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Stable unextendibility of vector bundles over the quaternionic projective spaces

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ABSTRACT. We study the stable unextendibility of vector bundles over the quaternionic projective space $\mathbf{H}P^n$ by making use of combinatorial properties of the Stiefel-Whitney classes and the Pontrjagin classes. First, we show that the tangent bundle of $\mathbf{H}P^n$ is not stably extendible to $\mathbf{H}P^{n+1}$ for $n \geq 2$, and also induce such a result for the normal bundle associated to an immersion of $\mathbf{H}P^n$ into $\mathbf{R}^{4n+k}$. Secondly, we show a sufficient condition for a quaternionic r-dimensional vector bundle over $\mathbf{H}P^n$ not to be stably extendible to $\mathbf{H}P^{n+l}$ for $r \leq n$ and $l > 0$, which is also a necessary condition when $r = n$ and $l = 1$.

1. Introduction and results

Let F be the real field $\mathbf{R}$, the complex field $\mathbf{C}$ or the quaternionic skew field $\mathbf{H}$. Then, an F-vector bundle V of dimension k over a base space B is called extendible to a space B' with $B \subset B'$ if there exists an F-vector bundle W of dimension k over B' whose restriction to B is isomorphic to V as F-vector bundles. That is, $i^*W \cong V$ for the inclusion map $i: B \to B'$. If i^*W is stably equivalent to V, namely $i^*W + m_F \cong V + m_F$ for a trivial F-vector bundle m_F of dimension $m \geq 0$, then V is called stably extendible to B' ([6]).

It is an interesting problem to determine when given vector bundles are stably extendible or not, which is related to some stable properties of vector bundles like geometrical dimensions or decompositions to line bundles (cf. [12], [2], [11], [9]). In this paper, we study the stable unextendibility of some vector bundles over the quaternionic projective space $\mathbf{H}P^n$.

Schwarzenberger ([4, Appendix 1]) has shown, as an application of the Riemann-Roch theorem, that the tangent bundle $T(\mathbf{C}P^n)$ of the complex projective space $\mathbf{C}P^n$ for $n \geq 2$ is not extendible to $\mathbf{C}P^{n+1}$ as $\mathbf{C}$-vector bundle. Kobayashi-Maki-Yoshida [7] has also shown that the tangent bundle $T(\mathbf{R}P^n)$ (resp. $T(L^n(p))$) of the real projective space $\mathbf{R}P^n$ (resp. the lens space $L^n(p)$ for

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an odd prime p) is not stably extendible to $\mathbf{R}P^{n+1}$ (resp. $L^{2n+2}(p)$) if $n \neq 1, 3$ or 7 (resp. $n \geq 2p-2$) as $\mathbf{R}$-vector bundle.

The tangent bundle $T(\mathbf{H}P^n)$ of the quaternionic projective space $\mathbf{H}P^n$ is an $\mathbf{R}$-vector bundle of dimension $4n$, and we show the following result applying the Stiefel-Whitney classes of $\mathbf{R}$-vector bundles over $\mathbf{H}P^n$.

THEOREM A. *For $n \geq 2$, $T(\mathbf{H}P^n)$ is not stably extendible to $\mathbf{H}P^{n+1}$.*

Since $T(\mathbf{H}P^1) = T(S^4)$ is stably trivial, it is stably extendible to $\mathbf{H}P^k$ for any $k \geq 2$. We remark that $T(\mathbf{H}P^1)$ is extendible to $\mathbf{H}P^2$ (see Lemma 2.4). Hence, the unextendibility of $T(\mathbf{H}P^n)$ to $\mathbf{H}P^{n+1}$ agrees with the stable unextendibility of it.

Stable extendibility of the normal bundles of $\mathbf{R}P^n$ is studied in [8], and in [9] it is remarked that the stable extendibility of the normal bundle associated to an immersion $f: M \subseteq \mathbf{R}^l$ of a manifold M does not depend on the map f but only on the existence of the immersion of M in $\mathbf{R}^l$. Let ν^k be the normal bundle associated to an immersion $\mathbf{H}P^n \subseteq \mathbf{R}^{4n+k}$ if it exists. Then, we obtain the following by a similar method used for the proof of Theorem A.

THEOREM B. *Assume that $k \leq 4n+3$ and $n = 2^m - 1$ for some $m \geq 2$. Then, if an immersion $\mathbf{H}P^n \subseteq \mathbf{R}^{4n+k}$ exists, its normal bundle ν^k is not stably extendible to $\mathbf{H}P^{n+1}$.*

Thomas [15] has studied the so called Chern vectors whose components are given from the Chern classes, and apply it to obtain some condition on the extendibility of $\mathbf{C}$-vector bundles over $\mathbf{C}P^n$. Analogously, we have the Pontrjagin classes of $\mathbf{H}$-vector bundles (cf. [13, Chapter V]), and we can consider the notion of the Pontrjagin vectors and apply it to study the stable unextendibility of $\mathbf{H}$-vector bundles over $\mathbf{H}P^n$. We need some notations to express the result.

First, let ξ be the canonical $\mathbf{H}$-line bundle over $\mathbf{H}P^n$, and $x \in H^4(\mathbf{H}P^n; \mathbf{Z})$ the Euler class of ξ. Then, the cohomology ring $H^*(\mathbf{H}P^n; \mathbf{Z})$ is isomorphic to the truncated polynomial ring $\mathbf{Z}[x]/(x^{n+1})$, and the i-th Pontrjagin class $P_i(V) = (-1)^i C_{2i}(c'(V))$ of an $\mathbf{H}$-vector bundle V can be represented as an integer $p_i(V)$ multiple of x^i, namely $P_i(V) = p_i(V)x^i$. Here, $c'(V)$ denotes the underlying $\mathbf{C}$-vector bundle of V, and $C_j(c'(V))$ is the j-th Chern class of it. Then, we define *the Pontrjagin vector* of V as the integral vector $(p_1(V), \dots, p_n(V)) \in \mathbf{Z}^n$.

Next, let $s_k : \mathbf{Z}^k \to \mathbf{Z}$ for $k \geq 1$ be the map defined recursively using the Newton's relations as follows: $s_1(m_1) = m_1$; for $k \geq 2$,

$$(1) \qquad s_k(m_1, \dots, m_k) = \sum_{i=1}^{k-1} (-1)^{i+1} m_i s_{k-i}(m_1, \dots, m_{k-i}) + (-1)^{k+1} k m_k.$$

Thirdly, let $g_k : \mathbf{Z}^k \to \mathbf{Z}$ for $k \geq 1$ be the map defined recursively by $g_1(m_1) = m_1$ and, for $k \geq 2$,

$$g_k(m_1, \dots, m_k) = g_{k-1}(m_2, \dots, m_k) - (k-1)^2 g_{k-1}(m_1, \dots, m_{k-1}). \tag{2}$$

Then, we show the following, and throughout the paper $a(i) = 1$ or 2 according as i is an even or odd integer.

