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Stable extendibility of the tangent bundles over lens spaces

Dedicated to Professor Takao Matumoto on his sixtieth birthday

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ABSTRACT. The purpose of this paper is to study the stable extendibility of the tangent bundle $\tau_n(p)$ of the $(2n+1)$-dimensional standard lens space $L^n(p)$ for odd prime p. We investigate the value of integer m for which $\tau_n(p)$ is stably extendible to $L^m(p)$ but not stably extendible to $L^{m+1}(p)$, and in particular we completely determine m for $p = 5$ or 7. A stable splitting of $\tau_n(p)$ and the stable extendibility of a Whitney sum of $\tau_n(p)$ are also discussed.

1. Introduction

Let F be the real number field R, the complex number field C or the quaternion number field H. For a subspace A of a space X, a t-dimensional F-vector bundle ζ over A is said to be *extendible* to X, if there is a t-dimensional F-vector bundle over X whose restriction to A is equivalent to ζ, that is, if ζ is equivalent to the induced bundle $i^*\eta$ of a t-dimensional F-vector bundle η over X under the inclusion map $i : A \to X$. Instead, if $i^*\eta$ is stably equivalent to ζ, namely $i^*\eta + m$ is equivalent to $\zeta + m$ for a trivial F-vector bundle m of dimension $m \geq 0$, ζ is called *stably extendible* to X (cf. [10], p. 20 and [4], p. 273).

Let $L^n(p) = S^{2n+1}/Z_p$ denote the $(2n+1)$-dimensional standard lens space mod p. For an R-vector bundle ζ over $L^n(p)$, we define an integer $s(\zeta)$ by

$$s(\zeta) = \max\{m \mid m \geq n \text{ and } \zeta \text{ is stably extendible to } L^m(p)\},$$

where $s(\zeta) = \infty$ if ζ is stably extendible to $L^m(p)$ for every $m \geq n$.

Let $\tau_n(p) = \tau(L^n(p))$ be the tangent bundle of $L^n(p)$. Then, concerning $s(\tau_n(p))$, the following theorems have been obtained.

THEOREM ([7], Theorem 5.3). *Let p be an integer* > 1. *Then,* $s(\tau_n(p)) = \infty$ *if* $n = 0, 1$ *or* 3.

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THEOREM ([8], Theorem 4.3). *Let p be an odd prime. Then,* $s(\tau_n(p)) < 2n+2$, *if* $n \geq 2p-2$.

THEOREM ([6], Theorem 1). $s(\tau_n(3)) = \infty$ *if and only if* $0 \leq n \leq 3$.

The purpose of this paper is to develop these results on the stable extendibility of the tangent bundle $\tau_n(p)$. Our main results are stated as follows.

THEOREM 1. *Let p be an odd prime. Then,* $s(\tau_n(p)) = 2n+1$ *if* $n \geq 2p-2$.

THEOREM 2. (1) *If* $0 \leq n \leq 5$, *then* $s(\tau_n(5)) = \infty$.
(2) *If* $n \geq 6$, *then* $s(\tau_n(5)) = 2n+1$.

THEOREM 3. (1) *If* $0 \leq n \leq 7$, *then* $s(\tau_n(7)) = \infty$.
(2) *If* $n \geq 8$, *then* $s(\tau_n(7)) = 2n+1$.

These theorems give support to our following conjecture.

CONJECTURE. For an odd prime p,

$$s(\tau_n(p)) = \infty \quad \text{for } 0 \leq n \leq p, \qquad \text{and} \qquad s(\tau_n(p)) = 2n+1 \quad \text{for } n \geq p+1.$$

We organize the paper as follows. In §2, we state some known results necessary to establish our results. In §3 we prove Theorem 1. In §4, we study $\tau_n(5)$ and $\tau_n(7)$ and prove Theorems 2 and 3. In §5, as a consequence of the preceding results, we give Theorem 4 concerning Schwarzenberger's property. In §6, we study the extendibility of the m-times Whitney sum $m\tau_n(p)$ of $\tau_n(p)$ for $m \geq 1$, and show in Proposition 6.1 the inequality

$$s(m\tau_n(p)) \geq m(2n+1) \qquad \text{or} \qquad s(m\tau_n(p)) \geq m(2n+1)-1$$

if m is an odd or even integer respectively. Then, in Theorem 5 we give some condition for

$$s(m\tau_n(p)) = m(2n+1) \qquad \text{or} \qquad m(2n+1)-1 \leq s(m\tau_n(p)) \leq m(2n+1)+1$$

to hold according as m is odd or even.

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2. Preliminary

For an odd prime p, the structures of the reduced K-ring $\tilde{K}(L^n(p))$ and the reduced KO-ring $\widetilde{KO}(L^n(p))$ are determined by Kambe [5].

Let η be the canonical C-line bundle over $L^n(p)$, the induced bundle from the canonical C-line bundle over the complex projective space CP^n under the projection $\pi : L^n(p) \to CP^n$, and $\sigma = \eta - 1$ its stable class in $\tilde{K}(L^n(p))$. Sometimes, we denote η (resp. σ) by η_n (resp. σ_n) to make it clear that η (resp. σ) is over $L^n(p)$.

