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On the K-Ring of the Orbit Manifold $(S^{2m+1}\times S^l)/D_n$ by the Dihedral Group D_n

Mitsunori IMAOKA and Masahiro SUGAWARA

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Introduction

The dihedral group D_n ($n\geqq 3$) of symmetries of the n-sided regular polygon is generated by two elements a and b with relations $a^n=b^2=abab=1$. Consider the unit spheres S^{2m+1} and S^l in the complex $(m+1)$-space and the real $(l+1)$-space. Then D_n operates freely on the product space $S^{2m+1}\times S^l$ by

$$a\cdot(z, x)=(z\exp(2\pi\sqrt{-1}/n), x), \qquad b\cdot(z, x)=(tz, -x),$$

where t is the conjugation, and the orbit manifold

$$D(m, l; n)=(S^{2m+1}\times S^l)/D_n=(L^m(n)\times S^l)/Z_2$$

is defined, where $L^m(n)=S^{2m+1}/Z_n$ is the standard lens space. The bordism group of D_n is studied by considering this manifold in [9].

The purpose of this note is to study the complex K-ring $K(D(m, l; n))$ ($m>0$, $l>0$) for odd n.

Let

$$\nu, \alpha\in\tilde{K}(D(m, l; n)) \qquad \text{and} \qquad \gamma\in\tilde{K}(D(m, 2l; n))$$

be the elements defined as follows: $\nu+1$ is the induced bundle of the canonical complex line bundle over the real projective space $RP(l)$ by the natural projection $D(m, l; n)\to S^l/Z_2=RP(l)$. $\alpha+\nu+2$ is the associated complex 2-plane bundle of the principal $U(2)$-bundle induced from the principal D_n-bundle $S^{2m+1}\times S^l\to D(m, l; n)$ by the natural inclusion $D_n\subset O(2)\subset U(2)$. γ is the image of $\sigma\otimes g^l$ by the induced homomorphism of the projection

$$D(m, 2l; n)\xrightarrow{q} D(m, 2l; n)/D(m, 2l-1; n)\xrightarrow[\approx]{h}(L^m(n)\times S^{2l})/(L^m(n)\times *),$$

where $\sigma\in\tilde{K}(L^m(n))$ is the stable class of the canonical complex line bundle over $L^m(n)$, and $g^l\in\tilde{K}(S^{2l})=Z$ is the canonical generator.

Denote the natural inclusions by

$$i: L^m(n)\subset D(m, l; n), \qquad k: RP(l)\subset D(m, l; n).$$

Also, consider the $2m$-skeleton $L_0^m(n)$ of the cell complex $L^m(n)=\cup_{i=0}^{2m+1}C^i$, and set

$$D_0(m, l; n) = (L_0^m(n)\times S^l)/Z_2 \subset D(m, l; n).$$

Denote the projection and the homeomorphism by

$$D(m, l; n)\xrightarrow{q_0} D(m, l; n)/D_0(m, l; n)\xrightarrow[\approx]{f} S^m \wedge (RP(m+l+1)/RP(m)),$$

where the last term is the suspension of the stunted real projective space.

By these notations, our result (Theorem 3.9) is stated by

THEOREM. *Assume that n is odd, $m>0$ and $l>0$. Then we have the direct sum decomposition*

$$\tilde{K}(D(m, l; n)) = A_{m,l}\oplus B_{m,l}\oplus Z_{2^{[l/2]}}\oplus \begin{cases} Z_{2^{[(l+1)/2]}} & \text{if } m \text{ is even,} \\ Z & \text{if } m \text{ and } l \text{ are odd,} \\ 0 & \text{otherwise,} \end{cases}$$

where $A_{m,l}\oplus B_{m,l}$ is the odd component and the summands are given as follows:

(i) *$A_{m,l}$ is the subring generated by α, and is isomorphic to the image of the complexification c: $\widetilde{KO}(L^m(n))\to\tilde{K}(L^m(n))$ by the induced homomorphism $i^!$.*

(ii) *$B_{m,2l+1}=0$ and $B_{m,2l}$ is the ideal generated by γ which satisfies $\gamma^2=0$, and the subgroup $A_{m,2l}\oplus B_{m,2l}$ is isomorphic to $\tilde{K}(L^m(n))$.*

(iii) *The third cyclic summand $Z_{2^{[l/2]}}$ is generated by ν, and is isomorphic to $\tilde{K}(RP(l))$ by the induced homomorphism $k^!$.*

(iv) *The rest is the image of $\tilde{K}(S^m\wedge RP(m+l+1)/RP(m)))$ by the induced homomorphism $q_0^! f^!$, which is monomorphic. Its generator ν_m satisfies $\nu_m^2=0$ and $\nu\nu_{2m}=-2\nu_{2m}$, $\nu\nu_{2m+1}=0$.*

The partial result for odd m, l and odd prime n is obtained in [8].

In § 1, we prepare some preliminary results on the cell structures and the integral cohomology groups of $D(m, l; n)$, $D_0(m, l; n)$, and on the homeomorphisms h, f and the double covering π: $L^m(n)\times S^l\to D(m, l; n)$. In § 2, we are concerned with (iii), and notice that the order of $\tilde{K}(D_0(m, l; n)/RP(l))$ is a divisor of $n^{[m/2]}$ or n^m according as l is odd or even. Also, we consider the above elements α and γ, and study their images by $i^!$ and $\pi^!$. Using these results and the known results for $\tilde{K}(L^m(n))$, we study $A_{m,l}$ of (i) by $i^!$ and $B_{m,2l}$ of (ii) by $\pi^!$, and prove the theorem by the exact sequence of $(D(m, l; n), D_0(m, l; n), RP(l))$ in § 3. Finally, we are concerned with the special case that n is an odd prime p (Corollary 3.14), using the known results for $\tilde{K}(L^m(p))$ in [10].

§ 1. Preliminaries

The dihedral group D_n $(n\geqq 3)$ of order $2n$ is the subgroup of the orthogonal group $O(2)$ generated by

$$a=\begin{pmatrix} \cos(2\pi/n) & \sin(2\pi/n) \\ -\sin(2\pi/n) & \cos(2\pi/n) \end{pmatrix} \quad \text{and} \quad b=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

with relations $a^n=b^2=abab=1$. These elements a and b generate the cyclic subgroups Z_n and Z_2 of order n and 2, respectively, and D_n is a split extension of Z_n by Z_2.

