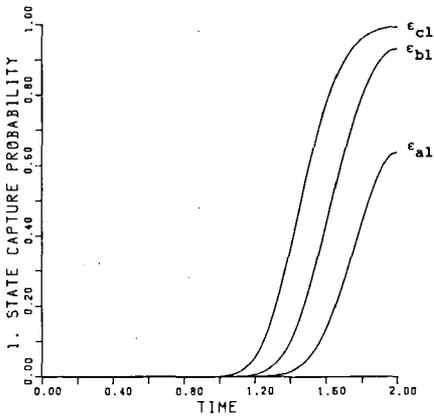


Fig. 9. Capture probability PC_1 of the 1. state variable for the capture distances : $\epsilon_{a1} = 0.1568$
 $\epsilon_{b1} = 0.3136$
 $\epsilon_{c1} = 0.4704$

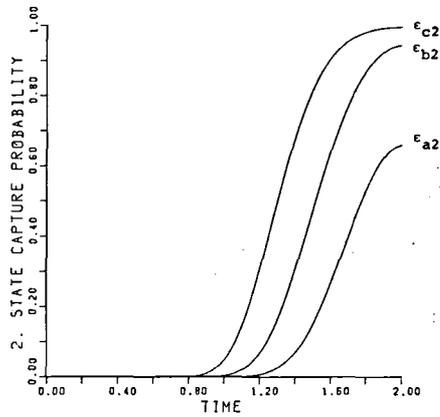
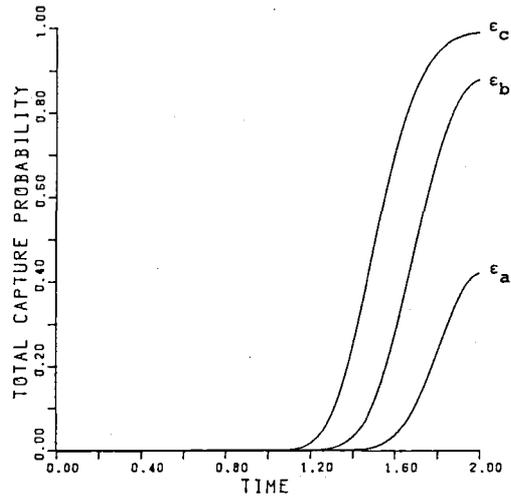


Fig. 10. Capture probability PC_2 of the 2. state variable for the capture distances : $\epsilon_{a2} = 0.2217$
 $\epsilon_{b2} = 0.4433$
 $\epsilon_{c2} = 0.6650$

Fig.11. Total capture probability *PCT*

6. Validity of the Reachable Region Approach

Let us consider the following deterministic pursuit-evasion game :

Example 6.1

$$\dot{x}_p = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} x_p + \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} u \quad (6.1)$$

$$\dot{x}_e = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} x_e + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} v \quad (6.2)$$

with energy constraints :

$$E_p = 3, \quad E_e = 2$$

The play begins at initial time $t_0 = 0$ and at initial positions :

$$x_p(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad x_e(0) = \begin{bmatrix} 5 \\ 5 \end{bmatrix} \quad (6.3)$$

We will obtain the game termination time T_0 and the game termination point x_f :

