New Iterative Algorithm for Algebraic Riccati Equation Related to H_∞ Control Problem of Singularly Perturbed Systems

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***Abstract*—In this note, we present the solution to the algebraic Riccati equation (ARE) with indefinite sign quadratic term related to the H_∞ control problem for singularly perturbed system by means of a Kleinman's type algorithm. The resulting algorithm is very efficient from the numerical point of view because the ARE is solvable even if the quadratic term has an indefinite sign. Moreover, the resulting iterative algorithm is quadratically convergent. We also present a new algorithm for solving the generalized algebraic Lyapunov equation (GALE) on the basis of the fixed point algorithm.**

***Index Terms*—Fixed point algorithm, H_∞control, Kleinman algorithm, singularly perturbed systems.**

I. Introduction

H_∞ control problems for linear singularly perturbed systems were considered in many papers [1]–[9]. In particular, a great deal of studies on the composite controller design for singularly perturbed systems in H_∞ sense have been made [2], [3], [6], [8].

In order to obtain the optimal controller, we must solve the algebraic Riccati equation (ARE). Various reliable approaches to the theory of the ARE have been well documented in many literatures (see e.g., [11]–[14]). These methods consist of the invariant subspace approach which is based on the Hamiltonian matrix [11], [12] and the general matrix pencil technique which is based on the extended Hamiltonian pencil [13], [14] (in particular, the reference [14] is the most complete reference to date dealing with ARE by means of the matrix pencils). However, such approaches are not adequate to the singularly perturbed systems because of high dimension and numerical stiffness [10].

The recursive algorithm for the solution of ARE of singularly perturbed systems have been developed in many literatures (see, e.g., [15]). From a practical point of view, it has been shown that the recursive algorithm is very effective to solve the ARE when the system matrices are functions of a small perturbation parameter ε. However, the recursive algorithm converge only to the approximation solution. Moreover, such an algorithm is the linear convergence. On the other hand, the exact slow–fast decomposition method for solving the singularly perturbed systems has been proposed (see, for example, [7] and the references therein). However, in order to obtain the exact solution, ones need the same workspace compared with with the full-order ARE for calculating the inverse matrix.

In this paper, we study the numerical solution to the ARE with indefinite sign quadratic term related to the H_∞ control problem of singularly perturbed systems. The objective of this paper is to extend the convergence result of [17] to the ARE with indefinite sign quadratic term. Our new idea is to set the initial condition to the solutions of the reduced-order ARE. Because of such a choice, we can prove that our iterative algorithm converges to a unique solution of the ARE

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with indefinite sign quadratic term by using Newton–Kantorovich theorem [16]. Also, while the classical recursive algorithm is of the linear convergence property, the new iterative algorithm achieves the quadratic convergence property since the resulting algorithm is based on the Kleinman algorithm [17]. Furthermore, we also present a new algorithm for solving the generalized algebraic Lyapunov equation (GALE). The proof of the algorithm is based on the fixed point theorem. As another important feature, it is easy to construct an $O(\varepsilon^{2^i})$ high-order approximation controller compared with the existing methods (e.g., [2], [3], [7], [20]).

II. Problem Statement and Preliminaries

The AREs of singularly perturbed system corresponding to H_∞ control problem [4], [6] have the following form:

$$A_\varepsilon^{\mathrm{T}} P_\varepsilon + P_\varepsilon A_\varepsilon + \gamma^{-2} P_\varepsilon G_\varepsilon G_\varepsilon^{\mathrm{T}} P_\varepsilon - \left(P_\varepsilon B_\varepsilon + C_1^{\mathrm{T}} D_{12}\right) \cdot \left(D_{12}^{\mathrm{T}} D_{12}\right)^{-1} \left(B_\varepsilon^{\mathrm{T}} P_\varepsilon + D_{12}^{\mathrm{T}} C_1\right) + C_1^{\mathrm{T}} C_1 = 0 \tag{1}$$

$$A_\varepsilon W_\varepsilon + W_\varepsilon A_\varepsilon^{\mathrm{T}} + \gamma^{-2} W_\varepsilon C_1^{\mathrm{T}} C_1 W_\varepsilon - \left(W_\varepsilon C_2^{\mathrm{T}} + G_\varepsilon D_{21}^{\mathrm{T}}\right) \cdot \left(D_{21} D_{21}^{\mathrm{T}}\right)^{-1} \left(C_2 W_\varepsilon + G_{21} B_\varepsilon^{\mathrm{T}}\right) + G_\varepsilon G_\varepsilon^{\mathrm{T}} = 0 \tag{2}$$

where ε is a small positive parameter. Let us introduce the following matrices:

$$\begin{aligned}
P_\varepsilon &= P_\varepsilon(\gamma) = \begin{bmatrix} P_{11}(\varepsilon, \gamma) & \varepsilon P_{21}(\varepsilon, \gamma)^{\mathrm{T}} \\ \varepsilon P_{21}(\varepsilon, \gamma) & \varepsilon P_{22}(\varepsilon, \gamma) \end{bmatrix} \\
W_\varepsilon &= W_\varepsilon(\gamma) = \begin{bmatrix} W_{11}(\varepsilon, \gamma) & W_{12}(\varepsilon, \gamma) \\ W_{12}(\varepsilon, \gamma)^{\mathrm{T}} & \varepsilon^{-1} W_{22}(\varepsilon, \gamma) \end{bmatrix} \\
A_\varepsilon &= \begin{bmatrix} A_{11} & A_{12} \\ \varepsilon^{-1} A_{21} & \varepsilon^{-1} A_{22} \end{bmatrix} \qquad A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \\
B_\varepsilon &= \begin{bmatrix} B_1 \\ \varepsilon^{-1} B_2 \end{bmatrix} \qquad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \\
G_\varepsilon &= \begin{bmatrix} G_1 \\ \varepsilon^{-1} G_2 \end{bmatrix} \qquad G = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} \\
C_1 &= [\, C_{11} \quad C_{12} \,] \qquad C_2 = [\, C_{21} \quad C_{22} \,].
\end{aligned}$$

In addition, dimensions of block matrices are as follows:

$$\begin{aligned}
P_{11} &= P_{11}^{\mathrm{T}} \qquad W_{11} = W_{11}^{\mathrm{T}} \qquad A_{11} \in \mathbf{R}^{n_1 \times n_1} \\
P_{22} &= P_{22}^{\mathrm{T}} \qquad W_{22} = W_{22}^{\mathrm{T}} \\
A_{22} &\in \mathbf{R}^{n_2 \times n_2} \qquad n_1 + n_2 = n \\
B_1 &\in \mathbf{R}^{n_1 \times p} \qquad B_2 \in \mathbf{R}^{n_2 \times p} \\
G_1 &\in \mathbf{R}^{n_1 \times q} \qquad G_2 \in \mathbf{R}^{n_2 \times q} \\
C_{11} &\in \mathbf{R}^{r \times n_1} \qquad C_{12} \in \mathbf{R}^{r \times n_2} \\
C_{21} &\in \mathbf{R}^{s \times n_1} \qquad C_{22} \in \mathbf{R}^{s \times n_2}.
\end{aligned}$$

The remaining matrices are constant matrices of appropriate demensions. For technical simplification, we shall make the following basic assumptions.

Assumption 1:

1) $D_{12}^{\mathrm{T}} D_{12}$ and $D_{21} D_{21}^{\mathrm{T}}$ are nonsingular.
2)

$$\begin{aligned}
\operatorname{rank} \begin{bmatrix} A - sI_n & B \\ C_1 & D_{12} \end{bmatrix} &= n + p, \forall s \in \mathrm{C}^+ \\
\operatorname{rank} \begin{bmatrix} A - sI_n & G \\ C_2 & D_{21} \end{bmatrix} &= n + q, \forall s \in \mathrm{C}^+.
\end{aligned}$$

Assumption 2:

1) The pair $(A_{22},\ B_2)$ is stabilizable and $(C_{12},\ A_{22})$ is observable.
2)

$$\begin{aligned}
\operatorname{rank} \begin{bmatrix} sI_{n_1} - A_{11} & -A_{12} & B_1 \\ -A_{21} & -A_{22} & B_2 \end{bmatrix} &= n,\ \forall s \in \mathrm{C}^+ \\
\operatorname{rank} \begin{bmatrix} sI_{n_1} - A_{11}^{\mathrm{T}} & -A_{21}^{\mathrm{T}} & C_{11}^{\mathrm{T}} \\ -A_{12}^{\mathrm{T}} & -A_{22}^{\mathrm{T}} & C_{12}^{\mathrm{T}} \end{bmatrix} &= n,\ \forall s \in \mathrm{C}^+.
\end{aligned}$$

Assumption 3:

1) The pair $(\,A_{22}^{\mathrm{T}},\ C_{22}^{\mathrm{T}}\,)$ is stabilizable and $(\,G_2^{\mathrm{T}},\ A_{22}^{\mathrm{T}}\,)$ is observable.
2)

$$\begin{aligned}
\operatorname{rank} \begin{bmatrix} sI_{n_1} - A_{11}^{\mathrm{T}} & -A_{21}^{\mathrm{T}} & C_{21}^{\mathrm{T}} \\ -A_{12}^{\mathrm{T}} & -A_{22}^{\mathrm{T}} & C_{22}^{\mathrm{T}} \end{bmatrix} &= n,\ \forall s \in \mathrm{C}^+ \\
\operatorname{rank} \begin{bmatrix} sI_{n_1} - A_{11} & -A_{12} & G_1 \\ -A_{21} & -A_{22} & G_2 \end{bmatrix} &= n,\ \forall s \in \mathrm{C}^+.
\end{aligned}$$

