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Stably extendible vector bundles over the quaternionic projective spaces

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Dedicated to Professor Fuichi Uchida on his 60th birthday

ABSTRACT. We show that, if a quaternionic k-dimensional vector bundle γ over the quaternionic projective space HP^n is stably extendible and its non-zero top Pontrjagin class is not zero mod 2, then γ is stably equivalent to the Whitney sum of k quaternionic line bundles provided $k \leq n$.

1. Introduction and results

Let F denote the field of the complex numbers **C**, that of the real numbers **R** or the skew field of the quaternionic numbers **H**, and FP^n the n-dimensional F-projective space. Two F-vector bundles V and W over a finite complex B are said to be stably equivalent if the Whitney sums $V \oplus I_a$ and $W \oplus I_b$ for some trivial F-vector bundles I_a and I_b are isomorphic as F-vector bundles.

The purpose of this paper is to study Schwarzenberger's property for vector bundles over the quaternionic projective space HP^n. Schwarzenberger ([Sc], [Hi]) has shown the fact that a k-dimensional F-vector bundle V over FP^n for $F = \mathbf{R}$ or $\mathbf{C}$ is stably equivalent to a Whitney sum of k F-line bundles if V is extendible, that is, if V is the restriction of a F-vector bundle over FP^m for any $m \geq n$. For the **C**-vector bundles over CP^n, proofs have been given by [Re] and [AM]. As for the **R**-vector bundles over RP^n, the stable splitting is also true under the assumption that V is the restriction of a vector bundle over RP^m for sufficiently large m ([Sc]). Some related results concerning vector bundles over the lens spaces are found in [KMY], [KM]. Our results mean that some additional conditions seem necessary for the quaternionic vector bundles over HP^n.

We remark that the extendible condition can be slightly weakened. A k-dimensional F-vector bundle γ over FP^n is called *stably extendible* if for

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each $m \geq n$ there exists a k-dimensional F-vector bundle $\tilde{\gamma}_m$ over FP^m whose restriction to FP^n is stably equivalent to γ as F-vector bundles. Then, the original result due to Schwarzenberger is also valid if we only assume that the vector bundle is stably extendible instead of extendible.

Let ξ be the canonical quaternionic line bundle over HP^n, and $X = P_1(\xi)$ the first symplectic Pontrjagin class of ξ. Here, the symplectic Pontrjagin class $P_i(\zeta) \in H^{4i}(B; \mathbf{Z})$ for a quaternionic k-dimensional vector bundle ζ over a space B is given by $P_i(\zeta) = (-1)^i C_{2i}(c'(\zeta))$, the Chern class of the underlying complex vector bundle $c'(\zeta)$ of ζ up to sign. Also we denote the total symplectic Pontrjagin class of ζ by $P(\zeta) = 1 + P_1(\zeta) + \cdots + P_k(\zeta)$. Then, the cohomology ring $H^*(HP^n; \mathbf{Z})$ is isomorphic to a truncated polynomial ring $\mathbf{Z}[X]/(X^{n+1})$. Our results are stated as follows:

THEOREM A. *Let $k \leq n$. If a quaternionic k-dimensional vector bundle γ over HP^n is stably extendible, then $P(\gamma) = \prod_{i=1}^{k}(1 + m_i^2 X)$ for some $m_i \in \mathbf{Z}$.*

THEOREM B. *Let γ be a stably extendible quaternionic k-dimensional vector bundle over HP^n for $k \leq n$. If $P_m(\gamma) \equiv X^m \pmod 2$ for some $0 \leq m \leq k$ and $P_i(\gamma) = 0$ for any $i > m$, then γ is stably equivalent to a Whitney sum $\gamma(1) \oplus \cdots \oplus \gamma(m)$ of some quaternionic line bundles $\gamma(1), \dots, \gamma(m)$ over HP^n.*

We remark that some counter example arises if the condition $P_m(\gamma) \equiv X^m \pmod 2$ is removed in Theorem B, as follows:

PROPOSITION C. *Let γ be a quaternionic vector bundle stably equivalent to $(\xi \otimes_{\mathbf{H}} \xi^*) \otimes_{\mathbf{R}} \mathbf{H}$, the quaternification of $\xi \otimes_{\mathbf{H}} \xi^*$, over HP^n for $n \geq 2$, where ξ^* is the quaternionic conjugate bundle of ξ. Then, γ is stably extendible and its total Pontrjagin class is $P(\gamma) = (1 + 4X)^2$, but γ is not stably equivalent to any Whitney sum of less than or equal to n numbers of quaternionic line bundles.*

As for a stably extendible complex vector bundle ρ over HP^n, similar results with Theorems A and B hold if the Pontrjagin classes are replaced by the Chern classes $C_i(\rho)$ and quaternionic line bundles $\gamma(i)$ by some complex 2-dimensional vector bundles.

