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Symplectic Pontrjagin numbers and homotopy groups of $MSp(n)$

Mitsunori IMAOKA

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Introduction

In [10] and [11], E. Rees and E. Thomas have studied the divisibility of some Chern numbers of the complex cobordism classes and the homotopy groups of $MU(n)$. The purpose of this paper is to study the symplectic cobordism theory by using their methods.

Let $MSp(n)$ be the Thom space of the universal symplectic vector bundle over the classifying space $BSp(n)$, and $MSp=\{MSp(n), \varepsilon_n\}$ be the Thom spectrum of the symplectic cobordism theory, where $\varepsilon_n: \Sigma^4 MSp(n)\to MSp(n+1)$ is the structure map. Let $b_n: MSp(n)\to \Omega^{4N}MSp(n+N)$ be the adjoint map of the composition $\varepsilon_{n,N}: \Sigma^{4N}MSp(n)\to MSp(n+N)$ of $\Sigma^i\varepsilon_{n+i}$, where $N\geqq n>0$. Converting b_n into a fibering with fiber F_n, we consider the fibering

$$F_n \longrightarrow MSp(n) \xrightarrow{b_n} \Omega^{4N}MSp(n+N). \tag{1}$$

Then F_n is $(8n-2)$-connected, and we can determine the cohomology groups of F_n in dimensions less than $12n-2$ (see Proposition 2.15).

Let $P_i\in H^{4i}(BSp)$ be the i-th symplectic Pontrjagin class. For a symplectic cobordism class $u\in\pi_{4k}(MSp)$ and a class $P_{i_1}\cdots P_{i_j}\in H^{4k}(BSp)$ with $\sum_{t=1}^{j} i_t=k$, $P_{i_1}\cdots P_{i_j}[u]$ denotes the Pontrjagin number of u for a class $P_{i_1}\cdots P_{i_j}$.

Our first purpose is to obtain the divisibility of some Pontrjagin numbers of the symplectic cobordism classes by making use of the cohomology groups of F_n. As a concrete result, we have the following theorem (see Theorem 3.8):

THEOREM I. *Let $n\geqq 1$. Then*
(i) $P_n[u]\equiv 0 \bmod 8$ *for any* $u\in\pi_{4n}(MSp)$.
(ii) $P_1P_n[u]-((n+4)/2)P_{n+1}[u]\equiv 0 \bmod 24$ *for any* $u\in\pi_{4n+4}(MSp)$.

The divisibility of Pontrjagin numbers of some symplectic cobordism classes has been studied in [14], [13], [3], [6] to investigate the structure of $\pi_*(MSp)$. For the divisibility (i) of the above theorem, E. E. Floyd [3] has proved it with some restriction by using the alternative method, and some application of the method of Floyd is considered in [4].

The second purpose of this paper is to study the homotopy groups

$\pi_{8n-1}(MSp(n))$ and $\pi_{8n+3}(MSp(n))$ by using the fibering (1) and some examples of the symplectic cobordism classes. Our second results are stated as follows (see Corollaries 4.4, 4.5 and Theorems 4.6, 4.7):

THEOREM II. (i) *Let $m(n)$ be the greatest common measure of* $\{(1/8)P_n[u] \mid u \in \pi_{4n}(MSp)\}$. *Then the induced homomorphism*

$$b_{n*}: \pi_{8n-1}(MSp(n)) \longrightarrow \pi_{4n-1}(MSp)$$

of b_n in (1) *is epimorphic and its kernel is a cyclic group of order* $4m(n)$ *generated by the Whitehead product* $[i, i]$ *for the homotopy class* i *of the natural inclusion* $S^{4n} \to MSp(n)$.

(ii) *If* $2\pi_{4n-1}(MSp)=0$ *and* n *is not a power of* 2, *then* b_{n*} *in* (i) *is split epimorphic, that is,*

$$\pi_{8n-1}(MSp(n)) \cong Z_{4m(n)} \oplus \pi_{4n-1}(MSp).$$

(iii) *$m(n)$ is a power of 2 for $n \neq 1$, 3, and $m(1)=m(3)=3$.*

(iv) *$m(n)=1$ if $n=2^k+2^l-1$ or 2^k+2^l ($k, l \geqq 0$) and $n \neq 1$, 3.*

THEOREM III. (i) *$\pi_{8n+3}(MSp(n))$ ($n \geqq 3$) has no p-torsion for any odd prime p.*

(ii) *The homomorphism $b_{n*}: \pi_{8n+3}(MSp(n)) \to \pi_{4n+3}(MSp)$ is epimorphic for $n \geqq 1$.*

(iii) *If $n=2^k+2^l-1$ ($k, l \geqq 1$), then b_{n*} in* (ii) *is isomorphic, that is,* $\pi_{8n+3}(MSp(n)) \cong \pi_{4n+3}(MSp)$.

We notice that the assumption $2\pi_{4n-1}(MSp)=0$ in Theorem II (ii) is valid for $n \leqq 8$ by the result of D. M. Segal [12].

This paper is organized as follows. In § 1 we summarize the necessary lemmas concerning the iterated cohomology suspension investigated by R. J. Milgram [5]. In § 2 we study the cohomology groups of F_n, and in § 3 we state the divisibility of some Pontrjagin numbers and prove Theorem I. In § 4 we consider the homotopy exact sequence concerning $\pi_{8n-1}(MSp(n))$ and $\pi_{8n+3}(MSp(n))$ and state Theorems II and III. In § 5 we prepare some symplectic cobordism classes and prove these theorems.

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§ 1. Preliminaries

In this section, we summarize some necessary lemmas concerning the iterated cohomology suspension studied by R. J. Milgram [5].

Let Y be an $(r-1)$-connected CW-complex, and $j\colon Y\to\Omega^k\Sigma^k Y$ be the natural inclusion. Then Milgram [5; Th. 1. 11] proved that the cofiber $\Omega^k\Sigma^k Y/Y$ of j is homotopy equivalent in dimensions less than $3r-1$ to the space $S^{k-1}\rtimes_T Y\wedge Y$, where $S^{k-1}\rtimes_T Y\wedge Y$ is the quotient space of $S^{k-1}\times(Y\wedge Y)$ by the identification of (x, y_1, y_2) with $(-x, y_2, y_1)$ and $(x, *)$ with the base point.

When $Y=\Omega^k X$ for a $(k+r-1)$-connected CW-complex X, we can consider the evaluation map $e\colon \Sigma^k\Omega^k X\to X$ and the fibering

$$F \xrightarrow{i} \Sigma^k Y \xrightarrow{e} X \qquad (Y=\Omega^k X).$$

Then the inclusion $j\colon Y\to\Omega^k\Sigma^k Y$ is a section of the fibering $\Omega^k F\xrightarrow{\Omega^k i}\Omega^k\Sigma^k Y\xrightarrow{\Omega^k e} Y$, and we have the maps $F\xleftarrow{e}\Sigma^k\Omega^k F\xrightarrow{\Sigma^k(q\cdot\Omega^k i)}\Sigma^k(\Omega^k\Sigma^k Y/Y)$, where $q\colon \Omega^k\Sigma^k Y\to \Omega^k\Sigma^k Y/Y$ is the canonical projection. Since these maps are $(k+3r-1)$-equivalent, we have the following lemma (cf. Proof of [5; Cor. 4.4]).

LEMMA 1.1. *In dimensions less than $k+3r-1$, F is homotopy equivalent to $\Sigma^k(S^{k-1}\rtimes_T\Omega^k X\wedge\Omega^k X)$.*

Take X to be the Thom space $MSp(n+N)$ of the universal symplectic vector bundle over $BSp(n+N)$. Then we have the fibering

$$F(e)\xrightarrow{i}\Sigma^{4N}\Omega^{4N}MSp(n+N)\xrightarrow{e} MSp(n+N). \tag{1.2}$$

Hereafter we shall take integers n and N to satisfy $N\geqq n>0$. By Lemma 1.1, we have

COROLLARY 1.3. *In dimensions less than $4N+12n-1$, $F(e)$ is homotopy equivalent to $\Sigma^{4N}\Gamma(n, N)$, where we use the notation*

$$\Gamma(n, N) = S^{4N-1}\rtimes_T\Omega^{4N}MSp(n+N)\wedge\Omega^{4N}MSp(n+N).$$

Put $\Lambda=Z$ or Z_p (p: prime). By this corollary, we have the isomorphisms

$$H^{i+4N}(F(e);\Lambda) = H^i(\Gamma(n, N);\Lambda) \qquad \text{for} \quad i\leqq 12n-2.$$

Therefore the Serre cohomology exact sequence of (1.2) turns out to the exact sequence

$$\begin{aligned}(1.4)\quad \cdots \longrightarrow H^{i-1}(\Gamma(n, N);\Lambda)\xrightarrow{\tau} H^{i+4N}(MSp(n+N);\Lambda)\xrightarrow{\sigma}\\ H^i(\Omega^{4N}MSp(n+N);\Lambda)\xrightarrow{j} H^i(\Gamma(n, N);\Lambda)\longrightarrow\cdots \qquad (i\leqq 12n-2),\end{aligned}$$

where τ, σ and j are the transgression, the induced homomorphisms e^* and i^* composed with the suspension isomorphisms respectively, and σ is known to be the iterated cohomology suspension.

We shall use the following notations:

(1.5) (i) By a series $R=(r_1, r_2,\dots)$, we mean that r_i's are non negative integers with the condition $r_i=0$ $(i\geqq m)$ for some $m\geqq 1$, and this condition will be denoted by $R<m$.

(ii) For a series $R=(r_1, r_2,\dots)$, we set $|R|=\sum_{i\geqq 1} ir_i$.

(iii) For series $R=(r_1, r_2,\dots)$ and $S=(s_1, s_2,\dots)$, $R>S$ means that $r_i=s_i$ $(i>m)$ and $r_m>s_m$ for some $m\geqq 1$.

Let $P_i\in H^{4i}(BSp)$ be the universal i-th symplectic Pontrjagin class. Then it is known that $H^*(BSp(n+N))=Z[\![P_1,\dots, P_{n+N}]\!]$. We set $P^R=P_1^{r_1}P_2^{r_2}\cdots\in H^{4|R|}(BSp)$ for a series $R=(r_1, r_2,\dots)$.

Let $U\in H^{4(n+N)}(MSp(n+N))$ be the Thom class of $MSp(n+N)$, and consider the composition

$$V\colon H^{i-4n}(BSp(n+N))\xrightarrow[\cong]{U} H^{i+4N}(MSp(n+N))\xrightarrow{\sigma} H^i(\Omega^{4N}MSp(n+N)),$$

where U is the Thom isomorphism given by $U(x)=Ux$ and σ is the iterated cohomology suspension in (1.4). Here σ is isomorphic for $i\leqq 8n-1$, and $H^*(\Omega^{4N}MSp(n+N))$ for $*\leqq 8n-1$ is the free abelian group with basis $\{V(P^R)\mid |R|<n\}$, where

$$V(P^R) = \sigma(UP^R)\in H^{4(n+|R|)}(\Omega^{4N}MSp(n+N)). \tag{1.6}$$

The following lemma is an immediate consequence of [5; Prop. 3.1] (cf. [11; (2.1)]), where $\langle\theta, \theta'\rangle$ and $e^i\cdot\theta\otimes\theta$ are the notations used in [11].

LEMMA 1.7. (i) *The cohomology group $H^i(\Gamma(n, N))$ for $i\leqq 12n-2$ is a direct sum of some copies of Z and Z_2. A basis of its free part consists of the following classes*:

$$\langle V(P^R), V(P^S)\rangle\in H^{8n+4(|R|+|S|)}(\Gamma(n, N)) \ \textit{with}\ R>S,\ |R|+|S|\leqq n-1,$$

$$1\cdot V(P^R)\otimes V(P^R)\in H^{8n+8|R|}(\Gamma(n, N)) \quad \textit{with}\ 2|R|\leqq n-1.$$

A basis of its Z_2-summands consists of the following classes:

$$e^{2k}\cdot V(P^R)\otimes V(P^R)\in H^{8n+2k+8|R|}(\Gamma(n, N)) \ \textit{with}\ k\geqq 1,\ 2k+8|R|\leqq 4n-2.$$

(ii) *A basis of $H^i(\Gamma(n, N); Z_2)$ for $i\leqq 12n-2$ consists of the* mod 2 *reductions of the classes given in* (i) *and moreover the classes*

$$e^{2k+1}\cdot V(P^R)\otimes V(P^R)\in H^{8n+2k+1+8|R|}(\Gamma(n, N); Z_2)$$
$$\textit{with}\ k\geqq 0,\ 2k+8|R|\leqq 4n-4.$$

We remark that the classes $u\cdot\theta\otimes\theta$ in [11; (2.1)] do not appear in $H^i(\Gamma(n, N))$ for $i\leqq 12n-2$, since $N\geqq n$.

By the above lemma, we have $H^j(\Gamma(n, N))=0$ if j is odd and $j<12n-2$, and the following

LEMMA 1.8. (i) *The sequence* (1.4) *for* $\Lambda=Z$ *and* $i\leqq 12n-2$ *is short exact*:

$$0 \longrightarrow H^{i+4N}(MSp(n+N)) \xrightarrow{\sigma} H^i(\Omega^{4N}MSp(n+N)) \xrightarrow{j} H^i(\Gamma(n, N)) \longrightarrow 0.$$

(ii) $H^i(\Omega^{4N}MSp(n+N))=0$ *if* i *is odd and* $i\leqq 12n-2$.

For the maps j and τ in (1.4), we have the following lemma by [5; Th. 4.6] (cf. [11; (2.10), (2.5)]):

LEMMA 1.9. (i) *In the integral cohomology groups,*

$$j(V(P^R)V(P^S)) = \begin{cases} \langle V(P^R), V(P^S)\rangle & \text{if } R > S, \\ 2(1\cdot V(P^R)\otimes V(P^R)) & \text{if } R = S. \end{cases}$$

(ii) *In the* mod 2 *cohomology groups,*

$$\tau(e^{4j-1}\cdot V(P^R)\otimes V(P^R)) = Sq^{4(n+j+|R|)}(UP^R).$$

The next lemma can be proved by a similar argument to E. Recs and E. Thomas [11; 2.4, 2.6, 2.8].