Theorem C. *A necessary and sufficient condition for an integral vector $\boldsymbol{q} = (q_1, \dots, q_n)$ to be a Pontrjagin vector of an* $\mathbf{H}$*-vector bundle over* $\mathbf{H}P^n$ *is that the congruences*

$$g_i(s_1(q_1), \dots, s_i(q_1, \dots, q_i)) \equiv 0 \qquad \left(\mathrm{mod}\ \frac{(2i)!}{a(i)}\right)$$

hold for all i with $1 \leq i \leq n$.

Now, for an $\mathbf{H}$-vector bundle α of dimension r over $\mathbf{H}P^n$ with $r \leq n$, we set

$$s_k = s_k(p_1(\alpha), \dots, p_k(\alpha)) \quad \text{for } 1 \leq k \leq r, \qquad \text{and} \tag{3}$$

$$t_{r+i} = s_{r+i}(p_1(\alpha), \dots, p_r(\alpha), \underbrace{0, \dots, 0}_{i}) \quad \text{for } i \geq 1. \tag{4}$$

Then, we have the following result from Theorem C.

Theorem D. *An* $\mathbf{H}$*-vector bundle α of dimension r over* $\mathbf{H}P^n$ *with $r \leq n$ is not stably extendible to* $\mathbf{H}P^m$ *for $m > n$ if*

$$g_{r+i}(s_1, \dots, s_r, t_{r+1}, \dots t_{r+i}) \not\equiv 0 \qquad \left(\mathrm{mod}\ \frac{(2(r+i))!}{a(r+i)}\right)$$

for some i with $1 \leq i \leq m - r$.

When an $\mathbf{H}$-vector bundle α over $\mathbf{H}P^n$ is of dimension n, α is stably extendible to $\mathbf{H}P^m$ if and only if it is extendible to $\mathbf{H}P^m$, and we have the following.

Corollary E. *An* $\mathbf{H}$*-vector bundle α of dimension n over* $\mathbf{H}P^n$ *is not extendible to* $\mathbf{H}P^{n+1}$ *if and only if the following holds:*

$$g_{n+1}(s_1, \dots, s_n, t_{n+1}) \not\equiv 0 \qquad \left(\mathrm{mod}\ \frac{(2(n+1))!}{a(n+1)}\right).$$

The paper is organized as follows. In §2 we prove Theorems A and B, and in §3 we study the Pontrjagin vectors and prove Theorem C. In §4, we prove Theorem D and Corollary E, and, as an example, we show in Proposition 4.1 some condition under which an $\mathbf{H}$-vector bundle of dimension n stably equivalent to $(n+k)\xi$ for some $k > 0$ is not extendible.

2. Proof of Theorems A and B

Let $KO^i(B), K^i(B)$ and $KSp^i(B)$ for $i \in \mathbf{Z}$ be the reduced real, complex and symplectic K-group of a compact space B, respectively, which are often denoted by $\widetilde{KO}^i(B)$ and so on. Then $KSp^0(B)$ is an abelian group formed by stable classes $[V - (\dim V)_{\mathbf{H}}]$ of virtual dimensions 0 for $\mathbf{H}$-vector bundles V over B, and $KSp^0(B) = KO^4(B)$ by definition. We refer to the classical works (cf. [1], [14, Chapter 11, 13]) for the general properties on these K-theories and use them without comments.

By the multiplications induced from the tensor products of vector bundles, $KO^{4*}(B) = \bigoplus_{i \in \mathbf{Z}} KO^{4i}(B)$ is a graded $KO^{4*}(S^0)$-algebra. To assign an $\mathbf{R}$-vector bundle (resp. $\mathbf{H}$-vector bundle) V the $\mathbf{C}$-vector bundle $V \otimes_{\mathbf{R}} \mathbf{C}$ (resp. the underlying $\mathbf{C}$-vector bundle $c'(V)$) induces a homomorphism $c: KO^0(B) \to K^0(B)$ (resp. $c: KO^4(B) = KSp^0(B) \to K^0(B)$), and c is extended to a ring homomorphism $c: KO^{4*}(B) \to K^0(B)$ using the Bott periodicity, which also satisfies $c(\alpha y) = c(\alpha)c(y)$ for $\alpha \in KO^{4i}(S^0)$ and $y \in KO^{4j}(B)$. Here, $\alpha y \in KO^{4(i+j)}(B)$ denotes the element defined by the $KO^{4*}(S^0)$-algebra structure of $KO^{4*}(B)$, and we regard $c(\alpha) \in K^0(S^0)$ as an integer by $K^0(S^0) \cong \mathbf{Z}$. If $KO^{4i}(B)$ is a free abelian group for some $i \in \mathbf{Z}$, then $c: KO^{4i}(B) \to K^0(B)$ is a monomorphism.

As for the ring $KO^{4*}(S^0)$, we have an isomorphism $KO^{4*}(S^0) \cong \mathbf{Z}[u, v, w]/(u^2 - 4v, vw - 1)$, where $u \in KO^{-4}(S^0)$, $v \in KO^{-8}(S^0)$ and $w = v^{-1} \in KO^8(S^0)$. Then, as is known, we can take u and v to satisfy $c(u) = 2$ and $c(v) = 1$ regarding $K^0(S^0)$ as $\mathbf{Z}$. We set $g_{2i} = v^i \in KO^{-8i}(S^0)$ and $g_{2i+1} = uv^i \in KO^{-8i-4}(S^0)$ for $i \in \mathbf{Z}$, and thus we have $c(g_{2i}) = 1$ and $c(g_{2i+1}) = 2$.