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2. Proofs

Let $F(t) = t^r - m_1 t^{r-1} + \cdots + (-1)^{r-1} m_{r-1} t + (-1)^r m_r \in \mathbf{Z}[t]$ be a polynomial, and $F(t) = \prod_{i=1}^{r}(t - z_i)$ its factorization by complex numbers $z_i \in \mathbf{C}$. We set $s_j(F) = \sum_{i=1}^{r} z_i^j$ for $j \geq 1$. Then, $\{s_j(F)\}$ satisfies Newton's relations

$$\sum_{i=0}^{j-1}(-1)^i m_i s_{j-i}(F) = (-1)^{j+1} j m_j \tag{2.1}$$

for any $j \geq 1$, where $m_0 = 1$ and $m_i = 0$ if $i > r$. Thus, all $s_j(F)$ are integers. Concerning the linear factorization of a polynomial, Feit-Rees has shown the following fact.

THEOREM 1 [FR]. *If $s_j(F) \equiv s_{j+p-1}(F) \pmod{p}$ for $1 \leq j \leq r$ and for all but a finite number of primes p, then $F(t)$ is a product of linear factors in $\mathbf{Z}[t]$, that is, $z_i \in \mathbf{Z}$ for $1 \leq i \leq r$.*

Rees [Re] and Adams-Mahmud [AM] have shown that this property is effective to prove Schwarzenberger's property for complex vector bundles over CP^n. Their methods are also valid if the assumption of the extendibility is weakened to stably extendibility, and we have the following, where $x = C_1(\xi_{\mathbf{C}}) \in H^2(CP^n; \mathbf{Z})$ is the first Chern class of the canonical complex line bundle $\xi_{\mathbf{C}}$ over CP^n:

PROPOSITION 2. *Let $k \leq n$. Then, we have the following:*

(1) *If ρ is a stably extendible complex k-dimensional vector bundle over CP^n, then the total Chern class $C(\rho)$ of ρ factorizes as $C(\rho) = \prod_{i=1}^{k}(1 + a_i x)$ for some integers a_i.*

(2) *If γ is a stably extendible quaternionic k-dimensional vector bundle over HP^n, then the total symplectic Pontrjagin class $P(\gamma)$ of γ factorizes as $P(\gamma) = \prod_{i=1}^{k}(1 + b_i X)$ for some integers b_i.*

PROOF. We shall prove (2) since the proof of (1) is similar and simpler. Thus, assume that γ is a stably extendible quaternionic k-dimensional vector bundle over HP^n for $k \leq n$. The i-th symplectic Pontrjagin class of γ is denoted by $P_i(\gamma) = u_i X^i$ for some $u_i \in \mathbf{Z}$, where $u_0 = 1$ and $u_i = 0$ for $i > k$. Then, for the polynomial $Q(t) = \sum_{i=0}^{k}(-1)^i u_i t^{k-i} \in \mathbf{Z}[t]$, we define integers s_j as $s_j(Q)$ of (2.1) for $m_i = u_i$. That is, $s_0 = 1$ and s_j for $j \geq 1$ is defined recursively by

$$\sum_{i=0}^{j-1}(-1)^i u_i s_{j-i} = (-1)^{j+1} j u_j. \tag{2.2}$$

When we need to distinguish $\{s_j\}$ for different γ's, we denote s_j by $s_j(\gamma)$.

Remark that the total Pontrjagin class $P(\gamma)$ is equal to $(-1)^k X^k Q(-1/X)$. Hence, if $Q(t)$ is shown to be a product of linear factors in $\mathbf{Z}[t]$, then $P(\gamma)$ turns out to be a product of linear factors in $\mathbf{Z}[X]$ as required. Thus, by Theorem 1, it suffices to show that for all but a finite number of primes p it holds

$$s_j \equiv s_{j+p-1} \pmod{p} \qquad \text{for } 1 \leq j \leq k. \tag{2.3}$$

Let p be any prime with $p > k(2k-1)$, and m an integer satisfying $m \geq \max(k+p, n)$. By the assumption, there exists a quaternionic k-dimensional vector bundle γ_m over HP^m whose restriction to HP^n is stably equivalent to γ. Let $f_m : HP^m \to BSp$ be the classifying map of the virtual bundle $\gamma_m - I_k$, and $i : HP^n \to HP^m$ the inclusion. Thus, $f = f_m i : HP^n \to BSp$ is the classifying map for $\gamma - I_k$. For the universal symplectic Pontrjagin classes $P_i \in H^{4i}(BSp; \mathbf{Z})$, we define a class $\tilde{s}_j \in H^{4j}(BSp; \mathbf{Z})$ for $j \geq 1$ recursively as in (2.1) by the relations

$$\sum_{i=0}^{j-1}(-1)^i P_i \tilde{s}_{j-i} = (-1)^{j+1} j P_j. \tag{2.4}$$

Since $s_j(\gamma_m) = s_j(\gamma)$, $f_m^*(\tilde{s}_j) = s_j X^j$ by (2.2) and (2.4).