LEMMA 1.10. *For* $i<3n$, *the cohomology group* $H^{4i}(\Omega^{4N}MSp(n+N))$ *is a free abelian group.*

PROOF. $H^{4i}(\Omega^{4N}MSp(n+N))$ has no odd torsion by Lemma 1.7. We prove that

$$(*) \qquad \tau\colon H^{4i-1}(\Gamma(n, N); Z_2) \longrightarrow H^{4N+4i}(MSp(n+N); Z_2)$$

is monomorphic if $i < 3n$.

Then $H^{4i-1}(\Omega^{4N}MSp(n+N); Z_2)=0$ by the exact sequence (1.4), and hence $H^{4i}(\Omega^{4N}MSp(n+N))$ has no 2-torsion by the universal coefficient theorem. Thus we have the lemma.

Now we prove (*). A basis of $H^{4i-1}(\Gamma(n, N); Z_2)$ $(i<3n)$ consists of the classes $\alpha_R=e^{4i-8(n+|R|)-1}\cdot V(P^R)\otimes V(P^R)$ with $4i-8(n+|R|)-1>0$ by Lemma 1.7 (ii), and

$$\tau(\alpha_R) = Sq^{4t}(UP^R), \quad t = i - n - |R|,$$

by Lemma 1.9 (ii). We have $Sq^iU=UP_j$ if $i=4j$, $=0$ otherwise. Hence, by the Cartan formula, the Wu formula $Sq^{4s}P_j=\sum_l\binom{j-s+l-1}{s}P_{s-l}P_{j+l}$ and the condition $2(n+|R|)<i<3n$, we see that

$$\tau(\alpha_R) = Sq^{4t}(UP^R) = UP_t P^R + \textstyle\sum_{S<t} m_S P^S, \quad R < t = i - n - |R|,$$

for some integers m_S, where $<$ is the notation in (1.5) (i). Therefore we see that τ is monomorphic and $(*)$ is proved. q.e.d.

The formulas for the cohomology operations on $H^*(\Gamma(n, N); Z_2)$ are given by Milgram [5; Th. 3.7.] (cf. [11; (2.3)]) as follows:

LEMMA 1.11. (i) $Sq^{4i}\langle V(P^R), V(P^S)\rangle = \sum_{0\leqq r<i/2}\langle Sq^{4r}V(P^R), Sq^{4(i-r)}V(P^S)\rangle$,

$$Sq^j\langle V(P^R), V(P^S)\rangle = 0 \quad \textit{if} \quad j \not\equiv 0 \mod 4.$$

(ii) $$Sq^{4i}(1\cdot V(P^R)\otimes V(P^R)) = \textstyle\sum_{0\leqq r<i/2}\langle Sq^{4r}V(P^R), Sq^{4(i-r)}V(P^R)\rangle + \sum_{j\geqq 0}\binom{n+|R|-j}{i-2j} e^{4i-8j}\cdot Sq^{4j}V(P^R)\otimes Sq^{4j}V(P^R),$$

$$Sq^j(1\cdot V(P^R)\otimes V(P^R)) = 0 \quad \textit{if} \quad j \not\equiv 0 \mod 4.$$

(iii) *For* $k \geqq 1$,

$$Sq^i(e^k\cdot V(P^R)\otimes V(P^R)) = \textstyle\sum_{j,r\geqq 0}\binom{k}{r}\binom{4(n+|R|-j)}{i-r-8j} e^{k+i-8j}\cdot Sq^{4j}V(P^R)\otimes Sq^{4j}V(P^R).$$

Especially, we have

COROLLARY 1.12. *For* $k\geqq 1$,

$$Sq^1(e^k\cdot V(P^R)\otimes V(P^R)) = ke^{k+1}\cdot V(P^R)\otimes V(P^R),$$

$$Sq^2(e^k\cdot V(P^R)\otimes V(P^R)) = \binom{k}{2} e^{k+2}\cdot V(P^R)\otimes V(P^R),$$

$$Sq^4(e^k\cdot V(P^R)\otimes V(P^R)) = \left(\binom{k}{4} + n + |R|\right) e^{k+4}\cdot V(P^R)\otimes V(P^R).$$

Let X be a $(k+r-1)$-connected space and $r\geqq 2$. The evaluation map $e_i\colon \Sigma^i\Omega^i X\to X$ is the composition of the evaluation maps $e'\colon \Sigma^j\Omega^j X\to \Sigma^{j-1}\Omega^{j-1}X$ $(i\geqq j\geqq 1)$, and we have the commutative diagram

$$\text{(1.13)} \qquad \begin{tikzcd} F(e_k) \arrow[r] \arrow[d, "f_1"] & \Sigma^k\Omega^k X \arrow[r, "e_k"] \arrow[d, "e'"] & X \arrow[d, equal] \\ F(e_{k-1}) \arrow[r] & \Sigma^{k-1}\Omega^{k-1}X \arrow[r, "e_{k-1}"] & X, \end{tikzcd}$$

where $F(e_i)$ $(i=k-1, k)$ are the fibers of the respective fiberings and f_1 is the restriction of e' to the fiber. If we identify $F(e_i)$ with $\Sigma^i(S^{i-1}\rtimes_T\Omega^i X\wedge\Omega^i X)$ in dimensions less than $i+3r-1$ by Lemma 1.1, then we see that f_1 is identified with the composition of

$$\Sigma^k(S^{k-1}\rtimes_T Y\wedge Y)\xrightarrow{\Sigma^k\tau_1}\Sigma^k\Omega(S^{k-2}\rtimes_T\Sigma Y\wedge\Sigma Y)$$
$$\xrightarrow{\tilde{e}}\Sigma^{k-1}(S^{k-2}\rtimes_T\Omega^{k-1}X\wedge\Omega^{k-1}X),$$

where $Y=\Omega^kX$ and τ_1 is the natural map $\Omega^k\Sigma^kY/Y\to\Omega(\Omega^{k-1}\Sigma^{k-1}(\Sigma Y)/\Sigma Y)$ (see [5; § 2]) with the identifications $\Omega^i\Sigma^iW/W\simeq S^{i-1}\rtimes_T W\wedge W(W=Y,\Sigma Y)$ and $\tilde{e}$ is the map induced by the evaluation maps.

In the diagram (1.13), set $X=MSp(n+N)$ $(N\geqq n+2)$ and $k=4N,\dots,4N-3$ to obtain the commutative diagram

$$\begin{tikzcd} F(e_{4N}) \arrow[r] \arrow[d, "f"] & \Sigma^{4N}\Omega^{4N}MSp(n+N) \arrow[r, "e_{4N}"] \arrow[d, "e''"] & MSp(n+N) \arrow[d, equal] \\ F(e_{4N-4}) \arrow[r] & \Sigma^{4N-4}\Omega^{4N-4}MSp(n+N) \arrow[r, "e_{4N-4}"] & MSp(n+N), \end{tikzcd} \tag{1.14}$$

where $e''=(e')^4$ and $f=(f_1)^4$. Let σ': $H^{i+4N-4}(MSp(n+N))\to H^i(\Omega^{4N-4}MSp(n+N))$ be the iterated cohomology suspension. Then, by using the identifications of $F(e_{4N})$ with $\Sigma^{4N}\Gamma(n, N)$, $F(e_{4N-4})$ with $\Sigma^{4N-4}\Gamma(n+1, N-1)$ and f_1 with $\tilde{e}(\Sigma^k\tau_1)$ as is stated above, we have the following lemma by [5; Th. 3.8] on τ_1:

LEMMA 1.15. *Set $V'(x)=\sigma'(Ux)$. Then*

$$f^*(e^k\cdot V'(P^R)\otimes V'(P^R))=e^{k+4}\cdot V(P^R)\otimes V(P^R)\quad \textit{for any}\quad k\geqq 0,$$
$$f^*(\langle V'(P^R), V'(P^S)\rangle)=0.$$

§ 2. The cohomology groups of F_n

The structure map ε_n: $\Sigma^4 MSp(n)\to MSp(n+1)$ in the Thom spectrum $MSp=\{MSp(n), \varepsilon_n\}$ of the symplectic cobordism theory is defined to be the map induced by the bundle map of $\xi_n\oplus 1$ to ξ_{n+1}, where ξ_i is the universal symplectic vector bundle over $BSp(i)$ and 1 means the trivial symplectic line bundle. Consider the composition $\varepsilon_{n,N}$: $\Sigma^{4N}MSp(n)\to MSp(n+N)$ of $\Sigma^j\varepsilon_{n+i}$, and its adjoint map $b_{n,N}$: $MSp(n)\to\Omega^{4N}MSp(n+N)$. Converting $b_{n,N}$ into a fibering with fiber $F_{n,N}$, we consider the fibering

$$F_{n,N}\longrightarrow MSp(n)\xrightarrow{b_{n,N}}\Omega^{4N}MSp(n+N). \tag{2.1}$$

For any $N'>N\geqq n\geqq 1$, the homotopy groups, the cohomology groups of $F_{n,N}$ and $F_{n,N'}$ are naturally isomorphic in dimensions less than $12n-2$, because ε_{n+N} is $(8n+8N+6)$-equivalent. Therefore, for a positive integer n, we shall take an integer N large enough to satisfy $N\geqq n$; and we denote simply by

$$b_n=b_{n,N}\quad\text{and}\quad F_n=F_{n,N},$$

and investigate the cohomology groups of F_n in dimensions less than $12n-2$. We remark that F_n is $(8n-2)$-connected.

Let $I_n \subset H^*(BSp)$ be the ideal generated by $\{P_i \mid i>n\}$, and

(2.2) $UI_n^j \subset H^{4(n+N)+j}(MSp(n+N))$ be the subgroup generated by $\{UP^R \mid P^R \in I_n^j = I_n \cap H^j(BSp)\}$.

Then we have

LEMMA 2.3. (i) *The composition*

$$e(\Sigma^{4N}b_n)\colon \Sigma^{4N}MSp(n) \longrightarrow \Sigma^{4N}\Omega^{4N}MSp(n+N) \longrightarrow MSp(n+N)$$

is homotopic to $\varepsilon_{n,N}$, where e is the evaluation map in (1.2).

(ii) *The following commutative diagram of four short exact sequences holds for $i \leqq 12n-2$:*

$$\begin{tikzcd}
UI_n^{i-4n} \arrow[r, tail, "\tilde{\sigma}"] \arrow[d, tail, "\cap"] & H^{i-1}(F_n) \arrow[r, two heads, "\tilde{j}"] \arrow[d, tail, "\tau"] & H^i(\Gamma(n, N)) \arrow[d, equal] \\
H^{i+4N}(MSp(n+N)) \arrow[r, tail, "\sigma"] \arrow[d, "\varepsilon_{n,N}^*"] & H^i(\Omega^{4N}MSp(n+N)) \arrow[r, two heads, "j"] \arrow[d, "b_n^*"] & H^i(\Gamma(n, N)) \\
H^{i+4N}(\Sigma^{4N}MSp(n)) & H^i(MSp(n)). \arrow[l, "\Sigma^{4N}", "\cong"'] &
\end{tikzcd}$$

Here the central horizontal sequence is the one in Lemma 1.8 (i), *the central vertical sequence is the Serre cohomology exact sequence of the fibering* (2.1), *where τ denotes its transgression, $\tilde{\sigma}$ is the restriction of σ, and $\tilde{j}$ is the composition $j\tau$.*

PROOF. (i) is clear by definition.

(ii) The left hand vertical sequence is exact by the definition (2.2) of UI_n^j. By (i), the lower square commutes. Since $\varepsilon_{n,N}^*$ is epimorphic, so is b_n^*, and the central vertical sequence is short exact. Since the central horizontal sequence is short exact as is shown in Lemma 1.8 (i), so is the upper one by the 9 lemma.

q.e.d.

Consider the central vertical exact sequence in Lemma 2.3 (ii):

$$0 \longrightarrow H^{i-1}(F_n) \xrightarrow{\tau} H^i(\Omega^{4N}MSp(n+N)) \xrightarrow{b_n^*} H^i(MSp(n)) \longrightarrow 0.$$

LEMMA 2.4. *In $H^{8n+4i}(\Omega^{4N}MSp(n+N))$ for $i<n$, the following elements belong to* $\operatorname{Ker} b_n^* = \operatorname{Im}\tau$:

(1) $V(P^R)V(P^S) - V(P_nP^RP^S)$ *for any series R and S with $|R|+|S|=i$,*

(2) $V(P_{n+k}P^R)$ *for any k and any series R with $k \geqq 1$ and $k+|R|=i$.*

PROOF. (1) Let $\tilde{U}$ denote the Thom class of $MSp(n)$. Then, by Lemma 2.3,

$$b_n^*(V(P^R)V(P^S)) = b_n^*\sigma(UP^R)\cdot b_n^*\sigma(UP^S) = (\Sigma^{4N})^{-1}\varepsilon_{n,N}^*(UP^R)\cdot(\Sigma^{4N})^{-1}\varepsilon_{n,N}^*(UP^S)$$
$$= \tilde{U}P^R\cdot\tilde{U}P^S = \tilde{U}P_nP^RP^S = b_n^*(V(P_nP^RP^S)).$$

(2) By the condition, $x=P_{n+k}P^R\in I_n$ and $V(x)=\sigma(Ux)=\tau\tilde{\sigma}(x)\in \operatorname{Im}\tau$.

q.e.d.