Let $r : \tilde{K}(X) \to \widetilde{KO}(X)$ and $c : \widetilde{KO}(X) \to \tilde{K}(X)$ be the homomorphisms induced by the real restriction and the complexification of the vector bundles, respectively. We set $\bar{\sigma} = r(\sigma)$ in $\widetilde{KO}(L^n(p))$. Also, let $L_0^n(p)$ denote the $2n$-skeleton of $L^n(p)$ as in [5].

Then, we shall use the following result, where $[x]$ denotes the largest integer m with $m \leq x$ for a real number x.

THEOREM 2.1 ([5], Theorem 2, Lemma 3.4).
(1) *We have the following isomorphism of abelian groups:*

$$\widetilde{KO}(L^n(p)) \cong \begin{cases} \widetilde{KO}(L_0^n(p)) & \text{if } n \not\equiv 0 \mod 4, \\ Z_2 + \widetilde{KO}(L_0^n(p)) & \text{if } n \equiv 0 \mod 4. \end{cases}$$

(2) *Let* $q = (p-1)/2$ *and* $n = s(p-1) + r$ $(0 \leq r < p-1)$. *Then,*

$$\widetilde{KO}(L_0^n(p)) = (Z_{p^{s+1}})^{[r/2]} + (Z_{p^s})^{q-[r/2]},$$

and the direct summand $(Z_{p^{s+1}})^{[r/2]}$ *and* $(Z_{p^s})^{q-[r/2]}$ *are generated additively by* $\bar{\sigma}^1, \dots, \bar{\sigma}^{[r/2]}$ *and* $\bar{\sigma}^{[r/2]+1}, \dots, \bar{\sigma}^q$ *respectively. Moreover, the ring structure is given by*

$$\bar{\sigma}^{q+1} = \sum_{i=1}^{q} \frac{-(2q+1)}{2i-1} \binom{q+i-1}{2i-2} \bar{\sigma}^i, \qquad \bar{\sigma}^{[n/2]+1} = 0,$$

where $\binom{a}{b}$ *denotes a binomial coefficient.*

We also apply the following property.

LEMMA 2.2 ([5], Lemma 3.5(2)). *The homomorphism* $c : \widetilde{KO}(L_0^n(p)) \to \tilde{K}(L_0^n(p))$ *is a monomorphism.*

The following theorem due to Sjerve [11] is crucial in our proof, where $\pi_m : S^{2m+1} \to L^m(p)$ is the natural projection.

THEOREM 2.3 ([11], Theorem A). *If* $\zeta \in \widetilde{KO}(L^m(p)) \cap \ker \pi_m^*$, *then the geometrical dimension of* ζ *satisfies* $\text{g.dim}\, \zeta \leq 2\left[\frac{m}{2}\right] + 1$.

3. Proof of Theorem 1

By Theorem 2.3, we have the following.

PROPOSITION 3.1. *For any* $n \geq 1$, $s(\tau_n(p)) \geq 2n + 1$.

PROOF. Let $m \geq n$ be an integer. Since $r(\eta_m) - 2 \in \ker \pi_m^* \subset \widetilde{KO}(L^m(p))$ for the projection $\pi_m : S^{2m+1} \to L^m(p)$, where $r : \tilde{K}(L^m(p)) \to \widetilde{KO}(L^m(p))$ is the homomorphism mentioned in §2, we have

$$\text{g.dim}((n+1)(r(\eta_m) - 2)) \leq 2\left[\frac{m}{2}\right] + 1$$

by Theorem 2.3. Thus, there is a $\left(2\left[\frac{m}{2}\right] + 1\right)$-dimensional vector bundle β over $L^m(p)$ satisfying that $(n+1)r(\eta_m)$ is stably equivalent to β. When $m = 2n+1$, we have $2\left[\frac{m}{2}\right] + 1 = 2n+1$ and thus β is of dimension $2n+1$. Hence, $(n+1)r(\eta_{2n+1})$ is stably equivalent to $\beta + 1$, and $\tau_n(p)$ is stably equivalent to $i^*\beta$ since $\tau_n(p) + 1 = (n+1)r(\eta_n)$ is stably equivalent to $i^*\beta + 1$. Therefore, $\tau_n(p)$ is stably extendible to $L^{2n+1}(p)$, and we have the required inequality $s(\tau_n(p)) \geq 2n+1$. □

PROOF OF THEOREM 1. By Theorem 4.3 in [8], we have $s(\tau_n(p)) < 2n+2$ as we described in §1. Thus, by Proposition 3.1, we obtain the required result. □

REMARK 3.2. Proposition 3.1 is a special case of Theorem 4.2 in [7]. Therefore, Theorem 1 is originally due to Kobayashi-Maki-Yoshida ([7], [8]).