By the naturality of the cohomology operation $\mathscr{P}^2 : H^*(Y; \mathbf{Z}/p) \to H^{*+4(p-1)}(Y; \mathbf{Z}/p)$, we have $\mathscr{P}^2 f_m^*(\tilde{s}_j) = f_m^*(\mathscr{P}^2 \tilde{s}_j)$. Moreover, for $1 \leq j \leq k$,

$$\mathscr{P}^2 f_m^*(\tilde{s}_j) = s_j \mathscr{P}^2(X^j) = \binom{2j}{2} s_j X^{j+p-1};$$

$$f_m^*(\mathscr{P}^2 \tilde{s}_j) = f_m^*\left(\binom{2j}{2}\tilde{s}_{j+p-1}\right) = \binom{2j}{2} s_{j+p-1} X^{j+p-1}.$$

Since $1 \leq \binom{2j}{2} < p$ and $j + p - 1 \leq m$ for $1 \leq j \leq k$ by the assumption, we obtain (2.3), which completes the proof. □

PROOF OF THEOREM A. Let γ be a stably extendible quaternionic k-dimensional vector bundle over HP^n for $k \leq n$. Then, by Proposition 2(2), we have $P(\gamma) = \prod_{i=1}^{k}(1 + b_i X)$ for some $b_i \in \mathbf{Z}$. Thus, in order to complete the proof, it is sufficient to show that each integer b_i is a square.

Let $q : CP^{2n+1} \to HP^n$ be the canonical projection, and $c'(\gamma)$ denote underlying complex vector bundle of γ, that is, the complexification of γ. Then, $q^*c'(\gamma)$ is a stably extendible complex $2k$-dimensional vector bundle over CP^{2n+1}. Hence, by Proposition 2(1), the total Chern class of $q^*c'(\gamma)$ is written as $C(q^*c'(\gamma)) = \prod_{i=1}^{2k}(1 + a_i x)$ for some integers a_i. On the other hand, we have

$$C(q^*c'(\gamma)) = q^*(C(c'(\gamma))) = q^*\left(\prod_{i=1}^{k}(1 - b_i X)\right) = \prod_{i=1}^{k}(1 - b_i x^2),$$

since $C_{2j}(c'(\gamma)) = (-1)^j P_j(\gamma), C_{2j+1}(c'(\gamma)) = 0$ and $q^*(X) = x^2$ by definitions. Thus, comparing these two expressions of $C(q^*c'(\gamma))$, we conclude that $b_i = m_i^2$, $1 \leq i \leq k$, for some integers m_i. □

In order to establish Theorem B, we need the following result.

THEOREM 3 ([Su], [FG]). *The degree of non-zero self map $f : HP^\infty \to HP^\infty$ is an odd square, that is, $f^*(X) = (2h+1)^2 X$ for some integer h, and conversely, for any integer h, there exists a self map f of HP^∞ whose degree is $(2h+1)^2$.*

The symplectic Pontrjagin classes determine the stably equivalent classes of quaternionic vector bundles over HP^n as follows:

LEMMA 4. *Quaternionic vector bundles V and W over HP^n are stably equivalent as quaternionic vector bundles if and only if they have the same symplectic Pontrjagin classes $P(V) = P(W)$.*

PROOF. The only if part is clear by the stable property $P(V \oplus I_a) = P(V)$ of the symplectic Pontrjagin class. We assume that $P(V) = P(W)$ for quaternionic vector bundles V, W over HP^n. Let $\widetilde{KSp}(HP^n)$ be the reduced symplectic K-group of HP^n, which is isomorphic to the based homotopy group $[HP^n, BSp]$. Then, by the definition of the symplectic K-group, it suffices to show that the virtual bundles $\alpha = V - \dim V$ and $\beta = W - \dim W$ represent the same class of $\widetilde{KSp}(HP^n)$. The Pontrjagin character $ph : \widetilde{KSp}(HP^n) \to H^*(HP^n; Q)$ has the form of $ph(\alpha) = 2\sum_{k=1}^n \alpha^*(\tilde{s}_k)/(2k)!$. Here, we regard α as an element of $[HP^n, BSp]$, and $\tilde{s}_k \in H^{4k}(BSp; \mathbf{Z})$ are the classes of (2.4). Since $\alpha^*(\tilde{s}_k) = s_k(V)$ is determined by $P(V)$, the equality $ph(\alpha) = ph(\beta)$ follows from the assumption that $P(V) = P(W)$.