$$T_0 = 2.0143, \quad x_f = \begin{bmatrix} 2.0588 \\ 0.8027 \end{bmatrix}$$

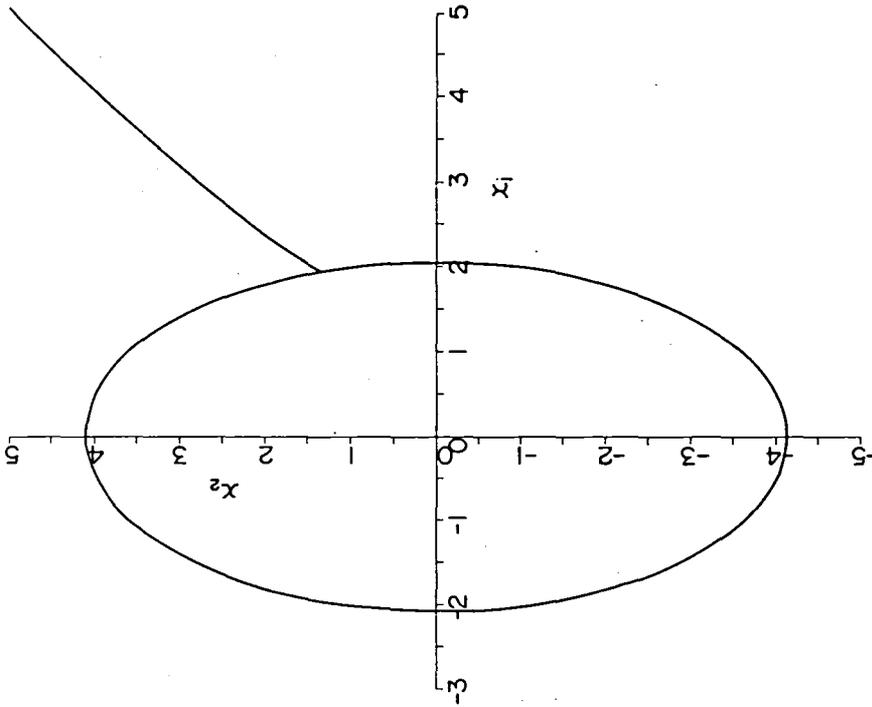


Fig.13. Reachable region of P for $T = 1.4$ and state trajectory of x , of E from Fig.12, but drawn only until time $T = 1.4$

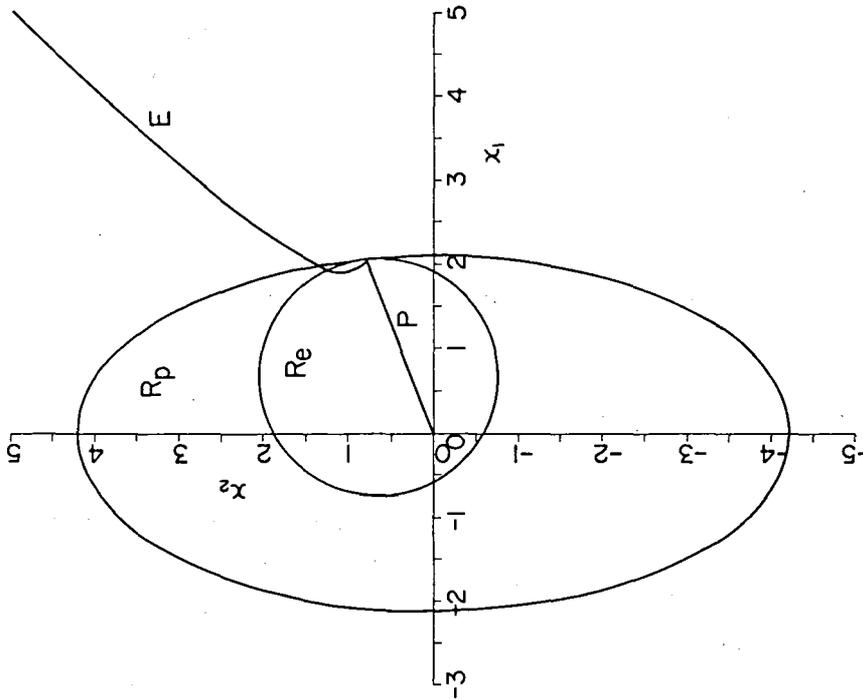


Fig.12. State trajectories of x , of P and x , of E and reachable regions of P and E for time T^0

Fig.12 shows the state trajectories and reachable regions for T^0 and x_f of pursuer and evader. Fig.13 shows the reachable region of the pursuer for time $T = 1.4$ and the same state trajectory of the evader as in Fig.12, but drawn only until time $T = 1.4$.

As we see, already at time $T = 1.4$ the pursuer could intercept the evader if the latter tries to reach x_f in time $T = 2.0413$. Therefore the found solution with $x_f = (2.0588, 0.8027)^T$ and $T^0 = 2.0143$ is not optimal for the pursuer.

To understand, why the reachable region approach does not give a correct solution for this example, we have to see that reachable regions for systems with initial point equal zero and reachable regions for systems with initial point not equal zero, have different properties.

Let us therefore consider the following example :

Example 6.2

$$\dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u \quad (6.4)$$

with energy constraint :

$$E = 2$$

The initial time is $t_0 = 0$ and the initial point is :

Example 6.2.1

$$x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Example 6.2.2

$$x(0) = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

The reachable regions for times $T_6 > T_5 > T_4 > T_3 > T_2 > T_1$ are shown for both examples 6.2.1 and 6.2.2 in Figs.14 and 15.

Generally, we can say : if the initial point is zero, then the symmetry points of all reachable regions are identically with the initial point of the system and the reachable region for a time T_2 contains completely the reachable region for a time T_1 , if $T_2 > T_1$.