4. Stable extendibility of $\tau_n(5)$ and $\tau_n(7)$

Let p be an odd prime. Hereafter, we use the same notation α to denote the stable class of α in $\widetilde{KO}(L^n(p))$ (resp. $\tilde{K}(L^n(p))$) for a real (resp. complex) vector bundle α over $L^n(p)$. Also, we simply denote by $\alpha = \beta$ that vector bundle α and β are stably equivalent.

Using ring structures of $KO(L^n(p))$ and $K(L^n(p))$ for an odd prime p, we have the following lemma, where and hereafter we denote $r(\eta)$ or $c(r(\eta))$ simply by $r\eta$ or $cr\eta$ for the homomorphisms $r : K(L^n(p)) \to KO(L^n(p))$ and $c : KO(L^n(p)) \to K(L^n(p))$.

LEMMA 4.1. *In* $KO(L^n(p))$,

$$(r\eta)^2 = r(\eta^2) + 2, \qquad (r\eta)^3 = r(\eta^3) + 3r\eta.$$

PROOF. Recall that $cr\eta = \eta + \eta^{-1}$ (cf. [3], Proposition 11.3, p. 191). Since $c : KO(L^n(p)) \to K(L^n(p))$ is a ring homomorphism, we have $c(r(\eta^2)) = \eta^2 + \eta^{-2}$ and $c((r\eta)^2) = (cr\eta)^2 = (\eta + \eta^{-1})^2 = \eta^2 + \eta^{-2} + 2$. Then, by Lemma 2.2, $(r\eta)^2 = r(\eta^2) + 2$ in $KO(L^n(p))$. In the same way, $c(r(\eta^3)) = \eta^3 + \eta^{-3}$ and $c((r\eta)^3) = (cr\eta)^3 = (\eta + \eta^{-1})^3 = \eta^3 + \eta^{-3} + 3(\eta + \eta^{-1})$. Thus, we have $(r\eta)^3 = r(\eta^3) + 3r\eta$, and complete the proofs. □

Since $\tau_n(p)$ is stably trivial for $n = 0$ or 1 (cf. [7]), we have

LEMMA 4.2.

$$s(\tau_n(p)) = \infty \qquad \textit{for } n = 0 \textit{ or } 1.$$

Concerning $\tau_n(5)$ for $2 \le n \le 5$, we have the following.

PROPOSITION 4.3. *The following stable equivalences hold:*

$$\tau_2(5) = 2r(\eta^2) + 1, \quad \tau_3(5) = r(\eta^2) + 5, \quad \tau_4(5) = 9 \quad \textit{and} \quad \tau_5(5) = r\eta + 9.$$

Hence, $s(\tau_n(5)) = \infty$ *for* $2 \le n \le 5$.

PROOF. Let $n = 2$ or 3. Then, by Theorem 2.1, $\widetilde{KO}(L^n(5)) = Z_5\{\bar{\sigma}\}$ and $\bar{\sigma}^2 = 0$. Thus, we have $5r\eta - 10 = 0$ and $(r\eta)^2 - 4r\eta + 4 = 0$. Then, using Lemma 4.1, we obtain $r(\eta^2) + r\eta - 4 = 0$. Since $\tau_n(5) = (n+1)r\eta - 1$, we have

$$\tau_2(5) = 3r\eta - 1 = -2r\eta + 9 = 2r(\eta^2) + 1;$$

$$\tau_3(5) = 4r\eta - 1 = -r\eta + 9 = r(\eta^2) + 5.$$

Similarly, for $n = 4$ or 5, $\widetilde{KO}(L_0^n(5)) = Z_5\{\bar{\sigma}, \bar{\sigma}^2\}$ and thus $5r\eta - 10 = 0$. Then, we have $\tau_4(5) = 5r\eta - 1 = 9$, $\tau_5(5) = 6r\eta - 1 = r\eta + 9$. Thus, we have $s(\tau_n(5)) = \infty$ for $n = 2, 3, 4$ or 5 as is required, since $r(\eta^2)$ and $r\eta$ over $L^n(5)$ are extendible to $L^m(5)$ for every $m \ge n$. □

REMARK 4.4. According to Yoshida [12], $L^3(p)$ has a tangent 5-field. Hence, $\tau_3(p) = \beta \oplus 5$ for a 2-plane bundle β in general.