Now, let $X = [\xi - 1_{\mathbf{H}}] \in KSp^0(\mathbf{H}P^n) = KO^4(\mathbf{H}P^n)$ be the stable class of the canonical $\mathbf{H}$-line bundle ξ over $\mathbf{H}P^n$. Then, $c(X) = [c'(\xi) - 2_{\mathbf{C}}] \in K^0(\mathbf{H}P^n)$ by the definition of c. Since $H^*(\mathbf{H}P^n; \mathbf{Z}) \cong \mathbf{Z}[x]/(x^{n+1})$ and $KO^{4i+1}(S^0) = 0$ for any $i \in \mathbf{Z}$, the Atiyah-Hirzebruch spectral sequence

$$E_2^{p,q} = \tilde{H}^p(\mathbf{H}P^n; \mathbf{Z}) \otimes KO^q(S^0) \Rightarrow KO^{p+q}(\mathbf{H}P^n)$$

collapses. Remark that $E_2^{4p,-4p} = \mathbf{Z}\{x^p \otimes g_p\}$ for any p with $1 \leq p \leq n$. Then, $g_1 X \in KO^0(\mathbf{H}P^n)$ is represented by $x \otimes g_1$ modulo $\mathbf{Z}\{x^i \otimes g_i\}$ for all i with $2 \leq i \leq n$ in the E_2-term of the spectral sequence, because $i^*(g_1 X) \in KO^0(\mathbf{H}P^1) \cong \mathbf{Z}$ is a generator for the inclusion $i: \mathbf{H}P^1 = S^4 \to \mathbf{H}P^n$ and the E_2-term $E_2^{p,q}(1) = \tilde{H}^p(\mathbf{H}P^1; \mathbf{Z}) \otimes KO^q(S^0)$ of the Atiyah-Hirzebruch spectral sequence on $\mathbf{H}P^1$ satisfies that $E_2^{4,-4}(1) = \mathbf{Z}\{x \otimes g_1\}$ and $E_2^{p,-p}(1) = 0$ if $p \neq 4$. Since $(g_1 X)^{2m+\varepsilon} = 4^m g_{2m+\varepsilon} X^{2m+\varepsilon}$ in $KO^0(\mathbf{H}P^n)$ and $(x \otimes g_1)^{2m+\varepsilon} = 4^m x^{2m+\varepsilon} \otimes g_{2m+\varepsilon}$ in $E_2^{4(2m+\varepsilon), -4(2m+\varepsilon)}$ for $\varepsilon = 0$ or 1, $g_p X^p$ is represented by $x^p \otimes g_p$ modulo $\mathbf{Z}\{x^i \otimes g_i\}$ for all i with $p + 1 \leq i \leq n$ in the E_2-term of the spectral sequence. In this way, we see that $KO^0(\mathbf{H}P^n)$ is a free abelian group with basis $g_1X, g_2X^2, \dots, g_nX^n$. That is, we have

$$KO^0(\mathbf{H}P^n) \cong \mathbf{Z}\{g_1X, g_2X^2, \dots, g_nX^n\}. \tag{5}$$

The tangent bundle $T(\mathbf{H}P^n)$ of $\mathbf{H}P^n$ satisfies

$$T(\mathbf{H}P^n) + \xi \otimes_{\mathbf{H}} \xi^* \cong (n+1)\xi_{\mathbf{R}}, \tag{6}$$

where ξ^* and $\xi_{\mathbf{R}}$ denote the quaternionic conjugate bundle and the underlying real vector bundle of ξ, respectively. Then, we have the following.

Lemma 2.1. *In* $KO^0(\mathbf{H}P^n)$, $[T(\mathbf{H}P^n) - 4n_{\mathbf{R}}] = (n-1)g_1X - g_2X^2$.

Proof. We remark that the underlying $\mathbf{C}$-vector bundle $c'(\xi)$ of ξ is self conjugate and $c(\xi_{\mathbf{R}}) \cong 2c'(\xi)$. Also, there is a relation $c(\xi \otimes_{\mathbf{H}} \xi^*) = c'(\xi) \otimes_{\mathbf{C}} c'(\xi) = c'(\xi)^2$. Thus, we have the following equalities:

$$\begin{aligned} c(g_1X) &= c(g_1)c(X) = 2[c'(\xi) - 2_{\mathbf{C}}] = c([\xi_{\mathbf{R}} - 4_{\mathbf{R}}]); \\ c(g_2X^2) &= c(g_2)c(X)^2 = [c'(\xi) - 2_{\mathbf{C}}]^2 = [c'(\xi)^2 - 4_{\mathbf{C}}] - 4[c'(\xi) - 2_{\mathbf{C}}] \\ &= c([\xi \otimes_{\mathbf{H}} \xi^* - 4_{\mathbf{R}}] - 2g_1X). \end{aligned}$$

Since the homomorphism $c : KO^0(\mathbf{H}P^n) \to K^0(\mathbf{H}P^n)$ is a monomorphism, we have $[\xi_{\mathbf{R}} - 4_{\mathbf{R}}] = g_1X$ and $[\xi \otimes_{\mathbf{H}} \xi^* - 4_{\mathbf{R}}] = 2g_1X + g_2X^2$. Hence, by (6),

$$[T(\mathbf{H}P^n) - 4n_{\mathbf{R}}] = (n+1)[\xi_{\mathbf{R}} - 4_{\mathbf{R}}] - [\xi \otimes_{\mathbf{H}} \xi^* - 4_{\mathbf{R}}] = (n-1)g_1X - g_2X^2$$

as is required. □

Let $w(V) = 1 + w_1(V) + \cdots \in H^*(B; \mathbf{Z}/2)$ be the total Stiefel-Whitney class of an $\mathbf{R}$-vector bundle V over a compact space B. Then, by the stable and multiplicative properties of the Stiefel-Whitney classes, we can also have the Stiefel-Whitney class $w(\alpha) \in H^*(B; \mathbf{Z}/2)$ of $\alpha \in KO^0(B)$. As for the elements g_iX^i of $KO^0(\mathbf{H}P^n)$ in (5), we have the following lemma, where we denote the mod 2 reduction of $x \in H^4(\mathbf{H}P^n; \mathbf{Z})$ by the same letter x.

Lemma 2.2. $w(g_1X) = 1 + x$, $w(g_2X^2) = (1+x)^{-2}$ *and* $w(g_mX^m) \equiv 1 \pmod{x^{m+1}}$ *for* $m \geq 3$.

Proof. First, we shall prove the following congruence, where k is a positive integer and $\varepsilon = 0$ or 1:

$$C(c'(\xi)^{2k+\varepsilon}) \equiv 1 + \varepsilon x^{4^k} \quad (\mathrm{mod}\ 2). \tag{7}$$

Here, $C(c'(\xi)^{2k+\varepsilon})$ denotes the total Chern class of the $(2k+\varepsilon)$-fold tensor product of $c'(\xi)$.

Let η be the canonical $\mathbf{C}$-line bundle over $\mathbf{C}P^{2n}$. Then, for the canonical projection $p : \mathbf{C}P^{2n} \to \mathbf{H}P^n$, we have $p^*(c'(\xi)) = \eta + \bar{\eta}$, where $\bar{\eta}$ is the complex conjugate bundle of η, and $p^*(x) = t^2$ for the Euler class $t \in H^2(\mathbf{C}P^{2n};\mathbf{Z})$ of η. Since $\eta\bar{\eta} = \eta \otimes_{\mathbf{C}} \bar{\eta} = 1_{\mathbf{C}}$, we have

$$p^*(c'(\xi)^{2k+1}) = \sum_{i=0}^{k} \binom{2k+1}{i} (\eta^{2(k-i)+1} + \bar{\eta}^{2(k-i)+1}).$$

Since $C(\eta^j) = 1 + jt$, $C(\bar{\eta}^j) = 1 - jt$ and $p^* : H^*(\mathbf{H}P^n;\mathbf{Z}) \to H^*(\mathbf{C}P^{2n};\mathbf{Z})$ is a monomorphism, we have the congruence

$$C(c'(\xi)^{2k+1}) = \prod_{i=0}^{k} (1 - (2(k-i)+1)^2 x)^{a_i} \equiv (1+x)^a \qquad (\text{mod } 2)$$

for $a_i = \binom{2k+1}{i}$ and $a = \sum_{i=0}^{k} a_i = 4^k$, and thus we have (7) for $\varepsilon = 1$. The congruence (7) for $\varepsilon = 0$ is similarly shown.