If the initial point is not equal zero, then the symmetry points of the reachable regions move along the trajectory of the uncontrolled system. The reachable region for a time T_2 does not generally contain completely the reachable region of a time T_1 , if $T_2 > T_1$.

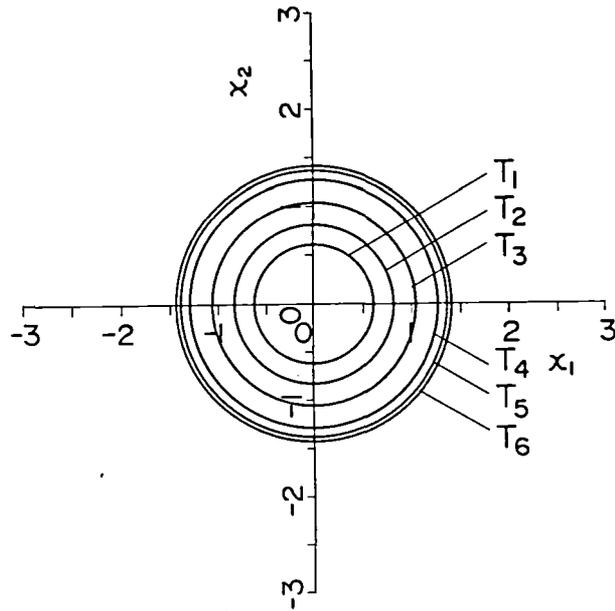


Fig.14. Reachable regions of example 6.2.1 for times $T_6 > T_5 > T_4 > T_3 > T_2 > T_1$

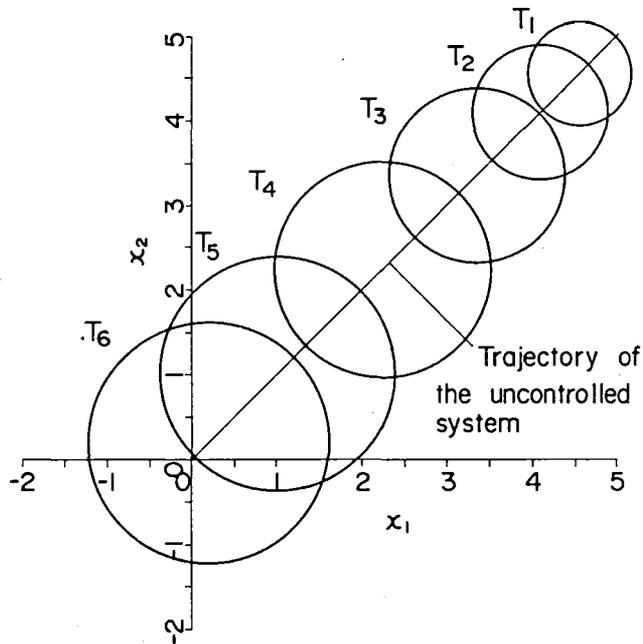


Fig.15. Reachable regions of example 6.2.2 for times $T_6 > T_5 > T_4 > T_3 > T_2 > T_1$

Because of this, the boundaries of the reachable regions can have points of intersection, which yield two different times in which these points can be reached, using the optimal controls and the same amount of energy.

We can show this for example 6.1. The boundary of the reachable region of the evader in example 6.1. has the following equation :

$$\frac{2}{1-e^{-2T}} [(x_{f1} - x_{e1}(0) e^{-T})^2 + (x_{f2} - x_{e2}(0) e^{-T})^2] = E_e^2 \quad (6.5)$$

If we solve this equation for the time, using the data of example 6.1 : $x_{f1} = 2.0588$, $x_{f2} = 0.8027$, $E_e = 2$, we will get two results : $T_1 = 2.0143$ the time we got as optimal game termination time with the reachable region approach and $T_2 = 0.8735$. The reachable regions for both times are shown together with the state trajectories in Fig.16. Fig.17 shows the reachable region for $T_1 = 2.0143$ and the general appearance of the state trajectories with final points on the boundary of this reachable region.

The boundary of the reachable region of the pursuer of example 6.1 is expressed by the following equation :

$$\frac{2}{1-e^{-2T}} [x_p^2 + \frac{1}{4} x_{f2}^2] = E_p^2 \quad (6.6)$$

If we set $x_{f1} = 2.0588$ and $x_{f2} = 0.8027$ in this equation and in equation (6.5) and calculate the functions $E_p(T)$ and $E_e(T)$, we will get the results shown in Fig.18. Here we can also see that for the evader with $E_e = 2$ two different times T are possible. This phenomenon has been already mentioned in Ref. [7].

