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Extendibility of negative vector bundles over the complex projective space

Dedicated to the memory of Professor Masahiro Sugawara

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ABSTRACT. By Schwarzenberger's property, a complex vector bundle of dimension t over the complex projective space $\mathbf{C}P^n$ is extendible to $\mathbf{C}P^{n+k}$ for any $k \geq 0$ if and only if it is stably equivalent to a Whitney sum of t complex line bundles. In this paper, we show some conditions for a negative multiple of a complex line bundle over $\mathbf{C}P^n$ to be extendible to $\mathbf{C}P^{n+1}$ or $\mathbf{C}P^{n+2}$, and its application to unextendibility of a normal bundle of $\mathbf{C}P^n$.

1. Introduction and results

An m-dimensional vector bundle V over a space A is called extendible to a space $B \supset A$ when there exists an m-dimensional vector bundle over B whose restriction to A is isomorphic to V. Classically, Schwarzenberger [11], [4, Appendix I] studied extendibility of vector bundles over the real or complex projective spaces. Related results were obtained by Rees [3], [10] and Adams–Mahmud [1]. Extendibility of vector bundles over the real projective spaces and the standard lens spaces are studied extensively by Kobayashi-Maki-Yoshida [8], [9] and so on, and that of vector bundles over the quaternionic projective spaces by [6], [7].

We consider only complex vector bundles, and thus a k-dimensional vector bundle means a $\mathbf{C}^k$-vector bundle. Let ξ be the canonical line bundle over the complex projective space $\mathbf{C}P^n$, and for an integer m

$$\xi^m = \underbrace{\xi \otimes \cdots \otimes \xi}_{m} \quad \text{if } m > 0; \qquad \xi^0 = \underline{\mathbf{C}}^1; \qquad \xi^m = \underbrace{\bar{\xi} \otimes \cdots \otimes \bar{\xi}}_{-m} \quad \text{if } m < 0,$$

where $\bar{\xi}$ is the complex conjugate bundle of ξ and $\underline{\mathbf{C}}^1$ is the trivial line bundle. Then, any line bundle over $\mathbf{C}P^n$ is isomorphic to ξ^m for some m.

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There exists a vector bundle $-\xi^m$ over CP^n which satisfies $\xi^m \oplus (-\xi^m) \oplus \underline{C}^j = \underline{C}^k$ for trivial vector bundles $\underline{C}^j$ and $\underline{C}^k$ of some dimensions j and k. Then, $-\xi^m$ is uniquely determined up to stable equivalence, that is, if γ satisfies the relation, then $\gamma \oplus \underline{C}^{j'} = (-\xi^m) \oplus \underline{C}^{k'}$ for some j' and k'. A vector bundle $-l\xi^m$ with an integer $l > 0$ is the Whitney sum of l numbers of $-\xi^m$. Then, we can take $-l\xi^m$ as an n-dimensional vector bundle over CP^n by the following stability property (cf. [5, Chapter 9, Section 1]):

PROPOSITION 1.1 (Stability property). *For any m-dimensional vector bundle α over CP^n with $m \geq n$, there exists an n-dimensional vector bundle β satisfying $\alpha = \beta \oplus \underline{C}^{m-n}$. In addition, β is unique for the stably equivalent class of α.*

By Schwarzenberger [4, Appendix I], if a t-dimensional vector bundle α over CP^n is extendible to CP^{n+k} for any $k \geq 0$, then α is stably equivalent to a Whitney sum of t line bundles. On the other hand, since the K-group of CP^n is additively generated by the stably equivalent classes of line bundles ξ^m for $0 \leq m \leq n$ (cf. [2]), any vector bundle over CP^n is stably equivalent to a Whitney sum of line bundles and vector bundles $-\xi^k$. Our main purpose of this paper is to determine conditions when an n-dimensional vector bundle $-l\xi^m$ over CP^n is extendible to CP^{n+1} or CP^{n+2}.

Thomas [14] has characterized the so-called Chern vectors of vector bundles over CP^n, which is applicable to our problem. Using such combinatorial relations of Chern classes, we show the following, where $\binom{a}{b}$ denotes a binomial coefficient.