PROPOSITION 4.5.

$$s(\tau_n(5)) = 2n + 1 \qquad \textit{for } n = 6 \textit{ or } 7.$$

PROOF. Let $n = 6$ or 7. By Proposition 3.1, we have $s(\tau_n(5)) \ge 2n + 1$. To establish the opposite inequality, we suppose that $\tau_n(5)$ is stably extendible to $L^{2n+2}(5)$, and derive a contradiction from the hypothesis. Thus, there is a $(2n+1)$-dimensional vector bundle α over $L^{2n+2}(5)$ satisfying that $\tau_n(5)$ is stably equivalent to $i^*\alpha$ for the inclusion map $i : L^n(5) \to L^{2n+2}(5)$. By Theorem 2.1, $\widetilde{KO}(L^{2n+2}(5))$ is generated additively by $\bar{\sigma}$ and $\bar{\sigma}^2$ modulo a 2-torsion. Thus, we can put $\alpha - (2n+1) = a\bar{\sigma} + b\bar{\sigma}^2 + \delta$ in $\widetilde{KO}(L^{2n+2}(5))$, where δ is zero or a 2-torsion element. Then, since $i^*\delta = 0$ in $\widetilde{KO}(L^n(5)) = Z_{5^2}\{\bar{\sigma}\} + Z_5\{\bar{\sigma}^2\}$, we have $i^*\alpha - (2n+1) = a\bar{\sigma} + b\bar{\sigma}^2$ in $\widetilde{KO}(L^n(5))$.

Since $i^*\alpha = \tau_n(5)$ and $\tau_n(5) - (2n+1) = (n+1)\bar{\sigma}$, we have

$$\begin{cases} a \equiv n+1 \mod 5^2, \\ b \equiv 0 \mod 5. \end{cases}$$

Hence, we can put

$$\begin{cases} a = 5k + a_1 & \text{with } k \equiv 1 \mod 5, \\ b = 5l \end{cases}$$

for some integers k and l, where $a_1 = 2$ and 3 when $n = 6$ and 7 respectively.

Since $K(L^{2n+2}(5))$ has no 2-torsion (cf. [5]), $c\delta = 0$. Then, we have

$$\begin{aligned} c\alpha - (2n+1) &= ac\bar{\sigma} + bc\bar{\sigma}^2 = a((\eta + \eta^{-1}) - 2) + b((\eta + \eta^{-1})^2 - 4(\eta + \eta^{-1}) + 4) \\ &= (a - 4b)(\eta + \eta^{-1}) + b(\eta^2 + \eta^{-2}) - (2a - 6b). \end{aligned}$$

Let $C_i(\gamma)$ denote the i-th Chern class of a complex vector bundle γ, and $C(\gamma) = 1 + C_1(\gamma) + \cdots$ the total Chern class. We denote $C_i(\gamma)$ and $C(\gamma)$ in the Z_5-coefficient cohomology group by the same letters. Then, for $x = C_1(\eta)$,

$$\bigoplus_{i \geq 0} H^{2i}(L^{2n+2}(5); Z_5) \cong Z_5[x]/(x^{2n+3})$$

as graded algebras (cf. [11]), and we have

$$C(c\alpha) = C(\eta + \eta^{-1})^{a-4b} C(\eta^2 + \eta^{-2})^b = (1 - x^2)^{a-4b}(1 - 4x^2)^b.$$

Since $a - 4b = 5(k - 4l) + a_1$ with $k \equiv 1 \mod 5$ and $b = 5l$, and since $n = 6$ or 7,

$$\begin{aligned} C(c\alpha) &= (1 - x^2)^{a_1}((1 - x^2)^5)^{k-4l}((1 - 4x^2)^5)^l \\ &= (1 - x^2)^{a_1}(1 - x^{10})^{k-4l}(1 - 4^5x^{10})^l \\ &= (1 - x^2)^{a_1}(1 - (k - 4l)x^{10})(1 - 4^5lx^{10}) \\ &= (1 - x^2)^{a_1}(1 - kx^{10}) \\ &= (1 - x^2)^{a_1}(1 - x^{10}) \\ &= 1 - a_1x^2 + \cdots + (-1)^{a_1+1}x^{10+2a_1}. \end{aligned}$$

Since $10 + 2a_1 = 2n + 2$, we have $C_{2n+2}(c\alpha) \neq 0$ which contradicts that α is $(2n + 1)$-dimensional. Thus, we have completed the proof. □

PROOF OF THEOREM 2. We obtain (1) by Lemma 4.2 and Proposition 4.3, and (2) by Theorem 1 and Proposition 4.5. □

Next, we consider the proof of Theorem 3, but we can proceed similarly to Theorem 2.

PROPOSITION 4.6. *We have the following stable equivalences:*

$$\tau_2(7) = r(\eta^3) + r\eta + 1, \qquad \tau_3(7) = r(\eta^3) + 2r\eta + 1,$$
$$\tau_4(7) = 2r(\eta^3) + 2r(\eta^2) + 1, \qquad \tau_5(7) = 2r(\eta^3) + 2r(\eta^2) + r\eta + 1,$$
$$\tau_6(7) = 13 \quad \textit{and} \quad \tau_7(7) = r\eta + 13.$$

Hence, $s(\tau_n(7)) = \infty$ *for* $2 \le n \le 7$.