Let S^{2m+1} and S^l be the unit spheres in the complex $(m+1)$-space C^{m+1} and real $(l+1)$-space R^{l+1}, respectively. Then D_n operates freely on the product space $S^{2m+1}\times S^l$ by

$$a\cdot(z, x)=(z\exp(2\pi\sqrt{-1}/n), x), \qquad b\cdot(z, x)=(tz, \tau x)$$

for $(z, x)\in S^{2m+1}\times S^l$, where tz is the conjugation of z and $\tau x=-x$ is the antipodal point of x. In this note, we study the orbit manifold

$$D(m, l; n)=(S^{2m+1}\times S^l)/D_n.$$

Since Z_n operates trivially on the second factor S^l, we have

$$D(m, l; n)=(L^m(n)\times S^l)/Z_2, \tag{1.1}$$

where $L^m(n)=S^{2m+1}/Z_n$ is the standard lens space mod n, and the action of Z_2 is given by $b\cdot([z], x)=([tz], \tau x)$. Therefore, we have the fibering

$$L^m(n)\xrightarrow{i} D(m, l; n)\xrightarrow{p} RP(l), \tag{1.2}$$

where $RP(l)=S^l/Z_2$ is the real l-dimensional projective space, i is the inclusion and p is the projection. Also, we have the double covering

$$\pi: L^m(n)\times S^l\longrightarrow D(m, l; n). \tag{1.3}$$

The lens space $L^m(n)$ has the cell decomposition

$$L^m(n)=C^0\cup C^1\cup\cdots\cup C^{2m}\cup C^{2m+1}, \quad \partial(C^{2i+1})=0, \quad \partial(C^{2i})=nC^{2i-1},$$

which is invariant under the conjugation t. Also, S^l has the cell decomposition $\{D^j_\pm | 0\leqq j\leqq l\}$ such that $S^j=\bar{D}^j_+\cup\bar{D}^j_-\supset\bar{D}^j_+\cap\bar{D}^j_-=S^{j-1}$, and

LEMMA 1.4. [9, p. 338] *$D(m, l; n)$ is the cell complex with the cells defined by*

$$(C^i, D^j)=\pi(C^i\times D^j_+) \qquad \textit{for} \quad 0\leqq i\leqq 2m+1, \quad 0\leqq j\leqq l,$$

which have the boundary relations

$$\partial(C^{2i+1}, D^j) = ((-1)^i + (-1)^{j+1})(C^{2i+1}, D^{j-1}),$$

$$\partial(C^{2i}, D^j) = n(C^{2i-1}, D^j) + ((-1)^i + (-1)^j)(C^{2i}, D^{j-1}).$$

We consider the $2m$-skeleton

$$L_0^m(n) = C^0 \cup C^1 \cup \cdots \cup C^{2m} \tag{1.5}$$

of $L^m(n)$ of the above, and the subcomplex

$$D_0(m, l; n) = (L_0^m(n) \times S^l)/Z_2 \subset D(m, l; n) \tag{1.6}$$

with cells $\{(C^i, D^j) | 0 \leqq i \leqq 2m,\ 0 \leqq j \leqq l\}$, and we consider naturally

$$D(i, j; n) \subset D_0(m, j; n) \subset D(m, l; n) \qquad \text{for } i < m,\ j \leqq l,$$

by $L^i(n) \subset L_0^m(n) \subset L^m(n)$ and $S^j \subset S^l$. It is clear that

LEMMA. 1.7. *$D_0(0, l; n) = RP(l)$, and the inclusion*

$$k: RP(l) \longrightarrow D(m, l; n)$$

is a right inverse of p in (1.2).

By the cell structure of Lemma 1.4, the subcomplex

$$X_{i,j} = D_0(i-1, l; n) \cup D_0(i, j; n) \qquad (i \leqq m,\ j \leqq l)$$

of $D_0(m, l; n)$ has the following cell structure:

$$X_{i,j} = X_{i,j-2} \cup (C^{2i-1}, D^{j-1}) \cup (C^{2i}, D^{j-1}) \cup (C^{2i-1}, D^j) \cup (C^{2i}, D^j)$$

for $j \geqq 1$ and even $i+j$, with the bounday relations

$$\partial(C^{2i}, D^j) = n(C^{2i-1}, D^j) - 2\varepsilon(C^{2i}, D^{j-1}),$$

$$\partial(C^{2i-1}, D^j) = 2\varepsilon(C^{2i-1}, D^{j-1}), \qquad \partial(C^{2i}, D^{j-1}) = n(C^{2i-1}, D^{j-1}),$$

$$\partial(C^{2i-1}, D^{j-1}) = 0, \qquad (\varepsilon = \pm 1);$$

$$X_{i,j} = X_{i,j-1} \cup (C^{2i-1}, D^j) \cup (C^{2i}, D^j)$$

for $j = 0$ and even i or $j = l$ and odd $i+l$, with the boundary relations

$$\partial(C^{2i}, D^j) = n(C^{2i-1}, D^j), \qquad \partial(C^{2i-1}, D^j) = 0.$$

Therefore, it is easy to see that the reduced integral cohomology groups $\tilde{H}^*$ are

$$\tilde{H}^*(X_{i,j}) = \tilde{H}^*(X_{i,j-2}) \qquad \text{for } j \geqq 1, \text{ even } i+j \text{ and odd } n;$$

$$\tilde{H}^k(X_{i,j}) = \tilde{H}^k(X_{i,j-1}) \oplus \begin{cases} Z_n & \text{if } k = 2i+j \\ 0 & \text{otherwise} \end{cases}$$

for $j=0$ and even i or $j=l$ and odd $i+l$. Also, we have

$$D(m, l; n) = D_0(m, l; n) \cup \cup_{j=0}^{l} (C^{2m+1}, D^j),$$

$\partial(C^{2m+1}, D^j) = \pm 2(C^{2m+1}, D^{j-1})$ if $j \geqq 1$ and $m+j$ is odd, $=0$ otherwise.

Using these facts, we have easily

LEMMA. 1.8. *Assume that n is odd.*

(i) $$\tilde{H}^i(D_0(m, l; n), RP(l)) = \begin{cases} (Z_n)^{a_i} & \text{if } 0 < i \leqq 2m+l, \\ 0 & \text{otherwise,} \end{cases}$$

where $(Z_n)^a$ means the direct sum of a copies of Z_n, and

$$0 \leqq a_i \leqq 1, \quad \sum a_{2i} = \sum a_{2i-1} = [m/2] \qquad \text{if } l \text{ is odd,}$$

$$0 \leqq a_i \leqq 2, \quad a_{2i-1} = 0, \quad \sum a_{2i} = m \qquad \text{if } l \text{ is even.}$$

(ii) $\tilde{H}^{2i}(D(2m+1, 2l+1; n), RP(2l+1)) = Z$ *if* $2i = 4m+2l+4$,

$= \tilde{H}^{2i}(D_0(2m+1, 2l+1; n), RP(2l+1))$ *if* $2i \neq 4m+2l+4$.