THEOREM 1.2. *Let n, l and m be integers with $n > 0$ and $l > 0$, and $-l\xi^m$ be the n-dimensional vector bundle over CP^n. Then, the following hold:*

(1) *$-l\xi^m$ is extendible to CP^{n+1} if and only if the following congruence holds:*

$$\binom{n+l}{n+1} m^{n+1} \equiv 0 \pmod{n!}.$$

(2) *If $-l\xi^m$ is extendible to CP^{n+2}, then the congruence in* (1) *and the following congruence hold:*

$$l\left(m - \binom{n+2}{2}\right)\binom{n+l}{n} m^{n+1} \equiv 0 \pmod{(n+2)!}.$$

Conversely, when n is odd, $-l\xi^m$ is extendible to CP^{n+2} if the above two congruences hold.

Thus, if one of the congruences in Theorem 1.2 does not hold, then $-l\xi^m$ over CP^n is not stably equivalent to a Whitney sum of less than or equal to

n numbers of line bundles, because the latter is extendible to CP^{n+k} for any $k \geq 0$.

We also remark that the stable extendibility, introduced in [6], of the n-dimensional vector bundle $-l\xi^m$ over CP^n is the same as extendibility of it by stability property (Proposition 1.1).

Let $q(n)$ denote the product of all distinct primes less than or equal to n, that is,

$$q(n) = \prod_{\text{prime } p \leq n} p.$$

Then, in special cases, Theorem 1.2 is expressed as follows:

COROLLARY 1.3. *Assume that* $m \equiv 0 \pmod{q(n)}$. *Then, for the n-dimensional vector bundle $-l\xi^m$ over CP^n with $n > 0$ and $l > 0$, the following hold:*

(1) *$-l\xi^m$ is extendible to CP^{n+1}.*

(2) *When n is odd, if $n+2$ is not a prime or $m \equiv 0 \pmod{n+2}$, then $-l\xi^m$ is extendible to CP^{n+2}.*

(3) *When $n+2$ is a prime and $m \not\equiv 0 \pmod{n+2}$, $-l\xi^m$ is extendible to CP^{n+2} if and only if $l \not\equiv 1 \pmod{n+2}$.*

COROLLARY 1.4. *Let $-\xi^m$ be the n-dimensional vector bundle over CP^n for $n > 0$. Then, the following hold:*

(1) *$-\xi^m$ is extendible to CP^{n+1} if and only if $m \equiv 0 \pmod{q(n)}$.*

(2) *If $-\xi^m$ is extendible to CP^{n+2}, then $m \equiv 0 \pmod{q(n+2)}$ or $m \equiv 0 \pmod{q(n)}$ according as $n+2$ is a prime or not. When n is odd, the converse holds.*

Let $\nu(CP^n)$ be a normal bundle of CP^n in the sense that $\nu(CP^n)$ is a complex vector bundle satisfying that $T(CP^n) \oplus \nu(CP^n)$ is stably equivalent to a trivial vector bundle, where $T(CP^n)$ is the complex tangent bundle of CP^n. Then, $\nu(CP^n)$ exists and is unique up to stable equivalence, and the following holds:

LEMMA 1.5. *For $n \geq 2$, $\nu(CP^n)$ is not stably equivalent to any Whitney sum of line bundles over CP^n.*

Thus, by Schwarzenberger's property, any choice of normal bundle $\nu(CP^n)$ for $n \geq 2$ is not extendible to CP^{n+k} for some $k > 0$. Now, by stability property, we can take $\nu(CP^n)$ as an n-dimensional vector bundle over CP^n. Then, applying Theorem 1.2, we show the following:

THEOREM 1.6. *The n-dimensional normal bundle $\nu(\mathbf{C}P^n)$ is not extendible to $\mathbf{C}P^{n+1}$ for $n \geq 3$. $\nu(\mathbf{C}P^1) = \xi^2$ is extendible to $\mathbf{C}P^k$ for any $k \geq 1$, and $\nu(\mathbf{C}P^2)$ is extendible to $\mathbf{C}P^3$ but not extendible to $\mathbf{C}P^4$.*

The paper is organized as follows: In § 2 we prepare some necessary properties about Chern vectors studied in [14], and in § 3 we prove Theorem 1.2 and Corollaries 1.3 and 1.4. § 4 is devoted to the proof of Lemma 1.5 and Theorem 1.6.