PROOF. First, let $n = 2$ or 3. Then, $\widetilde{KO}(L^n(7)) = Z_7\{\bar{\sigma}\}$, $\bar{\sigma}^2 = 0$ and $\bar{\sigma}^3 = 0$ by Theorem 2.1. Thus, we have $7r\eta - 14 = 0$, $(r\eta)^2 - 4r\eta + 4 = 0$ and $(r\eta)^3 - 6(r\eta)^2 + 12r\eta - 8 = 0$. Then, using Lemma 4.1 and these three equations, we obtain $r(\eta^3) + 5r\eta - 12 = 0$. Since $\tau_n(7) = (n+1)r\eta - 1$ in $KO(L^n(7))$, we have

$$\tau_2(7) = 3r\eta - 1 = -4r\eta + 13 = r(\eta^3) + r\eta + 1;$$

$$\tau_3(7) = 4r\eta - 1 = -3r\eta + 13 = r(\eta^3) + 2r\eta + 1.$$

Next, let $n = 4$ or 5. Then, $\widetilde{KO}(L_0^n(7)) = Z_7\{\bar{\sigma}, \bar{\sigma}^2\}$ and $\bar{\sigma}^3 = 0$ by Theorem 2.1. Thus, we have $7r\eta - 14 = 0$, $7(r\eta)^2 - 28r\eta + 28 = 0$ and $(r\eta)^3 - 6(r\eta)^2 + 12r\eta - 8 = 0$. Then, using Lemma 4.1 and these three equations, we obtain $r(\eta^3) + r(\eta^2) + r\eta - 6 = 0$. Since $\tau_n(7) = (n+1)r\eta - 1$ in $KO(L^n(7))$, we have

$$\tau_4(7) = 5r\eta - 1 = -2r\eta + 13 = 2r(\eta^3) + 2r(\eta^2) + 1;$$

$$\tau_5(7) = 6r\eta - 1 = -r\eta + 13 = 2r(\eta^3) + 2r(\eta^2) + r\eta + 1.$$

Similarly, for $n = 6$ or 7, we also have $7r\eta - 14 = 0$ by Theorem 2.1. Thus, we have $\tau_6(7) = 7r\eta - 1 = 13$ and $\tau_7(7) = 8r\eta - 1 = r\eta + 13$. Hence, $s(\tau_n(7)) = \infty$ for $2 \le n \le 7$ as is required, since $r(\eta^3)$, $r(\eta^2)$ and $r\eta$ over $L^n(7)$ are extendible to $L^m(7)$ for every $m \ge n$. □

PROPOSITION 4.7.

$$s(\tau_n(7)) = 2n + 1 \qquad \textit{for } 8 \le n \le 11.$$

PROOF. Let $n = 8, 9, 10$ or 11. By Proposition 3.1, we have $s(\tau_n(7)) \ge 2n + 1$. We suppose that $\tau_n(7)$ is stably extendible to $L^{2n+2}(7)$, and derive a contradiction from the hypothesis. Thus, there is a $(2n+1)$-dimensional vector bundle α over $L^{2n+2}(7)$ satisfying that $\tau_n(7)$ is stably equivalent to $i^*\alpha$. By Theorem 2.1, $\widetilde{KO}(L^n(7))$ and $\widetilde{KO}(L^{2n+2}(7))$ are both generated additively by $\bar{\sigma}$, $\bar{\sigma}^2$ and $\bar{\sigma}^3$ modulo a 2-torsion. Thus, we can put $\alpha - (2n+1) = a\bar{\sigma} + b\bar{\sigma}^2 + d\bar{\sigma}^3 + \delta$, where δ is zero or a 2-torsion element. Then, since $i^*\delta = 0$ in $\widetilde{KO}(L^n(7))$, we have $i^*\alpha - (2n+1) = a\bar{\sigma} + b\bar{\sigma}^2 + d\bar{\sigma}^3$ in

$$\widetilde{KO}(L_0^n(7)) = \begin{cases} Z_{7^2}\{\bar{\sigma}\} + Z_7\{\bar{\sigma}^2, \bar{\sigma}^3\} & n = 8 \text{ or } 9, \\ Z_{7^2}\{\bar{\sigma}, \bar{\sigma}^2\} + Z_7\{\bar{\sigma}^3\} & n = 10 \text{ or } 11. \end{cases}$$

Since $i^*\alpha = \tau_n(7)$ and $\tau_n(7) - (2n+1) = (n+1)\bar{\sigma}$, we have

$$\begin{cases} a \equiv n+1 \mod 7^2, \\ b \equiv 0 \mod 7 \ (n = 8, 9), \ \mod 7^2 \ (n = 10, 11), \\ d \equiv 0 \mod 7. \end{cases}$$

Hence, we can put

$$\begin{cases} a = 7k + a_1 \quad \text{with } k \equiv 1 \mod 7, \\ b = 7l, \\ d = 7h \end{cases}$$

for some integers k, l and h, where $a_1 = 2, 3, 4$ or 5 according as $n = 8, 9, 10$ or 11. Consider the complexification of α. Then,

$$\begin{aligned} c\alpha - (2n+1) &= ac\bar{\sigma} + bc\bar{\sigma}^2 + dc\bar{\sigma}^3 \\ &= a((\eta + \eta^{-1}) - 2) + b((\eta + \eta^{-1})^2 - 4(\eta + \eta^{-1}) + 4) \\ &\quad + d((\eta + \eta^{-1})^3 - 6(\eta + \eta^{-1})^2 + 12(\eta + \eta^{-1}) - 8) \\ &= (a - 4b + 15d)(\eta + \eta^{-1}) + (b - 6d)(\eta^2 + \eta^{-2}) + d(\eta^3 + \eta^{-3}) \\ &\quad - (2a - 6b + 20d). \end{aligned}$$

