HIROSHIMA MATH. J.
26 (1996), 181–188

Generalized Bernoulli numbers on the *KO*-theory

Dedicated to Professor Yasutoshi Nomura on his 60th birthday

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(Received August 31, 1994)
(Revised November 14, 1994)

ABSTRACT. The Bernoulli number defined on the generalized cohomology theory is studied, mainly focusing it on complex unoriented theories. We give a concrete formula about it on the *KO*-theory for the stunted quaternionic quasi-projective space, and apply the formula to represent a factorization of the double transfer map concerning such projective spaces.

Introduction

In this paper, I study the Bernoulli numbers defined on the generalized cohomology theory, and represent some concrete formulas of them concerning the quaternionic quasi-projective spaces. Significant combination of the geometry with the classical Bernoulli numbers has been shown by Bott [6] and Adams [1] in the study of the *J*-theory. Extendending such utility, Miller [8] has introduced a generalized sense of Bernoulli numbers by giving them for each formal group law over a complex oriented theory, and Ray [10] has discussed some related articles. Our purpose here is to make such treatment of the Bernoulli numbers applicable also to complex unoriented theories. We pick up a typical case of the real *KO*-theory, and show effectiveness of our definition.

In § 1, we prepare some characteristic classes of vector bundles and give our definition of the Bernoulli numbers. In § 2, we describe the *KO*-theoretical Bernoulli numbers for the vector bundles which define the quaternionic quasi-projective spaces. The result is summarized in Proposition 2.5. In § 3, we apply the result of § 2 to a factorization of the double transfer maps combined with the quaternionic quasi-projective spaces. The contents of this section are related to [7], and our main result is Theorem 3.8.

1991 *Mathematics Subject Classification.* 55P42, 55R12, 55N20, 55Q10, 55N15.
Key words and phrases. Bernoulli numbers; *KO*-theory; quasi-projective spaces; double transfer map.

1. Bernoulli numbers of vector bundles

We refer to [2] on the concepts of the stable homotopy category and the generalized cohomology theories, and make the conventional use of notations about them. Let E be a ring spectrum with the unit $\iota: S^0 \to E$. We denote by $E_* = \sum_i E_i$ the coefficient ring $\pi_*(E)$ of E. Now, assume that a vector bundle α over a finite complex B is orientable and E-orientable. Then, the orientability of α gives a Thom class $U_\alpha^H \in H^a(B^\alpha; Z)$, which is uniquely determined up to sign, for the ordinary integral cohomology theory HZ, and the E-orientability means that there is a Thom class $U_\alpha^E \in E^a(B^\alpha)$ in the E-cohomology theory. Here, B^α denotes the Thom space of α, and a is the fiber dimension of α. For the maps $\eta_R: E = S^0 \wedge E \xrightarrow{\iota \wedge 1} HZ \wedge E$ and η_L: $HZ = HZ \wedge S^0 \xrightarrow{1 \wedge \iota} HZ \wedge E$ induced from the respective units, both images $(\eta_R)_*(U_\alpha^E)$ and $(\eta_L)_*(U_\alpha^H)$ of U_α^E and U_α^H in $(HZ \wedge E)^a(B^\alpha)$ are Thom classes of α in the $HZ \wedge E$-cohomology theory.

DEFINITION 1.1. $sh^E(\alpha) \in (HZ \wedge E)^0(B_+)$ is an element defined by the relation $(\eta_R)_*(U_\alpha^E) = (\eta_L)_*(U_\alpha^H) sh^E(\alpha)$, where the right hand side of the equality is the image of $sh^E(\alpha)$ under the Thom isomorphism $(HZ \wedge E)^0(B_+) \to (HZ \wedge E)^a(B^\alpha)$ defined by $(\eta_L)_*(U_\alpha^H)$.

By definition, $sh^E(\alpha)$ is in $1 + (HZ \wedge E)^0(B)$, and sh^E is multiplicative in the sense that $sh^E(\alpha_1 \oplus \alpha_2) = sh^E(\alpha_1) sh^E(\alpha_2)$. Later, we will treat the case that E is the real K-theory KO, where we will see that $sh^{KO}(\alpha)$ corresponds to the characteristic class $sh(\alpha)$ as in [1].