The projection π of (1.3) defines naturally the homeomorphism

$$h\colon D(m, l; n)/D(m, l-1; n) \underset{\approx}{\overset{\pi}{\leftarrow}} (L^m(n) \times \bar{D}^l_+)/(L^m(n) \times S^{l-1}) \approx (L^m(n) \times S^l)/(L^m(n) \times *). \tag{1.9}$$

We consider the diagram

$$\begin{array}{ccccccc} L \times S = L^m(n) \times S^{2l} & \xrightarrow{\pi} & D(m, 2l; n) & \xrightarrow{q} & D(m, 2l; n)/D(m, 2l-1; n) \\ \downarrow{\scriptstyle q_1} & & & & \approx\downarrow{\scriptstyle h} \\ X = (L \times S)/(L \times *) & \xrightarrow{\rho} & X \vee X & \xrightarrow{\lambda} & X \vee X & \xrightarrow{\nabla} & X \end{array} \tag{1.10}$$

where π, q and q_1 are the projections, h is the one of (1.9),

$$\rho\colon (L \times S)/(L \times *) \longrightarrow (L \times (S \vee S))/(L \times *) = X \vee X$$

is the map induced from $\mathrm{id} \times \rho\colon L \times S \to L \times (S \vee S)$ of the comultiplication $\rho\colon S \to S/S^{2l-1} = S \vee S$, and ∇ is the folding map.

LEMMA. 1.11. *There exists such a homeomorphism λ that the diagram* (1.10) *is commutative and $\lambda = \mathrm{id} \vee (t \times \tau_1)$, where $t \times \tau_1$ is the induced map of*

$$t \times \tau_1\colon L^m(n) \times S^{2l} \longrightarrow L^m(n) \times S^{2l}$$

(*t is the conjugation*) *and the degree of* τ_1 *is equal to* -1.

PROOF. Consider the commutative diagram

$$\begin{array}{ccccccc} L\times S & \xrightarrow{q_2} & (L\times S)/(L\times S') = Y_+ \vee Y_- & \xrightarrow{\mathrm{id}\vee(t\times\tau)} & Y_+ \vee Y_+ & \xrightarrow{\nabla} & Y_+ \\ \downarrow{\scriptstyle q_1} & & \downarrow{\scriptstyle \varphi_+\vee\varphi_-} & & \downarrow{\scriptstyle \varphi_+\vee\varphi_+} & & \downarrow{\scriptstyle \varphi_+} \\ X & \xrightarrow{\rho} & X\vee X & \xrightarrow{\mathrm{id}\vee(t\times\tau_1)} & X\vee X & \xrightarrow{\nabla} & X, \end{array}$$

where $S\supset D_\pm = \bar{D}^{2l}_\pm \supset S' = S^{2l-1}$, $Y_\pm = (L\times D_\pm)/(L\times S')$, and the maps are as follows: q_1, q_2 are projections, and $\varphi_\pm$, $t\times\tau$, $t\times\tau_1$ are the maps induced from

$$\mathrm{id}\times\varphi_\pm : L\times D_\pm \to L\times S, \quad t\times\tau : L\times D_- \to L\times D_+, \quad t\times\tau_1 : L\times S \to L\times S,$$

respectively, where $\varphi_\pm$ is the restriction of the relative homeomorphism $\varphi_\pm$: $(S, D_\mp)\to(S, *)$ of degree 1, t is the conjugation, τ is the antipodal map and $\tau_1 = \varphi_+\tau\varphi_-^{-1}$.

Then we have the lemma, since $\varphi_+\nabla(\mathrm{id}\vee(t\times\tau))q_2 = hq\pi$ by the definition of h of (1.9) and the degree of $\tau_1 = \varphi_+\tau\varphi_-^{-1}$ is $(-1)^{2l+1} = -1$.

q.e.d.

We have also the following

LEMMA 1.12. *There is a homeomorphism*

$$f:\ D(m, l; n)/D_0(m, l; n) \approx (S^m\times RP(m+l+1))/(*\times RP(m+l+1)\cup S^m\times RP(m))$$
$$= S^m\wedge(RP(m+l+1)/RP(m)),$$

where the last term is the suspension of the stunted real projective space.

PROOF. Consider the relative homeomorphisms and homeomorphisms

$$\varphi:\ (D^{2m+1}_+, S^{2m}) \longrightarrow (L^m(n), L^m_0(n)), \quad D^m\times D^{m+1} \xrightarrow[\approx]{\theta} D^{2m+1} \xleftarrow[\approx]{p_+} D^{2m+1}_+,$$

$$g:\ (D^{m+1}\times S^l, S^m\times S^l) \longrightarrow (S^{m+l+1}, S^m),$$

defined as follows: D^{2m+1}_+ is the upper hemi-sphere of S^{2m+1} and

$$\varphi(z_0, \ldots, z_{m-1}, r\exp(\pi s\sqrt{-1})) = [z_0, \ldots, z_{m-1}, r\exp(2\pi s\sqrt{-1}/n)];$$

D^j is the unit disc in R^j and p_+ is the projection;

$$\theta(u, v) = \max(|u|, |v|)(|u|^2+|v|^2)^{-1/2}(u_1, v_1, \ldots, u_m, v_m, v_{m+1}),$$

where $u=(u_1, \ldots, u_m)$, $v=(v_1, \ldots, v_m, v_{m+1})$; and

$$g(v, x) = (v, (1-|v|^2)^{1/2}x).$$

Then, it is easy to see that the homeomorphism

$$f\colon (L^m(n)\times S^l)/(L^m_0(n)\times S^l)\approx (D^m\times S^{m+l+1})/(S^{m-1}\times S^{m+l+1}\cup D^m\times S^m)$$
$$\approx (S^m\times S^{m+l+1})/(*\times S^{m+l+1}\cup S^m\times S^m)$$

is obtained by the composition $(\mathrm{id}\times g)((\theta^{-1}p_+\varphi^{-1})\times \mathrm{id})$, and f is equivariant with respect to the Z_2-actions, where Z_2 acts on $L^m(n)\times S^l$ by (1.1) and on $D^m\times S^{m+l+1}$ by $b\cdot(u, y)=(u, -y)$. Therefore f induces the desired homeomorphism f. q.e.d.

Finally, we consider the diagram

$$\begin{array}{ccccc} D(m,l;n) & \xrightarrow{d} & D(m,l;n)\times D(m,l;n) & \xrightarrow{fq_0\times p} & ((S\times RP)/Z)\times RP(l) \\ \downarrow{\scriptstyle q_0} & & & & \downarrow{\scriptstyle \mathrm{id}\times\mathrm{id}\times i_0} \\ D(m,l;n)/D_0(m,l;n) & \xrightarrow[\approx]{f} & (S\times RP)/Z & \xrightarrow{\mathrm{id}\times d} & (S\times RP\times RP)/(Z\times RP) \end{array} \tag{1.13}$$

where $S=S^m$, $RP=RP(m+l+1)$, $Z=*\times RP\cup S\times RP(m)$, and q_0 is the projection, f is the homeomorphism of the above lemma, d means the diagonal map, p is the projection in (1.2) and $i_0\colon RP(l)\to RP$ is the inclusion given by $i_0[x]=[0, x]$.