Especially, $(V(P^R))^2-V(P_n(P^R)^2)$ $(2|R|<n)$ is contained in $\operatorname{Ker} b_n^*$ by the above lemma for $R=S$. On the other hand, its j-image is $2(1\cdot V(P^R)\otimes V(P^R))$ by Lemma 1.9 (i). Therefore, by the commutative diagram in Lemma 2.3 (ii), we see the following

(2.5) For any series R with $2|R|<n$, there are elements $b'\in H^{8(n+|R|)-1}(F_n)$ and $v'\in I_n^{4n+8|R|}$ such that $\tilde{j}(b')=1\cdot V(P^R)\otimes V(P^R)$ and

$$V(v') = \sigma(Uv') = 2\tau(b') - (V(P^R))^2 + V(P_n(P^R)^2).$$

LEMMA 2.6. *For any series R with $2|R|<n$, we can take elements*

$$b(R)(=b_{n,N}(R))\in H^{8i-1}(F_n) \text{ and } v_n(R)(=v_{n,N}(R))\in I_n^{4i+4|R|} \quad (i=n+|R|)$$

such that

(i) $\tilde{j}(b(R)) = 1\cdot V(P^R)\otimes V(P^R) +$ *a torsion element,*

(ii) $\tau(b(R)) = (1/2)\{(V(P^R))^2 - V(P_n(P^R)^2) + V(v_n(R))\}$ *in* $H^{8i}(\Omega^{4N}MSp(n+N))$, *and that $v_n(R)$ for $0<2|R|\leqq n-3$ satisfies the conditions*

(iii) $v_n(R)=P_{n+|R|}P^R+\sum_{S<n+|R|}m_SP^S$ *for some integers* m_S,

(iv) $Uv_n(R)=Sq^{4i}(UP^R)+UP_n(P^R)^2$ *in* $H^{4N+8i}(MSp(n+N);Z_2)$.

PROOF. By the Cartan formula, we have

$$Sq^{4i}(UP^R) = Ux + UP_n(P^R)^2,\quad x = P_iP^R + \textstyle\sum_{l=1}^{|R|-1}P_{i-l}Sq^{4l}(P^R).$$

Hence we can take an element $v_n(R)\in I_n^{4i+4|R|}$ such that its mod 2 reduction is x and that it satisfies (iii), and then it satisfies (iv) also.

Now, let $b'\in H^{8i}(F_n)$ and $v'\in I_n^{4i+4|R|}$ be elements in (2.5). Then we can prove the lemma by showing

(2.7) $V(v')\equiv V(v_n(R)) \bmod 2$ for $R\neq 0$ (the 0-series) with $2|R|\leqq n-3$.

In fact, there is an element $y=(1/2)\{V(v')-V(v_n(R))\}$ by (2.7), and $b_n^*(y)=0$ since $H^{8i}(MSp(n))$ has no torsion. Thus we see that $b(R)=b'-\tau^{-1}(y)$ and $v_n(R)$ satisfy (i) and (ii) by (2.5). When $R=0$ or $2|R|>n-3$, $b(R)=b'$ and $v_n(R)=v'$ are the desired element.

To show (2.7), consider the commutative diagram ($8i\leqq 12n-9$, $n\geqq 2$)

$$\begin{tikzcd}
H^{8i-1}(\Gamma(n, N); Z_2) \arrow[r, "\tau"] \arrow[d, "f^*"] & H^{8i+4N}(MSp(n+N); Z_2) \arrow[d, equal] & \\
H^{8i-5}(\Gamma(n-1, N+1); Z_2) \arrow[r, "\tau'"] & H^{8i+4N}(MSp(n+N); Z_2) \arrow[r, "\sigma"] & H^{8i}(\Omega^{4N}MSp(n+N); Z_2) \arrow[d, "\sigma''"] \\
 & \arrow[r, "\sigma'"] & H^{8i-4}(\Omega^{4N+4}MSp(n+N); Z_2).
\end{tikzcd}$$

Here two exact sequences are the ones in (1.4) for $\Lambda=Z_2$, f^* is the homomorphism in Lemma 1.15, and σ, σ', σ'' are the iterated cohomology suspensions. Since $(V(P^R))^2-V(P_n(P^R)^2)+V(v')=0$ in $H^{8i}(\Omega^{4N}MSp(n+N); Z_2)$ by (2.5), we have $V'(P_n(P^R)^2)+V'(v')=0$ where $V'(x)=\sigma'(Ux)$. Hence there is a class $z\in H^{8i-5}(\Gamma(n-1, N+1); Z_2)$ satisfying $\tau'(z)=UP_n(P^R)^2+Uv'$. Since $v'\in I_n$, we have $z=e^3\cdot V'(P^R)\otimes V'(P^R)+\sum_{T,l\geqq 1}\lambda_{l,T}e^{8l+3}\cdot V'(P^T)\otimes V'(P^T)$ for some $\lambda_{l,T}\in Z_2$, by Lemmas 1.7 (ii) and 1.9. These two equalities imply

$$UP_n(P^R)^2 + Uv' = Sq^{4i}(UP^R) + \tau(z') \ (z' = \textstyle\sum_{T,l\geqq 1}\lambda_{l,T}e^{8l-1}\cdot V(P^T)\otimes V(P^T))$$

by Lemmas 1.15 and 1.9. Therefore the mod 2 reduction of $Uv'-Uv_n(R)$ is equal to $\tau(z')$ by (iv), and hence that of $V(v')-V(v_n(R))$ is equal to $\sigma\tau(z')=0$. Thus we see (2.7). q.e.d.

LEMMA 2.8. *Assume $N\geqq 3n-2$. Then, for any integer $k\geqq 1$ and any series R with $k+2|R|\leqq n-1$, there is an element $c(4k, R)\in H^{8(n+|R|)+4k-1}(F_n)$ satisfying*

$$\tau(c(4k, R)) = (1/2)\{-V(P_{n+k}(P^R)^2) + V(v_{n+k}(R))\},$$

where $v_{n+k}(R)=v_{n+k,N-k}(R)\in I_{n+k}^{4(n+k)+8|R|}$ is an element in Lemma 2.6 and it satisfies

$$v_{n+k}(R) = P_{n+k+|R|}P^R + \textstyle\sum_{S<n+k+|R|} m_S P^S \quad \textit{for some integers } m_S.$$

PROOF. Consider the commutative diagram

$$\begin{tikzcd}
F_n \arrow[r] \arrow[d, "\tilde{b}"] & MSp(n) \arrow[r, "b_n"] \arrow[d, "b_{n,k}"] & \Omega^{4N}MSp(n+N) \arrow[d, equal] \\
\Omega^{4k}F_{n+k} \arrow[r] & \Omega^{4k}MSp(n+k) \arrow[r, "\Omega^{4k}b_{n+k}"] & \Omega^{4N}MSp(n+N),
\end{tikzcd}$$

where $\tilde{b}$ is the restriction of $b_{n,k}$. Then we have the commutative diagram

$$(*)\quad\begin{tikzcd}
H^{i+4k-1}(F_{n+k}) \arrow[r, "\sigma", "\cong"'] \arrow[d, tail, "\tau"] & H^{i-1}(\Omega^{4k}F_{n+k}) \arrow[r, "\tilde{b}^*"] & H^{i-1}(F_n) \arrow[d, tail, "\tau"] \\
H^{i+4k}(\Omega^{4(N-k)}MSp(n+N)) \arrow[rr, "\sigma"] & & H^{i}(\Omega^{4N}MSp(n+N)),
\end{tikzcd}$$

where σ's are the iterated cohomology suspensions. Furthermore,

$$\sigma\{(V'(P^R))^2 - V'(P_{n+k}(P^R)^2) + V'(v_{n+k}(R))\} = - V(P_{n+k}(P^R)^2) + V(v_{n+k}(R)),$$

where $v_{n+k}(R)$ satisfies (iii) in Lemma 2.6, i.e., the last equality in the lemma, since $2|R| \leqq n+k-3$. Thus $c(4k, R) = \tilde{b}^*\sigma(b_{n+k,N-k}(R))$ satisfies the desired equality by Lemma 2.6. q.e.d.

Now we can define the following classes $a(R, S)$, $b(R)$, $c(2i, R)$ and $d(k, R)$ in $H^*(F_n)$ $(* \leqq 12n-3)$:

(2.9) $a(R, S) \in H^{8n+4(|R|+|S|)-1}(F_n)$ for series R and S with $R>S$ and $|R|+|S| \leqq n-1$ satisfying

$$\tau(a(R, S)) = V(P^R)V(P^S) - V(P_n P^R P^S), \qquad \text{(cf. Lemma 2.4)}.$$

(2.10) $b(R) \in H^{8(n+|R|)-1}(F_n)$ in Lemma 2.6 for a series R with $2|R| \leqq n-1$ satisfying

$$\tau(b(R)) = (1/2)\{(V(P^R))^2 - V(P_n(P^R)^2) + V(v_n(R))\}.$$

(2.11) $c(4k, R) \in H^{8(n+|R|)+4k-1}(F_n)$ in Lemma 2.8 for an integer $k \geqq 1$ and a series R with $k+2|R| \leqq n-1$ satisfying

$$\tau(c(4k, R)) = (1/2)\{-V(P_{n+k}(P^R)^2) + V(v_{n+k}(R))\}, \qquad (N \geqq 3n-2).$$

(2.12) $c(4k+2, R) \in H^{8(n+|R|)+4k+1}(F_n)$ for an integer $k \geqq 0$ and a series R with $k+2|R| \leqq n-1$ satisfying

$$\bar{j}(c(4k+2, R)) = e^{4k+2} \cdot V(P^R) \otimes V(P^R),$$

where $\bar{j} = j\tau\colon H^{i-1}(F_n) \to H^i(\Gamma(n, N))$ in Lemma 2.3 (ii) is isomorphic if $i \equiv 1$ mod 4.

(2.13) $d(k, R) \in H^{8n+4|R|+4k-1}(F_n)$ for an integer $k \geqq 1$ and a series R with $k \leqq |R| < n-1-k$ satisfying

$$\tau(d(k, R)) = V(P_{n+k}P^R), \quad \text{(cf. Lemma 2.4)}.$$

For the epimorphism $\bar{j} = j\tau\colon H^{i-1}(F_n) \to H^i(\Gamma(n, N))$ $(i \leqq 12n-2)$ in Lemma 2.3 (ii), we have the following

LEMMA 2.14. (i) $\bar{j}(a(R, S)) = \langle V(P^R), V(P^S)\rangle$,

$\bar{j}(b(R)) = 1 \cdot V(P^R) \otimes V(P^R) +$ *a linear combination of* $e^{4l} \cdot V(P^S) \otimes V(P^S)$ $(l \geqq 1)$,

$\bar{j}(c(4k, R)) = e^{4k} \cdot V(P^R) \otimes V(P^R) +$ *a linear combination of*

$$e^{4(k+l)} \cdot V(P^S) \otimes V(P^S) \ (l \geqq 1).$$

Especially, for the case $R=0$ (*the* 0-*series*),

$$\tilde{j}(b(0))=1\cdot V\otimes V,\quad \tilde{j}(c(4k,0))=e^{4k}\cdot V\otimes V\quad (V=V(1)).$$

(ii) *The set of* $d(k, R)$ *in* (2.13) *and* $2c(4k, T)$ *of* $c(4k, T)$ *in* (2.11) *forms a basis of* $\operatorname{Ker}\tilde{j}$.

PROOF. (i) By (2.9) and Lemma 1.9, we have

$$\tilde{j}(a(R,S))=j(V(P^R)V(P^S)-V(P_nP^RP^S))=\langle V(P^R),V(P^S)\rangle.$$

The second equality is in Lemma 2.6 (i). By the definition of f in (1.14), we have the commutative diagram

$$\begin{array}{ccc} H^{i+4k}(\Omega^{4(N-k)}MSp(n+N)) & \xrightarrow{j} & H^{i+4k}(\Gamma(n+k,N-k)) \\ \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle(f^*)^k} \\ H^i(\Omega^{4N}MSp(n+N)) & \xrightarrow{\quad j\quad} & H^i(\Gamma(n,N)) \end{array}$$

for $i\leqq 12n-2$, where $N\geqq 3n-2$, j's are the homomorphisms in (1.4) and σ is the iterated cohomology suspension. By (*) in the proof of Lemma 2.8, this diagram, Lemmas 2.6 (i) and 1.15, we have

$$\begin{aligned}\tilde{j}(c(4k,R)) &= j\sigma\tau(b_{n+k}(R))=(f^*)^k\tilde{j}(b_{n+k}(R))\\ &=(f^*)^k(1\cdot V'(P^R)\otimes V'(P^R)+\cdots)=e^{4k}\cdot V(P^R)\otimes V(P^R)+\cdots.\end{aligned}$$

(ii) It holds that $d(k,R)=\tilde{\sigma}(UP_{n+k}P^R)$ where $n+k\leqq n+|R|$ by (2.13). Also,

$$2c(4k,R)=\tilde{\sigma}(UP_{n+k+|R|}P^R+\textstyle\sum_{S<n+k+|R|}m_SUP^S)\quad \text{for some integers } m_S,$$

where $n+k+|R|>n+|R|$ by (2.11) and Lemma 2.8. Since $\tilde{\sigma}$ is monomorphic and $\operatorname{Ker}\tilde{j}=\operatorname{Im}\tilde{\sigma}$, these facts and the definition of UI_n imply (ii). q.e.d.

Now, by using the upper short exact sequence in Lemma 2.3 (ii) and Lemmas 1.7 and 2.14, we see immediately the following

PROPOSITION 2.15. *Let* $j\leqq n-1$. *Then*

(i) $H^{8n+4j-1}(F_n)$ *is a free abelian group with basis consisting of the following classes*:

$a(R,S)$ *in* (2.9) *with* $|R|+|S|=j$, $b(R)$ *in* (2.10) *with* $2|R|=j$,

$c(4k,R)$ *in* (2.11) *with* $k+2|R|=j$, $d(k,R)$ *in* (2.13) *with* $k+|R|=j$.

(ii) $H^{8n+4j+1}(F_n)$ *is isomorphic to a direct sum of some copies of* Z_2 *with basis consisting of* $c(4k+2,R)$ *in* (2.12) *with* $k+2|R|=j$.

(iii) $H^{8n+4j-2}(F_n)=H^{8n+4j}(F_n)=0$.