Assume that $H^*(B_+; Q)$ has a basis $\{u_k\}_k$ as a vector space, where Q is the field of the rational numbers. Then, using this basis, we define the Bernoulli numbers $B_k^E(\alpha) \in E_{|u_k|} \otimes Q$ of α to be the elements satisfying

$$sh^E(\alpha) = \sum_k B_k^E(\alpha) u_k . \tag{1.2}$$

When $E = K$, the complex K-theory, and $\alpha = -\gamma$ for the canonical complex line bundle γ over the complex projective space CP^n, we get $B_i^K(-\gamma) = t^i B_i / i!$ up to sign for the classical Bernoulli numbers B_i and the Bott class $t \in K_2$. Here, we take $U_\gamma^K = t^{-1}(\gamma - 1) \in K^2(CP^{n+1})$, which determines $sh^K(\gamma)$ and hence $sh^K(-\gamma) = sh^K(\gamma)^{-1}$, and the basis $\{u^i | i \geq 0\}$ of $H^*(CP^n; Q)$ for the Euler class $u = e(\gamma) \in H^2(CP^n; Z)$ of γ.

In [8], it is effectively used the concept of the Bernoulli numbers with respect to each formal group law over a complex oriented ring spectrum E. The above example on the K-theory is a typical one which corresponds to the multiplicative formal group law, and such Bernoulli numbers defined for a formal group law are included in our definition by taking the following way: the bundle $-\gamma$, the Thom class U_γ^E which is associated with the Euler class determined by the formal group law, and the basis $\{u^i | i \geq 0\}$ of $H^*(CP^n; Q)$.

By our definition of the Bernoulli numbers, it is also possible to consider the case of the complex unoriented theories, like KO. The following is obvious from the properties of $sh^E(\alpha)$.

LEMMA 1.3.

(1) *Let α be as above, and $f: D \to B$ a map between finite complexes. Then, by taking $f^*(U_\alpha^E)$ as the Thom class of the induced vector bundle $f^*(\alpha)$ and a basis $\{v_m\}_m$ of $H^*(D_+; Q)$, we have the relation $B_m^E(f^*(\alpha)) = A_f(B_k^E(\alpha))_m$ between the matrices, where A_f is the matrix representing $f^*: H^*(B; Q) \to H^*(D; Q)$ with respect to the given bases.*

(2) *When $\alpha = \alpha_1 \oplus \alpha_2$ over B, we have $B_k^E(\alpha_1 \oplus \alpha_2) = \sum_{k_1, k_2} a_{k,(k_1,k_2)} B_{k_1}^E(\alpha_1) B_{k_2}^E(\alpha_2)$ if $u_{k_1} u_{k_2} = \sum_k a_{k,(k_1,k_2)} u_k$.*

2. Quaternionic quasi-projective spaces

Let H be the skew field of the quaternionic numbers, and ξ the canonical quaternionic line bundle over the quaternionic projective space HP^k for each non-negative integer k. Let $x = e(\xi) \in H^4(HP^k; Z)$ be the Euler class of ξ, and take $X = \xi - \underline{H}^1 \in KO^4(HP^\infty)$ as the KO-Euler class of ξ. Then, it holds that $H^*(HP^k; Z) \cong Z[x]/(x^{k+1})$ and $KO^*(HP^k) \cong Z[X]/(X^{k+1})$.

Now, the tensor product $\xi \otimes_H \bar{\xi}$ of ξ and its quaternionic conjugate bundle $\bar{\xi}$ has a non-zero section, and thus it is isomorphic to $\zeta \oplus \underline{R}^1$ for a 3-dimensional real vector bundle ζ. The quaternionic quasi-projective space Q_n is defined to be the Thom space $(HP^{n-1})^\zeta$ of ζ. Since HP^{n-1} is 3-connected, ζ is orientable and KO-orientable. Let $U \in H^3(Q_n; Z)$ and $U^{KO} \in KO^3(Q_n)$ be the respective Thom class of ζ. Then, through the Thom isomorphisms, $H^*(Q_n; Z)$ and $KO^*(Q_n)$ are the free $H^*(HP^{n-1}; Z)$ and $KO^*(HP^{n-1})$ modules with generators U and U^{KO}, respectively. We assume that, for a KO-orientable vector bundle α, like ζ, we take the Thom class U_α^{KO} as the one of the Atiyah–Bott–Shapiro's sense [4].