Recall that $c(X) = [c'(\xi) - 2_{\mathbf{C}}]$. Then, for $\varepsilon = 0$ or 1, we have

$$(8) \qquad C(c(X)^{2k+\varepsilon}) \equiv 1 \qquad (\text{mod}(2, x^{4^k})).$$

In fact, for $\varepsilon = 0$, $c(X)^{2k} = \sum_{i=0}^{2k} b_i (-2)^i [c'(\xi)^{2k-i} - 2_{\mathbf{C}}^{2k-i}]$, where $b_i = \binom{2k}{i}$. Then, using (7), we have the following congruences:

$$C(c(X)^{2k}) \equiv \prod_{j=0}^{k-1} (1 + x^{4^{k-j-1}})^{-2^{2j+1} b_{2j+1}} \equiv (1 + x^{4^k})^d \qquad (\text{mod } 2),$$

where $d = -(1/2) \sum_{j=0}^{k-1} b_{2j+1}$ is an integer. Thus, we have $C(c(X)^{2k}) \equiv 1$ $(\text{mod}(2, x^{4^k}))$ as is required in (8) for $\varepsilon = 0$. The congruence (8) for $\varepsilon = 1$ is similarly shown.

Now, we conclude the proof of the congruence $w(g_m X^m) \equiv 1 \pmod{x^{m+1}}$ for $m \geq 3$. Generally, for an element $\alpha = [V - (\dim V)_{\mathbf{R}}] \in KO^0(B)$, we have $C(c(\alpha)) = C(c(V)) \equiv w(V)^2 = w(\alpha)^2$ (mod 2). Thus,

$$\begin{aligned} w(g_m X^m)^2 &\equiv C(c(g_m X^m)) = C(c(g_m) c(X)^m) = C(a(m) c(X)^m) \\ &= C(c(X)^m)^{a(m)} \qquad (\text{mod } 2), \end{aligned}$$

where $a(m) = 1$ or 2 according as m is an even or odd integer. When $m \geq 3$, $C(c(X)^m)^{a(m)} \equiv 1$ $(\text{mod}(2, x^{2m+2}))$ by (8) since $2m + 2 \leq 2^m$. Therefore, $w(g_m X^m)^2 \equiv 1 \pmod{x^{2m+2}}$. This congruence is valid on $\mathbf{H}P^N$ for any $N \geq 1$ by the same reason, in particular on $\mathbf{H}P^{2n}$. Hence, we have $w(g_m X^m) \equiv 1$ $(\text{mod } x^{m+1})$ on $\mathbf{H}P^n$ for $m \geq 3$ as is required.

In the case of $m = 1$ and 2, we have

$$\begin{aligned} w(g_1X)^2 &\equiv C(c(g_1)c(X)) = C(2[c'(\xi) - 2_{\mathbf{C}}]) = C(c'(\xi))^2 \\ &\equiv (1+x)^2 \qquad (\mathrm{mod}\ 2), \\ w(g_2X^2)^2 &\equiv C(c(g_2)c(X)^2) = C([c'(\xi) - 2_{\mathbf{C}}]^2) = C(c'(\xi)^2)C(c'(\xi))^{-4} \\ &\equiv (1+x)^{-4} \qquad (\mathrm{mod}\ 2), \end{aligned}$$

which is still valid on $\mathbf{H}P^{2n}$. Thus, $w(g_1X) = 1 + x$ and $w(g_2X^2) = (1+x)^{-2}$ on $\mathbf{H}P^n$, and we have completed the proof. □

COROLLARY 2.3. *Let γ be an $\mathbf{R}$-vector bundle over $\mathbf{H}P^{n+1}$ for $n \geq 2$. If the restriction $i^*\gamma$ over $\mathbf{H}P^n$ satisfies $i^*\gamma + s_{\mathbf{R}} \cong T(\mathbf{H}P^n) + t_{\mathbf{R}}$ for some $s, t \geq 0$, then it follows $w(\gamma) = (1+x)^{n+1}$ in $H^*(\mathbf{H}P^{n+1}; \mathbf{Z}/2)$.*

PROOF. The kernel of the homomorphism $i^* : KO^0(\mathbf{H}P^{n+1}) \to KO^0(\mathbf{H}P^n)$ is a free abelian group of rank 1 with generator $g_{n+1}X^{n+1}$ by (5). Thus, by Lemma 2.1, the stable class of γ satifies

$$[\gamma - r_{\mathbf{R}}] = (n-1)g_1X - g_2X^2 + ag_{n+1}X^{n+1} \qquad \text{in } KO^0(\mathbf{H}P^{n+1})$$

for some integer a, where r is the dimension of γ. Then, using Lemma 2.2, we have

$$w(\gamma) = w(g_1X)^{n-1}w(g_2X^2)^{-1}w(g_{n+1}X^{n+1})^a = (1+x)^{n+1}$$

as is required. □

Now we complete the proofs of Theorem A and Theorem B.

PROOF OF THEOREM A. Assume that $T(\mathbf{H}P^n)$ is stably extendible to $\mathbf{H}P^{n+1}$ for some $n \geq 2$. Then, there exists a $4n$-dimensional $\mathbf{R}$-vector bundle β over $\mathbf{H}P^{n+1}$ whose restriction to $\mathbf{H}P^n$ is stably equivalent to $T(\mathbf{H}P^n)$. Then, by Corollary 2.3, we have $w(\beta) = (1+x)^{n+1}$, which contradicts that β is of dimension $4n$. Thus, we have the required result. □

PROOF OF THEOREM B. Assume that $n = 2^m - 1$ for $m \geq 2$ and $k \leq 4n + 3$. Since $v_k + T(\mathbf{H}P^n)$ is equivalent to the trivial bundle $(4n+k)_{\mathbf{R}}$, we have

$$[v^k - k_{\mathbf{R}}] = -[T(\mathbf{H}P^n) - 4n_{\mathbf{R}}] = g_2X^2 - (n-1)g_1X$$

in $KO^0(\mathbf{H}P^n)$ by Lemma 2.1. By the same reason as in Corollary 2.3, if there exists a k-dimensional $\mathbf{R}$-vector bundle γ over $\mathbf{H}P^{n+1}$ satisfying that $i^*\gamma$ is stably equivalent to v^k, then $w(\gamma) = (1+x)^{-(n+1)}$, and thus

$$w_{4(n+1)}(\gamma) = \binom{2n+1}{n+1}x^{n+1}.$$

Hence, when $n = 2^m - 1$, $w_{4(n+1)}(\gamma) \neq 0$ which contradicts that γ is of dimension k with $k \leq 4n+3$. Thus, we have completed the proof. $\square$

As the last of this section, we show the following mentioned in §1.