LEMMA 1.14. *The diagram* (1.13) *is homotopy commutative.*

PROOF. For the map g of the above proof, the diagram

$$\begin{array}{ccccc} D^{m+1}\times S^l & \xrightarrow{d} & D^{m+1}\times S^l\times D^{m+1}\times S^l & \xrightarrow{g\times p} & S^{m+l+1}\times S^l \\ \downarrow{\scriptstyle g} & & & & \downarrow{\scriptstyle \mathrm{id}\times i_0} \\ S^{m+l+1} & & \xrightarrow{\quad d\quad} & & S^{m+l+1}\times S^{m+l+1} \end{array}$$

(p is the projection and i_0 is the inclusion given by $i_0(x)=(0, x)$) is homotopy commutative by the homotopy H_s given by

$$H_s(v, x)=(g(v, x), (sv, (1-|sv|^2)^{1/2}x)).$$

Since $H_s(S^m\times S^l)\subset S^m\times S^{m+l+1}$ and $H_s(-v, -x)=-H_s(v, x)$, we see easily by the above proof that H_s induces the desired homotopy of (1.13). q.e.d.

REMARK 1.15. The orthogonal group $O(2)$ acts freely on $S^{2m+1}\times S^l$ by

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\cdot(z, x)=(z\exp(\theta\sqrt{-1}), x), \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\cdot(z, x)=(tz, \tau x).$$

Therefore, there is the natural projection

$$p'\colon D(m, l; n)\longrightarrow (S^{2m+1}\times S^l)/O(2)=D(m, l)$$

to the Dold manifold $D(m, l)$. The homeomorphisms h of (1.9) and f of Lemma

1.12 are analogous to those for the Dold manifold of [7, Prop. 2], and Lemma 1.14 is similar to [7, Lemma 1].

§ 2. Some elements of $\tilde{K}(D(m, l; n))$ for odd n

In the rest of this note, we assume that n *is odd*, and study the complex K-group of the manifold $D(m, l; n)$ $(m>0, l>0)$ of (1.1).

Let $K(X)$ be the K-ring of the complex vector bundles over a finite CW-complex X, and $\tilde{K}(X)$ be the reduced K-ring. It is well known that a map $f: X \to Y$ induces naturally the ring homomorphisms

$$f^!: K(Y) \longrightarrow K(X), \quad f^!: \tilde{K}(Y) \longrightarrow \tilde{K}(X)$$

and the Puppe exact sequence

$$\tilde{K}(X) \xleftarrow{f^!} \tilde{K}(Y) \longleftarrow \tilde{K}(C_f) \longleftarrow \tilde{K}^1(X) \xleftarrow{f^!} \tilde{K}^1(Y)$$

where C_f is the mapping cone of f and $\tilde{K}^1(X)=\tilde{K}^{-1}(X)=\tilde{K}(S^1 \wedge X)$. Also, there is the Atiyah-Hirzebruch spectral sequence $\{E_r^{p,q}\}$ for $\tilde{K}(X)$, such that $E_2^{p,q}=\tilde{H}^p(X; K^q(*))(K^{2i}(*)=Z, K^{2i-1}(*)=0)$ and $E_\infty^{p,q}$ is the graded group associated to $\tilde{K}^{p+q}(X)=\tilde{K}(S^{p+q} \wedge X)$ (cf. [4, § 2]).

Consider the induced homomorphisms

$$\tilde{K}(RP(l)) \xrightarrow{p^!} \tilde{K}(D(m, l; n)) \xrightarrow{j^!} \tilde{K}(D_0(m, l; n)) \tag{2.1}$$

of p in (1.2) and the inclusion j of (1.6). It is proved in [1, Th. 7.3] that

(2.2) $\tilde{K}(RP(l)) = Z_{2[l/2]}$ *is generated by* v,

and v is the stable class of the complexification of the canonical real line bundle over $RP(l)$. Define

$$v = p^! v \in \tilde{K}(D(m, l; n)), \quad v = j^! v \in \tilde{K}(D_0(m, l; n)). \tag{2.3}$$

Then, by Lemma 1.7 and (2.2), we have immediately

PROPOSITION 2.4. *There is the commutative diagram*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \tilde{K}(D(m, l; n)/RP(l)) & \longrightarrow & \tilde{K}(D(m, l; n)) & \xrightarrow{k^!} & \tilde{K}(RP(l)) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle j^!} & & \downarrow{\scriptstyle j^!} & & \| & & \\
0 & \longrightarrow & \tilde{K}(D_0(m, l; n)/RP(l)) & \longrightarrow & \tilde{K}(D_0(m, l; n)) & \xrightarrow{k^!} & \tilde{K}(RP(l)) & \longrightarrow & 0
\end{array}$$

of the split exact sequences, where j, k are the inclusions. The subrings of $\tilde{K}(D(m, l; n))$ and $\tilde{K}(D_0(m, l; n))$ generated by v of (2.3) *are $Z_{2[l/2]}$ mapped isomorphically onto $\tilde{K}(RP(l))$ by $k^!$, and they are isomorphic by $j^!$.*

The following lemma is proved easily by using the Atiyah-Hirzebruch spectral sequence and Lemma 1.8, where $\sharp G$ means the number of elements of G.

LEMMA 2.5. (i) $\sharp\tilde{K}^i(D_0(m,l;n)/RP(l))$ $(i=0,1)$ *is a divisor of* $n^{[m/2]}$ *or* n^m *according as* l *is odd or even.*

(ii) $\sharp$(*the torsion part of* $\tilde{K}(D(2m+1,2l+1;n)/RP(2l+1))$) *is a divisor of* n^m.

Now, we consider the (unitary) representation ring $R(D_n)$ of D_n. It is well known that D_n (n: odd) has two representations 1 and χ_0 of degree 1 and $(n-1)/2$ representations χ_i $(1\leqq i\leqq(n-1)/2)$ of degree 2, which are given by

$$\chi_0(a)=1,\quad \chi_0(b)=-1;$$

$$\chi_i(a)=\begin{pmatrix}\exp(2\pi i\sqrt{-1}/n) & 0\\ 0 & \exp(-2\pi i\sqrt{-1}/n)\end{pmatrix},\quad \chi_i(b)=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \tag{2.6}$$

(cf. [5, p. 339]). We notice that the following is proved easily.

(2.7) *The inclusion* $D_n\subset O(2)\subset U(2)$ *is equivalent to* χ_1 *in* $R(D_n)$.

REMARK 2.8. It is easy to see that the multiplication is given by

$$\chi_0\chi_0=1,\quad \chi_0\chi_i=\chi_i,\quad \chi_i\chi_i=\chi_{2i}+\chi_0+1,\quad \chi_i\chi_j=\chi_{i+j}+\chi_{i-j}$$

for $i\geqq 1$, $j\geqq 1$ and $i\neq j$, where $\chi_{-i}=\chi_{n-i}=\chi_i$. Therefore, we see that the reduced representation ring $\tilde{R}(D_n)$ is generated by χ_0-1 and $\chi_1-\chi_0-1$.