For the mod 2 cohomology of F_n, we can define the class

(2.16) $c(4k+1, R) \in H^{8(n+|R|)+4k}(F_n; Z_2)$ for an integer $k \geqq 0$ and a series R with $k+2|R| \leqq n-1$ satisfying

$$\tilde{j}(c(4k+1, R)) = e^{4k+1} \cdot V(P^R) \otimes V(P^R),$$

where $\tilde{j}$: $H^{i-1}(F_n; Z_2) \to H^i(\Gamma(n, N); Z_2)$ is isomorphic for $i \equiv 1 \bmod 4$.

By the same way as the above proposition, we have

LEMMA 2.17. *The* mod 2 *reductions of* $a(R, S)$, $b(R)$, $c(2k, R)$, $d(k, R)$ *in* (2.9–13) *and* $c(4k+1, R)$ *in* (2.16) *form a basis of* $H^i(F_n; Z_2)$ *for* $i \leqq 12n-3$.

We can study the cohomology operations on $H^*(F_n; Z_p)$ for $* \leqq 12n-3$. We remark that the transgression τ: $H^{i-1}(F_n; Z_p) \to H^i(\Omega^{4N}MSp(n+N); Z_p)$ is monomorphic for $i \leqq 12n-2$, by the proof of Lemma 2.3 (ii).

When p is an odd prime, the operation P^i on $H^*(F_n; Z_p)$ for $*+2i(p-1) \leqq 12n-3$ is completely determined by Proposition 2.15 and (2.9–13), because we can compute $\tau(P^i x) = P^i \tau(x)$ for any $x \in H^*(F_n; Z_p)$ and τ is monomorphic. Consider the operation Sq^i on $H^*(F_n; Z_2)$ for $*+i \leqq 12n-3$. Then we can determine $Sq^i x$ for $x = a(R, S)$, $d(k, R)$, by the same way as above. For $x = c(4k+j, R)$ $(j=1, 2)$, we can compute $\tilde{j}(Sq^i x) = Sq^i \tilde{j}(x)$ by Lemma 1.11 (iii), (2.16) and (2.12). For $y = b(R)$, $c(4k, R)$, we have $\tilde{j}(Sq^i y) = Sq^i \tilde{j}(y) = 0$ if $i \not\equiv 0 \bmod 4$, by lemmas 1.11 (ii), (iii) and 2.14. Since $\tilde{j}$: $H^{i-1}(F_n; Z_2) \to H^i(\Gamma(n, N); Z_2)$ $(i \leqq 12n-2)$ is monomorphic if $i \not\equiv 0 \bmod 4$, $Sq^i c(4k+j, R)$ for $j = 1,2$ and $i + j \not\equiv 0 \bmod 4$ can be determined and $Sq^i b(R) = Sq^i c(4k, R) = 0$ if $i \not\equiv 0 \bmod 4$.

Consequently we have the following

LEMMA 2.18. (i) $Sq^i a(R, S) = Sq^i b(R) = Sq^i c(4k, R) = Sq^i(d(k, R)) = 0$ *if* $i \not\equiv 0 \bmod 4$.

(ii) $Sq^1 c(4k+1, R) = c(4k+2, R)$.

In the case $R=0$ (the 0-series), we have the following

LEMMA 2.19. (i) $Sq^4 b(0) = a((1), 0) + nc(4, 0)$,

$$P^1 b(0) = -(a((1), 0) + (n+1)c(4, 0)) \qquad \textit{for} \quad p = 3.$$

(ii) $Sq^i c(4k+1, 0) = c(4k+2, 0)$ *if* $i = 1$,

$= 0$ *if* $i = 2$, $= (n+k)c(4k+5, 0)$ *if* $i = 4$,

$Sq^i c(4k+2, 0) = 0$ *if* $i = 1$, $= c(4k+4, 0)$ *if* $i = 2$, $= (n+k)c(4k+6, 0)$ *if* $i = 4$,

$Sq^i c(4k, 0) = 0$ *if* $i = 1, 2$, $= (n+k)c(4k+4, 0) + \tilde{\sigma}(UP_1 P_{n+k})$ *if* $i = 4$, *where $\tilde{\sigma}$ is the homomorphism in Lemma* 2.3 (ii).

PROOF. First, we prove the formula for $P^1 b(0)$. When $p=3$, it holds $P^1 V=-V(P_1)$ and $P^1 P_n=(n+1)P_{n+1}-P_1 P_n$, where $V=V(1)$. By (2.9–11), $\tau(a((1),0))=V(P_1)V-V(P_1 P_n)$, $\tau(b(0))=(1/2)(V^2-V(P_n))$ and $\tau(c(4,0))=(1/2)V(P_{n+1})$. By these relations, we have $\tau(P^1 b(0))=-\tau(a((1),0)+(n+1)c(4,0))$. Since τ is monomorphic, we have the desired formula for $P^1 b(0)$.

Next, by using Lemmas 1.11, 2.14 and (2.16), we have $\bar{j}(Sq^4 b(0))=\langle V(P_1),V\rangle+ne^4\cdot V\otimes V=\bar{j}(a((1),0)+nc(4,0))$ and we can compute $\bar{j}(Sq^i c(k,0))=Sq^i\bar{j}(c(k,0))$ for $i=1,2,4$ and $k=1,2$. Since $\bar{j}\colon H^{i-1}(F_n;Z_2)\to H^i(\Gamma(n,N);Z_2)$ is monomorphic for $i\leqq 8n+6$, we have the desired formulas for $Sq^4 b(0)$, $Sq^i c(k,0)$ ($i=1,2,4$ and $k=1,2$).

To obtain the formulas for $Sq^i c(4k+j,0)$, consider the commutative diagram (*) in the proof of Lemma 2.8. Then, we see that

$$(2.20)\qquad \begin{aligned}&\tilde{b}^*\sigma(b'(0))=c(4k,0),\quad \tilde{b}^*\sigma(c'(i,0))=c(4k+i,0)\ (i\geqq 1),\\ &\tilde{b}^*\sigma(a'((1),0))=-\tilde{\sigma}(UP_1P_{n+k}),\end{aligned}$$

where $b'(0)$, $c'(i,0)$ and $a'((1),0)$ are the classes in $H^*(F_{n+k})$. In fact, the first equality is seen there. We have the second equality for $i\equiv 0 \bmod 4$ and the last equality by considering $\tau\tilde{b}^*\sigma(x)=\sigma\tau(x)$ for $x=c'(4j,0)$, $a'((1),0)$ and by Lemma 2.8 and (2.9). By considering $j\tau\tilde{b}^*\sigma(c'(i,0))$ for $i\equiv 1,2 \bmod 4$ and by using the commutative diagram in the proof of Lemma 2.14 (i), we have the second equality for $i\equiv 1,2 \bmod 4$ by Lemma 1.15. In (2.20), we have $Sq^4c'(1,0)=(n+k)c'(5,0)$, for example, by the formula for $Sq^4c'(1,0)$. Thus, by the naturality of the cohomology operation, we obtain the desired formulas for $Sq^i c(4k+j,0)$. q.e.d.

§3. Symplectic Pontrjagin numbers

For a symplectic cobordism class $u\in\pi_t(MSp)$ and a class $y\in H^j(BSp)$, let $y[u]$ be the Pontrjagin number of u for the class y.

To study the divisibility of some Pontrjagin numbers, consider a fixed element

$$(0)\qquad x_0=x+x'\in H^t(F_n)\qquad (t=8n+4j-1 \text{ with } j<n),$$

where x is one of the classes $a(R,S)$, $b(R)$, $c(4k,R)$ and $d(k,R)$ given in Proposition 2.15 (i) and x' is a linear combination of another classes. Then we can take the following steps (1)–(4):

(1) Take a basis $\{x_i\}$ of $H^t(F_n)$ which includes x_0, and let $\{\bar{x}_i\}$ be the dual basis of $H_t(F_n)$.

(2) Take a suitable cell decomposition of F_n, and denote its i-skeleton by $F_n^{(i)}$.

(3) For an integer $l\geqq 2$ and the Hurewicz homomorphism $H^{(l)}: \pi_t(F_n/F_n^{(t-l)}) \to H_t(F_n/F_n^{(t-l)}) \cong H_t(F_n)$, set

$$H^{(l)}(v) = \sum_i k_i^{(l)}(v)\bar{x}_i \qquad \text{for} \quad v \in \pi_t(F_n/F_n^{(t-l)}),$$

where $k_i^{(l)}(v)$ are integers.

(4) Let $\alpha(l)$ be the greatest common measure of $\{k_0^{(l)}(v) \mid v \in \pi_t(F_n/F_n^{(t-l)})\}$.

Now we have the following basic lemma.

LEMMA 3.1. *Assume that the class* $x_0 = x + x' \in H^t(F_n)$ *in* (0) *satisfies*

$$\tau(x_0) = \sum_T (\lambda_T/2)V(P^T) + X \qquad \text{for some integers} \quad \lambda_T,$$

where $\tau: H^t(F_n) \to H^{t+1}(\Omega^{4N}MSp(n+N))$ *is the transgression in Lemma* 2.3 (ii), *and X is a sum of decomposable terms. Then*

$$\sum_T \lambda_T P^T[u] \equiv 0 \mod 2\alpha(l) \qquad \text{for any} \quad u \in \pi_i(MSp),$$

where $\alpha(l)$ *is the integer given in the step* (4).

PROOF. Consider the following commutative diagram:

$$\begin{array}{ccccccc}
\pi_{t-4n+1}(MSp) & \xleftarrow{\cong} & \pi_{t+1}(\Omega^{4N}MSp(n+N)) & \xrightarrow{\partial} & \pi_t(F_n) & \xrightarrow{q_*} & \pi_t(F_n/F_n^{(t-l)}) \\
\downarrow H & & \downarrow H & & \downarrow H & & \downarrow H^{(l)} \\
H_{t-4n+1}(MSp) & \xleftarrow{\sigma} & H_{t+1}(\Omega^{4N}MSp(n+N)) & \xrightarrow{\tau} & H_t(F_n) & \underset{\cong}{\xrightarrow{q_*}} & H_t(F_n/F_n^{(t-l)}).
\end{array} \tag{3.2}$$

Here ∂ and τ are the connecting map and the transgression of the fibering (2.1) respectively, σ is the iterated homology suspension, q is the natural projection and H's are the Hurewicz homomorphisms. For any class $u \in \pi_{t-4n+1}(MSp)$, let $u' \in \pi_{t+1}(\Omega^{4N}MSp(n+N))$ be the class corresponding to u under the isomorphism in (3.2). By the above step (3), $H^{(l)}q_*\partial(u') = \sum_i k_i^{(l)}(q_*\partial(u'))\bar{x}_i$. Taking the Kronecker pairing, we have

$$\langle H^{(l)}q_*\partial(u'), x_0\rangle = k_0^{(l)}(q_*\partial(u')) \equiv 0 \mod \alpha(l).$$

On the other hand, by (3.2) and the assumption, we have

$$2\langle H^{(l)}q_*\partial(u'), x_0\rangle = 2\langle H(u'), \tau(x_0)\rangle = \sum_T \lambda_T \langle H(u'), \sigma(UP^T)\rangle = \sum_T \lambda_T P^T[u].$$

Thus we have the lemma. q.e.d.

By this lemma, if we can take a basis of $H^t(F_n)$ in (1) and a cell decomposition of F_n in (2) which enable us to compute $k_i^{(l)}(v)$ in (3) and $\alpha(l)$ in (4) for a fixed element x_0 in (0), then we have the divisibility of some Pontrjagin number. Here we shall consider the case $x_0 = a((1), 0) + (n+4)c(4, 0)$ or $c(4, 0)$.

We remark that F_n is $(8n-2)$-connected. By Proposition 2.15 and Lemmas 2.18 and 2.19, we have

LEMMA 3.3. (i) *For* $n \geqq 1$,

$$H^{8n-1}(F_n) = Z\langle b(0)\rangle, \quad H^{8n}(F_n) = 0, \quad H^{8n+1}(F_n) = Z_2\langle c(2, 0)\rangle.$$

(ii) *For* $n \geqq 2$,

$$H^{8n+2}(F_n) = H^{8n+4}(F_n) = 0, \quad H^{8n+3}(F_n) = Z\langle a'\rangle \oplus Z\langle c(4, 0)\rangle,$$

$$H^{8n+5}(F_n) = Z_2\langle c(6, 0)\rangle,$$

where $a' = a((1), 0) + (n+4)c(4, 0)$.

(iii) $Sq^4 b(0) = a'$, $P^1 b(0) = -a'$ *for* $p=3$,

$Sq^i b(0) = 0$ *if* $1 \leqq i \leqq 7$ *and* $i \neq 4$,

$Sq^i c(1, 0) = c(2, 0)$ *if* $i=1$, $=0$ *if* $i=2$, $=nc(5, 0)$ *if* $i=4$,

$Sq^i c(2, 0) = 0$ *if* $i=1$, $=c(4, 0)$ *if* $i=2$, $=nc(6, 0)$ *if* $i=4$,

$Sq^i(a') = Sq^i c(4, 0) = 0$ *if* $1 \leqq i \leqq 3$.

By this lemma, we have immediately the following

LEMMA 3.4. *Let* $n \geqq 2$. *Then we can take a complex* K *given by*

$$K = S^{8n-1} \cup_{\phi_0} e^{8n} \cup_{\phi_1} e^{8n+1} \cup_{\psi_1 \vee \psi_2} (e_1^{8n+3} \vee e_2^{8n+3}) \cup_{\phi_4} e^{8n+4} \cup_{\phi_5} e^{8n+5}$$

and a map $f: K \to F_n$, *which satisfy the following* (i)–(ii):

(i) $f_*: H_i(K) \to H_i(F_n)$ *is isomorphic for* $i \leqq 8n+4$.

(ii) *The cells* e_1^{8n+3} *and* e_2^{8n+3} *correspond to the cohomology classes* $f^*(a')$ *and* $f^*(c(4, 0))$ *respectively.*

PROPOSITION 3.5. *Let* $n \geqq 2$. *Then*

(i) $\pi_{8n-1}(F_n) = Z$, $\pi_{8n}(F_n) = \pi_{8n+1}(F_n) = Z_2 \oplus Z_2$, $\pi_{8n+2}(F_n) = 0$,

$\pi_{8n+3}(F_n) = Z \oplus Z$ (*resp.* $Z \oplus Z \oplus Z_2$) *if* n *is odd* (*resp. even*).