2. Chern vectors of negative line bundles

Let $x \in H^2(\mathbf{C}P^n; \mathbf{Z})$ be the Euler class of the canonical line bundle ξ over $\mathbf{C}P^n$. Then, the cohomology ring $H^*(\mathbf{C}P^n; \mathbf{Z})$ is isomorphic to the truncated polynomial ring $\mathbf{Z}[x]/(x^{n+1})$, and the i-th Chern class $C_i(V)$ of a vector bundle V over $\mathbf{C}P^n$ is represented as an integer $c_i(V)$ multiple of x^i, namely $C_i(V) = c_i(V)x^i$. Then, the Chern vector of V is defined to be an integral vector $(c_1(V), \dots, c_n(V)) \in \mathbf{Z}^n$.

As for the Chern vector of $-l\xi^m$, we have the following:

LEMMA 2.1. *The Chern vector of $-l\xi^m$ with $l > 0$ over $\mathbf{C}P^n$ is equal to*

$$\left(-lm, \binom{l+1}{2}m^2, \dots, (-1)^i\binom{l+i-1}{i}m^i, \dots, (-1)^n\binom{l+n-1}{n}m^n\right).$$

PROOF. Let $C(V) = \sum_{i \geq 0} C_i(V)$ be the total Chern class of a vector bundle V. Then, since $C(V)$ is multiplicative and $C(\xi^m) = 1 + mx$ (cf. [4, § 4]),

$$C(-l\xi^m) = (1+mx)^{-l} = \sum_{i=0}^{n}\binom{-l}{i}m^i x^i = \sum_{i=0}^{n}(-1)^i\binom{l+i-1}{i}m^i x^i,$$

and we have the required Chern vector. □

Next, let $s_k : \mathbf{Z}^k \to \mathbf{Z}$ for $k \geq 1$ be a map defined recursively using the Newton's formula as follows: $s_1(m_1) = m_1$; for $k \geq 2$,

$$(2.1) \qquad s_k(m_1, \dots, m_k) = \sum_{i=1}^{k-1}(-1)^{i+1}m_i s_{k-i}(m_1, \dots, m_{k-i}) + (-1)^{k+1}km_k.$$

Also, for a vector bundle V over $\mathbf{C}P^n$, we set

$$(2.2) \qquad s_k(V) = s_k(c_1(V), \dots, c_k(V)).$$

Then, $s_k(V)$ for $1 \leq k \leq n$ is additive, that is, $s_k(V \oplus W) = s_k(V) + s_k(W)$ holds for vector bundles V and W over $\mathbf{C}P^n$ (cf. [4, § 10]), and obviously $s_k(\underline{\mathbf{C}}^j) = 0$ for a trivial vector bundle $\underline{\mathbf{C}}^j$.

For the line bundle ξ^m over $\mathbf{C}P^n$, since $c_1(\xi^m) = m$ and $c_i(\xi^m) = 0$ for $i \geq 2$, we have $s_k(\xi^m) = m^k$ for $k \geq 1$ by definition. Hence, for the vector bundle $-l\xi^m$ over $\mathbf{C}P^n$, we have the following:

LEMMA 2.2. *$s_k(-l\xi^m) = -lm^k$ for $1 \leq k \leq n$.*

Let $f_k : \mathbf{Z}^k \to \mathbf{Z}$ for an integer $k \geq 1$ be a map defined recursively by $f_1(m_1) = m_1$ and for $k \geq 2$

$$f_k(m_1, \dots, m_k) = f_{k-1}(m_2, \dots, m_k) - (k-1)f_{k-1}(m_1, \dots, m_{k-1}).$$

The following is straightforward from the definition.