It is well known in [3], [4] that a controller which stabilizes the singularly perturbed system with disturbance attenuation level measured by γ does exist if and only if (1) and (2) admit the positive–semidefinite solutions $P_\varepsilon(\gamma)$ and $W_\varepsilon(\gamma)$, respectively, such that

a) $A_\varepsilon + G_\varepsilon F_1 + B_\varepsilon F_2$ with $F_1 = F_1(\varepsilon) = \gamma^{-2} G_\varepsilon^{\mathrm{T}} P_\varepsilon$, $F_2 = F_2(\varepsilon) = (D_{12}^{\mathrm{T}} D_{12})^{-1} (B_\varepsilon P_\varepsilon + D_{12}^{\mathrm{T}} C_1)$ is stable;
b) $A_\varepsilon + H_{1\varepsilon} C_1 + H_{2\varepsilon} C_2$ with $H_{1\varepsilon} = \gamma^{-2} W_\varepsilon C_1^{\mathrm{T}}$, $H_{2\varepsilon} = \left(W_\varepsilon C_2^{\mathrm{T}} + G_\varepsilon D_{21}^{\mathrm{T}}\right) \left(D_{21} D_{21}^{\mathrm{T}}\right)^{-1}$ is stable;
c) $\rho(P_\varepsilon W_\varepsilon) < \gamma^2$, where $\rho(\cdot)$ is the spectral radius.

In order to solve the AREs (1) and (2) with indefinite sign quadratic term, we introduce the following useful lemma for the generalized algebraic Riccati equation (GARE) [20].

Lemma 1: The AREs (1) and (2) are equivalent to the following GAREs (3) and (4), respectively

$$\begin{aligned}
\mathcal{F}_1(P) &= A^{\mathrm{T}} P + P^{\mathrm{T}} A + \gamma^{-2} P^{\mathrm{T}} G G^{\mathrm{T}} P - \left(P^{\mathrm{T}} B + C_1^{\mathrm{T}} D_{12}\right) \tilde{D}_1 \left(B^{\mathrm{T}} P + D_{12}^{\mathrm{T}} C_1\right) + C_1^{\mathrm{T}} C_1 = 0 && \text{(3a)} \\
P_\varepsilon &= \Pi_\varepsilon P = P^{\mathrm{T}} \Pi_\varepsilon && \text{(3b)} \\
\mathcal{F}_2(W) &= A W^{\mathrm{T}} + W A^{\mathrm{T}} + \gamma^{-2} W C_1^{\mathrm{T}} C_1 W^{\mathrm{T}} - \left(W C_2^{\mathrm{T}} + G D_{21}^{\mathrm{T}}\right) \tilde{D}_2 \left(C_2 W^{\mathrm{T}} + D_{21} G^{\mathrm{T}}\right) + G G^{\mathrm{T}} = 0 && \text{(4a)} \\
W_\varepsilon &= \Pi_\varepsilon^{-1} W = W^{\mathrm{T}} \Pi_\varepsilon^{-1} && \text{(4b)}
\end{aligned}$$

where

$$\begin{aligned}
\Pi_\varepsilon &= \operatorname{diag}(\, I_{n_1} \quad \varepsilon I_{n_2} \,) \quad P = \begin{bmatrix} P_{11} & \varepsilon P_{21}^{\mathrm{T}} \\ P_{21} & P_{22} \end{bmatrix} \\
W &= \begin{bmatrix} W_{11} & W_{12} \\ \varepsilon W_{12}^{\mathrm{T}} & W_{22} \end{bmatrix} \\
\tilde{D}_1 &= \left(D_{12}^{\mathrm{T}} D_{12}\right)^{-1} \qquad \tilde{D}_2 = \left(D_{21} D_{21}^{\mathrm{T}}\right)^{-1} \\
P_{11} &= P_{11}^{\mathrm{T}} \qquad P_{22} = P_{22}^{\mathrm{T}} \quad W_{11} = W_{11}^{\mathrm{T}} \quad W_{22} = W_{22}^{\mathrm{T}} \\
A &= \Pi_\varepsilon A_\varepsilon \qquad B = \Pi_\varepsilon B_\varepsilon \quad G = \Pi_\varepsilon G_\varepsilon.
\end{aligned}$$