Recall that $\bigoplus_{i \geq 0} H^{2i}(L^{2n+2}(7); Z_7) \simeq Z_7[x]/(x^{2n+3})$ as graded algebras, where $x = C_1(\eta)$. Then, we have

$$\begin{aligned} C(c\alpha) &= C(\eta + \eta^{-1})^{a-4b+15d} C(\eta^2 + \eta^{-2})^{b-6d} C(\eta^3 + \eta^{-3})^d \\ &= (1 - x^2)^{a-4b+15d}(1 - 4x^2)^{b-6d}(1 - 9x^2)^d. \end{aligned}$$

Since $a - 4b + 15d = 7(k - 4l + 15h) + a_1$ with $k \equiv 1 \mod 7$, $b - 6d = 7(l - 6h)$ and $d = 7h$, we have

$$\begin{aligned} C(c\alpha) &= (1 - x^2)^{a_1}((1 - x^2)^7)^{k-4l+15h}((1 - 4x^2)^7)^{l-6h}((1 - 9x^2)^7)^h \\ &= (1 - x^2)^{a_1}(1 - x^{14})^{k-4l+15h}(1 - 4^7x^{14})^{l-6h}(1 - 9^7x^{14})^h \\ &= (1 - x^2)^{a_1}(1 - (k - 4l + 15h)x^{14})(1 - 4(l - 6h)x^{14})(1 - 2hx^{14}) \\ &= (1 - x^2)^{a_1}(1 - (k - 9h)x^{14})(1 - 2hx^{14}) \\ &= (1 - x^2)^{a_1}(1 - (k - 7h)x^{14}) \\ &= (1 - x^2)^{a_1}(1 - x^{14}) \\ &= 1 - a_1x^2 + \cdots + (-1)^{a_1+1}x^{14+2a_1}. \end{aligned}$$

Since $14 + 2a_1 = 2n + 2$, we have $C_{2n+2}(c\alpha) \neq 0$, which contradicts that α is $(2n+1)$-dimensional. Thus, we obtain the required result. □

PROOF OF THEOREM 3. We have (1) by Lemma 4.2 and Proposition 4.6, and (2) by Theorem 1 and Proposition 4.7. □

5. Application to stably splitting problem

A splitting (resp. stably splitting) problem of vector bundles can be stated: When is a given k-plane bundle equivalent (resp. stably equivalent) to a sum of k line bundles? Concerning this, the following result is called Schwarzenberger's property.

THEOREM ([1], [2], [9], [10]). *Let $F = C$ or R. If a k-dimensional F-vector bundle ζ over FP^n is extendible to FP^m for every $m > n$, then ζ is stably equivalent to the Whitney sum of k numbers of F-line bundles.*

We remark that the theorem is also valid if the condition for extendibility is changed to that for stably extendibility (cf. [8], [4]). Then, some related results are shown as follows:

THEOREM ([4], Theorem B). *If a k-dimensional H-vector bundle ζ over HP^n is stably extendible to HP^m for every $m > n$ and its top non-zero Pontrjagin class is not zero* mod 2, *then ζ is stably equivalent to the Whitney sum of k numbers of H-line bundles provided $k \leq n$.*

THEOREM ([8], Theorem B). *If a k-dimensional vector bundle ζ over $L^n(3)$ is stably extendible to $L^m(3)$ for every $m > n$, then ζ is stably equivalent to the Whitney sum of $\left[\frac{k}{2}\right]$ numbers of 2-plane bundles.*

We have another answer from Lemma 5.2 in [7], Theorems 2 and 3 and Propositions 4.3 and 4.6.

THEOREM 4. *Let $p = 5$ or 7 and $n \geq 1$. Then, $\tau_n(p)$ is stably equivalent to the Whitney sum of $\left[\frac{2n+1}{2}\right]$ numbers of 2-plane bundles if and only if $s(\tau_n(p)) = \infty$ holds.*

6. Study on $m\tau_n(p)$

Let $m\tau_n(p)$ be the m-times Whitney sum of the tangent bundle $\tau_n(p)$. We have the following in the similar way to the proof of Proposition 3.1.

PROPOSITION 6.1. *Let $m \geq 1$. Then, for any $n \geq 1$, we have*

$$s(m\tau_n(p)) \geq m(2n+1) \qquad \text{or} \qquad s(m\tau_n(p)) \geq m(2n+1) - 1$$

if m is an odd or even integer respectively.