LEMMA 2.3. (1) *f_k is a linear map, that is, for $x, y \in \mathbf{Z}^k$ and $r, s \in \mathbf{Z}$,*

$$f_k(rx + sy) = rf_k(x) + sf_k(y).$$

(2) $f_k(1, 0, 0, \dots, 0) = (-1)^{k-1}(k-1)!$.
(3) $f_k(0, \dots, 0, 1) = 1$, $f_k(0, \dots, 0, 1, 0) = -\binom{k}{2}$ *for* $k \geq 2$.
(4) ([14, Lemma 3.3(i)]). *For any integer j,*

$$f_k(j, j^2, \dots, j^k) = \prod_{i=0}^{k-1}(j - i).$$

Using the maps f_k, Thomas has shown the following.

THEOREM 2.4 ([14, Theorem A, Proposition 3.5]).
(1) *An integral vector $(m_1, \dots, m_n)$ is a Chern vector of a vector bundle over $\mathbf{C}P^n$ if and only if $f_k(s_1, \dots, s_k) \equiv 0 \pmod{k!}$ for $1 \leq k \leq n$, where $s_i = s_i(m_1, \dots, m_i)$.*
(2) *An n-dimensional vector bundle α over $\mathbf{C}P^n$ is extendible to $\mathbf{C}P^{n+1}$ if and only if the following congruence holds:*

$$f_{n+1}(s_1(\alpha), \dots, s_n(\alpha), s_{n+1}(\alpha)) \equiv 0 \pmod{(n+1)!}.$$

Some part of this theorem are slightly generalized as follows:

PROPOSITION 2.5. *If an n-dimensional vector bundle α over $\mathbf{C}P^n$ is extendible to $\mathbf{C}P^{n+k}$ for some $k \geq 1$, then the following congruences hold:*

$$f_{n+i}(s_1(\alpha), \dots, s_n(\alpha), s_{n+1}(\alpha), \dots, s_{n+i}(\alpha)) \equiv 0 \pmod{(n+i)!}$$

for any i with $1 \leq i \leq k$. Furtheremore, when n is odd and $k = 2$, the converse holds.

PROOF. If α is extendible to an n-dimensional vector bundle β over $\mathbf{C}P^{n+k}$, then $c_j(\beta) = c_j(\alpha)$ for any $j \geq 1$. Thus, $s_j(\beta) = s_j(\alpha)$ for any $j \geq 1$. Hence, applying Theorem 2.4(1) to β, we have the first required result.

As for the converse, we assume that n is odd and the congruences hold for $k = 2$. Then, by Theorem 2.4(1) and the stability property, there exists an $(n+2)$-dimensional vector bundle γ over $\mathbf{C}P^{n+2}$, which satisfies $c_i(\gamma) = c_i(\alpha)$ for any $i \geq 1$. In particular, we have $c_{n+1}(\gamma) = c_{n+2}(\gamma) = 0$. Then, by Thomas [15, Theorem 3.5], γ has two linearly independent sections, and hence there exists an n-dimensiona vector bundle β over $\mathbf{C}P^{n+2}$ satisfying $\gamma = \beta \oplus \underline{\mathbf{C}}^2$. Then, $c_i(\beta) = c_i(\gamma) = c_i(\alpha)$ for all $i \geq 1$. Since the cohomology group $H^*(\mathbf{C}P^n; \mathbf{Z})$ has no torsion, two vector bundles over $\mathbf{C}P^n$ which have the same Chern classes are stably equivalent. Thus, the restriction of β over $\mathbf{C}P^n$ is stably equivalent to α. Since α and the restriction of β are both n-dimensional vector bundles over $\mathbf{C}P^n$, they are isomorphic by stability property, which completes the proof of the converse. □

3. Proof of Theorem 1.2 and its corollaries

First, we prove Theorem 1.2 using the results in the last section.