Let $g_i \in KO_{4i}$ be the Bott generator, and put $a(i) = 1$ or 2 according as i is even or odd. Then, $g_i/a(i) = (g_1/2)^i$ holds in $KO_* \otimes Q$. Let $ph = ch \circ c$: $KO \to K \to HQ$ be the Pontrjagin character. The classical characteristic class $sh(\alpha)$ for a KO-orientable vector bundle α is defined by $ph(U_\alpha^{KO}) = U_\alpha^H sh(\alpha)$ (cf. [6], [1]). $(\eta_R)_*(U^{KO})$ corresponds to $ph(U^{KO})$ under the isomorphism $(HZ \wedge KO)^3(Q_n) \to H^*(Q_n; Q)$, and, if $sh(\alpha) = \sum_i t_i x^i$ for $t_i \in Q$, then $sh^{KO}(\alpha) = \sum_i (g_i/a(i)) t_i x^i$. Now, for a power series $g(z) = (2 \sinh(\sqrt{z}/2))^2 = \sum_{i \geq 0} r_i z^{i+1}$ for $r_i \in Q$, we put

$$(2.1)\qquad G(x) = \sum_{i\geq 0} \frac{g_i}{a(i)} r_i x^{i+1} = \frac{2}{g_1} g\left(\frac{g_1}{2}x\right) \quad \text{in} \quad (HZ \wedge KO)^4(HP^{n-1}),$$

where $(HZ \wedge KO)^*(HP^{n-1}) \cong (KO_* \otimes Q)[x]/(x^n)$. Since $ph(U_\xi^{KO}) = ph(X) = g(x) = U_\xi^H(g(x)/x)$, we have $sh(\xi) = g(x)/x$, and thus

$$(2.2)\qquad sh^{KO}(\xi) = \frac{G(x)}{x}.$$

Also, we have the following, where $dG(x)/dx$ is the derivative of $G(x)$:

LEMMA 2.3.

$$sh^{KO}(\zeta) = \frac{dG(x)}{dx}.$$

PROOF. It is enough to prove that

$$(2.4)\qquad sh(\xi \otimes_H \bar{\xi}) = \sum_{i\geq 0} \frac{1}{(2i+1)!} x^i,$$

since the right hand side of the equation is equal to $dg(x)/dx$ and $sh(\zeta) = sh(\xi \otimes_H \bar{\xi})$. Let $\kappa\colon HP^{n-1} \to BSO(4)$ be the classifying map of $\xi \otimes_H \bar{\xi}$, and $BT^2 \xrightarrow{i} BU(2) \xrightarrow{i} BSO(4)$ the canonical maps, where T^2 is the maximal torus of $U(2)$ and we have $H^*(BT^2; Z) \cong Z[x_1, x_2]$. Then,

$$SH = (\sinh(x_1/2)/(x_1/2))(\sinh(x_2/2)/(x_2/2))$$

is in the image of the monomorphism $(ri)^*\colon H^*(BSO(4); Q) \to H^*(BT^2; Q)$, and by [1] it follows that $sh(\xi \otimes_H \bar{\xi}) = \kappa^*((ri)^*)^{-1}(SH)$. Let $P_i \in H^{4i}(BSO(4))$ be the Pontrjagin class. Then, we see that $\kappa^*(P_1) = 4x$ and $\kappa^*(P_i) = 0$ for $i \geq 2$. Also, we have $(ri)^*(P_1) = x_1^2 + x_2^2$ and $(ri)^*(P_2) = (x_1 x_2)^2$. Then, it is straightforward to obtain (2.4) from these data.