Consider the inclusions

$$Z_n\xrightarrow{i}D_n\xleftarrow{k}Z_2$$

and the representations χ, χ' of Z_n, Z_2 defined by

$$\chi(a)=\exp(2\pi\sqrt{-1}/n),\quad \chi'(b)=-1,$$

respectively. Then, for the induced homomorphisms

$$R(Z_n)\xleftarrow{i^*}R(D_n)\xrightarrow{k^*}R(Z_2) \tag{2.9}$$

of the above inclusions, we have the following by definition.

LEMMA 2.10. $i^*\chi_0=1$, $i^*\chi_1=\chi+t\chi$, $k^*\chi_0=\chi'$, $k^*\chi_1=\chi'+1$, *where* t *is the conjugation.*

In general, a principal G-bundle

$$X\longrightarrow X/G$$

defines the natural ring homomorphisms

$$\xi: R(G) \longrightarrow K(X/G), \qquad \xi: \tilde{R}(G) \longrightarrow \tilde{K}(X/G)$$

($\tilde{R}(G)$ is the reduced representation ring) as follows (cf. [4, §4.5]): For a representation ω of G of degree n, $\xi(\omega)$ is the associated complex n-plane bundle of the principal $U(n)$-bundle induced from the given principal G-bundle $X \to X/G$ by the group homomorphism $\omega: G \to U(n)$.

Taking the principal D_n, Z_n, Z_n-bundles

$$S^{2m+1} \times S^l \longrightarrow D(m, l; n), \quad S^{2m+1} \times S^l \longrightarrow L^m(n) \times S^l, \quad S^{2m+1} \longrightarrow L^m(n),$$

we have the commutative diagram

$$\begin{array}{ccccc} \tilde{R}(Z_n) & = & \tilde{R}(Z_n) & \xleftarrow{i^*} & \tilde{R}(D_n) \\ \downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle \xi} \\ \tilde{K}(L^m(n)) & \xleftarrow{i^!} & \tilde{K}(L^m(n) \times S^l) & \xleftarrow{\pi^!} & \tilde{K}(D(m, l; n)) \end{array} \tag{2.11}$$

by the naturality of ξ, where i^* is the one in (2.9), $i: L^m(n) \subset L^m(n) \times S^l$ and π is the projection of (1.3). Therefore, we have the following commutative diagram, by taking also the principal Z_2-bundle $S^l \to RP(l)$:

$$\begin{array}{ccccc} \tilde{R}(Z_n) & \xleftarrow{i^*} & \tilde{R}(D_n) & \xrightarrow{k^*} & \tilde{R}(Z_2) \\ \downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle \xi} \\ \tilde{K}(L^m(n)) & \xleftarrow{i^!} & \tilde{K}(D(m, l; n)) & \xrightarrow{k^!} & \tilde{K}(RP(l)), \end{array} \tag{2.12}$$

where the upper homomorphisms are the reduced ones of (2.9), i is the inclusion of (1.2) and $k^!$ is the one in Proposition 2.4.

Now, we consider the elements

$$\alpha = \xi(\chi_1 - \chi_0 - 1) \in \tilde{K}(D(m, l; n)), \quad \alpha = j^!\alpha \in \tilde{K}(D_0(m, l; n)), \tag{2.13}$$

where χ_1 and χ_0 are the ones of (2.6) and $j^!$ is in (2.1). Let

$$\sigma \in \tilde{K}(L^m(n)) \tag{2.14}$$

be the stable class of the canonical complex line bundle over $L^m(n)$ whose first Chern class is the generator of $H^2(L^m(n)) = Z_n$.

LEMMA 2.15. *For the lower homomorphisms of* (2.12), *we have* $i^!\alpha = \sigma + t\sigma$ ($t: \tilde{K} \to \tilde{K}$ *is the conjugation*), $k^!\alpha = 0$.

PROOF. The desired results follow from the commutativity of (2.12), Lemma 2.10 and the equality $\xi(\chi) = \sigma + 1$ which is proved easily by definition (cf. [3, §2 and Appendix (3)]). q.e.d.

The equality $\xi(\chi')=v+1\in K(RP(l))$ holds by the same way as the one of the above proof, and so we see that

$$v=\xi(\chi_0-1)\in\tilde{K}(D(m,l;n)) \qquad \text{for the element } v \text{ of (2.3)},$$

considering the projections $p\colon D_n\to Z_2$ and p in (1.2). Therefore, (2.7) shows that

(2.16) $\alpha+v+2\in\tilde{K}(D(m,l;n))$ *is the associated complex 2-plane bundle of the principal $U(2)$-bundle induced from the principal D_n-bundle $S^{2m+1}\times S^l\to D(m,l;n)$ by the natural inclusion $D_n\subset O(2)\subset U(2)$.*

The following is an immediate consequence of the naturality of ξ.

LEMMA 2.17. *The elements α of* (2.13) *are natural with respect to the inclusions $D(m',l';n)\subset D_0(m,l';n)\subset D(m,l;n)$ for $m'<m$, $l'\leqq l$.*

Let

$$A_{m,l}\subset\tilde{K}(D(m,l;n)),\quad A_{m,l,0}\subset\tilde{K}(D_0(m,l;n)) \tag{2.18}$$

be the subrings generated by α of (2.13). Then

LEMMA 2.19. *$\sharp A_{m,l}$ and $\sharp A_{m,l,0}$ are divisors of $n^{[m/2]}$.*

PROOF. In the lower exact sequence in Proposition 2.4, we see that $k^!\alpha=0$ by Lemma 2.15, and that $\sharp A_{m,2l+1,0}$ is a divisor of $n^{[m/2]}$ by Lemma 2.5 (i). Therefore, since $A_{m,2l+1}$ is the image of $A_{m+1,2l+1,0}$ by the above lemma, $\sharp A_{m,2l+1}$ is a divisor of $n^{[(m+1)/2]}$, and so of $n^{[m/2]}$ by using Lemma 2.5 (ii) if m is odd. These and the naturality of the above lemma show the desired results for even l. q.e.d.

We consider the induced homomorphisms

$$K(L^m(n))\otimes\tilde{K}(S^{2l})=\tilde{K}((L^m(n)\times S^{2l})/(L^m(n)\times *))$$
$$\xrightarrow{h^!}\tilde{K}(D(m,2l;n)/D(m,2l-1;n))\xrightarrow{q^!}\tilde{K}(D(m,2l;n))$$

of the homeomorphism h of (1.9) and the projection q, where the first equality is obtained by the Künneth formula

$$\tilde{K}(L^m(n)\times S^{2l})=K(L^m(n))\otimes\tilde{K}(S^{2l})\oplus\tilde{K}(L^m(n))\otimes Z \quad \text{(cf. [2])}. \tag{2.20}$$

Consider the elements

$$\gamma=q^!h^!(\sigma\otimes g^l)\in\tilde{K}(D(m,2l;n)),\quad \gamma=j^!\gamma\in\tilde{K}(D_0(m,2l;n)), \tag{2.21}$$

where $\sigma\in\tilde{K}(L^m(n))$ is the one of (2.14), $g^l\in\tilde{K}(S^{2l})=Z$ is the canonical generator, and $j^!$ is the one in (2.1).