(ii) *We can take a basis* $\{u(3), v(3)\}$ *of a free part of* $\pi_{8n+3}(F_n)$ *to satisfy* $H(u(3)) = 24\bar{a}'$ *and* $H(v(3)) = 4\bar{c}(4, 0)$, *where* $H: \pi_{8n+3}(F_n) \to H_{8n+3}(F_n)$ *is the Hurewicz homomorphism and* $\{\bar{a}', \bar{c}(4, 0)\}$ *is the dual basis of* $\{a', c(4, 0)\}$ *in Lemma* 3.3.

PROOF. By Lemma 3.4, we prove the proposition for K in Lemma 3.4 instead of F_n.

It is obvious that $\pi_{8n-1}(K) = Z$ and $K^{(8n)} = S^{8n-1} \vee S^{8n}$. If $q_1: K^{(8n)} \to S^{8n-1}$ and $q_2: K^{(8n)} \to S^{8n}$ are the respective projections, then $q_1\phi_1$ is homotopic to the constant map and deg $q_2\phi_1=2$ since $Sq^2b(0)=0$ and $Sq^1c(1,0)=c(2,0)$ by Lemma 3.3 (iii). Hence we have $K'=K^{(8n+1)}=K^{(8n+2)}=S^{8n-1}\vee(S^{8n}\cup_2 e^{8n+1})$, and the split exact sequence

$$(*)\qquad 0\longrightarrow \pi_{8n+i}(S^{8n-1}) \underset{q_*}{\rightleftarrows} \pi_{8n+i}(K') \xrightarrow{p_*} \pi_{8n+i}(S^{8n}\cup_2 e^{8n+1})\longrightarrow 0$$

where p and q are the projections. Therefore $\pi_{8n}(K)=\pi_{8n+1}(K)=Z_2\oplus Z_2$. Furthermore

(3.6) $\pi_{8n+2}(K')\cong\pi_{8n+2}(S^{8n-1})\oplus\pi_{8n+2}(S^{8n}\cup_2 e^{8n+1})=Z_{24}\oplus Z_4$ (cf. [2; 4.1]), and for the attaching maps ψ_1 and ψ_2, $q_*\psi_1$ and $p_*\psi_2$ generate the first and second summands respectively, and the orders of $p_*\psi_1$ and $q_*\psi_2$ are divisors of 2 and 4 respectively.

To prove the latter half of (3.6), we consider the commutative diagram

$$\begin{tikzcd}
S^{8n+2} \arrow[r] & S^{8n-1} \arrow[r] & S^{8n-1}\cup_{q\psi_k} e^{8n+3} \\
S^{8n+2} \arrow[u, equal] \arrow[d, equal] \arrow[r, "\psi_k"] & K^{(8n+1)} \arrow[u, "q"] \arrow[d, "\pi"] \arrow[r] & K^{(8n+1)}\cup_{\psi_k} e^{8n+3} \arrow[u, "\tilde q"] \arrow[d, "\tilde\pi"] \\
S^{8n+2} \arrow[r] & S^{8n+1} \arrow[r] & S^{8n+1}\cup_{\pi\psi_k} e^{8n+3}
\end{tikzcd}$$

for $k=1$ and 2, where π is the natural projection, and $\tilde q$ and $\tilde\pi$ are the maps defined by q and π respectively. Consider the mod 2 and mod 3 cohomology groups of this diagram. Then, since $Sq^4b(0)=a'$ and $P^1b(0)=-a'$ for $p=3$ by Lemma 3.3 (iii), we see that $q_*\psi_1$ is a generator of $\pi_{8n+2}(S^{8n-1})=Z_{24}$ and the order of $q_*\psi_2$ is a divisor of 4. Since $Sq^2c(2,0)=c(4,0)$ by Lemma 3.3 (iii), $\pi_*\psi_1=0$ and $\pi_*\psi_2\neq 0$ in $\pi_{8n+2}(S^{8n+1})=Z_2$. Hence the order of $p_*\psi_1$ is at most 2 and $p_*\psi_2$ is a generator of $\pi_{8n+2}(K'/S^{8n-1})=\pi_{8n+2}(S^{8n}\cup_2 e^{8n+1})=Z_2$, by the fact that $\pi'_*\colon \pi_{8n+2}(S^{8n+2}\cup_2 e^{8n+1})\to\pi_{8n+2}(S^{8n+1})$ is epimorphic, where π' is the restriction of π (cf. [2; 4.1]). These imply the latter half of (3.6).

$\pi_{8n+2}(K)=0$ follows immediately from (3.6).

Consider the exact sequence

$$\pi_{8n+3}(S_1^{8n+2}\vee S_2^{8n+2})\xrightarrow{(\psi_1\vee\psi_2)_*}\pi_{8n+3}(K')\xrightarrow{i_*}\pi_{8n+3}(K^{(8n+3)})$$
$$\xrightarrow{\partial}\pi_{8n+2}(S_1^{8n+2}\vee S_2^{8n+2})\xrightarrow{(\psi_1\vee\psi_2)_*}\pi_{8n+2}(K').$$

Then $\operatorname{Im}\partial=\operatorname{Ker}(\psi_1\vee\psi_2)_*=Z\langle 24\iota_1\rangle\oplus Z\langle 4\iota_2\rangle$ by (3.6), where ι_j is a generator of $\pi_{8n+2}(S_j^{8n+2})$ $(j=1,2)$. On the other hand, $p_*\colon\pi_{8n+3}(K')\to\pi_{8n+3}(S^{8n}\cup_2 e^{8n+1})$ in $(*)$ for $i=3$ is isomorphic since $\pi_{8n+3}(S^{8n-1})=0$. Furthermore $\pi_{8n+3}(S^{8n}\cup_2 e^{8n+1})=Z_2\oplus Z_2$ and one of its generators is $\tilde\eta\eta$, where $\tilde\eta\in\pi_{8n+2}$

$(S^{8n} \cup_2 e^{8n+1})=Z_4$ and $\eta \in \pi_{8n+3}(S^{8n+2})=Z_2$ are generators (cf. [2; 4.1]). By (3.6), we can take $\tilde{\eta}=p_*\psi_2$ and then $p_*\psi_1=2\varepsilon\tilde{\eta}$. Thus $p_*\psi_{1*}(\eta_1)=2\varepsilon\tilde{\eta}\eta=0$ and $p_*\psi_{2*}(\eta_2)=\tilde{\eta}\eta$ for the generator $\eta_j \in \pi_{8n+3}(S_j^{8n+2})=Z_2$. These imply that $\operatorname{Im} i_*=Z_2$ and $\pi_{8n+3}(K^{(8n+3)})=Z\oplus Z\oplus Z_2$.

Now, the attaching map ϕ_4 is contained in $\operatorname{Ker}\partial=\operatorname{Im} i_*=Z_2$ by the last two equalities in Lemma 3.3 (iii). Furthermore, since $Sq^4c(1,0)=nc(5,0)$ by Lemma 3.3 (iii), we see that $\phi_4 \neq 0$ if n is odd and $\phi_4=0$ if n is even. Thus we see that the desired results for $\pi_{8n+3}(K)$.

By the above exact sequence, we can take a basis $\{u(3), v(3)\}$ of $\pi_{8n+3}(K)/\mathrm{Tor}=\pi_{8n+3}(K^{(8n+3)})/\mathrm{Tor}$ to satisfy $\partial u(3)=24\iota_1$ and $\partial v(3)=4\iota_2$. These imply that $H(u(3))=24\bar{a}'$, and $H(v(3))=4\bar{c}(4,0)$, and we complete the proof. q.e.d.

REMARK 3.7. For $n=1$, $\pi_{8n+i}(F_n)$ $(i=-1,0,1)$ are the same as the ones given in Proposition 3.5.

By Lemmas 3.1, 3.3 and Proposition 3.5, we have the following

THEOREM 3.8. *Let* $n \geqq 1$. *Then*

(i) $P_n[u]\equiv 0 \mod 8$ *for any* $u \in \pi_{4n}(MSp)$.

(ii) $P_1P_n[u]-((n+4)/2)P_{n+1}[u]\equiv 0 \mod 24$ *for any* $u \in \pi_{4n+4}(MSp)$.

PROOF. For $n=1$, (i) and (ii) follow from the results of [7], [6] on $\pi_4(MSp)$ and $\pi_8(MSp)$. Let $n \geqq 2$. We consider the case that $x_0=a'$ or $c(4,0)$ and $l=5$ in Lemma 3.1. By Lemma 3.3 (ii), we can take a basis $\{a', c(4,0)\}$ of $H^{8n+3}(F_n)$. When $x_0=c(4,0)$, we see that $\alpha(5)$ is a multiple of 4 by Proposition 3.5 and $\tau(c(4,0))=(1/2)V(P_{n+1})$ by (2.11), hence (i) follows from Lemma 3.1. When $x_0=a'$, $\alpha(5)$ is a multiple of 24 by Proposition 3.5 and $\tau(a')=-V(P_1P_n)+((n+4)/2)V(P_{n+1})+V(P_1)V$ by (2.9) and (2.11), hence (ii) follows from Lemma 3.1. q.e.d.

REMARK 3.9. In addition to Proposition 3.5 (i), the homotopy groups $\pi_i(F_n)$ can be determined for $i \leqq 8n+6$ by the results due to S. Oka.

§4. Homotopy groups of $MSp(n)$

In the rest of this paper, we study the homotopy groups $\pi_{8n-1}(MSp(n))$ and $\pi_{8n+3}(MSp(n))$ for $n \geqq 1$.

Consider the homotopy exact sequence of the fibering (2.1):

$$(4.1) \qquad \cdots \longrightarrow \pi_i(MSp) \xrightarrow{\partial} \pi_{i+4n-1}(F_n) \longrightarrow \pi_{i+4n-1}(MSp(n))$$
$$\xrightarrow{b_{n*}} \pi_{i-1}(MSp) \xrightarrow{\partial} \pi_{i+4n-2}(F_n) \longrightarrow \cdots \qquad (i \leqq 8n-2),$$

where we identify $\pi_i(MSp)$ with $\pi_{i+4n}(\Omega^{4N}MSp(n+N))$ since $N\geqq n$. Because F_n is $(8n-2)$-connected, b_{n*} is isomorphic for $i\leqq 4n-1$ and epimorphic for $i=4n$.

PROPOSITION 4.2. (i) *For* $\partial\colon \pi_{4n}(MSp)\to\pi_{8n-1}(F_n)=Z$ $(n\geqq 1)$ *(see Proposition* 3.5 (i) *and Remark* 3.7), *it holds that*

$$\partial u = \pm(1/2)P_n[u] \quad \textit{for any} \quad u\in\pi_{4n}(MSp).$$

(ii) *For* $\partial\colon \pi_{4n+4}(MSp)\to\pi_{8n+3}(F_n)=Z\langle u(3)\rangle\oplus Z\langle v(3)\rangle\oplus \mathrm{Tor}$ $(n\geqq 2)$ *(see Proposition* 3.5), *it holds that*

$$\partial u = ((1/24)(-P_1P_n[u]+((n+4)/2)P_{n+1}[u]),\ (1/8)P_{n+1}[u])$$

for any $u\in\pi_{4n+4}(MSp)$, *where* $(k, l)=ku(3)+lv(3)+a$ *torsion element.*

PROOF. We shall prove (ii). (i) can be proved similarly.

For $u\in\pi_{4n+4}(MSp)=\pi_{8n+4}(\Omega^{4N}MSp(n+N))$, set $\partial u=(k, l)$. Then $H(\partial u)=24k\bar{a}'+4l\bar{c}(4, 0)$ by Proposition 3.5 (ii). Thus, by taking the Kronecker pairing, we have

$$24k = \langle H(\partial u), a'\rangle = \langle H(u), \tau(a')\rangle = -P_1P_n[u]+((n+4)/2)P_{n+1}[u],$$
$$4l = \langle H(\partial u), c(4, 0)\rangle = \langle H(u), \tau(c(4, 0))\rangle = (1/2)P_{n+1}[u],$$

since $\tau(a')=V(P_1)V-V(P_1P_n)+((n+4)/2)V(P_{n+1})$, and $\tau(c(4, 0))=(1/2)V(P_{n+1})$ by (2.9) and (2.11). Hence we have the desired result. q.e.d.

The Pontrjagin number $P_n[u]$ is a multiple of 8 for any $u\in\pi_{4n}(MSp)$ $(n\geqq 1)$ by Theorem 3.8 (i). Thus we set

(4.3) $\qquad m(n)=\mathrm{g.c.m.}\,\{(1/8)P_n[u]\mid u\in\pi_{4n}(MSp)\}\quad$ for $\quad n\geqq 1$.

COROLLARY 4.4. *The kernel of the epimorphism* $b_{n*}\colon \pi_{8n-1}(MSp(n))\to\pi_{4n-1}(MSp)$ *is a cyclic group of order* $4m(n)$ *generated by the Whitehead product* $[i, i]$ *for the homotopy class* i *of the natural inclusion* $i\colon S^{4n}\to MSp(n)$.

PROOF. By Proposition 4.2 (i), the definition (4.3) and the exact sequence (4.1), we see that $\mathrm{Ker}\,b_{n*}$ is a cyclic group of order $4m(n)$. Consider the commutative diagram

$$\begin{tikzcd} F(i_1) \arrow[r, "j'"] \arrow[d, "\bar{i}"] & S^{4n} \arrow[r, "i_1"] \arrow[d, "i"] & \Omega S^{4n+1} \arrow[d, "i'"] \\ F_n \arrow[r, "j"] & MSp(n) \arrow[r, "b_n"] & \Omega^{4N}MSp(n+N). \end{tikzcd}$$

Here i_1 denotes the natural inclusion and $F(i_1)$ is the fiber, and i' is the composition $\Omega S^{4n+1} \to \Omega^{4N} S^{4(n+N)} \to \Omega^{4N} MSp(n+N)$ of the natural inclusions. It holds that $\pi_{8n-1}(F(i_1)) = Z$ and $j'_*(1) = \pm[\iota, \iota]$ for a generator $\iota \in \pi_{4n}(S^{4n})$ by the definition of the Whitehead product. Since $\bar{i}_*: \pi_{8n-1}(F(i_1)) \to \pi_{8n-1}(F_n)$ is isomorphic, the kernel of $b_{n*}: \pi_{8n-1}(MSp(n)) \to \pi_{4n-1}(MSp)$ is generated by $[i, i]$ by the naturality. q.e.d.