By (2.2) and Lemma 2.3, $sh^{KO}(\zeta + (m-1)\xi) = (G(x)/x)^{m-1} dG(x)/dx$, and thus we have the following by (1.2):

PROPOSITION 2.5. *As for the Bernoulli numbers $B_i^{KO}(\zeta + (m-1)\xi)$, we have the relation*

$$\left(\frac{G(x)}{x}\right)^{m-1} \frac{dG(x)}{dx} = \sum_{i\geq 0} B_i^{KO}(\zeta + (m-1)\xi) x^i \qquad \text{for any} \quad m \in Z.$$

Before we apply Proposition 2.5 in the next section, it is convenient to prepare the next notation for the Thom class of $\zeta \oplus (m-1)\xi$. For an integer m, we denote by QP_m^{n+m} the Thom space $(HP^n)^{\zeta \oplus (m-1)\xi}$, which is called a stunted quaternionic quasi-projective space. For a positive integer m, it is homeomorphic to Q_{n+m}/Q_{m-1} (cf. [3]). Then, we have the canonical maps

$q\colon Q_{n+m} \to QP_m^{n+m}$ and $q\colon QP_m^{n+m} \to Q_{n+m}$ according as $m > 0$ and $m \leq 0$. Let U_m^{KO} be the KO-Thom class of $\zeta \oplus (m-1)\xi$. Then, the following is easily shown by taking the Pontrjagin character on the both sides of the equations.

LEMMA 2.6. *$q^*(U_m^{KO}) = U^{KO}X^{m-1}$ if $m > 0$, and $q^*(U^{KO}) = U_m^{KO}X^{1-m}$ if $m \leq 0$.*

By this lemma, it is possible that, with the notation $U^{KO}X^j$ for any $j \geq m-1$, we should regard $U^{KO}X^{i+m-1}$ as $U_m^{KO}X^i$ for any $i \geq 0$ and $m \in Z$, as in [8]. Then, $KO^*(QP_m^{n+m})$ is a free $KO^*(HP^n)$-module with a generator $U^{KO}X^{m-1}$ for any $m \in Z$.

3. Application

In [8], [5] and [7], some factorizations of transfer maps are discussed. Such factorization certainly exists for the transfer map combined with the quaternionic quasi-projective space, and we describe it by applying Proposition 2.5.

From the S^3-principal bundle $p\colon S^{4n-1} \to HP^{n-1}$, a stable map $\tau\colon QP_{m+1}^{n+m} \to S^{4m}$ called the S^3-transfer map is constructed by a transfer construction. Our necessary knowledge about τ is not the construction of it but the fact that its fiber spectrum is QP_m^{n+m} and that it is compatible with n. Therefore, by omitting n, we denote QP_m^{n+m} simply by QP_m, and then we have the cofibering

$$S^{4m-1} \xrightarrow{i} QP_m \xrightarrow{j} QP_{m+1} \xrightarrow{\tau} S^{4m}. \tag{3.1}$$

Since the Thom class $U_m^H \in H^{4m-1}(QP_m; Z)$ of $\zeta + (m-1)\xi$ can be considered as an element of the stable cohomotopy group $\pi^{4m-1}(QP_m; Q)$ with Q-coefficient through the Hurewicz isomorphism $h^H\colon \pi^{4m-1}(QP_m; Q) \to H^{4m-1}(QP_m; Q)$, we get the following diagram which is stably homotopy commutative up to sign:

$$\begin{array}{ccccccc} S^{4m-1} & \xrightarrow{i} & QP_m & \xrightarrow{j} & QP_{m+1} & \xrightarrow{\tau} & S^{4m} \\ \| & & \downarrow{\scriptstyle U_m^H} & & \downarrow{\scriptstyle \bar{u}_1} & & \| \\ S^{4m-1} & \xrightarrow{i_Q} & S^{4m-1}Q & \xrightarrow{\rho_Z} & S^{4m-1}Q/Z & \xrightarrow{\beta} & S^{4m}, \end{array} \tag{3.2}$$

where the lower sequence is the cofibering of the Moore spectra associated with the exact sequence $0 \to Z \to Q \to Q/Z \to 0$.