Furthermore we see that for time $T = 1.25$ a minimum $E_e = 1.311$ of energy is needed to reach $x_f = (2.0588, 0.8027)^T$. The reachable region of the evader for $T = 1.25$ and $E_e = 1.311$ and the state trajectory are shown in Fig.19.

Summing up these results we will come to the following conclusions. The reachable regions as given in (2.15) and (2.16) were obtained in [3] by finding the minimal norm of the controls, which take the state of a linear system to a point $x(T)$ in state space in a given time T . Therefore the meaning of this reachable regions is that they are, for a fixed time T , functions $x_p(E_p)$ and $x_e(E_e)$ respectively. And in this sense they really have, for systems with initial point equal zero as well as for systems with initial point not equal zero, the properties of a reachable region. That is : The reachable region for an energy E_2 contains completely the reachable region for an energy E_1 , if $E_2 > E_1$, or in other words : if the boundary of the reachable region (equality holds in (2.15) and (2.16)) is reached with a energy E_2 , then for an arbitrary point inside this boundary we can say, it is reached with energy E_1 , $E_2 > E_1$, and an arbitrary point outside the boundary is reached with energy E_3 , $E_3 > E_2$.

We can see this if we calculate the reachable regions for examples 6.2.1 and 6.2.2 for a fixed time T and different values of energy E . The results are shown in Figs. 20 and 21.

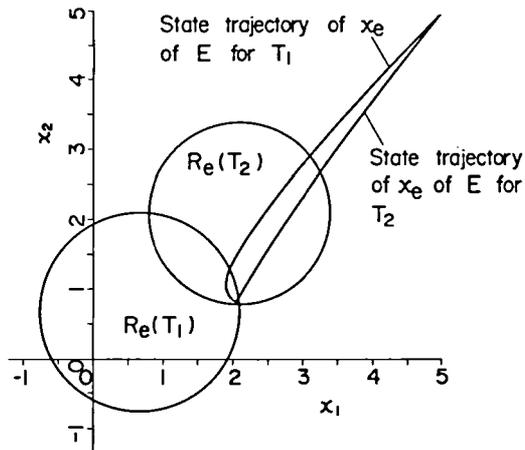


Fig.16. Reachable regions for times $T_1 = 2.0143$ and $T_2 = 0.8735$ of E and state trajectories of x , of E for T_1 and T_2

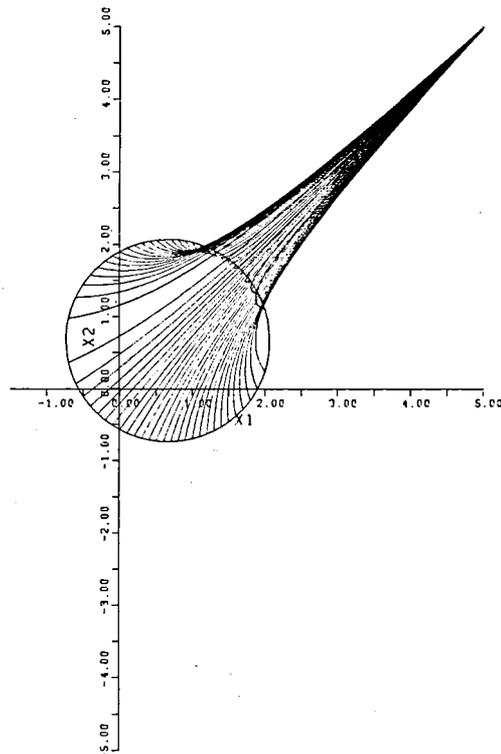


Fig.17. Reachable region for $T_1 = 2.0143$ of E and state trajectories with final points on the boundary of this reachable region

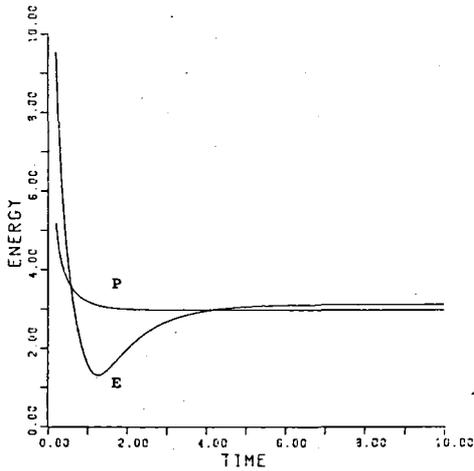


Fig.18. Functions $E_p(T)$ and $E_e(T)$ of example 6.1 for $x_{f1} = 2.0588$ and $x_{f2} = 0.8027$