LEMMA 2.22. (i) *γ is the element of odd order and $\gamma^2=0$.*

(ii) *The elements γ are natural with respect to the inclusions $D(m', 2l; n) \subset D_0(m, 2l; n) \subset D(m, 2l; n)$ for $m'<m$.*

(iii) $$i^!\gamma = 0, \qquad k^!\gamma = 0,$$

for the lower homomorphisms of (2.12).

PROOF. (i) is easy to see since σ is of odd order (cf. [11, Prop. 2.6 (i)]) and $(g^l)^2=0$ $(l>0)$, and the others are seen easily. q.e.d.

Let

$$B_{m,2l} \subset \tilde{K}(D(m, 2l; n)), \qquad B_{m,2l,0} \subset \tilde{K}(D_0(m, 2l; n)) \tag{2.23}$$

be the subgroups generated by the elements $\gamma\alpha^{i-1}$ $(i \geqq 1)$, where α and γ are the ones of (2.13) and (2.21). To study these subgroups, we use the induced homomorphism

$$\pi^!: \tilde{K}(D(m, 2l; n)) \longrightarrow \tilde{K}(L^m(n) \times S^{2l}) \tag{2.24}$$

of the double covering π of (1.3), where the range is given by (2.20).

LEMMA 2.25. (i) *$\pi^!$ is monomorphic on $B_{m,2l}$.*

(ii) $$\pi^!(\gamma\alpha^{i-1}) = (\sigma - t\sigma)(\sigma + t\sigma)^{i-1} \otimes g^l.$$

PROOF. (i) The desired result follows immediately from Lemma 2.22 (i) and the fact that the order of $\operatorname{Ker}\pi^!$ is a power of 2 by [3, Prop. 2.11].

(ii) $$\pi^!\gamma = \pi^! q^! h^!(\sigma \otimes g^l) = q_1^! \rho^! \lambda^! \nabla^!(\sigma \otimes g^l)$$
$$= q_1^! \rho^!(\sigma \otimes g^l, t\sigma \otimes (-g^l)) = (\sigma - t\sigma) \otimes g^l,$$

by (2.21), Lemma 1.11 and the definition of ρ in (1.10). Also, we have $\pi^!\alpha = (\sigma + t\sigma) \otimes 1$ using the right square in (2.11) where the equality $\xi(\chi) = (\sigma+1) \otimes 1$ holds by the same way as the equality in the proof of Lemma 2.15. Therefore, we have the desired equality by the product formula in (2.20). q.e.d.

REMARK 2.26. The K-ring of the Dold manifold $D(m, l)$, stated in Remark 1.15, is studied in [6, 7], by considering the generators

$$v_1, \alpha \in \tilde{K}(D(m, l)), \qquad \gamma \in \tilde{K}(D(m, 2l)).$$

We notice that the equalities $p'^! v_1 = v$, $p'^! \alpha = \alpha$, $p'^! \gamma = \gamma$ are proved easily by definition, where p' is the projection in Remark 1.15.

§ 3. Proof of the main theorem

The following are known for the K-rings of the lens space $L^m(n)$ and its subcomplex $L_0^m(n)$ of (1.5) (cf. [11, Lemma 2.4 (i), Prop. 2.6, 2.11]).

(3.1) *The ring $\tilde{K}(L^m(n))$ is generated by σ of (2.14) with relations $(1+\sigma)^n=1$, $\sigma^{m+1}=0$, and contains exactly n^m elements. Also, $\tilde{K}(L^m(n))=\tilde{K}(L_0^m(n))$ by the isomorphism induced by the inclusion $j: L_0^m(n)\subset L^m(n)$.*

(3.2) *The complexification*

$$c: \widetilde{KO}(L_0^m(n))\longrightarrow \tilde{K}(L_0^m(n)) = \tilde{K}(L^m(n))$$

is monomorphic, and its image

$$C_m = c(\widetilde{KO}(L_0^m(n)) = c(\widetilde{KO}(L^m(n))$$

is the subring of $\tilde{K}(L^m(n))$ generated by $\sigma+t\sigma$, and contains exactly $n^{[m/2]}$ elements.

LEMMA 3.3. $$\tilde{K}(L^m(n)) = C_m\oplus D_m,$$

where D_m is the subgroup of $\tilde{K}(L^m(n))$ generated by the elements

$$(\sigma-t\sigma)(\sigma+t\sigma)^{i-1} \qquad (i\geqq 1).$$

PROOF. Consider the real restriction $r: \tilde{K}(L_0^m(n))\to\widetilde{KO}(L_0^m(n))$. Since

$$cr((\sigma-t\sigma)(\sigma+t\sigma)^{i-1}) = (1+t)((\sigma-t\sigma)(\sigma+t\sigma)^{i-1}) = 0$$

and c is monomorphic, we see that $r(D_m)=0$. This shows that $2a=rca=0$ if $ca\in C_m\cap D_m$, and so we have $C_m\cap D_m=0$. Since $1+t\sigma=(1+\sigma)^{-1}$ by definition, we have $t\sigma=-\sigma(1+\sigma)^{-1}$ and so $(\sigma-t\sigma)^2=(\sigma+t\sigma)^2+4(\sigma+t\sigma)$. This shows that $C_m\oplus D_m$ is a subring of $\tilde{K}(L^m(n))$, and we have the desired result by (3.1) since $\sigma=((\sigma+t\sigma)+(\sigma-t\sigma))/2\in C_m\oplus D_m$. q.e.d.

For the canonical generator $g^l\in\tilde{K}(S^{2l})=Z$, we denote by

(3.4) $$D_m\otimes g^l\subset \tilde{K}(L^m(n)\times S^{2l})$$

the image of D_m in the above lemma by the isomorphism

$$\otimes g^l: K(L^m(n))\approx K(L^m(n))\otimes\tilde{K}(S^{2l}) \ (\subset\tilde{K}(L^m(n)\times S^{2l})).$$

To prove our main theorem, we study more precisely $A_{m,l}$, $A_{m,l,0}$ of (2.18) and $B_{m,2l}$, $B_{m,2l,0}$ of (2.23), using the above facts and the commutative diagram

$$\begin{array}{ccccc} \tilde{K}(L^m(n)) & \xleftarrow{i^!} & \tilde{K}(D(m,l;n)) & \xrightarrow{\pi^!} & \tilde{K}(L^m(n)\times S^l) \\ =\downarrow{\scriptstyle j^!} & & \downarrow{\scriptstyle j^!} & & =\downarrow{\scriptstyle (j\times \mathrm{id})^!} \\ \tilde{K}(L_0^m(n)) & \xleftarrow{i_0{}^!} & \tilde{K}(D_0(m,l;n)) & \xrightarrow{\pi_0{}^!} & \tilde{K}(L_0^m(n)\times S^l), \end{array} \tag{3.5}$$

where i and π are the maps of (1.2) and (1.3), i_0 and π_0 are their restrictions, j's are the inclusions, and the right square is used for even l.