PROOF OF THEOREM 1.2. Let α be the $(n+2)$-dimensional vector bundle $-l\xi^m$ over $\mathbf{C}P^{n+2}$. Then, by Lemmas 2.1 and 2.2,

$$c_{n+j}(\alpha) = (-1)^{n+j}\binom{l+n+j-1}{n+j}m^{n+j} \qquad \text{and} \qquad s_{n+j}(\alpha) = -lm^{n+j}$$

for $j = 1, 2$. Thus, for the vector bundle $-l\xi^m$ over $\mathbf{C}P^n$, $s_i(-l\xi^m) = -lm^i$ for $1 \leq i \leq n$, and, by (2.1) and (2.2),

$$\begin{aligned} s_{n+1}(-l\xi^m) &= s_{n+1}(\alpha) - (-1)^n(n+1)c_{n+1}(\alpha) \\ &= -lm^{n+1} + (n+1)\binom{l+n}{n+1}m^{n+1}. \end{aligned}$$

$$\begin{aligned} s_{n+2}(-l\xi^m) &= s_{n+2}(\alpha) - (-1)^n c_{n+1}(\alpha)s_1(\alpha) - (-1)^{n+1}(n+2)c_{n+2}(\alpha) \\ &= -lm^{n+2} - l\binom{l+n}{n+1}m^{n+2} + (n+2)\binom{l+n+1}{n+2}m^{n+2}. \end{aligned}$$

Now, we consider the extendibility of $-l\xi^m$ to $\mathbf{C}P^{n+1}$ in (1). Using Lemma 2.3,

$$\begin{aligned} &f_{n+1}(s_1(-l\xi^m), \dots, s_{n+1}(-l\xi^m)) \\ &\quad = -lf_{n+1}(m, \dots, m^{n+1}) + (n+1)\binom{l+n}{n+1}m^{n+1}f_{n+1}(0, \dots, 0, 1) \\ &\quad = -l\prod_{i=0}^{n}(m-i) + (n+1)\binom{n+l}{n+1}m^{n+1}. \end{aligned}$$

But, concerning the first term of the last equation,

$$\prod_{i=0}^{n}(m-i) = (n+1)!\binom{m}{n+1} \equiv 0 \pmod{(n+1)!}.$$

Hence, by Theorem 2.4(2), $-l\xi^m$ is extendible to CP^{n+1} if and only if the following congruence holds:

$$\binom{n+l}{n+1}m^{n+1} \equiv 0 \pmod{n!}, \tag{3.1}$$

which is the required result of (1).

As for the extendibility of $-l\xi^m$ to CP^{n+2} in (2), we can proceed similarly. Using Lemma 2.3,

$$\begin{aligned} f_{n+2}(s_1,\dots,s_{n+2}) &= -l\prod_{i=0}^{n+1}(m-i) - (n+1)\binom{l+n}{n+1}m^{n+1}\binom{n+2}{2} \\ &\quad + \left(-l\binom{l+n}{n+1} + (n+2)\binom{l+n+1}{n+2}\right)m^{n+2} \\ &= -l(n+2)!\binom{m}{n+2} + l\left(m - \binom{n+2}{2}\right)\binom{n+l}{n}m^{n+1}, \end{aligned}$$

where $s_i = s_i(-l\xi^m)$. Hence, by Proposition 2.5, if $-l\xi^m$ is extendible to CP^{n+2}, then the congruence (3.1) and the following congruence hold:

$$l\left(m - \binom{n+2}{2}\right)\binom{n+l}{n}m^{n+1} \equiv 0 \pmod{(n+2)!}.$$

Also, the converse holds by Proposition 2.5 when n is odd. Thus, we have completed the proof. □

In order to prove Corollaries 1.3 and 1.4, we prepare some notations. For a prime p, let $v_p(m) = a$ for an integer m if $m = p^a b$ and b is an integer prime to p, and $\alpha_p(k)$ for an integer $k \geq 1$ be the sum $\sum_{i=0}^{j} a_i$ of the coefficients in the p-adic expansion $k = \sum_{i=0}^{j} a_i p^i$, where $0 \leq a_i \leq p-1$. Then, the following is known, but we give a proof briefly.