LEMMA 2.4. *The tangent bundle* $T(\mathbf{H}P^1)$ *of* $\mathbf{H}P^1$ *is extendible to* $\mathbf{H}P^2$.

PROOF. Let $[X, Y]$ denote the homotopy set of maps from a space X to a space Y, and BG the classifying space of a group G. Then, for the inclusion map $i : \mathbf{H}P^1 \to \mathbf{H}P^2$, we shall show that $i^* : [\mathbf{H}P^2, BO(4)] \to [\mathbf{H}P^1, BO(4)]$ is surjective, which establishes that $T(\mathbf{H}P^1)$ is extendible to $\mathbf{H}P^2$. As is known, we have an isomorphism $Spin(4) \cong Sp(1) \times Sp(1)$ of the Lie groups, and thus $BSpin(4) = BSp(1) \times BSp(1) = \mathbf{H}P^\infty \times \mathbf{H}P^\infty$. Then, for $i = 1$ or 2, we have the canonical bijection $[\mathbf{H}P^i, BO(4)] \approx [\mathbf{H}P^i, BSpin(4)] \approx [\mathbf{H}P^i, \mathbf{H}P^\infty] \times [\mathbf{H}P^i, \mathbf{H}P^\infty] \approx [\mathbf{H}P^i, \mathbf{H}P^2] \times [\mathbf{H}P^i, \mathbf{H}P^2]$, where the last bijection is induced by the cellular approximation. Thus, it is sufficient to show that $i^* : [\mathbf{H}P^2, \mathbf{H}P^2] \to [\mathbf{H}P^1, \mathbf{H}P^2]$ is surjective.

Let $\varphi : S^7 \to \mathbf{H}P^1$ be the attaching map of the top cell of $\mathbf{H}P^2$. Then, the cofiber sequsence $S^7 \xrightarrow{\varphi} \mathbf{H}P^1 \xrightarrow{i} \mathbf{H}P^2$ induces the exact sequence of the homotopy sets $[\mathbf{H}P^2, \mathbf{H}P^2] \xrightarrow{i^*} [\mathbf{H}P^1, \mathbf{H}P^2] \xrightarrow{\varphi^*} [S^7, \mathbf{H}P^2]$. However, $[\mathbf{H}P^1, \mathbf{H}P^2] = [S^4, \mathbf{H}P^2]$ is a free abelian group $\mathbf{Z}$ with a base of the homotopy class of i, and thus $\varphi^* = 0$. Hence, i^* is surjective, and we have completed the proof. $\square$

3. Pontrjagin vectors

First, we define an $\mathbf{H}$-vector bundles $\xi(k)$ recursively as follows:

$$\xi(1) = \xi;$$
$$\xi(2i) = (\xi(2i-1) \otimes_{\mathbf{H}} \xi^*) \otimes_{\mathbf{R}} 1_{\mathbf{H}} \qquad \text{for } i \geq 1;$$
$$\xi(2i+1) = (\xi(2i-1) \otimes_{\mathbf{H}} \xi^*) \otimes_{\mathbf{R}} \xi \qquad \text{for } i \geq 1.$$

Here, the $\mathbf{H}$-vector bundle structures of $\xi(2i)$ and $\xi(2i+1)$ are given by $1_{\mathbf{H}}$ and ξ, respectively. Thus, $\xi(k)$ is an $\mathbf{H}$-vector bundle of dimension $2^k/a(k)$, where $a(k) = 1$ or 2 according as k is an even or odd integer. We have $c'(\xi(2i)) = c(\xi(2i-1) \otimes_{\mathbf{H}} \xi^*) \otimes_{\mathbf{C}} 2_{\mathbf{C}} = 2c'(\xi(2i-1)) \otimes_{\mathbf{C}} c'(\xi)$ and $c'(\xi(2i+1)) = c(\xi(2i-1) \otimes_{\mathbf{H}} \xi^*) \otimes_{\mathbf{C}} c'(\xi) = c'(\xi(2i-1)) \otimes_{\mathbf{C}} c'(\xi)^2$. For instance, $c'(\xi(2)) = 2c'(\xi)^2$ and $c'(\xi(3)) = c'(\xi)^3$. Thus, proceeding recursively, we have the relation

$$c'(\xi(k)) = \frac{2}{a(k)} c'(\xi)^k \tag{9}$$

for any $k \geq 1$. Furtheremore, we have

LEMMA 3.1. *$\{[\xi(i)-(2^i/a(i))_{\mathbf{H}}] \mid 1 \le i \le n\}$ forms a basis of the free abelian group $KSp^0(\mathbf{H}P^n)$.*

PROOF. By the similar reason as for (5), $KSp^0(\mathbf{H}P^n) = KO^4(\mathbf{H}P^n)$ is a free abelian group with basis $\{X, g_1X^2, \dots, g_{n-1}X^n\}$. Let $c: KO^{4*}(\mathbf{H}P^n) \to K^0(\mathbf{H}P^n)$ be the homomorphism mentioned in the first part of §2. Then, we have

$$c(g_{i-1}X^i) = c(g_{i-1})c(X)^i = a(i-1)[c'(\xi)-2_{\mathbf{C}}]^i = \frac{2}{a(i)}[c'(\xi)-2_{\mathbf{C}}]^i$$
$$= \sum_{j=1}^{i}\binom{i}{j}(-2)^{i-j}\frac{a(j)}{a(i)}\left(\frac{2}{a(j)}[c'(\xi)^j-2_{\mathbf{C}}^j]\right)$$

for any i with $1 \le i \le n$. Here, we have $(2/a(j))[c'(\xi)^j - 2_{\mathbf{C}}^j] = c([\xi(j) - (2^j/a(j))_{\mathbf{H}}])$ for $j \ge 1$ by (9), and the homomorphism $c: KSp^0(\mathbf{H}P^n) \to K^0(\mathbf{H}P^n)$ is a monomorphism. Thus, each $g_{i-1}X^i$ is written using $[\xi(j) - (2^j/a(j))_{\mathbf{H}}]$ for $1 \le j \le i$ as follows:

$$g_{i-1}X^i = [\xi(i)-(2^i/a(i))_{\mathbf{H}}] + \sum_{j=1}^{i-1} b_j[\xi(j)-(2^j/a(j))_{\mathbf{H}}]$$

for some integers b_j. Since $\{g_{i-1}X^i \mid 1 \le i \le n\}$ is a basis of $KSp^0(\mathbf{H}P^n)$, $\{[\xi(i)-(2^i/a(i))_{\mathbf{H}}] \mid 1 \le i \le n\}$ is also a basis of it from these equalities. Thus, we obtain the required result. □