PROPOSITION 3.6. (i) *The subrings $A_{m,l}$ and $A_{m,l,0}$ of* (2.18), *generated by α, are mapped isomorphically by the above $i^!$ and $i_0^!$ onto the subring C_m of* (3.2), *generated by $\sigma+t\sigma$, where $i^!\alpha = i_0^!\alpha = \sigma+t\sigma$.*

(ii) *$A_{m,l}$ and $A_{m,l,0}$ are isomorphic by $j^!$ in* (3.5).

(iii) $$\tilde{K}(D_0(m,2l+1;n)) = A_{m,2l+1,0}\oplus Z_{2^l},$$

where Z_{2^l} is the subring generated by v given in Proposition 2.4.

PROOF. (i) and (ii) Since the equalities of (i) hold by Lemma 2.15, we have the epimorphisms $A_{m,l}\xrightarrow{j^!}A_{m,l,0}\xrightarrow{i_0{}^!}C_m$. Therefore, we have the desired results by (3.2) and Lemma 2.19.

(iii) The result follows from (i), Proposition 2.4 and Lemma 2.5 (i). q.e.d.

PROPOSITION 3.7. (i) *$B_{m,2l}$ and $B_{m,2l,0}$ of* (2.23), *generated by $\{\gamma\alpha^{i-1}|i\geqq 1\}$, are mapped isomorphically onto the subgroup $D_m\otimes g^l$ of* (3.4) *by $\pi^!$ and $\pi_0^!$ in* (3.5), *where $\pi^!(\gamma\alpha^{i-1}) = \pi_0^!(\gamma\alpha^{i-1}) = (\sigma-t\sigma)(\sigma+t\sigma)^{i-1}\otimes g^l$.*

(ii) *They are isomorphic by $j^!$ in* (3.5).

(iii) $$A_{m,2l}\cap B_{m,2l} = 0, \qquad A_{m,2l,0}\cap B_{m,2l,0} = 0,$$

where $A_{m,2l}$ and $A_{m,2l,0}$ are the ones of the above proposition.

(iv) $$\tilde{K}(D_0(m,2l;n)) = A_{m,2l,0}\oplus B_{m,2l,0}\oplus Z_{2^l},$$

where Z_{2^l} is the subring generated by v given in Proposition 2.4.

PROOF. (i) and (ii) follow from Lemma 2.25, the definition of D_m in Lemma 3.3 and the right commutative square of (3.5).

(iii) follows from (i) of the above proposition and $i^!\gamma=0$ of Lemma 2.22 (iii).

(iv) By (i), (iii), Proposition 3.6 (i) and Lemma 3.3, we have

$$\tilde{K}(D_0(m,2l;n))\supset A_{m,2l,0}\oplus B_{m,2l,0}\cong C_m\oplus D_m = \tilde{K}(L^m(n)).$$

Since $k^!\alpha = k^!\gamma = 0$ by Lemmas 2.15 and 2.22 (iii), we have the desired result by Proposition 2.4, Lemma 2.5 (i) and (3.1). q.e.d.

The following for the stunted real projective space is known [1, Th. 7.3].

$$\tilde{K}(RP(m+l+1)/RP(m)) = \begin{cases} Z_{2^{[(l+1)/2]}} & \text{if } m \text{ is even,} \\ Z\oplus Z_{2^{[l/2]}} & \text{if } m \text{ is odd,} \end{cases} \tag{3.8}$$

$$\tilde{K}^1(RP(m+l+1)/RP(m)) = Z \quad \text{if } m+l \text{ is even,} \quad = 0 \quad \text{if } m+l \text{ is odd.}$$

(The results for $\tilde{K}^1$ are seen by the Atiyah-Hirzeburch spectral sequence.)

Now, we are ready to prove our main theorem.

THEOREM 3.9. *Suppose that n is odd. Then the reduced K-ring of $D(m, l; n)$ $(m>0, l>0)$ of* (1.1) *is given by the direct sum decomposition*

$$\tilde{K}(D(m, l; n)) = A_{m,l}\oplus B_{m,l}\oplus Z_{2^{[l/2]}}\oplus \begin{cases} Z_{2^{[(l+1)/2]}} & \text{if } m \text{ is even,} \\ Z & \text{if } m \text{ and } l \text{ are odd,} \\ 0 & \text{otherwise,} \end{cases}$$

where $A_{m,l}\oplus B_{m,l}$ is the odd component and the summands are given as follows:

(i) *$A_{m,l}$ is the subring generated by the element α of* (2.13), *and is given in Proposition* 3.6 (i).

(ii) *$B_{m,2l+1}=0$, and $B_{m,2l}$ is the ideal generated by the element γ of* (2.21) *which satisfies $\gamma^2=0$, and is given in Proposition* 3.7 (i). *Also the subgroup $A_{m,2l}\oplus B_{m,2l}$ is isomorphic to $\tilde{K}(L^m(n))$.*

(iii) *The third summand $Z_{2^{[l/2]}}$ is the subring generated by v of* (2.3) *and is given in Proposition* 2.4.

(iv) *The rest is the monomorphic image of*

$$\tilde{K}(S^m\wedge(RP(m+l+1)/RP(m)))$$

by $q_0^! f^!$, where $D(m, l; n) \xrightarrow{q_0} D(m, l; n)/D_0(m, l; n) \xrightarrow[\approx]{f} S^m\wedge(RP(m+l+1)/RP(m))$ are the projection and the homeomorphism of Lemma 1.12. *Its generator v_m satisfies $v_m^2=0$, $v_{2m}v=-2v_{2m}$ and $v_{2m+1}v=0$.*

PROOF. Consider the exact sequence of $(D, D_0, RP(l))$:

$$\tilde{K}^1(D_0/RP(l))\longrightarrow\tilde{K}(D/D_0)\xrightarrow{q_0^!}\tilde{K}(D/RP(l))\xrightarrow{j^!}\tilde{K}(D_0/RP(l))\longrightarrow\tilde{K}^1(D/D_0),$$

where $D=D(m, l; n)$, $D_0=D_0(m, l; n)$. By (3.8) and the isomorphism $f^!$, we see that $\tilde{K}(D/D_0)$ is given by the last summand of the theorem and that the order of the torsion of $K^1(D/D_0)$ is a power of 2. By these facts and Lemma 2.5 (i), we have the exact sequence

$$0\longrightarrow\tilde{K}(D/D_0)\xrightarrow{q_0^!}\tilde{K}(D/RP(l))\xrightarrow{j^!}\tilde{K}(D_0/RP(l))\longrightarrow 0.$$

Therefore, we have the desired direct sum decomposition by Propositions 3.6,

3.7 and 2.4, where $B_{m,2l}$ is the one of (2.23) and $A_{m,2l}\oplus B_{m,2l}\cong\tilde{K}(L^m(n))$ is seen by the proof of Proposition 3.7 (iv). We see that $B_{m,2l}$ is equal to the ideal generated by γ, since $A_{m,2l}\oplus B_{m,2l}$ is the odd component.