Henceforce, we assume that the matter we discuss is all localized at 2. By (3.2), τ factors through $S^{4m-1}Q/Z$ which is equal to $\Sigma^{4m-1}N_1$ for the first state $\delta_1\colon N_1 \to S^1$ of the chromatic filtration by [9]. Let h^{KO}; $\pi^*(-; \Lambda) \to$

$KO^*(-;\Lambda)$ be the KO-Hurewicz homomorphism. Since $j^*: KO^{4m-1}(QP_{m+1}; Q) \to KO^{4m-1}(QP_m; Q)$ is a monomorphism, $h^{KO}(\bar{u}_1) \in KO^{4m-1}(QP_{m+1}; Q/Z)$ is determined by $\rho_Z(h^{KO}(U_m^H))$, where ρ_Z denotes the mod Z reduction in the KO-cohomology groups. First, we describe the formula of $h^{KO}(U_m^H)$.

We put $f(z) = (2\sinh^{-1}(\sqrt{z}/2))^2 = \sum_{j\geq 0} s_j z^{j+1}$ for $s_j \in Q$, and define

$$(3.3) \qquad F(X) = \sum_{j\geq 0} \frac{g_j}{a(j)} s_j X^{j+1} = \frac{2}{g_1} f\left(\frac{g_1}{2} X\right)$$

as an element of $KO^4(HP^n; Q)$. Then we have

LEMMA 3.4.

$$h^{KO}(U_m^H) = U_m^{KO}\left(\frac{F(X)}{X}\right)^{m-1} \frac{dF(X)}{dX}.$$

PROOF. By the same way as the notation $U_m^{KO} = U^{KO}X^{m-1}$, we can write $U_m^H = Ux^{m-1}$ for any $m \in Z$. Recall that $g(x) = (2\sinh(\sqrt{x}/2))^2$ and then $ph(X) = g(x)$. Thus, $ph(F(X)) = f(g(x)) = x$. Since $sh(\zeta) = dg(x)/dx$ as in the proof of Lemma 2.3, $ph(U^{KO}) = U dg(x)/dx$, and thus $ph(U^{KO} dF(X)/dX) = U$. Hence,

$$(3.5) \qquad ph\left(U^{KO}F(X)^k \frac{dF(X)}{dX}\right) = Ux^k = U_m^H x^{k-m+1}$$

for any $k \geq m-1$. Since $(ph)^{-1}(U_m^H) = h^{KO}(U_m^H)$, by taking $k = m-1$ in (3.5), we get the required result.

Before proceeding to a factorization of the double transfer map, we remark that $h^{KO}(U_m^H) - U_m^{KO} \in \operatorname{Ker}(i^*) = \operatorname{Im}(j^*)$ for the maps i and j in (3.1). Thus, there is an element $V_m \in KO^{4m-1}(QP_{m+1}; Q)$ with $j^*(V_m) = h^{KO}(U_m^H) - U_m^{KO}$. Since j^* is injective, V_m is uniquely determined by the given relation, and we can denote $V_m = h^{KO}(U_m^H) - U_m^{KO}$. We notice that $h^{KO}(\bar{u}_1) = \rho_Z(V_m)$, and the following is clear from Lemma 3.4:

COROLLARY 3.6.

$$V_m = U_m^{KO}\left(\left(\frac{F(X)}{X}\right)^{m-1} \frac{dF(X)}{dX} - 1\right).$$

The double transfer map τ_2 of τ is defined to be $\tau \wedge \tau = (\tau \wedge 1)(1 \wedge \tau)$: $QP_{m+1} \wedge QP_{n+1} \to S^{4(m+n)}$ for any $m, n \in Z$. Let $N_2 \xrightarrow{\delta_2} \Sigma N_1 \xrightarrow{\delta_1} S^2$ be the first two stages of the chromatic filtration (cf. [9]). Then, by [7; Th. 2.8], the double transfer map τ_2 factors through N_2 as follows:

THEOREM 3.7. *There is a map* $\bar{u}_2: QP_{m+1} \wedge QP_{n+1} \to \Sigma^{4(m+n)-2} N_2$ *which* *makes the following diagram stably homotopy commutative up to sign:*

$$\begin{array}{ccccc} QP_{m+1}\wedge QP_{n+1} & \xrightarrow{1\wedge t} & QP_{m+1}\wedge S^{4n} & \xrightarrow{t\wedge 1} & S^{4m}\wedge S^{4n} \\ \Big\downarrow{\scriptstyle \bar{u}_2} & & \Big\downarrow{\scriptstyle \bar{u}_1} & & \Big\| \\ \Sigma^{4(m+n)-2}N_2 & \xrightarrow{\delta_2} & \Sigma^{4(m+n)-1}N_1 & \xrightarrow{\delta_1} & S^{4(m+n)} . \end{array}$$

In this paper, we omit the details of this factorization, and refer to [7] on its application to the transfer images. Here, we only remark that the map $\bar{u}_2$ is well described by an element $\tilde{u}\in KO^{4(m+n)-2}(QP_{m+1}\wedge QP_n; Q)$, by [7; §2], and we show in the next theorem that $\tilde{u}$ can be represented by the Bernoulli numbers.

THEOREM 3.8.

$$\tilde{u}=U_m^{KO}\left(\left(\frac{F(X)}{X}\right)^{m-1}\frac{dF(X)}{dX}-1\right)\otimes U_n^{KO}+\sum_{k,l>0}\Gamma_{k,l}U_m^{KO}h_{m,k}(X)\otimes U_n^{KO}h_{n,l}(X),$$

where $\Gamma_{k,l}=(9^l-1)/(9^{k+l}-1)$ *and* $h_{i,j}(X)$ *is given by*

$$h_{i,j}(X)=B_j^{KO}(\zeta+(i-1)\xi)F(X)^j\left(\frac{F(X)}{X}\right)^{i-1}\frac{dF(X)}{dX}.$$

PROOF. We put $B_k^m=B_k^{KO}(\zeta+(m-1)\xi)$ for brevity. By the proof of [7; Prop. 2.4], $\tilde{u}$ is given by

$$\tilde{u}=V_m\otimes U_n^{KO}-\sum_{k,l>0}\Gamma_{k,l}A_k\otimes B_l . \tag{3.9}$$

Here, V_m is the element of Corollary 3.6, and A_k and B_l are given respectively by the relations $V_m=\sum_{i>0}A_i$ with $\psi^3 A_i=9^i A_i$ and $U_n^{KO}=\sum_{j\geq 0}B_j$ with $\psi^3 B_j=9^j B_j$ for the stable Adams operation ψ^3. The first term on the right hand side of the required equality follows from Corollary 3.6, and thus we have only to check that A_i and B_j are given by the required formulas. We can regard the equation of Proposition 2.5 as the one with variable x, and thus, replacing x by $F(X)$ and using that $G(F(X))=X$, we have

$$\frac{X^{m-1}}{F(X)^{m-1}\dfrac{dF(X)}{dX}}=\sum_{k\geq 0}B_k^m F(X)^k .$$

Hence, $U_m^{KO}=U^{KO}X^{m-1}=\sum_{k\geq 0}U^{KO}B_k^m F(X)^{m+k-1}(dF(X)/dX)$. On the other hand, by (3.5), we have $ph(U^{KO}F(X)^{m+k-1}(dF(X)/dX))=Ux^{m+k-1}$. Thus, by these equations,

$$ph(U_m^{KO})=\sum_{k\geq 0}Uph(B_k^m)x^{m+k-1} . \tag{3.10}$$

Then, $ph(V_m) = ph(h^{KO}(U_m^H) - U_m^{KO}) = U_m^H - ph(U_m^{KO}) = -\sum_{k>0} Uph(B_k^m)x^{m+k-1}$. Hence, it follows that $ph(A_k) = -Uph(B_k^m)x^{m+k-1}$, and thus

$$A_k = -U^{KO}B_k^m F(X)^{m+k-1}\frac{dF(X)}{dX}. \tag{3.11}$$

Using (3.10) for n instead of m, and just by the same reason as above, we have

$$B_l = U^{KO}B_l^n F(X)^{n+l-1}\frac{dF(X)}{dX}. \tag{3.12}$$

Thus we complete the proof by (3.9), (3.11) and (3.12).

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