Let $p: \mathbf{C}P^{2n} \to \mathbf{H}P^n$ be the canonical projection, η the canonical $\mathbf{C}$-line bundle over $\mathbf{C}P^{2n}$ and $\bar{\eta}$ the complex conjugate bundle of η as in the proof of Lemma 2.2. Then, p induces a monomorphism $p^*: K^0(\mathbf{H}P^n) \to K^0(\mathbf{C}P^{2n})$, and we have the following.

LEMMA 3.2. *There exists a basis $\{Y_i \mid 1 \le i \le n\}$ of $KSp^0(\mathbf{H}P^n)$ which satisfies $p^*(c(Y_i)) = (2/a(i))[\eta^i + \bar{\eta}^i - 2_{\mathbf{C}}]$.*

PROOF. We put $Z_i = [\xi(i) - (2^i/a(i))_{\mathbf{H}}]$ for $1 \le i \le n$ to the basis of $KSp^0(\mathbf{H}P^n)$ in Lemma 3.1. Then, using the relation $p^*(c'(\xi)) = \eta + \bar{\eta}$ and (9), we have

$$p^*(c(Z_i)) = \frac{2}{a(i)}p^*([c'(\xi)^i - 2_{\mathbf{C}}^i]) = \frac{2}{a(i)}([(\eta+\bar{\eta})^i - 2_{\mathbf{C}}^i])$$
$$= \frac{2}{a(i)}[\eta^i + \bar{\eta}^i - 2_{\mathbf{C}}] + \sum_{0<j<i/2}\binom{i}{j}\frac{2}{a(i-2j)}[\eta^{i-2j} + \bar{\eta}^{i-2j} - 2_{\mathbf{C}}].$$

Then, since p^* and c are both monomorphisms, $\{(2/a(i))[\eta^i + \bar{\eta}^i - 2_{\mathbf{C}}] \mid 1 \leq i \leq n\}$ forms a basis of $p^*c(KSp^0(\mathbf{H}P^n))$, and thus we have a basis $\{Y_i \mid 1 \leq i \leq n\}$ of $KSp^0(\mathbf{H}P^n)$ which satisfies $p^*(c(Y_i)) = (2/a(i))[\eta^i + \bar{\eta}^i - 2_{\mathbf{C}}]$, as is required. □

As mentioned in §1, the Pontrjagin vector of an $\mathbf{H}$-vector bundle V over $\mathbf{H}P^n$ is defined to be an integral vector

$$p(V) = (p_1(V), p_2(V), \dots, p_n(V)) \in \mathbf{Z}^n,$$

where the integers $p_i(V)$ satisfy $P_i(V) = p_i(V)x^i$ for the Pontrjagin classes $P_i(V) \in H^{4i}(\mathbf{H}P^n; \mathbf{Z})$ of V, respectively.

Since the total Pontrjagin class $P(V) = 1 + P_1(V) + P_2(V) + \cdots$ satisfies the multiplicative property $P(V + W) = P(V)P(W)$ and $P(k_{\mathbf{H}}) = 1$, we can also define the Pontrjagin vector $p(\alpha) \in \mathbf{Z}^n$ of $\alpha = [V - \dim V_{\mathbf{H}}] \in KSp^0(\mathbf{H}P^n)$ by setting $p(\alpha) = p(V)$.

In (1), we have introduced a map $s_k : \mathbf{Z}^k \to Z$ defined by the Newton's relation as follows:

$$s_k(m_1, \dots, m_k) = \sum_{i=1}^{k-1} (-1)^{i+1} m_i s_{k-i}(m_1, \dots, m_{k-i}) + (-1)^{k+1} k m_k$$

for $m_i \in \mathbf{Z}$. Then, for an integral vector $q = (q_1, \dots, q_n) \in \mathbf{Z}^n$, we set

$$s(q) = (s_1(q_1), s_2(q_1, q_2), \dots, s_n(q_1, \dots, q_n)) \in \mathbf{Z}^n.$$

Then, it defines a monomorphism $s : \mathbf{Z}^n \to \mathbf{Z}^n$ between the free abelian groups $\mathbf{Z}^n$, and we set $s(\alpha) = s(p(\alpha))$ for $\alpha \in KSp^0(\mathbf{H}P^n)$. As for the basis $\{Y_i \mid 1 \leq i \leq n\}$ of $KSp^0(\mathbf{H}P^n)$ in Lemma 3.2, we have the following.

LEMMA 3.3. *$s(Y_k) = (2/a(k))(k^2, k^4, \dots, k^{2n})$ for $1 \leq k \leq n$.*

PROOF. By the definition of Y_k, we have $P(Y_k) = (1 + k^2 x)^{2/a(k)}$ for $k \geq 0$. Thus, using the Newton's relation, we have $s_i(Y_k) = (2/a(k))k^{2i}$ as is required. □

Let A be the $n \times n$ matrix whose j-th column is $s(Y_j)$ for $1 \leq j \leq n$. Hence, by Lemma 3.3, A is represented as

$$(10) \qquad A = \begin{pmatrix} 1 & 2 \cdot 2^2 & 3^2 & \cdots & (2/a(n)) \cdot n^2 \\ 1 & 2 \cdot 2^4 & 3^4 & \cdots & (2/a(n)) \cdot n^4 \\ 1 & 2 \cdot 2^6 & 3^6 & \cdots & (2/a(n)) \cdot n^6 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2 \cdot 2^{2n} & 3^{2n} & \cdots & (2/a(n)) \cdot n^{2n} \end{pmatrix}.$$

We also denote by $A(\boldsymbol{q})$ the $n \times (n+1)$ matrix whose first $n \times n$ submatrix is A and the last column is $\boldsymbol{s}(\boldsymbol{q})$. That is,

$$A(\boldsymbol{q}) = (A \;\; \boldsymbol{s}(\boldsymbol{q})).$$

Since $\{Y_i \mid 1 \leq i \leq n\}$ is a basis of $KSp^0(\mathbf{H}P^n)$ and $\boldsymbol{s} : \mathbf{Z}^n \to \mathbf{Z}^n$ is a monomorphism, a necessary and sufficient condition for an integral vector $\boldsymbol{q} \in \mathbf{Z}^n$ to be a Pontrjagin vector of some $\mathbf{H}$-vector bundle over $\mathbf{H}P^n$ is that $\boldsymbol{s}(\boldsymbol{q})$ is a linear combination of $\{\boldsymbol{s}(Y_i) \mid 1 \leq i \leq n\}$ with integer coefficients. It is equivalent to say that there exists an integral vector $\boldsymbol{b} = (b_1, \dots, b_n)$ satisfying

$$\boldsymbol{s}(\boldsymbol{q}) = \sum_{i=1}^{n} b_i \boldsymbol{s}(Y_i) = A\boldsymbol{b},$$

and thus we have the following.