Let $MU(2n)$ be the Thom space of the universal complex vector bundle over $BU(2n)$, and consider the map $c: MSp(n) \to MU(2n)$ induced by the inclusion $Sp(n) \subset U(2n)$. Then we have the following corollary, where $\nu_2(y)$ is the exponent of 2 in the prime power decomposition of a positive integer y:

COROLLARY 4.5. *Assume that*

(a) *n is not a power of 2 and $2\pi_{4n-1}(MSp) = 0$, or*

(b) $\nu_2(m(n)) + 2 = \nu_2(|\pi_{8n-1}(MU(2n))|)$.

Then the epimorphism $b_{n*}: \pi_{8n-1}(MSp(n)) \to \pi_{4n-1}(MSp)$ is split, that is,

$$\pi_{8n-1}(MSp(n)) \cong Z_{4m(n)} \oplus \pi_{4n-1}(MSp).$$

PROOF. Let $\tilde{F}_{2n} \to MU(2n) \xrightarrow{\tilde{b}_{2n}} \Omega^{4N} MU(2n+2N)$ be the fibering defined by the same way as (2.1), and consider the commutative digaram

$$\begin{array}{ccccc} F_n & \longrightarrow & MSp(n) & \xrightarrow{b_n} & \Omega^{4N} MSp(n+N) \\ \downarrow{\scriptstyle c'} & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle \Omega^{4N} c} \\ \tilde{F}_{2n} & \longrightarrow & MU(2n) & \xrightarrow{\tilde{b}_{2n}} & \Omega^{4N} MU(2n+2N) \end{array}$$

induced by c. We remark that $\tilde{F}_{2n}$ is $(8n-2)$-connected. Then we have the commutative digaram

$$\begin{array}{ccccccccc} \pi_{4n}(MSp) & \xrightarrow{\partial} & \pi_{8n-1}(F_n) & \longrightarrow & \pi_{8n-1}(MSp(n)) & \xrightarrow{b_{n*}} & \pi_{4n-1}(MSp) & \longrightarrow & 0 \\ \downarrow{\scriptstyle c_*} & & \downarrow{\scriptstyle c'_*} & & \downarrow{\scriptstyle c_*} & & \downarrow{\scriptstyle c_*} & & \\ \pi_{4n}(MU) & \xrightarrow{\tilde{\partial}} & \pi_{8n-1}(\tilde{F}_{2n}) & \longrightarrow & \pi_{8n-1}(MU(2n)) & \xrightarrow{\tilde{b}_{2n*}} & 0. & & \end{array}$$

In the first place, we notice that c'_* is isomorphic. By E. Rees and E. Thomas [11; § 2], $H^{8n-1}(\tilde{F}_{2n})$ is Z generated by α_1 which satisfies

$$\tilde{\tau}(\alpha_1) = (1/2)(\tilde{\sigma}(\tilde{U}c_{2n}) - (\tilde{\sigma}(\tilde{U}))^2),$$

where $\tilde{\tau}: H^{8n-1}(\tilde{F}_{2n}) \to H^{8n}(\Omega^{4N}M)$ and $\tilde{\sigma}: H^{8n+4N}(M) \to H^{8n}(\Omega^{4N}M)$ are the transgression and the iterated cohomology suspension respectively, and $\tilde{U} \in H^{4n+4N}(M)$ is the Thom class $(M = MU(2n+2N))$. The above equality and $\tau(b(0)) = (1/2)(V^2 - V(P_n))$ of (2.10) imply $\tau c'(\alpha_1) = -\tau(b(0))$ and so $c'^*(\alpha_1) = -b(0)$, because $c^*(c_{2n}) = \pm P_n$, $c^*(\tilde{U}) = \pm U$ and τ is monomorphic. Thus c'^*:

$H^{8n-1}(\tilde{F}_{2n})\to H^{8n-1}(F_n)$ is isomorphic and so is c'_* in the diagram.

Furthermore $\pi_{8n-1}(MU(2n))\cong\operatorname{Coker}\tilde{\partial}$ is a cyclic group of order 2^β where $\beta=\rho_0(2n)-1$ by [11; Th. A], and Coker $\partial=Z_{4m(n)}$ by Corollary 4.4. Thus we have the commutative diagram

$$(*)\qquad \begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Coker}\partial(=Z_{4m(n)}) & \longrightarrow & \pi_{8n-1}(MSp(n)) & \xrightarrow{b_{n*}} & \pi_{4n-1}(MSp) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle c''} & & \downarrow{\scriptstyle c_*} & & & & \\ & & \operatorname{Coker}\tilde{\partial}(=Z_{2^\beta}) & \cong & \pi_{8n-1}(MU(2n)), & & & & \end{array}$$

where c'' is the epimorphism induced by c'_*.

When (b) holds, c'' induces the isomorphism of the 2-torsion parts, hence the upper sequence in $(*)$ splits because $\pi_{4n-1}(MSp)$ is a 2-torsion group (cf. [15; 20. 40]).

Now we assume that (a) holds. Then $\beta\neq 0$ by the definition of $\rho_0(2n)$ ([11; Th. A]) and $\pi_{4n-1}(MSp)\otimes Z_2=\pi_{4n-1}(MSp)$. Hence, by tensoring Z_2 to $(*)$, we have the split exact sequence $0\to Z_2\to\pi_{8n-1}(MSp(n))\otimes Z_2\to\pi_{4n-1}(MSp)\to 0$. Therefore the upper sequence in $(*)$ splits as desired. q.e.d.

We shall prove the following theorems in the next section by preparing some symplectic cobordism classes.

THEOREM 4.6. *For the integer* $m(n)$ *in* (4.3), *the following* (i) *and* (ii) *hold*:
(i) $m(n)$ *is a power of* 2 *for* $n\neq 1, 3$, *and* $m(1)=m(3)=3$.
(ii) $m(n)=1$ *if* $n=2^k+2^l-1$ *or* 2^k+2^l $(k, l\geqq 0)$ *and* $n\neq 1, 3$.

THEOREM 4.7. (i) $\pi_i(MSp(n))$ $(i\leqq 8n+3)$ *has no p-torsion for any odd prime p, except for* $(n, i)=(1, 7), (1, 10), (1, 11), (2, 19)$ *and* $(3, 23)$.
(ii) $b_{n*}: \pi_{8n+3}(MSp(n))\to\pi_{4n+3}(MSp)$ *is epimorphic for* $n\geqq 1$.
(iii) *If* $n=2^k+2^l-1$ $(k, l\geqq 1)$, *then* b_{n*} *in* (ii) *is isomorphic, that is,* $\pi_{8n+3}(MSp(n))\cong\pi_{4n+3}(MSp)$.

§5. Symplectic cobordism classes

In this section, we examine the characteristic numbers of some symplectic cobordism classes to prove Theorems 4.6 and 4.7.

Let $\xi=\xi_1$ be the universal symplectic line bundle over the quaternion projective space $HP^\infty=BSp(1)$, and $\xi\otimes_C\xi\otimes_C\xi$ be the tensor product of ξ over $HP^\infty\times HP^\infty\times HP^\infty$ by taking ξ as the complex vector bundle. Then it is a symplectic vector bundle ξ^3 (cf. [14]), and so we denote its classifying map by

$$(5.1)\qquad \phi: Y=HP^\infty\times HP^\infty\times HP^\infty\longrightarrow BSp.$$

Let $P_1^{MSp}\in MSp^4(BSp)$ be the universal first Pontrjagin class and $P_1^{MSp}(\xi)\in$

$MSp^4(HP^\infty)$ be the Euler class of ξ in the symplectic cobordism theory. By using the projection $q_i\colon Y\to HP^\infty$ $(i=1, 2, 3)$ onto the i-th factor, we set $X_i=q_i^*P_1^{MSp}(\xi)\in MSp^4(Y)$. Then $MSp^*(Y)=MSp^*[\![X_1, X_2, X_3]\!]$, and we have an expansion

$$(5.2)\qquad \phi^*(P_1^{MSp}) = P_1^{MSp}(\xi_1^3) = \textstyle\sum_{i+j+k\geqq 1} a_{ijk}X_1^iX_2^jX_3^k$$

for some cobordism classes

$$(5.3)\qquad a_{ijk}\in \pi_{4(i+j+k-1)}(MSp).$$

We shall consider the Pontrjagin numbers $P_{i+j+k-1}[a_{ijk}]$ and $P_1P_{i+j+k-2}[a_{ijk}]$.

Consider the classes $x_i=q_i^*P_1(\xi)\in H^4(Y)$ $(i=1, 2, 3)$ where $Y=HP^\infty\times HP^\infty\times HP^\infty$. Then $H^*(Y)=Z[\![x_1, x_2, x_3]\!]$, and we have the following lemma, where $P^{\Delta_i}\in H^{4i}(BSp)$ denotes the primitive class defined inductively by $P^{\Delta_i}=\sum_{j=1}^{i-1}(-1)^{j+1}P_jP^{\Delta_{i-j}}+(-1)^{i+1}iP_i$, and $C(r, s, t)$ denotes $(r+s+t)!/r!s!t!$:

LEMMA 5.4. *For the induced homomorphism $\phi^*\colon H^*(BSp)\to H^*(Y)$ of ϕ in* (5.1),

$$\phi^*(P^{\Delta_i}) = 4\textstyle\sum_{k+l+m=i} C(2k, 2l, 2m)x_1^kx_2^lx_3^m.$$

PROOF. Let $c_i\in H^{2i}(BU)$ be the i-th Chern class, and $c^{\Delta_i}\in H^{2i}(BU)$ be the primitive class defined inductively by $c^{\Delta_i}=\sum_{j=1}^{i-1}(-1)^{j+1}c_jc^{\Delta_{i-j}}+(-1)^{i+1}ic_i$. Then, for the canonical map $c\colon BSp\to BU$, it holds $c^*(c^{\Delta_{2i}})=2P^{\Delta_i}$ by the definitions of P^{Δ_i} and c^{Δ_j}. Hence, by the definition of ϕ,

$$2\phi^*(P^{\Delta_i}) = 2P^{\Delta_i}(\xi^3) = c^{\Delta_{2i}}(\xi\otimes{}_C\xi\otimes{}_C\xi)\quad\text{in}\quad H^{4i}(Y).$$

Let η be the canonical complex line bundle over CP^∞, and $\bar\eta$ be the conjugate bundle of η. Then $\eta\oplus\bar\eta$ is a symplectic line bundle over CP^∞, and we denote its classifying map by $q\colon CP^\infty\to HP^\infty$. Set $Z=CP^\infty\times CP^\infty\times CP^\infty$. Then $H^*(Z)=Z[\![y_1, y_2, y_3]\!]$, where $y_i=q_i^*c_1(\eta)\in H^2(Z)$ $(i=1, 2, 3)$ for the projection $q_i\colon Z\to CP^\infty$ onto the i-th factor. For the homomorphism $(q\times q\times q)^*\colon H^*(Y)\to H^*(Z)$, we see that

$$\begin{aligned}(q\times q\times q)^*(c^{\Delta_{2i}}(\xi\otimes_C\xi\otimes_C\xi)) &= c^{\Delta_{2i}}((\eta\oplus\bar\eta)\otimes{}_C(\eta\oplus\bar\eta)\otimes{}_C(\eta\oplus\bar\eta))\\ &= 2\{(y_1+y_2+y_3)^{2i}+(y_1+y_2-y_3)^{2i}+(y_1-y_2+y_3)^{2i}+(y_1-y_2-y_3)^{2i}\}\\ &=8\textstyle\sum_{k+l+m=i} C(2k, 2l, 2m)y_1^{2k}y_2^{2l}y_3^{2m},\end{aligned}$$

by using the equality $c^{\Delta_i}(\sum_k\zeta_k)=\sum_k(c_1(\zeta_k))^i$ for line bundles ζ_k. Since $(q\times q\times q)^*$ is monomorphic and $(q\times q\times q)^*(x_k)=y_k^2$ for $k=1, 2, 3$, we have the desired result by the above equalities. q.e.d.

Now, for $E=H$ or MSp, let $\beta_j^E\in E_{4j}(HP^\infty)$ be the dual class of $(P_1^E(\xi))^j$ where $P_1^E(\xi)$ is the Euler class of ξ. Then the following holds (cf. [8], [15; § 16]):

(5.5) *$E_*(HP^\infty)$ is a free $\pi_*(E)$-module with basis $\{\beta_j^E\mid j\geqq 0\}$, and*

$$E_*(MSp)\cong\pi_*(E)[b_1^E, b_2^E,\dots],\quad b_j^E=i_*\beta_{j+1}^E\in E_{4j}(MSp),$$

where $i\colon HP^\infty\to\Sigma^4 MSp$ is the natural inclusion.

Let $(b)_l^k\in H_{4l}(MSp)$ denote the $4l$-dimensional component of $b^k=(1+b_1+b_2+\cdots)^k$, i.e.,

(5.6) $(1+b_1x+b_2x^2+\cdots)^k=\sum_{l\geqq 0}(b)_l^k x^l$, where $b_j=b_j^H$ in (5.5),

and let $H\colon\pi_*(MSp)\to H_*(MSp)$ be the Hurewicz homomorphism.

PROPOSITION 5.7. *For any non negative integers r, s, t with $r+s+t\geqq 1$,*

$$\sum H(a_{ijk})(b)_{r-i}^i(b)_{s-j}^j(b)_{t-k}^k=4C(2r, 2s, 2t)b_{r+s+t-1},$$

where the summation is taken over all $i, j, k\geqq 0$ with $i\leqq r, j\leqq s, k\leqq t$.