By definition, the Pontrjagin character ph is the composition of the complexification $c : \widetilde{KSp}(HP^n) \to \tilde{K}(HP^n)$ and the Chern character $ch : \tilde{K}(HP^n) \to H^*(HP^n; \mathbf{Q})$. The Chern character ch is injective since $H^*(HP^n; \mathbf{Z})$ has no torsion ([AH; 2.5 Corollary]). Also, the complexfication c is injective as is well known. Hence ph is injective, and thus $\alpha = \beta$, which completes the proof. □

REMARK 5. The above argument in the proof of Lemma 4 simply says that the map $g : \widetilde{KSp}(HP^n) \to \mathrm{Hom}(H_*(HP^n; \mathbf{Z}), H_*(BSp; \mathbf{Z}))$ defined by $g(\alpha) = \alpha_*$ is injective. In fact, $P(\alpha) = P(V)$ determines α as is shown, and $P_j(\alpha) = \langle P_j, \alpha_*(b_j)\rangle X^j$ is determined by α_*, where $b_i \in H_{4i}(HP^n; \mathbf{Z})$ is the dual homology class of X^i.

PROOF OF THEOREM B. Let γ be a stably extendible quaternionic k-dimensional vector bundle over HP^n for $k \le n$. By Theorem A we have $P(\gamma) = \prod_{i=1}^k (1 + m_i^2 X)$ for some integers m_i. But, each m_i must be either odd or zero by the assumption $P_m(\gamma) \equiv X^m \pmod 2$. By Theorem 3, there exists a quaternionic line bundle $\tilde{\gamma}(i)$ over HP^∞ with $P(\tilde{\gamma}(i)) = 1 + m_i^2 X$ for each $1 \le i \le k$. Let $\gamma(i)$ be the restriction of $\tilde{\gamma}(i)$ over HP^n. Then, we have $P(\gamma) = P(\gamma(1) \oplus \cdots \oplus \gamma(k))$, and thus the required result by Lemma 4.. □

PROOF OF PROPOSITION C. Let γ be stably equivalent to $(\xi \otimes_{\mathbf{H}} \xi^*) \otimes_{\mathbf{R}} \mathbf{H}$, the quaternionification of the bundle $\xi \otimes_{\mathbf{H}} \xi^*$, over HP^n for $n \geq 2$. Clearly, γ is stably extendible. The complexification $c'(\gamma)$ of γ is stably equivalent to $2c(\xi \otimes_{\mathbf{H}} \xi^*) \cong 2c'(\xi) \otimes_{\mathbf{C}} c'(\xi)$, where $c(\xi \otimes_{\mathbf{H}} \xi^*)$ denotes $(\xi \otimes_{\mathbf{H}} \xi^*) \otimes_{\mathbf{R}} \mathbf{C}$. Since the total Chern class $C(c'(\xi) \otimes_{\mathbf{C}} c'(\xi))$ is equal to $1 - 4X$, the total Pontrjagin class $P(\gamma)$ is equal to $(1 + 4X)^2$.

Suppose that γ is stably equivalent to $\bigoplus_{i=1}^{k} l_i$ for some quaternionic line bundles l_i and some $k \leq n$. If $P(l_i) = 1 + t_i X$ for $1 \leq i \leq k$, where $t_i \in \mathbf{Z}$, then we have the equality $(1 + 4X)^2 = \prod_{i=1}^{k}(1 + t_i X)$ in $H^*(HP^n) = \mathbf{Z}[X]/(X^{n+1})$. Since $k \leq n$, we may assume that $t_1 = t_2 = 4$ and $t_i = 0$ for $i \geq 3$. Then, we have a classifying map $f : HP^n \to HP^n$ of l_1, and thus $f^*(X) = 4X$. However, by Feder-Gitler ([FG]), if $g^*(X) = \lambda X$ holds for some map $g : HP^n \to HP^n$ for $n \geq 2$, then the integer λ satisfies $\lambda(\lambda - 1) \equiv 0 \pmod{24}$. Hence, $\lambda \neq 4$, which contradicts the existence of f. Therefore, γ cannot be stably decomposed into a sum of k numbers of quaternionic line bundles for $k \leq n$, which completes the proof. □

REMARK 6. We defined a k-dimensional F-vector bundle γ over FP^n to be stably extendible if it extends stably to some k-dimensional F-vector bundle $\tilde{\gamma}_m$ over FP^m for any $m \geq n$. The restriction of $\dim_F \tilde{\gamma}_m = k$ was needed in the proof of Theorem A. It is still open whether this restriction of dimension is actually necessary or not.

REMARK 7. In Theorem B and Proposition C, we discuss when a quaternionic vector bundle over HP^n is stably equivalent to the Whitney sum of less than or equal to n numbers of quaternionic line bundles. It is still open how it becomes if we are allowed to take the Whitney sum of more than n numbers of line bundles.

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