Proposition 3.4. *When $A(\boldsymbol{q})$ is transformed into an integral matrix $(B \;\; \boldsymbol{u})$ by row operations within integral matrices, a necessary and sufficient condition for an integral vector $\boldsymbol{q} = (q_1, \dots, q_n) \in \mathbf{Z}^n$ to be a Pontrjagin vector is that there exists an integral vector $\boldsymbol{b} \in \mathbf{Z}^n$ satisfying*

$$B\boldsymbol{b} = \boldsymbol{u}. \tag{11}$$

Let $g_k : \mathbf{Z}^k \to \mathbf{Z}$ be the map given in (2). Then, the following is easy by the induction on k.

Lemma 3.5. $g_k(m^2, m^4, \dots, m^{2k}) = \prod_{i=0}^{k-1}(m^2 - i^2)$.

Now, we transform $A(\boldsymbol{q})$ step by step through elementary row operations within integral matrices to make A an upper triangular matrix. As the first step, subtracting the i-th row from the $(i+1)$-th row proceeding upward consecutively from $i = n-1$ to $i = 1$, $A(\boldsymbol{q})$ is transformed into the following matrix, since $g_1(k_1) = k_1$ and $g_2(k_1, k_2) = k_2 - k_1$.

$$\begin{pmatrix} g_1(1) & 2g_1(2^2) & g_1(3^2) & \cdots & \frac{2}{a(n)} g_1(n^2) & g_1(s_1) \\ 0 & 2g_2(2^2, 2^4) & g_2(3^2, 3^4) & \cdots & \frac{2}{a(n)} g_2(n^2, n^4) & g_2(s_1, s_2) \\ 0 & 2g_2(2^4, 2^6) & g_2(3^4, 3^6) & \cdots & \frac{2}{a(n)} g_2(n^4, n^6) & g_2(s_2, s_3) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 2g_2(2^{2n-2}, 2^{2n}) & g_2(3^{2n-2}, 3^{2n}) & \cdots & \frac{2}{a(n)} g_2(n^{2n-2}, n^{2n}) & g_2(s_{n-1}, s_n) \end{pmatrix},$$

where we abbreviate $s_i(q_1, \dots, q_i)$ simply by s_i for $1 \leq i \leq n$.

As the next step, subtract the 2^2 times of i-th row from the $(i+1)$-th row proceeding upward consecutively from $i = n-1$ to $i = 2$. Then, the matrix is transformed into the following matrix, since $g_3(k_1, k_2, k_3) = g_2(k_2, k_3) - 2^2 g_2(k_1, k_2)$.

$$\begin{pmatrix} g_1(1) & 2g_1(2^2) & g_1(3^2) & \cdots & \frac{2}{a(n)} g_1(n^2) & g_1(s_1) \\ 0 & 2g_2(2^2, 2^4) & g_2(3^2, 3^4) & \cdots & \frac{2}{a(n)} g_2(n^2, n^4) & g_2(s_1, s_2) \\ 0 & 0 & g_3(3^2, 3^4, 3^6) & \cdots & \frac{2}{a(n)} g_3(n^2, n^4, n^6) & g_3(s_1, s_2, s_3) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & g_3(3^{2n-4}, 3^{2n-2}, 3^{2n}) & \cdots & \frac{2}{a(n)} g_3(n^{2n-4}, n^{2n-2}, n^{2n}) & g_3(s_{n-2}, s_{n-1}, s_n) \end{pmatrix}.$$

Proceeding such way, after $n-1$ steps we reach a matrix $M(\boldsymbol{q})$ whose first n columns form an upper triangular matrix, as follows.

LEMMA 3.6. *By elementary row operations within integral matrices, the matrix $A(\boldsymbol{q})$ is transformed into a matrix $M(\boldsymbol{q})$ whose (i, j) element $m_{i,j}$ satisfies*

$$m_{i,j} = \begin{cases} \frac{2}{a(j)} g_i(j^2, j^4, \dots, j^{2i}) & \text{for } 1 \le i \le j \le n, \\ 0 & \text{for } i > j, \\ g_i(s_1(q_1), \dots, s_i(q_1, \dots, q_i)) & \text{for } 1 \le i \le n \text{ and } j = n+1. \end{cases}$$

Using Lemma 3.5, the elements $m_{i,i}$ for $1 \le i \le n$ and $m_{i,j}$ for $1 \le i < j \le n$ can be written, respectively, as follows:

$$m_{i,i} = \frac{2}{a(i)} i^2(i^2 - 1) \dots (i^2 - (i-1)^2) = \frac{(2i)!}{a(i)}, \tag{12}$$

$$m_{i,j} = \frac{2}{a(j)} j^2(j^2 - 1) \dots (j^2 - (i-1)^2) = \frac{(2j)(j+i-1)!}{a(j)(j-i)!}. \tag{13}$$

Then, we have the following.

LEMMA 3.7. *For $1 \le i \le j \le n$, $m_{i,i}$ is a factor of $m_{i,j}$.*

PROOF. We show that $m_{i,j}/m_{i,i}$ is an integer. Using (12) and (13), we have

$$\frac{m_{i,j}}{m_{i,i}} = \frac{a(i)j}{a(j)i} \binom{j+i-1}{2i-1}.$$

By Feder and Gitler [3], it is shown that

$$\frac{j}{i} \binom{j+i-1}{2i-1} \in \mathbf{Z}.$$

Therefore, if j is even or if both i and j are odd, then $a(i)/a(j) = 1$ or 2 and thus $m_{i,j}/m_{i,i}$ is an integer. We also have

$$\frac{j}{i}\binom{j+i-1}{2i-1} = \frac{2j}{i+j}\binom{j+i}{2i}.$$

Hence, if j is odd and i is even, then the odd integer $i+j$ divides $j\binom{j+i}{2i}$. Thus, $\frac{m_{i,j}}{m_{i,i}}$ is still an integer, which concludes the proof. □

Now, we complete the proof of Theorem C.