LEMMA 3.1. *For a prime p and a positive integer k,*

$$v_p(k!) = \frac{k - \alpha_p(k)}{p-1}.$$

PROOF. When $k = 1$, it is clear. Thus, inductively, assume that the result is true for an integer $k \geq 1$. We put $k+1 = bp^t$ with $t \geq 0$ and

$b \not\equiv 0 \pmod{p}$. Then, $v_p(k+1) = t$ and $\alpha_p(k+1) = \alpha_p(b)$. Since $k = b-1$ if $t = 0$ and since

$$k = bp^t - 1 = (b-1)p^t + (p-1)p^{t-1} + \cdots + (p-1)p + (p-1)$$

if $t > 0$, we have $\alpha_p(k) = \alpha_p(b) - 1 + t(p-1)$. Thus, we have

$$\begin{aligned} v_p((k+1)!) = v_p(k!) + v_p(k+1) &= \frac{k - \alpha_p(k)}{p-1} + t \\ &= \frac{k - \alpha_p(b) + 1}{p-1} = \frac{(k+1) - \alpha_p(k+1)}{p-1}, \end{aligned}$$

which completes the induction. □

Let $q(n)$ be the product of all distinct primes less than or equal to a positive integer n, as is introduced in §1. Then, we have the following:

LEMMA 3.2. *For integers $k \geq 1$ and m, if $m^i \equiv 0 \pmod{k!}$ for some $i \geq 1$, then $m \equiv 0 \pmod{q(k)}$. Conversely, if $m \equiv 0 \pmod{q(k)}$, then $m^{k-1} \equiv 0 \pmod{k!}$.*

PROOF. First, assume that $m^i \equiv 0 \pmod{k!}$ for some $i \geq 1$. Then, $m \equiv 0 \pmod{p}$ for any prime $p \leq k$, and thus $m \equiv 0 \pmod{q(k)}$. Conversely, assume that $m \equiv 0 \pmod{q(k)}$, and let p be any prime with $p \leq k$. Then, $m \equiv 0 \pmod{p}$, and by Lemma 3.1 we have $v_p(k!) \leq k-1 \leq (k-1)v_p(m) = v_p(m^{k-1})$. Hence, we have $m^{k-1} \equiv 0 \pmod{k!}$, as is required. □

Now, we prove the corollaries.

PROOF OF COROLLARY 1.3. Assume that $m \equiv 0 \pmod{q(n)}$. As for (1), the congruence in Theorem 1.2(1) holds by Lemma 3.2, and we have the required result.

Concernig the proof of (2), we first assume that n is odd and $n+2$ is not a prime. Let p be any prime with $p \leq n$. We shall show

$$v_p((n+2)!) \leq v_p(m^{n+1}). \tag{3.2}$$

Then, since

$$l\binom{n+l}{n} m^{n+1} = \binom{n+l}{n+1}(n+1)m^{n+1}$$

and $(n+1)m^{n+1} \equiv 0 \pmod{(n+2)!}$ by (3.2), we obtain the required result in this case by Theorem 1.2(2). Now, we prove (3.2). We notice that $v_p(m^{n+1}) \geq n+1$ by the first assumption. We put $n+1 = ap^k + \sum_{i=0}^{k-1}(p-1)p^i$ for some $k \geq 0$ and $a \geq 0$ with $a \not\equiv p-1 \pmod{p}$, where we consider the second term of the right hand side of the equality is 0 when $k = 0$. Then, $\alpha_p(n+1) =$

$\alpha_p(a) + k(p-1)$ and $\nu_p(n+2) = k$, and thus we obtain (3.2) using lemma 3.1 as follows:

$$\nu_p((n+2)!) = \nu_p((n+1)!) + k = \frac{(n+1) - \alpha_p(a)}{p-1} \leq n+1 \leq \nu_p(m^{n+1}).$$

Next, assume that $m \equiv 0 \pmod{n+2}$ and $n+2$ is a prime. Then, since $n+1$ is not a prime, we have $m \equiv 0 \pmod{q(n+2)}$ by the assumptions $m \equiv 0 \pmod{q(n)}$ and $m \equiv 0 \pmod{n+2}$. Hence, $m^{n+1} \equiv 0 \pmod{(n+2)!}$ by Lemma 3.2, which establishes the congruence in Theorem 1.2(2), and thus we have (2).