PROOF. Consider the commutative diagram

$$\begin{array}{ccc} MSp^*(BSp) & \xrightarrow{\bar{h}} & (H\wedge MSp)^*(BSp)\cong H_*(MSp)\otimes H^*(BSp) \\ \downarrow{\scriptstyle\phi^*} & & \downarrow{\scriptstyle\phi^*} \qquad\qquad\qquad \downarrow{\scriptstyle 1\otimes\phi^*} \\ MSp^*(Y) & \xrightarrow{\bar{h}} & (H\wedge MSp)^*(Y)\cong H_*(MSp)\otimes H^*(Y), \end{array}$$

where $\bar{h}$ denotes the Boardman homomorphism. Then we have

$$(1\otimes\phi^*)\bar{h}(P_1^{MSp})=\bar{h}\phi^*(P_1^{MSp}). \tag{5.8}$$

The following relation holds (cf. [1], [8; (5.1)]):

$$\bar{h}(P_1^{MSp})=\textstyle\sum_{i\geqq 1}b_{i-1}P^{\Delta_i}, \tag{5.9}$$

where P^{Δ_i} is the primitive class in Lemma 5.4. By (5.9) and Lemma 5.4,

$$(1\otimes\phi^*)\bar{h}(P_1^{MSp})=4\textstyle\sum_{r+s+t=i\geqq 1}C(2r, 2s, 2t)b_{i-1}x_1^r x_2^s x_3^t.$$

On the other hand, by (5.9) and (5.6),

$$\bar{h}(X_k^j)=(\textstyle\sum_{i\geqq 1}b_{i-1}x_k^i)^j=\sum_{s\geqq j}(b)_{s-j}^j x_k^s.$$

By (5.2) and this equality, we have

$$\bar{h}\phi^*(P_1^{MSp})=\textstyle\sum_{r+s+t\geqq 1}(\sum H(a_{ijk})(b)_{r-i}^i(b)_{s-j}^j(b)_{t-k}^k)x_1^r x_2^s x_3^t.$$

Therefore, we have the proposition by (5.8). q.e.d.

For any class $u\in\pi_{4n}(MSp)$, its Hurewicz image $H(u)$ can be written as

$$H(u)=\sum\lambda_{r_1,r_2,\dots}b_1^{r_1}b_2^{r_2}\cdots\in H^*(MSp)=Z[b_1,b_2,\dots].$$

For our purpose, we denote simply the coefficient λ_n of b_1^n by (u) $(n\geqq 1)$ and $\lambda_{n-2,1}$ of $b_1^{n-2}b_2$ by $\langle u\rangle$ $(n\geqq 2)$.

Then we have

$$(5.10)\qquad P_n[u]=(u),\ P_1P_{n-1}[u]=\langle u\rangle+n(u)\qquad\text{for}\quad n\geqq 1.$$

These formulas can be proved by the same proof as that for MU given in [1; pp. 10–11], [15; pp. 401–402].

By comparing the coefficients of $b_1^{r+s+t-1}$ or $b_1^{r+s+t-3}b_2$ in the both sides of the equality in Proposition 5.7, and by the above notations ( ) and ⟨ ⟩, we have the following

LEMMA 5.11. *For $r, s, t\geqq 0$, the following hold, where summations are taken over $i, j, k\geqq 0$ with $i\leqq r$, $j\leqq s$, $k\leqq t$.*

$$\text{(i)}\quad \sum(a_{ijk})\binom{i}{r-i}\binom{j}{s-j}\binom{k}{t-k}=\begin{cases}4C(2r,2s,2t) & \text{if}\quad r+s+t\leqq 2,\\ 0 & \text{otherwise}.\end{cases}$$

$$\begin{aligned}\text{(ii)}\quad &\sum\Big\{\langle a_{ijk}\rangle\binom{i}{r-i}\binom{j}{s-j}\binom{k}{t-k}+(a_{ijk})\Big(i\binom{i-1}{r-i-2}\binom{j}{s-j}\binom{k}{t-k}\\ &+j\binom{i}{r-i}\binom{j-1}{s-j-2}\binom{k}{t-k}+k\binom{i}{r-i}\binom{j}{s-j}\binom{k-1}{t-k-2}\Big)\Big\}\\ &=\begin{cases}4C(2r,2s,2t) & \text{if}\quad r+s+t=3,\\ 0 & \text{otherwise}.\end{cases}\end{aligned}$$

PROPOSITION 5.12. (i) *For $i, j\geqq 1$,*

$$(a_{ij0})\equiv\begin{cases}8\mod 16 & \text{if } i \text{ and } j \text{ are powers of } 2,\\ 0\mod 16 & \text{otherwise}.\end{cases}$$

(ii) *For $i, j, k\geqq 1$, $(a_{ijk})=0$ and*

$$\langle a_{ijk}\rangle\equiv\begin{cases}8\mod 16 & \text{if } i, j, k \text{ are powers of } 2,\\ 0\mod 16 & \text{otherwise}.\end{cases}$$

We shall prove Proposition 5.12 by preparing the following two lemmas:

LEMMA 5.13. (i) $(a_{ijk})=0$ *if $i, j, k\geqq 1$.*

$$\text{(ii)}\qquad \sum\langle a_{ijk}\rangle\binom{i}{r-i}\binom{j}{s-j}\binom{k}{t-k}=\begin{cases}360 & \text{if}\quad r=s=t=1,\\ 0 & \text{otherwise},\end{cases}$$

for $r, s, t \geqq 1$, where the summation is taken over all $i, j, k \geqq 1$ with $i \leqq r$, $j \leqq s$, $k \leqq t$.

PROOF. (i) By Lemma 5.11 (i), $\sum \langle a_{ijk}\rangle \binom{i}{r-i}\binom{j}{s-j}\binom{k}{t-k}=0$ for any r, s, $t \geqq 1$, where the summation is taken over all i, j, $k \geqq 1$ with $i \leqq r$, $j \leqq s$, $k \leqq t$. Therefore we see (i) by the induction on $i+j+k$.

(ii) (i) and Lemma 5.11 (ii) imply (ii). q.e.d.

LEMMA 5.14. (i) *$\langle a_{ijk}\rangle$ is a multiple of $\langle a_{111}\rangle = 360$ for any $i, j, k \geqq 1$.*

(ii) *$\langle a_{ijk}\rangle = \langle a_{i'j'k'}\rangle$ for any permutation (i', j', k') of (i, j, k).*

(iii) *$\sum_{i=1}^{r} \langle a_{ist}\rangle \binom{i}{r-i}=0$ for $r \geqq 2$ and $s, t \geqq 1$.*

(iv) *Set $m_{ijk}=\langle a_{ijk}\rangle/360$. Then $m_{rst}=m_{r11}m_{s11}m_{t11}$ for $r, s, t \geqq 1$.*

(v) $m_{r11} = \begin{cases} 1 \mod 2 & \textit{if } r \textit{ is a power of } 2, \\ 0 \mod 2 & \textit{otherwise.} \end{cases}$

PROOF. By Lemma 5.13 (ii), we can prove (i) and (ii) by the induction on $i+j+k$, and (iii) by the induction on $s+t$. We can prove (iv) inductively on $r+s+t$ by using (iii) and (ii), and (v) inductively on r by using (iii) and the fact that $\binom{2^i}{r-2^i}$ for $r>2^i$ is odd if and only if $r=2^{i+1}$. q.e.d.

PROOF OF PROPOSITION 5.12. (ii) The first equality is proved in Lemma 5.13 (i). The second equality is an immediate consequence of Lemma 5.14 (iv), (v).

(i) By Lemma 5.11 (i), we see that for $r, s \geqq 1$,

$$\sum_{1 \leqq i \leqq r, 1 \leqq j \leqq s} \langle a_{ij0}\rangle \binom{i}{r-i}\binom{j}{s-j} = \begin{cases} 24 & \text{if } r=s=1 \\ 0 & \text{otherwise.} \end{cases}$$

By using this equality instead of Lemma 5.13 (ii), we can prove (i) by the same way as the above proof of the second equality in (ii). q.e.d.

Now we consider another example of symplectic cobordism classes defined by R. E. Stong [14] and N. Ray [9]. We follow the methods due to N. Ray.

The complex projective space CP^{2i-1} is a weakly almost symplectic manifold (see [14]), and so is the product $\prod_{i=1}^{2r} CP^{2n_i-1}$. Consider the composition

$$f\colon \prod_{i=1}^{2r} CP^{2n_i-1} \xrightarrow{j} (CP^\infty)^{2r} \xrightarrow{m} CP^\infty \xrightarrow{q} HP^\infty,$$

where j and q are canonical maps and m is the classifying map of the tensor product of $2r$ copies of the canonical complex line bundle η over CP^∞. Then we have a bordism class

$$[\prod_{i=1}^{2r} CP^{2n_i-1}, f] \in MSp_{4(n-r)}(HP^\infty)$$

for $n=\sum_{i=1}^{2r} n_i$. By (5.4), we have an expansion

$$[\prod_{i=1}^{2r} CP^{2n_i-1}, f] = \sum_{k\geq 0} a_k(n_1,\dots, n_{2r})\beta_k^{MSp}$$

for some classes

(5.15) $\qquad a_k(n_1,\dots, n_{2r})\in \pi_{4(n-r-k)}(MSp) \qquad (n = \sum_{i=1}^{2r} n_i,\ n_i \geqq 1).$

By the result of N. Ray [9; (3.1), (3.2)] for the computation of the Hurewicz image of these classes $a_k(n_1,\dots, n_{2r})$, we have the following proposition, where $C(\bar{j}_1,\dots, \bar{j}_l)=(\sum_{i=1}^{l}\bar{j}_i)!/\prod_{i=1}^{l}(\bar{j}_i)!$.

PROPOSITION 5.16. *For the Hurewicz homomorphism* $H\colon \pi_{4(n-r-k)}(MSp)\to H_{4(n-r-k)}(MSp)$, *it holds*

$$H(a_k(n_1,\dots, n_{2r})) = \sum C(\bar{j}_1,\dots, \bar{j}_{2r})(b)_{j_1}^{-n_1}\cdots(b)_{j_{2r}}^{-n_{2r}}(b)_{n-r-k-j}^{k},$$

where $n=\sum_{i=1}^{2r} n_i$, $j=\sum_{i=1}^{2r} j_i$, $\bar{j}_i=2(n_i-j_i)-1$ *and the summation is taken over all* $j_i\geqq 0$.

We notice that the coefficients of b_1^l and $b_1^{l-2}b_2$ in the $4l$-dimensional component of $(b)^{-m}$ are $(-1)^l\binom{m+l-1}{m-1}$ and $(-1)^{l-1}(l-1)\binom{m+l-2}{m-1}$ respectively. Therefore, by comparing the coefficients of b_1^{n-r-k} and $b_1^{n-r-k-2}b_2$ in the both sides of the above equality, and by using the notations ( ) and $\langle\ \rangle$ in (5.10), we see the following

LEMMA 5.17. *The following* (i) *and* (ii) *hold, where* $n=\sum_{i=1}^{2r} n_i$, $j=\sum_{i=1}^{2r} j_i$, $\bar{j}_i=2(n_i-j_i)-1$, $m=n-r-k-j$ *and the summations are taken over* $j_i\geqq 0$ *with* $j_i\leqq n_i-1$ $(1\leqq i\leqq 2r)$:

(i) $(a_k(n_1,\dots, n_{2r})) = \sum (-1)^j\binom{k}{m}C(\bar{j}_1,\dots, \bar{j}_{2r})\prod_{i=1}^{2r}\binom{n_i+j_i-1}{n_i-1}$.

(ii) $\langle a_k(n_1,\dots, n_{2r})\rangle$ *is equal to*

$$\sum(-1)^j C(\bar{j}_1,\dots, \bar{j}_{2r})\{(m-1)\binom{k}{m-1} - \binom{k}{m}\sum_{i=1}^{2r}(j_i-1)j_i/(n_i+j_i-1)\}\prod_{i=1}^{2r}\binom{n_i+j_i-1}{n_i-1}.$$

When $k=1$, we have the following

PROPOSITION 5.18. (i) $P_{n-r-1}[a_1(n_1,\dots, n_{2r})]$ $(n=\sum_{i=1}^{2r} n_i\geqq r+2)$ *is equal to*

$$\begin{cases} (-1)^{n+r}(2r)!\prod_{i=1}^{2r}\binom{2n_i-2}{n_i-1} & \text{if } r=1 \text{ and } n_1, n_2\geqq 2, \text{ or } r=2,\\ 0 & \text{otherwise.}\end{cases}$$

(ii) $P_1P_{n-3}[a_1(n_1, n_2)]=(-1)^{n+1}(n-5)\binom{2n_1-2}{n_1-1}\binom{2n_2-2}{n_2-1}$ $\quad (n=n_1+n_2)$

if $n_1, n_2 \geqq 2$.

PROOF. (i) The equality in Lemma 5.17 (i) for $k=1$ is

$$(a_1(n_1,\dots, n_{2r})) = \sum(-1)^j C(\bar{j}_1,\dots,\bar{j}_{2r}) \prod_{i=1}^{2r}\binom{n_i+j_i-1}{n_i-1},$$

where the summation is taken over $j_i \geqq 0$ $(1 \leqq i \leqq 2r)$ with $j_i \leqq n_i-1$ and $j=n-r-1$, $n-r-2$. Therefore the left hand side is 0 if $r \geqq 3$, because $j \geqq n-2r$.

Let $r=1$. If $n_1, n_2 \geqq 2$, then the summation in the above equality is taken over $(j_1, j_2)=(n_1-1, n_2-1)$, (n_1-1, n_2-2) and (n_1-2, n_2-1), and then $C(\bar{j}_1, \bar{j}_2)=$ 2, 4 and 4 respectively. Hence we have

$$(a_1(n_1, n_2)) = (-1)^{n-1}2\binom{2n_1-2}{n_1-1}\binom{2n_2-2}{n_2-1} \quad \text{for} \quad n_1, n_2 \geqq 2.$$

If $n_1=1$, then $n_2 \geqq 2$ and the summation is taken over $(j_1, j_2)=(0, n_2-1)$ and $(0, n_2-2)$. Hence we see that $(a_1(1, n_2))=0$. $(a_1(n_1, 1))=0$ holds similarly. Thus we have the desired equality for $P_{n-2}[a_1(n_1, n_2)]$ by (5.10).