$\gamma^2=0$ is seen in Lemma 2.22 (i). Also, $v_m^2=0$ and $vv_{2m+1}=0$ are clear. $v_{2m}v=-2v_{2m}$ is seen as follows: By definition and [1, Th. 7.3],

$$v_{2m}=q_0^!f^!(g^m\otimes v^{(2m+1)})$$

where $v^{(2m+1)}\in\tilde{K}(RP(2m+l+1)/RP(2m))$ is mapped to $v^{2m+1}\in\tilde{K}(RP(2m+l+1))$. Using the induced diagram for the $\tilde{K}$-rings of (1.13), which is commutative by Lemma 1.14, we have

$$\begin{aligned}v_{2m}v&=d^!(v_{2m}\otimes v)=d^!(fq_0\times i_0p)^!(g^m\otimes v^{2m+1}\otimes v)\\&=(fq_0)^!(g^m\otimes d^!(v^{2m+1}\otimes v))=-2v_{2m},\end{aligned}$$

as desired, since $p^!v=v$, $i_0^!v=v$ and $v^{2m+2}=-2v^{2m+1}$ by [1, Th 7.3]. q.e.d.

Finally, we are concerned with the special case that n is an odd prime.

Let $p=2q+1$ be an odd prime. The following is proved in [10, Th. 1, 2]:

(3.10) $\tilde{K}(L^m(p))=(Z_{p^{s+1}})^r\oplus(Z_{p^s})^{p-r-1}\qquad(m=s(p-1)+r,\ 0\leqq r<p-1)$

and the summands are generated by $\sigma, \sigma^2, \ldots, \sigma^{p-1}$, *respectively*;

(3.11) $C_m=c(\widetilde{KO}(L^m(p)))=(Z_{p^{s+1}})^{[r/2]}\oplus(Z_{p^s})^{q-[r/2]}$

and the summands are generated by $\sigma+t\sigma, (\sigma+t\sigma)^2, \ldots, (\sigma+t\sigma)^q$, *respectively*;

(3.12)
$$\sigma^{m+1}=(1+\sigma)^p-1=0,\quad(\sigma+t\sigma)^{[m/2]+1}=0,$$
$$(\sigma+t\sigma)^{q+1}=\textstyle\sum_{i=1}^q a_i(\sigma+t\sigma)^i,\quad a_i=\dfrac{-p}{2i-1}\binom{q+i-1}{2i-2}.$$

LEMMA 3.13. *The direct sum decomposition of* (3.10) *can be so taken that the summands are generated by*

$$\sigma-t\sigma,\quad\sigma+t\sigma,\quad(\sigma-t\sigma)(\sigma+t\sigma),\quad(\sigma+t\sigma)^2,\ldots,(\sigma-t\sigma)(\sigma+t\sigma)^{q-1},\quad(\sigma+t\sigma)^q,$$

respectively. Also, we have

$$(\sigma-t\sigma)(\sigma+t\sigma)^{[m/2]}=0,\quad(\sigma-t\sigma)(\sigma+t\sigma)^q=\textstyle\sum_{i=1}^q a_i(\sigma-t\sigma)(\sigma+t\sigma)^{i-1}.$$

PROOF. Since $\sigma^i(1+\sigma)^k=\sigma^i(1+\sigma)^{k-1}+\sigma^{i+1}(1+\sigma)^{k-1}$ and $\sigma^p(1+\sigma)^{k-1}=-\sum_{i=1}^{p-1}\binom{p}{i}\sigma^i(1+\sigma)^{k-1}$ by $(1+\sigma)^p-1=0$, we can take the summands of (3.10) so that they are generated by the elements

$$\sigma^i(1+\sigma)^k \qquad (1\leqq i\leqq p-1),$$

respectively, by the induction on k. Hence we can also take the summands of (3.10) generated by the elements

$$\sigma(1+\sigma)^{p-1},\quad \sigma^2(1+\sigma)^{p-1},\ldots,\quad \sigma^{2q-1}(1+\sigma)^{p-q},\quad \sigma^{2q}(1+\sigma)^{p-q},$$

respectively, obtained from the above generators for $k=q+1=p-q$ by adding repeatedly the neighboring generators. On the other hand,

$$(\sigma-t\sigma)(\sigma+t\sigma)^{i-1}=(2\sigma^{2i-1}+\sigma^{2i})(1+\sigma)^{p-i},\quad (\sigma+t\sigma)^i=\sigma^{2i}(1+\sigma)^{p-i},$$

since $1+t\sigma=(1+\sigma)^{-1}$ by definition and $(1+\sigma)^p=1$ by (3.1). Therefore, we have the first desired result from the last set of generators. The equalities in the lemma are proved by the same way as the proof of the last two equalities of (3.12) in [10, pp. 143–144]. q.e.d.

By these results and Theorem 3.9, we have immediately the following

COROLLARY 3.14. *Let $p=2q+1$ be an odd prime, and set $m=s(p-1)+r$, $0\leqq r<p-1$. Then $\tilde{K}(D(m,l;p))$ $(m>0,\ l>0)$ is given by Theorem 3.9 for $n=p$, where the summands $A_{m,l}$ and $B_{m,l}$ are given more precisely as follows:*

(i) $$A_{m,l}=(Z_{p^{s+1}})^{[r/2]}\oplus(Z_{p^s})^{q-[r/2]}$$

($(Z_k)^t$ means the direct sum of t copies of Z_k) and the summands are generated by the elements $\alpha,\ \alpha^2,\ldots,\ \alpha^q$, respectively.

(ii) $$A_{m,2l}\oplus B_{m,2l}=(Z_{p^{s+1}})^r\oplus(Z_{p^s})^{p-r-1}$$

and the summands are generated by $\gamma,\ \alpha,\ \gamma\alpha,\ \alpha^2,\ldots,\ \gamma\alpha^{q-1},\ \alpha^q$, respectively.

Furthermore, $\alpha^{[m/2]+1}=\gamma\alpha^{[m/2]}=0$ and

$$\alpha^{q+1}=\textstyle\sum_{i=1}^{q}a_i\alpha^i,\quad \gamma\alpha^q=\sum_{i=1}^{q}a_i\gamma\alpha^{i-1},\quad a_i=\dfrac{-p}{2i-1}\binom{q+i-1}{2i-2}.$$

We notice that a similar result for $n=p^2$ is obtained by using the known result for $\tilde{K}(L^m(p^2))$ [11, Th. 1.4, 1.7].

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Department of Mathematics,
Faculty of Science,
*Hiroshima University**)

*) The present address of the first named author is as follows: Department of Mathematics, College of Education, Wakayama University.