Partitioning for the ARE (3a) and letting $\varepsilon = 0$, we obtain the following equations:

$$\bar{A}_{11}^{\mathrm{T}} \bar{P}_{11} + \bar{P}_{11} \bar{A}_{11} + \bar{A}_{21}^{\mathrm{T}} \bar{P}_{21} + \bar{P}_{21}^{\mathrm{T}} \bar{A}_{21} - \bar{P}_{11} S_{11}^{\gamma} \bar{P}_{11} - \bar{P}_{21}^{\mathrm{T}} S_{22}^{\gamma} \bar{P}_{21} - \bar{P}_{11} S_{12}^{\gamma} \bar{P}_{21} - \bar{P}_{21}^{\mathrm{T}} S_{12}^{\gamma\mathrm{T}} \bar{P}_{11} + Q_{11} = 0 \tag{5a}$$

$$\bar{P}_{22}^{\mathrm{T}} \bar{A}_{21} + \bar{A}_{12}^{\mathrm{T}} \bar{P}_{11} + \bar{A}_{22}^{\mathrm{T}} \bar{P}_{21} - \bar{P}_{22} S_{12}^{\gamma\mathrm{T}} \bar{P}_{11} - \bar{P}_{22} S_{22}^{\gamma} \bar{P}_{21} + Q_{12}^{\mathrm{T}} = 0 \tag{5b}$$

$$\bar{A}_{22}^{\mathrm{T}} \bar{P}_{22} + \bar{P}_{22} \bar{A}_{22} - \bar{P}_{22} S_{22}^{\gamma} \bar{P}_{22} + Q_{22} = 0 \tag{5c}$$

where

$$\begin{aligned}\bar{A}^{\gamma} &= A - B\tilde{D}_1 D_{12}^{\mathrm{T}} C_1 = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} \\ S^{\gamma} &= B\tilde{D}_1 B^{\mathrm{T}} - \gamma^{-2} G G^{\mathrm{T}} = \begin{bmatrix} S_{11}^{\gamma} & S_{12}^{\gamma} \\ S_{12}^{\gamma\mathrm{T}} & S_{22}^{\gamma} \end{bmatrix} \\ Q &= C_1^{\mathrm{T}} \left(I_r - D_{12}\tilde{D}_1 D_{12}^{\mathrm{T}} \right) C_1 = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^{\mathrm{T}} & Q_{22} \end{bmatrix}\end{aligned}$$

and $\bar{P}_{11}$, $\bar{P}_{21}$ and $\bar{P}_{22}$ are 0-order solutions of the ARE (3a). The ARE (5c) will produce the unique positive–semidefinite stabilizing solution under assumption 2 if γ is large enough.

Let $\gamma_{1f} = \inf\{\gamma > 0|$ the ARE (5c) has a positive–semidefinite stabilizing solution$\}$.

Then, the matrix $\bar{A}_{22} - S_{22}^{\gamma}\bar{P}_{22}$ is nonsingular if we choose $\gamma > \gamma_{1f}$. Therefore, we obtain the following 0-order equations:

$$A_p^{\mathrm{T}\gamma}\bar{P}_{11} + \bar{P}_{11}A_p^{\gamma} - \bar{P}_{11}S_p^{\gamma}\bar{P}_{11} + Q_p^{\gamma} = 0 \tag{6a}$$

$$\bar{P}_{21} = -N_2^{\mathrm{T}} + N_1^{\mathrm{T}}\bar{P}_{11} \tag{6b}$$

$$\bar{A}_{22}^{\mathrm{T}}\bar{P}_{22} + \bar{P}_{22}\bar{A}_{22} - \bar{P}_{22}S_{22}^{\gamma}\bar{P}_{22} + Q_{22} = 0 \tag{6c}$$

where

$$\begin{aligned}T_0 &= T_1 - T_2 T_4^{-1} T_3 = \begin{bmatrix} A_p^{\gamma} & -S_p^{\gamma} \\ -Q_p^{\gamma} & -A_p^{\gamma\mathrm{T}} \end{bmatrix} \\ T_1 &= \begin{bmatrix} \bar{A}_{11} & -S_{11}^{\gamma} \\ -Q_{11} & -\bar{A}_{11}^{\mathrm{T}} \end{bmatrix} \\ T_2 &= \begin{bmatrix} \bar{A}_{12} & -S_{12}^{\gamma} \\ -Q_{12} & -\bar{A}_{21}^{\mathrm{T}} \end{bmatrix} \\ T_3 &= \begin{bmatrix} \bar{A}_{21} & -S_{12}^{\gamma\mathrm{T}} \\ -Q_{12}^{\mathrm{T}} & -\bar{A}_{12}^{\mathrm{T}} \end{bmatrix} \\ T_4 &= \begin{bmatrix} \bar{A}_{22} & -S_{22}^{\gamma} \\ -Q_{22} & -\bar{A}_{22}^{\mathrm{T}} \end{bmatrix} \\ N_2^{\mathrm{T}} &= D_4^{-\mathrm{T}}\hat{Q}_{12}^{\mathrm{T}} \qquad N_1^{\mathrm{T}} = -D_4^{-\mathrm{T}}D_2^{\mathrm{T}} \\ D_1 &= \bar{A}_{11} - S_{11}^{\gamma}\bar{P}_{11} - S_{12}^{\gamma}\bar{P}_{21} \\ D_3 &= \bar{A}_{21} - S_{12}^{\gamma\mathrm{T}}\bar{P}_{11} - S_{22}^{\gamma}\bar{P}_{21} \\ D_2 &= \bar{A}_{12} - S_{12}^{\gamma}\bar{P}_{22} \qquad D_4 = \bar{A}_{22} - S_{22}^{\gamma}\bar{P}_{22} \\ D_0 &= D_1 - D_2 D_4^{-1} D_3 \qquad \hat{Q}_{12} = Q_{12} + \bar{A}_{21}^{\mathrm{T}}\bar{P}_{22}.\end{aligned}$$