PROOF. For any integer $k \geq 1$, we have

$$\text{g.dim}(m(n+1)(r\eta_k - 2)) \leq 2\left[\frac{k}{2}\right] + 1$$

by Theorem 2.3. Thus, there is a $(2[\frac{k}{2}]+1)$-dimensional vector bundle β satisfying that $m(n+1)r\eta_k$ is stably equivalent to β. Let m be an odd (resp. even) integer. When $k = m(2n+1)$ (resp. $k = m(2n+1) - 1$), we have $2[\frac{k}{2}] + 1 = m(2n+1)$ (resp. $= m(2n+1) - 1$). Thus, $m(n+1)r\eta_{m(2n+1)}$ (resp. $m(n+1)r\eta_{m(2n+1)-1}$) is stably equivalent to $\gamma + m$ for the $m(2n+1)$-dimensional vector bundle $\gamma = \beta$ (resp. $= \beta + 1$). Then, $m\tau_n(p)$ is stably equivalent to $i^*(\gamma)$ since $m\tau_n(p) + m = m(n+1)r\eta_n$, and thus we have the required inequality $s(m\tau_n(p)) \geq m(2n+1)$ (resp. $s(m\tau_n(p)) \geq m(2n+1) - 1$). □

Now, in order to consider the case when $s(m\tau_n(p)) = m(2n+1)$ or $s(m\tau_n(p)) \leq m(2n+1) + 1$ holds in Proposition 6.1, we first define an integer $\varepsilon_p(t, l)$.

DEFINITION. For a non-negative integer t and a positive integer l, define an integer $\varepsilon_p(t, l)$ as follows.

$$\varepsilon_p(t, l) = \min\left\{ 2j \,\middle|\, 2\left[\frac{t}{2}\right] + 1 < 2j \text{ and } \binom{[\frac{t}{2}] + l}{j} \not\equiv 0 \bmod p \right\}.$$

Then, we have $t < \varepsilon_p(t, l) \leq 2[\frac{t}{2}] + 2l$ and $\varepsilon_p(t, 1) = 2[\frac{t}{2}] + 2$, and the following lemma.

LEMMA 6.2. *Let p be an odd prime and ζ a t-dimensional vector bundle over $L^n(p)$. If there is a positive integer l with $\varepsilon_p(t, l) \leq n$, then ζ is not stably equivalent to $([\frac{t}{2}] + l)r\eta$.*

PROOF. We write simply $\varepsilon(t, l)$ instead of $\varepsilon_p(t, l)$. For the Pontrjagin class of $([\frac{t}{2}] + l)r\eta$, we have

$$P_{\varepsilon(t,l)/2}\left(\left(\left[\frac{t}{2}\right] + l\right)r\eta\right) = \binom{[\frac{t}{2}] + l}{\frac{\varepsilon(t,l)}{2}} x^{\varepsilon(t,l)} \in H^{2\varepsilon(t,l)}(L^n(p); Z),$$

which is not zero by the definition of $\varepsilon(t, l)$ and the assumption $\varepsilon(t, l) \leq n$. However, since ζ is of dimension t and $[\frac{t}{2}] < \frac{\varepsilon(t,l)}{2}$, we have $P_{\varepsilon(t,l)/2}(\zeta) = 0$. Thus, ζ is not stably equivalent to $([\frac{t}{2}] + l)r\eta$, as is required. □

The following is also obtained using the calculation in the proof of Theorem 1.1 in [7].

PROPOSITION 6.3. *Let p be an odd prime, and ζ a t-dimensional vector bundle over $L^n(p)$. Assume that there is a positive integer l satisfying*

(1) *ζ is stably equivalent to* $([\frac{t}{2}]+l)r\eta$, *and*
(2) $p^{[n/(p-1)]} > [\frac{t}{2}]+l$.
Then, $s(\zeta) < \varepsilon_p(t,l)$.

PROOF. Here, we put $h = [\frac{t}{2}]+l$, and write $\varepsilon(t,l)$ instead of $\varepsilon_p(t,l)$. Then, by Lemma 6.2, $n < \varepsilon(t,l)$. Now, we suppose that ζ is stably extendible to $L^{\varepsilon(t,l)}(p)$, and derive a contradiction from the hypothesis. Thus, there exists a t-dimensional vector bundle α over $L^{\varepsilon(t,l)}(p)$ satisfying that $i^*\alpha$ is stably equivalent to $hr\eta$.