Lastly, we prove (3). Thus, we assume that $n+2$ is a prime and $m \not\equiv 0 \pmod{n+2}$. Then, since $n+1$ is even and $m^{n-1} \equiv 0 \pmod{n!}$ by the first assumption and Lemma 3.2, the following term in the second congruence in Theorem 1.2 satisfies

$$l\binom{n+2}{2}\binom{n+l}{n}m^{n+1} = \frac{n+1}{2}\binom{n+l}{n+1}(n+1)(n+2)m^{n+1} \equiv 0 \pmod{(n+2)!}.$$

Thus, by Theorem 1.2(2) and (1), $-l\xi^m$ is extendible to CP^{n+2} if and only if the congruence

$$l\binom{n+l}{n}m^{n+2} = \binom{n+l}{n+1}(n+1)m^{n+2} \equiv 0 \pmod{(n+2)!}$$

holds. Since $(n+1)m^{n+2} \equiv 0 \pmod{(n+1)!}$ and $(n+1)m^{n+2} \not\equiv 0 \pmod{(n+2)!}$, the congruence is equivalent to

$$\binom{n+l}{n+1} \equiv 0 \pmod{n+2}. \tag{3.3}$$

Then, putting $n+l = c(n+2)+d$ for some integers $c \geq 0$ and $0 \leq d \leq n+1$ and using a well known property of binomial coefficients modulo a prime (cf. [12, Lemma 2.6]), we have

$$\binom{n+l}{n+1} \equiv \binom{d}{n+1} \pmod{n+2}.$$

Hence, (3.3) holds if and only if $0 \leq d \leq n$, that is, if and only if $l \not\equiv 1 \pmod{n+2}$, and thus we have completed the proof. □

Proof of Corollary 1.4. As for (1), by Theorem 1.2(1), $-\xi^m$ over CP^n is extendible to CP^{n+1} if and only if the congruence $m^{n+1} \equiv 0 \pmod{n!}$ holds since $l = 1$ in this case. Then, the congruence is equivalent to the required congruence $m \equiv 0 \pmod{q(n)}$ by Lemma 3.2.

Concerning (2), assume first that $n+2$ is a prime. Then, if $m \equiv 0 \pmod{q(n+2)}$, then $m^{n+1} \equiv 0 \pmod{(n+2)!}$ by Lemma 3.2. Thus, $-\xi^m$ is extendible to CP^{n+2} by Theorem 1.2(2). Conversely, if $-\xi^m$ is extendible to CP^{n+2}, then $m^{n+1} \equiv 0 \pmod{n!}$ by the congruence in Theorem 1.2(1), and thus $m \equiv 0 \pmod{q(n)}$ by Lemma 3.2. Then, by Corollary 1.3(2) and (3), we have $m \equiv 0 \pmod{n+2}$ since $l = 1$, and thus $m \equiv 0 \pmod{q(n+2)}$ as is required. Similarly, when n is odd and $n+2$ is not a prime, $-\xi^m$ is extendible to CP^{n+2} if $m \equiv 0 \pmod{q(n)}$ by Corollary 1.3(2), and the converse follows from the congruence in Theorem 1.2(1) and Lemma 3.2. Thus, we have completed the proof. □

4. Unxtendibility of normal bundle

First, we prove Lemma 1.5 using the K-ring structure of CP^n.