Remark 1: The matrices A_p^{γ}, S_p^{γ} and Q_p^{γ} do not depend on $\bar{P}_{22}$ because their matrices can be computed by using T_m, $m = 1, \ldots, 4$ which is independent of $\bar{P}_{22}$ [8], [9], [18].

Let us define $\gamma_{1s} = \inf\{\gamma > 0|$ the ARE (6a) has a positive semidefinite stabilizing solution $\}$.

By following the similar steps, we obtain the following equations:

$$A_w^{\gamma}\bar{W}_{11} + \bar{W}_{11}A_w^{\gamma\mathrm{T}} - \bar{W}_{11}R_w^{\gamma}\bar{W}_{11} + M_w^{\gamma} = 0 \tag{7a}$$

$$\bar{W}_{12} = -L_2 + \bar{W}_{11}L_1 \tag{7b}$$

$$\hat{A}_{22}\bar{W}_{22} + \bar{W}_{22}\hat{A}_{22}^{\mathrm{T}} - \bar{W}_{22}R_{22}^{\gamma}\bar{W}_{22} + M_{22} = 0 \tag{7c}$$

where

$$\begin{aligned}\hat{A}^{\gamma} &= A - GD_{21}^{\mathrm{T}}\tilde{D}_2 C_2 = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix} \\ R^{\gamma} &= C_2^{\mathrm{T}}\tilde{D}_2 C_2 - \gamma^{-2}C_1^{\mathrm{T}}C_1 = \begin{bmatrix} R_{11}^{\gamma} & R_{12}^{\gamma} \\ R_{12}^{\gamma\mathrm{T}} & R_{22}^{\gamma} \end{bmatrix} \\ M &= G\left(I_q - D_{21}^{\mathrm{T}}\tilde{D}_2 D_{21} \right) G^{\mathrm{T}} = \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^{\mathrm{T}} & M_{22} \end{bmatrix} \\ H_0 &= H_1 - H_2 H_4^{-1} H_3 = \begin{bmatrix} A_w^{\gamma\mathrm{T}} & -R_w^{\gamma} \\ -M_w^{\gamma} & -A_w^{\gamma} \end{bmatrix} \\ H_1 &= \begin{bmatrix} \hat{A}_{11}^{\mathrm{T}} & -R_{11}^{\gamma} \\ -M_{11} & -\hat{A}_{11} \end{bmatrix} \\ H_2 &= \begin{bmatrix} \hat{A}_{21}^{\mathrm{T}} & -R_{12}^{\gamma} \\ -M_{12} & -\hat{A}_{12} \end{bmatrix} \\ H_3 &= \begin{bmatrix} \hat{A}_{12}^{\mathrm{T}} & -R_{12}^{\gamma\mathrm{T}} \\ -M_{12}^{\mathrm{T}} & -\hat{A}_{21} \end{bmatrix} \\ H_4 &= \begin{bmatrix} \hat{A}_{22}^{\mathrm{T}} & -R_{22}^{\gamma} \\ -M_{22} & -\hat{A}_{22} \end{bmatrix} \\ L_2^{\mathrm{T}} &= U_4^{-1}\hat{V}_{12} \quad L_1^{\mathrm{T}} = -U_4^{-1}U_3 \\ U_1 &= \hat{A}_{11} - \bar{W}_{12}R_{12}^{\gamma\mathrm{T}} - \bar{W}_{11}R_{11}^{\gamma\mathrm{T}} \\ U_2 &= \hat{A}_{12} - \bar{W}_{12}R_{22}^{\gamma\mathrm{T}} - \bar{W}_{11}R_{12}^{\gamma} \\ U_3 &= \hat{A}_{21} - \bar{W}_{22}R_{12}^{\gamma\mathrm{T}} \\ U_4 &= \hat{A}_{22} - \bar{W}_{22}R_{22}^{\gamma\mathrm{T}} \\ U_0 &= U_1 - U_2 U_4^{-1} U_3 \\ \hat{V}_{12} &= \bar{W}_{22}\hat{A}_{12}^{\mathrm{T}} + M_{12}^{\mathrm{T}}.\end{aligned}$$

The AREs (7c) and (7a) will produce the unique positive semidefinite stabilizing solution under assumption 3 if γ is large enough.