Proof of Theorem C. By Proposition 3.4, a necessary and sufficient condition for an integral vector $\boldsymbol{q} = (q_1, \dots, q_n)$ to be a Pontrjagin vector of an $\mathbf{H}$-vector bundle over $\mathbf{H}P^n$ is that there exists an integral vector $\boldsymbol{b}$ satisfying (11) when we take $(B\ \boldsymbol{u}) = M(\boldsymbol{q})$. Then, by Lemmas 3.6 and 3.7, such $\boldsymbol{b}$ exists if and only if each $m_{i,i}$ is a factor of $g_i(s_1(q_1), \dots, s_i(q_1, \dots, q_i))$ for $1 \le i \le n$. Thus, using (12), we obtain the required result. □

4. Proof of Theorem D and Corollary E

Now, we prove Theorem D and Corollary E using Theorem C.

Proof of Theorem D. Let α be an $\mathbf{H}$-vector bundle of dimension r over $\mathbf{H}P^n$ with $r \le n$, and assume that α is stably extendible to $\mathbf{H}P^m$ for some m with $m > n$. Then, there exists an $\mathbf{H}$-vector bundle β of dimension r over $\mathbf{H}P^m$ whose restriction to $\mathbf{H}P^n$ is stably equivalent to α. The Pontrjagin vector of β is represented as $\boldsymbol{p}(\beta) = (p_1(\alpha), \dots, p_r(\alpha), 0, \dots, 0)$. Hence, by Theorem C, we have $g_{r+i}(s_1, \dots, s_r, t_{r+1}, \dots, t_{r+i}) \equiv 0 \pmod{(2(r+i))!/a(r+i)}$ for any i with $1 \le i \le m-r$, where $s_1, \dots, s_r, t_{r+1}, \dots, t_m$ are the integers given in (3) and (4) for α under consideration. Thus, taking the contraposition, we have the required result. □

Proof of Corollary E. Let α be an $\mathbf{H}$-vector bundle of dimension n over $\mathbf{H}P^n$. Then, the extendibility of α is equivalent to the stable extendibility of it by a stability property of vector bundle (cf. [5, §8, Theorem 1.5]). Thus, it is sufficient to prove the converse of Theorem D for α when $r = n$ and $m = n+1$. We assume that $g_{n+1}(s_1, \dots, s_n, t_{n+1}) \equiv 0 \pmod{(2(n+1))!/a(n+1)}$. Then, by Theorem C, there exists an $\mathbf{H}$-vector bundle β over $\mathbf{H}P^{n+1}$ with $P_i(\beta) = P_i(\alpha)$ for $1 \le i \le n$ and $P_{n+1}(\beta) = 0$. By a stability property (cf. [5, §8, Theorem 1.2]), we can assume that β is of dimension $n+1$ as an $\mathbf{H}$-vector bundle. Then, the restriction $i^*\beta$ over $\mathbf{H}P^n$ is stably equivalent to $\alpha + 1_{\mathbf{H}}$, since they have the same Pontrjagin classes (cf. [6, Lemma 4]). On the other hand, regarding β as an oriented vector bundle, the Euler class of β is $P_{n+1}(\beta)$ up to sign, and thus the primary obstruction class to construct a cross section of the associated sphere bundle of β is $P_{n+1}(\beta)$ (cf. [10, Theorem 12.5]).

Hence, the equality $P_{n+1}(\beta)=0$ shows that β admits an everywhere nonzero section, and thus $\beta \cong \gamma + 1_{\mathbf{H}}$ for some $\mathbf{H}$-vector bundle γ of dimension n over $\mathbf{H}P^{n+1}$. Then, by the stability property again, it follows that $i^*\gamma \cong \alpha$. Thus, α is extendible to $\mathbf{H}P^{n+1}$, and we have completed the proof. □

As a special case of Corollary E, we have the following, where ξ denotes the canonical $\mathbf{H}$-line bundle as before.

PROPOSITION 4.1. *Let α be an $\mathbf{H}$-vector bundle of dimension n over $\mathbf{H}P^n$. If $\alpha + k_{\mathbf{H}} \cong (n+k)\xi$ for some $k \geq 1$, then the necessary and sufficient condition for α not to be extindible to $\mathbf{H}P^{n+1}$ is that the following holds:*

$$\binom{n+k}{n+1} \not\equiv 0 \qquad (\mathrm{mod}\ a(n)(2n+1)!).$$

PROOF. Put $\beta = (n+k)\xi$ over $\mathbf{H}P^{n+1}$. Then, $p_i(\alpha) = p_i(\beta)$ and $s_i(\alpha) = s_i(\beta)$ for $1 \leq i \leq n$. Since $P(\beta) = P((n+k)\xi) = (1+x)^{n+k}$, $p_i(\beta)$ for $1 \leq i \leq n+1$ is equal to the value of the i-th elementary symmetric polynomial with $n+k$ variables substituted 1 for all variables. Thus, using an algebraic property concerning the symmetric polynomial and the Newton polynomial, we have $s_i(\beta) = \underbrace{1^i + \cdots + 1^i}_{n+k} = n+k$ for $1 \leq i \leq n+1$. Then, by the definition of t_{n+1}, we have

$$\begin{aligned} t_{n+1} &= \sum_{i=1}^{n} (-1)^{i+1} p_i(\alpha) s_{n+1-i}(\alpha) = \sum_{i=1}^{n} (-1)^{i+1} p_i(\beta) s_{n+1-i}(\beta) \\ &= s_{n+1}(\beta) + (-1)^{n+1}(n+1)p_{n+1}(\beta) = (n+k) + (-1)^{n+1}(n+1)\binom{n+k}{n+1}. \end{aligned}$$

Since the map $g_k;\ \mathbf{Z}^k \to \mathbf{Z}$ is a homomorphism between the abelian groups and since $g_k(1,1,\dots,1) = 0$ and $g_k(0,0,\dots,0,1) = 1$ for any k, we have

$$\begin{aligned} & g_{n+1}(s_1, s_2, \dots, s_n, t_{n+1}) \\ &\quad = (n+k)g_{n+1}(1,1,\dots,1) + (-1)^{n+1}(n+1)\binom{n+k}{n+1} g_{n+1}(0,0,\dots,0,1) \\ &\quad = (-1)^{n+1}(n+1)\binom{n+k}{n+1}. \end{aligned}$$

Thus, by Corollary E, a necessary and sufficient condition for α not to be extendible to $\mathbf{H}P^{n+1}$ is that

$$\binom{n+k}{n+1} \not\equiv 0 \qquad (\mathrm{mod}\ a(n)(2n+1)!)$$

as is required. □

For example, when $1 \leq k \leq n$, a vector bundle α over $\mathbf{H}P^n$ of dimension n which is stably equivalent to $(n+k)\xi$ is not extendible to $\mathbf{H}P^{n+1}$.

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