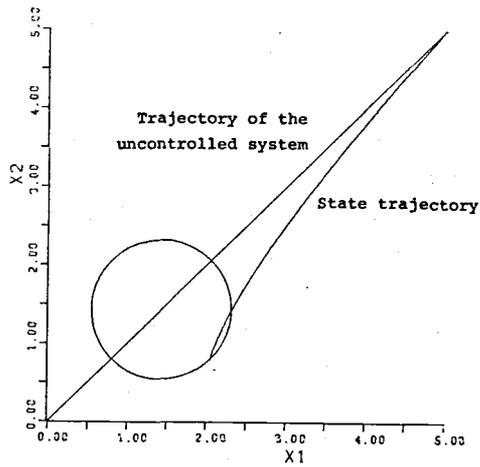


Fig.19. Reachable region for $T = 1.23$ and $E_e = 1.311$ and state trajectory of x_e of E

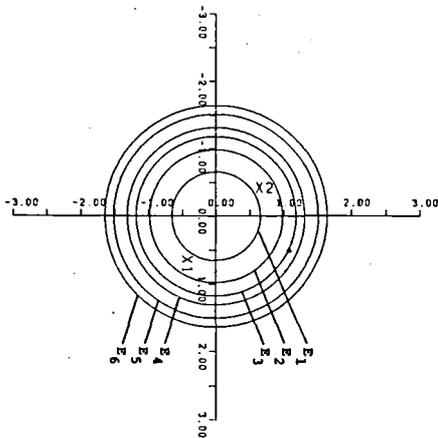


Fig.20. Reachable regions of example 6.2.1 for time $T = 1$ and energies $E_6 > E_5 > E_4 > E_3 > E_2 > E_1$

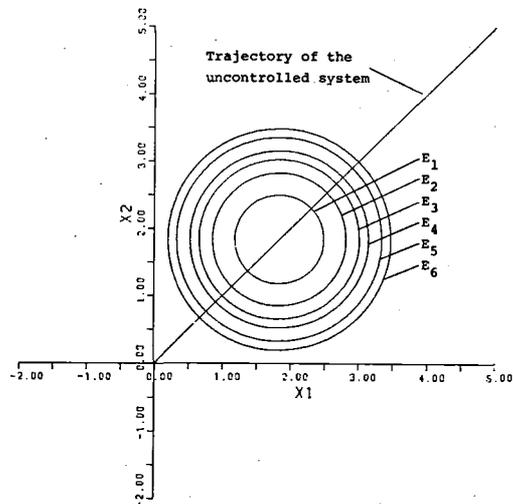


Fig.21. Reachable regions of example 6.2.2 for time $T = 1$ and energies $E_6 > E_5 > E_4 > E_3 > E_2 > E_1$

This reachable regions are therefore sets of final points for a certain time T , which do not state how this points are reached, that is, how the state trajectories will look like.

In the reachable region approach for pursuit-evasion games [3], however, the roles of time and energy were reversed. Here we give a fixed energy in order to get functions $x_p(T)$ and $x_e(T)$. This is possible only for systems with initial point equal zero. Because then the function $E(T)$ for a fixed point x in state space, is a one to one relation as we saw for $E_p(T)$ in Fig.18. And everything that was said before about the properties of the reachable region for the energy, holds here for the time. In the case that the initial point is not equal zero, the function $E(T)$ for a fixed point in state space is not a one to one relation as we saw for $E_e(T)$ in Fig.18.

Therefore, if we inverse it, the resulting configuration has no more the properties of a reachable region. We can see in Fig.15 that there are points of the inside of the boundary, which is reached in time T_2 , that can be reached in time T_1 , $T_1 < T_2$, and points, that can be reached in time T_3 , $T_3 > T_2$. The same holds for the outside. And because of this trajectories like the one of the evader in example 6.1 can occur.

7. Conclusion

Because the reachable region approach yields open-loop controls, its extension to the case with white noise disturbance can only be meaningful, if the disturbance is small. Otherwise closed-loop controls which take into account the actual noise disturbed trajectories of pursuer and evader will yield better results.

However, as we saw in chapter 6, we can not be sure, if the results of the reachable region approach are indeed optimal, without checking the resulting state trajectories. For further research it would therefore be of interest to reformulate the reachable region approach.

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