Let $\gamma_{2f} = \inf\{\gamma > \gamma_{1f}|$ the ARE (7c) has a positive–semidefinite stabilizing solution, and $\rho\left(\bar{P}_{22}\bar{W}_{22}\right) < \gamma^2\}$. Moreover, let us define $\gamma_{2s} = \inf\{\gamma > \gamma_{1s}|$ the ARE (7a) has a positive–semidefinite stabilizing solution, and $\rho\left(\bar{P}_{11}\bar{W}_{11}\right) < \gamma^2\}$.

As the results, for every $\gamma > \bar{\gamma} = \max\{\gamma_{1s}, \gamma_{1f}, \gamma_{2s}, \gamma_{2f}\}$, the ARE's (6) and (7) have the positive semidefinite stabilizing solutions if $\varepsilon > 0$ is small enough. Thus, we have the following result.

Lemma 2: Under the Assumptions 1–3, if we select a parameter $\gamma > \bar{\gamma} = \max\{\gamma_{1s}, \gamma_{1f}, \gamma_{2s}, \gamma_{2f}\}$, then there exists a small $\bar{\varepsilon} > 0$ such that for all $\varepsilon \in (0, \bar{\varepsilon})$, the ARE's (1) and (2) admits a positive–semidefinite solution, which can be written as

$$P_{\varepsilon} = \begin{bmatrix} \bar{P}_{11} + O(\varepsilon) & \varepsilon\bar{P}_{21}^{\mathrm{T}} + O(\varepsilon^2) \\ \varepsilon\bar{P}_{21} + O(\varepsilon^2) & \varepsilon\bar{P}_{22} + O(\varepsilon^2) \end{bmatrix} \tag{8}$$

$$W_{\varepsilon} = \begin{bmatrix} \bar{W}_{11} + O(\varepsilon) & \bar{W}_{12} + O(\varepsilon) \\ \bar{W}_{12}^{\mathrm{T}} + O(\varepsilon) & \varepsilon^{-1}(\bar{W}_{22} + O(\varepsilon)) \end{bmatrix}. \tag{9}$$

Proof: By using the implicit function theorem, Lemma 2 can be proved. The proof is omitted since it is similar to that of the references [4] and [20]. ■

Remark 2: We can prove Lemma 2 by using a method similar to that given in the proof of [4, Theorems 2.1, 2.2]. Note that the proof given in [4] is made on the invertible assumptionof A_{22}. However, this note improves the proof of Lemma 2 in the sense that the invertible assumption is not needed.

III. The New Iterative Algorithm

In this section, we establish an elegant and simple algorithm which converges globally to the positive–semidefinite symmetric solution of AREs (1) and (2). The algorithm is given in term of the standard GALE, which have to be solved iteratively. We present the new iterative algorithm based on the Kleinman algorithm. Here, we note that the Kleinman algorithm is based on the Newton type algorithm. In general, the stabilizable-detectable conditions will guarantee the convergence of the Kleinman algorithm for the standard linear-quadratic regulator type ARE to the positive semidefinite solutions. However, it is difficult to apply the Kleinman algorithm to the GAREs (3) and (4) because the matrix $S^{\gamma} = \gamma^{-2}GG^{\mathrm{T}} - B\tilde{D}_1 B^{\mathrm{T}}$ and/or $R^{\gamma} = \gamma^{-2}C_1^{\mathrm{T}}C_1 - C_2\tilde{D}_2 C_2^{\mathrm{T}}$ are in general indefinite.

We propose the following algorithm for solving the GAREs (3) and (4), respectively

$$\left(\bar{A}^{\gamma} - S^{\gamma}P^{(i)}\right)^{\mathrm{T}} P^{(i+1)} + P^{(i+1)\mathrm{T}}\left(\bar{A}^{\gamma} - S^{\gamma}P^{(i)}\right) + P^{(i)\mathrm{T}}S^{\gamma}P^{(i)} + Q = 0 \tag{10}$$

$$\begin{aligned}
&\Pi_\varepsilon P^{(i+1)} = P^{(i+1)\mathrm{T}}\Pi_\varepsilon \\
&\left(\hat{A}^\gamma - W^{(i)}R^\gamma\right)W^{(i+1)\mathrm{T}} + W^{(i+1)}(\hat{A}^\gamma - W^{(i)}R^\gamma)^{\mathrm{T}} \\
&\quad + W^{(i)}R^\gamma W^{(i)\mathrm{T}} + M = 0 \\
&W^{(i+1)}\Pi_\varepsilon = \Pi_\varepsilon W^{(i+1)\mathrm{T}} \\
&i = 0, 1, 2, 3, \ldots
\end{aligned} \tag{11}$$