For the case $r=2$, the summation in the first equality is taken over $j_i= n_i-1$ $(1 \leqq i \leqq 4)$ only, and then $C(\bar{j}_1,\dots,\bar{j}_4)=24$. Thus we have the desired equality for $P_{n-3}[a_1(n_1, n_2, n_3, n_4)]$.

(ii) By the equality in Lemma 5.17 (ii), and by a similar argument to (i), we see that

$$\langle a_1(n_1, n_2)\rangle = (-1)^n(n+1)\binom{2n_1-2}{n_1-1}\binom{2n_2-2}{n_2-1} \quad \text{if} \quad n_1, n_2 \geqq 2.$$

By (5.10), $P_1P_{n-3}[a_1(n_1, n_2)]=\langle a_1(n_1, n_2)\rangle+(n-2)(a_1(n_1, n_2))$ for $n \geqq 3$. Hence we have (ii) by the above equalities for $\langle a_1(n_1, n_2)\rangle$ and $(a_1(n_1, n_2))$. q.e.d.

COROLLARY 5.19. $P_{n-r-1}[a_1(n_1,\dots, n_{2r})]$ $(n-r \geqq 2)$ *is congruent to*

$$\begin{cases} 8 \mod 16 & \textit{if} \quad r=1 \textit{ and } n_1-1,\ n_2-1 \textit{ are powers of } 2, \textit{ or} \\ & \qquad r=2 \textit{ and } n_1=n_2=n_3=n_4=1, \\ 0 \mod 16 & \textit{otherwise}. \end{cases}$$

PROOF. It is sufficient to prove the corollary for the first two cases in Proposition 5.18 (i). We notice that $\nu_2\left(\binom{m}{n}\right)=\alpha(n)+\alpha(m-n)-\alpha(m)$ (cf. [10; (6)]), where $\alpha(y)$ is the number of 1's in the dyadic expansion of y. Thus, by Proposition 5.18 (i), we have

$$(*) \quad \begin{aligned} &\nu_2(|P_{n-2}[a_1(n_1, n_2)]|) = 1+\alpha(n_1-1)+\alpha(n_2-1) \quad \text{if} \quad n_1, n_2 \geqq 2, \\ &\nu_2(|P_{n-3}[a_1(n_1, n_2, n_3, n_4)]|) = 3+\textstyle\sum_{i=1}^4 \alpha(n_i-1). \end{aligned}$$

Hence $\nu_2(|P_{n-2}[a_1(n_1, n_2)]|)$ is at least 3, and is 3 if and only if n_1-1 and n_2-1 are powers of 2. Also $\nu_2(|P_{n-3}[a_1(n_1, n_2, n_3, n_4)]|)$ is 3 if and only if $n_i=1$ $(1\leqq i\leqq 4)$. Thus we have the corollary. q.e.d.

REMARK 5.20. In addition to Proposition 5.18, we can prove the following equalities by (5.10), Lemma 5.17 and routine computations:

$$\text{(i)}\quad P_1P_{n-r-2}[a_1(n_1,\dots, n_{2r})] = A\prod_{i=1}^{2r}\binom{2n_i-2}{n_i-1} \qquad (n=\textstyle\sum_{i=1}^{2r} n_i\geqq r+2),$$

where $A=(-1)^{n+1}(n-5)$ if $r=1$ and $n_1, n_2\geqq 2$, $=(-1)^n 12(n-4l-2)$ if $r=2$ (l is the number of i's with $n_i\geqq 2$), $=(-1)^n 720$ if $r=3$, $=0$ otherwise.

$$\text{(ii)}\quad P_{n-3}[a_2(n_1, n_2)] = (-1)^{n+1}2\{2-7(n-3)/(2n_1-3)(2n_2-3)\}\binom{2n_1-2}{n_1-1} \times\binom{2n_2-2}{n_2-1} \quad (n=n_1+n_2) \text{ if } n_1, n_2\geqq 3.$$

Now we can prove the following theorem which is Theorem III (i):

THEOREM 5.21. *$\pi_{8n+3}(MSp(n))$ $(n\geqq 3)$ has no p-torsion for any odd prime p.*

PROOF. Let $Q_p=\{l/m\,|\,(m, p)=1\}\subset Q$. Tensoring Q_p to (4.1) for $i=4n+4$ $(n\geqq 2)$, we have the exact sequence

$$\pi_{4n+4}(MSp)\otimes Q_p \xrightarrow{\partial\otimes 1} Q_p\oplus Q_p \longrightarrow \pi_{8n+3}(MSp(n))\otimes Q_p \longrightarrow 0,$$

since $\pi_{8n+3}(F_n)\otimes Q_p\cong Q_p\oplus Q_p$ $(n\geqq 2)$ by Proposition 3.5 (i) and $\pi_{4n+3}(MSp)$ is a 2-torsion group. Therefore it is sufficient to show that

$$(5.22)\qquad \partial\otimes 1\colon \pi_{4n+4}(MSp)\otimes Q_p \longrightarrow Q_p\oplus Q_p \qquad \text{is epimorphic for}\quad n\geqq 3.$$

Set $y_1=a_1(1, 1, 1, 1)$, $y_i=a_1(2, i)$ $(2\leqq i\leqq 6)$ and $z=a_1(3, 3)$. Then, by using Proposition 4.2, the equalities

$$(5.23)\qquad \begin{aligned} P_{k+l}[uv] &= P_k[u]P_l[v],\\ P_1P_{k+l-1}[uv] &= P_1P_{k-1}[u]P_l[v] + P_k[u]P_1P_{l-1}[v] \end{aligned}$$

for $u\in\pi_{4k}(MSp)$, $v\in\pi_{4l}(MSp)$ $(k, l\geqq 1)$ and Proposition 5.18, we see the following equalities for $k\geqq 0$, where $(a, b)=au(3)+bv(3)+$a torsion element:

$$\partial(y_1y_2^{k+1}) = ((-1)^{k+1}8^k\cdot 4\cdot(3k+5),\ \ (-1)^{k+1}8^{k+1}\cdot 3),$$

$$\partial(y_2^{k+2}) = ((-1)^k 8^k\cdot 4\cdot(k+3),\ \ (-1)^k 8^{k+1}),$$

$$\partial(y_2^k z) = ((-1)^{k+1}8^{k-1}\cdot 4\cdot 9\cdot(k+2),\ \ (-1)^{k+1}8^k\cdot 9),$$

$$\partial(y_2^k y_i) = \left((-1)^{k+i+1}2\cdot 8^{k-1}\cdot(k+2)\binom{2i-2}{i-1},\ \ (-1)^{k+i+1}4\cdot 8^{k-1}\binom{2i-2}{i-1}\right) \quad (i\geqq 3),$$

where $\binom{2i-2}{i-1}=6$ if $i=3$, $=4\cdot 5$ if $i=4$, $=2\cdot 5\cdot 7$ if $i=5$, $=4\cdot 3^2\cdot 7$ if $i=6$. Therefore, for $\partial\otimes 1$ in (5.21), we have the following equalities:

When $n=2k-1$ with $k\geqq 2$ and $p\neq 5$,

$$(-1)^k\partial\otimes 1\{(1/8^{k-2}\cdot 4)y_2^k+(1/8^{k-2}\cdot 5)y_2^{k-2}y_4\}=(1,0),$$
$$(-1)^{k+1}\partial\otimes 1\{(k/8^{k-1})y_2^k+((k+1)/8^{k-2}\cdot 2\cdot 5)y_2^{k-2}y_4\}=(0,1);$$

when $n=2k-1$ with $k\geqq 3$ and $p=5$,

$$(-1)^k\partial\otimes 1\{(1/8^{k-1})y_2^k-(1/8^{k-3}\cdot 2\cdot 7\cdot 9)y_2^{k-3}y_6\}=(1,0),$$
$$(-1)^{k+1}\partial\otimes 1\{((k-1)/8^{k-1}\cdot 2)y_2^k-((k+1)/8^{k-3}\cdot 4\cdot 7\cdot 9)y_2^{k-3}y_6\}=(0,1);$$

when $n=3$ and $p=5$,

$$\partial\otimes 1\{(1/4)y_2^2+(2/9)z\}=(1,0),\quad \partial\otimes 1\{-(1/4)y_2^2-(1/3)z\}=(0,1);$$

when $n=2k$ with $k\geqq 2$,

$$(-1)^{k+1}\partial\otimes 1\{(1/8^{k-1}\cdot 4)y_1y_2^k+(1/8^{k-2}\cdot 4)y_2^{k-1}y_3\}=(1,0),$$
$$(-1)^k\partial\otimes 1\{((k+36)/8^k)y_1y_2^k+((k+24)/8^{k-1})y_2^{k-1}y_3-(1/8^{k-2})y_2^{k-2}y_5\}=(0,1).$$

These equalities imply (5.22), and we have the desired result. q.e.d.

Now we prove Theorems 4.6 and 4.7.

PROOF OF THEOREM 4.6. (i) In the above proof, we have seen that

$$P_{2i}[a_1(2,2)^i]=(-8)^i\,(i\geqq 1),\ P_{2i+1}[a_1(2,2)^{i-1}a_1(2,3)]=(-1)^{i-1}3\cdot 8^i\,(i\geqq 1),$$
$$P_{2i+1}[a_1(2,2)^{i-2}a_1(2,5)]=(-1)^i 35\cdot 8^{i-1}\,(i\geqq 2).$$

Therefore the definition (4.3) of $m(n)$ implies that $m(2i)\,(i\geqq 1)$ is a power of 2 by the first equality, and so is $m(2i+1)\,(i\geqq 2)$ by the last two ones. $m(1)=m(3)=3$ follows from the result of [7], [6] on $\pi_4(MSp)$ and $\pi_{12}(MSp)$.

(ii) The desired result for $n=2^k+2^l-1$ (resp. 2^k+2^l) follows immediately from (i) and the fact that $P_n[a_{2^k2^l0}]$ (resp. $P_n[a_1(2^k+1,2^l+1)]$) is not a multiple of 16 by (5.10) and Proposition 5.12 (i) (resp. Corollary 5.19). q.e.d.

REMARK 5.24. Let $n\neq 1, 3$. Then by Theorem 4.6 (ii), $m(n)=1$ if $\alpha(n)\leqq 2$ or $\alpha(n+1)\leqq 2$ ($\alpha(t)$ is the number of 1's in the dyadic expansion of t). In general, the exponent $\nu_2(m(n))$ of $m(n)=2^{\nu_2(m(n))}$ can be estimated by the inequality

$$\rho_0(2n)-3\leqq \nu_2(m(n))\leqq \min\{\alpha(n),\alpha(n+1)\}-2 \ \text{ if } \alpha(n),\alpha(n+1)\geqq 3,$$

where $\rho_0(2n)=\min\{r\,|\,\alpha(2n+r)\leqq 2r\}$ is the number given in [11; Th. A]. But this inequality does not determine $\nu_2(m(n))$, since $\rho_0(2n)<\min\{\alpha(n), \alpha(n+1)\}$ there.

In fact, the first inequality is seen by (*) in the proof of Corollary 4.5. We note that for any t with $\alpha(t)\geqq 2$, $\alpha(t)\leqq\alpha(t_1)+\alpha(t_2)$ if $t=t_1+t_2$ and there are $t_1, t_2\geqq 1$ with $t=t_1+t_2$ and $\alpha(t)=\alpha(t_1)+\alpha(t_2)$. Thus we see the second inequality by (*) in the proof of Corollary 4.5 and by the equality $\nu_2(|P_n[a_2(t_1, t_2)]|)=1+\alpha(t_1)+\alpha(t_2)$ $(t_1+t_2=n+1;\ t_1, t_2\geqq 2)$ for odd n, which follows from Remark 5.20 (ii). The last inequality is seen easily.

PROOF OF THEOREM 4.7. (i) By Proposition 3.5 (i) and Remark 3.7, $\pi_i(F_n)$ has no p-torsion for $i\leqq 8n+3$ if $n\geqq 2$ and for $i\leqq 9$ if $n=1$. Furthermore $\pi_*(MSp)$ has no p-torsion. Thus (i) holds for $i\neq 8n-1, 8n-3$ by the exact sequence (4.1). For $i=8n-1$, (i) follows from Corollary 4.4. For $i=8n+3$, (i) is proved in Theorem 5.21.

(ii) If $n=1$, then $\pi_7(MSp)=0$ by [7], and (ii) is trivial. If $n\geqq 2$, then $\pi_{8n+2}(F_n)=0$ by Proposition 3.5 (i). Thus (ii) follows from the exact sequence (4.1).

(iii) Consider the exact sequence (4.1) for $i=4n+4$ and $n=2^k+2^l-1$ with $k, l\geqq 1$:

$$\pi_{4n+4}(MSp) \xrightarrow{\partial} Z\oplus Z \longrightarrow \pi_{8n+3}(MSp(n)) \xrightarrow{b_{n*}} \pi_{4n+3}(MSp) \longrightarrow 0,$$

where we identify $\pi_{8n+3}(F_n)$ with $Z\oplus Z$ by Proposition 3.5 (i). By Propositions 4.2 (ii), 5.12 (ii) and Corollary 5.19, we have

$$\partial(a_{2^k2^l1}) = (x, 0) \quad \text{and} \quad \partial(a_1(2^k+1, 2^l+1)) = (x', y)$$

for some integer x' and some odd integers x and y. These imply that Coker ∂ is a finite group and has no 2-torsion. By Theorem 5.21, Coker ∂ has no p-torsion for any odd prime p, hence Coker $\partial=0$, and b_{n*} is isomorphic. q.e.d.

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Department of Mathematics,
College of Education,
Wakayama University