Now, we apply the same methods used in the proof of Theorem 1.1 in [7]. The integers c_i used there are $c_1 = h$ and $c_i = 0$ for $2 \le i \le p-1$ in our case. Then, the total Pontrjagin class of $j^*\alpha$, where j is the inclusion map $j : L_0^{\varepsilon(t,l)}(p) \to L^{\varepsilon(t,l)}(p)$, is given as

$$P(j^*\alpha) = (1+x^2)^h \qquad \text{in } H^*(L_0^{\varepsilon(t,l)}(p); Z).$$

Here, the following equality is used to calculate the above Pontrjagin class as in [7]:

$$(1+i^2x^2)^{p^{[n/(p-1)]}} = 1 + i^{2p^{[n/(p-1)]}}x^{2p^{[n/(p-1)]}} = 1 \qquad \text{in } H^*(L_0^{\varepsilon(t,l)}(p); Z)$$

for $1 \le i \le \frac{p-1}{2}$, and it holds because $p^{[n/(p-1)]} > h$ from the assumption (2) and $2h \ge \varepsilon(t,l)$ as mentioned above. Then, from the total Pontrjagin class of $j^*\alpha$ and by the definition of $\varepsilon(t,l)$, we have

$$P_{\varepsilon(t,l)/2}(j^*\alpha) = \binom{h}{\frac{\varepsilon(t,l)}{2}} x^{\varepsilon(t,l)} \neq 0 \qquad \text{in } H^{2\varepsilon(t,l)}(L_0^{\varepsilon(t,l)}(p); Z),$$

which contradicts that $j^*\alpha$ is of dimension t and $t < \varepsilon(t,l)$. Thus, we have completed the proof. □

Then, we have the following.

THEOREM 5. *Let $m \ge 1$ and $n \ge 1$ be integers.*
(1) *If m is odd,*

$$p^{[n/(p-1)]} > m(n+1) \qquad \text{and} \qquad \binom{m(n+1)}{m(n+1)-\frac{m-1}{2}} \not\equiv 0 \mod p,$$

then $s(m\tau_n(p)) = m(2n+1)$.
(2) *If m is even,*

$$p^{[n/(p-1)]} > m(n+1) \qquad \text{and} \qquad \binom{m(n+1)}{mn+1+\frac{m}{2}} \not\equiv 0 \mod p,$$

then $s(m\tau_n(p)) = m(2n+1)-1$, $m(2n+1)$ *or* $m(2n+1)+1$.

PROOF. First, we assume that m is odd, and prove (1). By Proposition 6.1, we have $s(m\tau_n(p)) \geq m(2n+1)$. Thus, we assume further that

$$p^{[n/(p-1)]} > m(n+1) \quad \text{and} \quad \binom{m(n+1)}{\frac{m(2n+1)+1}{2}} = \binom{m(n+1)}{m(n+1)-\frac{m-1}{2}} \not\equiv 0 \mod p,$$

and prove the inequality $s(m\tau_n(p)) \leq m(2n+1)$. Consider $\varepsilon_p(m(2n+1), \frac{m+1}{2})$. Since $2\left[\frac{m(2n+1)}{2}\right] + 1 < m(2n+1)+1$, and by the latter assumption above, we have $\varepsilon_p(m(2n+1), \frac{m+1}{2}) \leq m(2n+1)+1$. Hence, by Proposition 6.3, we have $s(m\tau_n(p)) < \varepsilon_p(m(2n+1), \frac{m+1}{2}) \leq m(2n+1)+1$, and thus we have proved (1).

Next, we assume that m is even, and prove (2). By Proposition 6.1, we have $s(m\tau_n(p)) \geq m(2n+1)-1$. Thus, we further assume that

$$p^{[n/(p-1)]} > m(n+1) \qquad \text{and} \qquad \binom{m(n+1)}{\frac{m(2n+1)+2}{2}} = \binom{m(n+1)}{mn+1+\frac{m}{2}} \not\equiv 0 \mod p,$$

and prove $s(m\tau_n(p)) \leq m(2n+1)+1$. Then, since $2\left[\frac{m(2n+1)}{2}\right] + 1 < m(2n+1)+2$, and by the last assumption above, we have $\varepsilon_p(m(2n+1), \frac{m}{2}) \leq m(2n+1)+2$. Hence, by Proposition 6.3, $s(m\tau_n(p)) < m(2n+1)+2$, and thus we have proved (2) and completed the proof of Theorem 5. □

We illustrate the results of Theorems 5 for $p = 5$ or 7 and for $2 \leq m \leq 5$.

EXAMPLE. Let $n \geq 1$, and $p = 5$ or 7.

(1) If $n \geq 2p-2$, then $s(2\tau_n(p)) = 4n+1, 4n+2$ or $4n+3$.

(2) Assume that

$$\begin{cases} n \geq 3p-3 \text{ and } n+1 \not\equiv 0 \mod p & \text{for } p = 5, \\ n = 12, 14, 15 \text{ or } n \geq 3p-3 \text{ and } n+1 \not\equiv 0 \mod p & \text{for } p = 7. \end{cases}$$

Then, $s(3\tau_n(p)) = 6n+3$.

(3) Assume that $n \geq 3p-3$ and $n+1 \not\equiv 0 \mod p$. Then, $s(4\tau_n(p)) = 8n+3, 8n+4$ or $8n+5$.

(4) Assume that $n \geq 3p-3$. For $p = 5$, we have no information on $s(5\tau_n(5))$ from Theorem 5. For $p = 7$, if $\frac{1}{2}(5n+4)(5n+5) \not\equiv 0 \mod 7$, then $s(5\tau_n(7)) = 10n+5$.

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