with $P^{(0)} = \begin{bmatrix} \bar{P}_{11} & \varepsilon\bar{P}_{21}^{\mathrm{T}} \\ \bar{P}_{21} & \bar{P}_{22}\end{bmatrix}, W^{(0)} = \begin{bmatrix} \bar{W}_{11} & \bar{W}_{12} \\ \varepsilon\bar{W}_{12}^{\mathrm{T}} & \bar{W}_{22}\end{bmatrix}$. The Kleinman algorithm (10) can be constructed by setting $P^{(i+1)} = P^{(i)} + \Delta P^{(i)}$ and neglecting $\Delta P^{(i)\mathrm{T}}S^\gamma\Delta P^{(i)}$ term. By following the similar steps, we obtain the Kleinman algorithm (11). Kleinman algorithm is well-known and is widely used to find a solution of ARE, and its local convergence properties are well understood. We are concerned with good choices of the starting points which guarantee to find a required solution of a given GARE. Our new idea is to set the initial conditions to the matrices $P^{(0)}$ and $W^{(0)}$. Using Lemma 2, the fundamental idea is based on $\|P - P^{(0)}\| = O(\varepsilon)$ and $\|W - W^{(0)}\| = O(\varepsilon)$. Although the matrices S^γ and/or R^γ are in general indefinite, we can get the required solution with the rate of quadratic convergence by using the Kleinman algorithm. By using Newton–Kantorovich theorem, we now prove the existence of the unique solution for the GARE (3a). The main result of this section is as follows.

Theorem 1: Under the assumptions 1–3, if we select a parameter $\gamma > \bar{\gamma} = \max\{\gamma_{1s}, \gamma_{1f}, \gamma_{2s}, \gamma_{2f}\}$, then the new iterative algorithm (10) converges to the exact solution P^* of the GARE (3a) with the rate of quadratic convergence. The unique bounded solution $P^{(i)}$ of the GARE (3a) is in the neighborhood of the exact solution P^*. Furthermore, $P_\varepsilon^{(i)} = \Pi_\varepsilon^{\mathrm{T}}P^{(i)} = P^{(i)\mathrm{T}}\Pi_\varepsilon$ is positive semidefinite and $\bar{A}_\varepsilon^\gamma - S_\varepsilon^\gamma P_\varepsilon^{(i)}$ is stable for all i, $i = 0, 1, 2, \ldots$. That is, the following conditions are satisfied:

$$\left\|P^{(i)} - P^*\right\| = O(\varepsilon^{2^i}) \tag{12a}$$

$$\begin{aligned}
\left\|P^{(i)}\right\| &\le c < \infty \\
P_\varepsilon^{(i)} &= \Pi_\varepsilon P^{(i)} = P^{(i)\mathrm{T}}\Pi_\varepsilon \ge 0 \\
\mathrm{Re}\lambda\left[\bar{A}_\varepsilon^\gamma - S_\varepsilon^\gamma P_\varepsilon^{(i)}\right] &< 0
\end{aligned} \tag{12b}$$

where

$$P = P^* = \begin{bmatrix} P_{11} & \varepsilon P_{21}^{\mathrm{T}} \\ P_{21} & P_{22}\end{bmatrix}$$
$$P^{(i)} = \begin{bmatrix} P_{11}^{(i)} & \varepsilon P_{21}^{(i)\mathrm{T}} \\ P_{21}^{(i)} & P_{22}^{(i)}\end{bmatrix}.$$

Before proving this theorem, we will first establish a following useful lemma.

Lemma 3: Let us consider the GALE (13)

$$\mathcal{A}^{\mathrm{T}}\Xi + \Xi^{\mathrm{T}}\mathcal{A} + \varepsilon^j Q = 0 \tag{13}$$

where Ξ is the solution of the GALE (13) and $\mathcal{A}$ and Q known matrices defined by

$$\begin{aligned}
\Xi &= \begin{bmatrix} \Xi_{11}(\varepsilon) & \varepsilon\Xi_{21}^{\mathrm{T}}(\varepsilon) \\ \Xi_{21}(\varepsilon) & \Xi_{22}(\varepsilon)\end{bmatrix} \\
\mathcal{A} &= \begin{bmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ \mathcal{A}_{21} & \mathcal{A}_{22}\end{bmatrix} \qquad Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^{\mathrm{T}} & Q_{22}\end{bmatrix} \\
\Xi_{11} &= \Xi_{11}^{\mathrm{T}} \quad \mathcal{A}_{11} \quad Q_{11} = Q_{11}^{\mathrm{T}} \in \mathbf{R}^{n_1\times n_1} \\
\Xi_{22} &= \Xi_{22}^{\mathrm{T}} \quad e\mathcal{A}_{22} \quad Q_{22} = Q_{22}^{\mathrm{T}} \in \mathbf{R}^{n_2\times n_2}.
\end{aligned}$$