PROOF OF LEMMA 1.5. Let $X = [\xi - \underline{C}^1]$ be the stably equivalent class of ξ over CP^n. Then, the K-ring $K(CP^n)$ of CP^n is a truncated polynomial ring $Z[X]/(X^{n+1})$ (cf. [2]). The tangent bundle $T(CP^n)$ of CP^n satisfies $T(CP^n) \oplus \underline{C}^1 = (n+1)\bar{\xi} = (n+1)\xi^{-1}$ (cf. [13, Chapter V]). Thus, a normal bundle $\nu(CP^n)$ is stably equivalent to $-(n+1)\xi^{-1}$. Since $\xi \otimes \xi^{-1} = \underline{C}^1$, we have $(X+1)([\xi^{-1} - \underline{C}^1] + 1) = 1$ in $K(CP^n)$. Hence,

$$[\xi^{-1} - \underline{C}^1] = (X+1)^{-1} - 1 = \sum_{i=1}^{n} (-1)^i X^i,$$

and thus

$$[\nu(CP^n) - \underline{C}^N] = -(n+1)[\xi^{-1} - \underline{C}^1] \equiv (n+1)X - (n+1)X^2 \pmod{X^3},$$

where $n \geq 2$ and $N = \dim \nu(CP^n)$.

Now, we suppose that $\nu(CP^n)$ is stably equivalent to a Whitney sum $\xi^{k_1} \oplus \cdots \oplus \xi^{k_j}$ of line bundles, and induce a contradiction. Under the hypothesis, we have

$$[\nu(CP^n) - \underline{C}^N] = \sum_{i=1}^{j} (1+X)^{k_i} - j \equiv \sum_{i=1}^{j} k_i X + \sum_{i=1}^{j} \binom{k_i}{2} X^2 \pmod{X^3}.$$

Thus, comparing the coefficients of X and X^2 in the above two congruences,

$$\sum_{i=1}^{j} k_i = n+1 \quad \text{and} \quad \sum_{i=1}^{j} \binom{k_i}{2} = -(n+1).$$

But, these two equalities are not compatible since $\sum_{i=1}^{j} k_i^2 \neq -(n+1)$, and thus we have completed the proof. $\square$

Lastly, we prove Theorem 1.6.

PROOF OF THEOREM 1.6. Since the n-dimensional vector bundles $\nu(\mathbf{C}P^n)$ and $-(n+1)\xi^{-1}$ over $\mathbf{C}P^n$ are stably equivalent each other as is mentioned in the above, they are actually isomorphic by stability property.

The line bundle $\nu(\mathbf{C}P^1)$ is isomorphic to ξ^2 over $\mathbf{C}P^1$, because they have the same Chern classes. Thus, $\nu(\mathbf{C}P^1)$ is extendible to $\mathbf{C}P^k$ for any $k \geq 1$. As for the 2-dimensional vector bundle $\nu(\mathbf{C}P^2) = -3\xi^{-1}$, since the congruence in Theorem 1.2(1) is satisfied and the second congruence in Theorem 1.2(2) is not in the case of $n = 2$, $m = -1$ and $l = 3$, we have the required result.

Thus, we assume $n \geq 3$, and show that the n-dimensional vector bundle $-(n+1)\xi^{-1}$ is not extendible to $\mathbf{C}P^{n+1}$. By Theorem 1.2(1), it is sufficient to show

$$\binom{2n+1}{n+1} \not\equiv 0 \pmod{n!}.$$

But, the incongruence follows if we prove the inequality

$$\nu_2\binom{2n+1}{n+1} < \nu_2(n!). \tag{4.1}$$

As for the right hand side of (4.1), we have $\nu_2(n!) = n - \alpha_2(n)$ by Lemma 3.1. Since

$$\binom{2n+1}{n+1} = \frac{(2n+1)!}{(n+1)!n!} = \frac{2^n n!(2h+1)}{(n+1)!n!} = \frac{2^n(2h+1)}{(n+1)!}$$

for some integer $h > 0$, we have

$$\begin{aligned}\nu_2\binom{2n+1}{n+1} &= \nu_2(2^n) - \nu_2((n+1)!) \\ &= n - ((n+1) - \alpha_2(n+1)) = \alpha_2(n+1) - 1.\end{aligned}$$

Then, the following inequality is easily shown by the induction on $n \geq 3$:

$$\nu_2(n!) - \nu_2\binom{2n+1}{n+1} = n + 1 - \alpha_2(n) - \alpha_2(n+1) > 0.$$

Hence (4.1) holds, and thus we have completed the proof. $\square$

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