Assume that the matrix $\mathcal{A}_{22}$ is nonsingular and $\mathcal{A}_0 \equiv \mathcal{A}_{11} - \mathcal{A}_{12}\mathcal{A}_{22}^{-1}\mathcal{A}_{21}$ and $\mathcal{A}_{22}$ are stable. Then

$$\Xi = \sum_{k=j}^{\infty}\frac{\varepsilon^k}{k!}\begin{bmatrix} \Xi_{11}^{(k)}(0) & \varepsilon\Xi_{21}^{(k)\mathrm{T}}(0) \\ \Xi_{21}^{(k)}(0) & \Xi_{22}^{(k)}(0)\end{bmatrix} = O(\varepsilon^j) \tag{14}$$

where

$$\Xi_{lm}^{(k)}(0) = \frac{d^k}{d\varepsilon^k}\Xi_{lm}(\varepsilon)|_{\varepsilon=0} \quad lm = 11, 21, 22.$$

Proof: Partitioning for the GALE (13), we obtain the following equations:

$$\begin{aligned}
&\mathcal{A}_{11}^{\mathrm{T}}\Xi_{11} + \Xi_{11}\mathcal{A}_{11} + \mathcal{A}_{21}^{\mathrm{T}}\Xi_{21} \\
&\quad + \Xi_{21}^{\mathrm{T}}\mathcal{A}_{21} + \varepsilon^j Q_{11} = 0
\end{aligned} \tag{15a}$$

$$\begin{aligned}
&\Xi_{11}\mathcal{A}_{12} + \Xi_{21}^{\mathrm{T}}\mathcal{A}_{22} + \mathcal{A}_{21}^{\mathrm{T}}\Xi_{22} \\
&\quad + \varepsilon\mathcal{A}_{11}^{\mathrm{T}}\Xi_{21}^{\mathrm{T}} + \varepsilon^j Q_{12} = 0
\end{aligned} \tag{15b}$$

$$\begin{aligned}
&\mathcal{A}_{22}^{\mathrm{T}}\Xi_{22} + \Xi_{22}\mathcal{A}_{22} + \varepsilon\left(\mathcal{A}_{12}^{\mathrm{T}}\Xi_{21}^{\mathrm{T}} + \Xi_{21}\mathcal{A}_{12}\right) \\
&\quad + \varepsilon^j Q_{22} = 0.
\end{aligned} \tag{15c}$$

Setting $\varepsilon = 0$ for the above (15), since $\mathcal{A}_0$ and $\mathcal{A}_{22}$ are stable we deduce that $\Xi_{lm}(0) = \Xi_{lm}^{(0)}(0) = 0$, $lm = 11, 21, 22$. We now obtain the derivative of (15) at parameter ε as follows:

$$\begin{aligned}
&\mathcal{A}_{11}^{\mathrm{T}}\Xi_{11}^{(1)}(\varepsilon) + \Xi_{11}^{(1)}(\varepsilon)\mathcal{A}_{11} + \mathcal{A}_{21}^{\mathrm{T}}\Xi_{21}^{(1)}(\varepsilon) \\
&\quad + \Xi_{21}^{(1)\mathrm{T}}(\varepsilon)\mathcal{A}_{21} + j\varepsilon^{j-1}Q_{11} = 0 \\
&\Xi_{11}^{(1)}(\varepsilon)\mathcal{A}_{12} + \Xi_{21}^{(1)\mathrm{T}}(\varepsilon)\mathcal{A}_{22} + \mathcal{A}_{21}^{\mathrm{T}}\Xi_{22}^{(1)}(\varepsilon) \\
&\quad + \mathcal{A}_{11}^{\mathrm{T}}\Xi_{21}^{\mathrm{T}}(\varepsilon) + \varepsilon\mathcal{A}_{11}^{\mathrm{T}}\Xi_{21}^{(1)\mathrm{T}}(\varepsilon) + j\varepsilon^{j-1}Q_{12} = 0 \\
&\mathcal{A}_{22}^{\mathrm{T}}\Xi_{22}^{(1)}(\varepsilon) + \Xi_{22}^{(1)}(\varepsilon)\mathcal{A}_{22} + \varepsilon\left(\mathcal{A}_{12}^{\mathrm{T}}\Xi_{21}^{(1)\mathrm{T}}(\varepsilon) + \Xi_{21}^{(1)}(\varepsilon)\mathcal{A}_{12}\right) \\
&\quad + \mathcal{A}_{12}^{\mathrm{T}}\Xi_{21}^{\mathrm{T}}(\varepsilon) + \Xi_{21}(\varepsilon)\mathcal{A}_{12} + j\varepsilon^{j-1}Q_{22} = 0.
\end{aligned}$$

Using $\Xi_{lm}^{(0)}(0) = 0$ and the fact that $\mathcal{A}_0$ and $\mathcal{A}_{22}$ are stable, we get $\Xi_{lm}^{(1)}(0) = 0$. By following the similar steps, we have $\Xi_{lm}^{(k)}(0) = 0, k = 0, 1, 2, j - 1$. Note that the exact proof is done by using mathematical induction. On the other hand, it is well known that the matrix Ξ possess a power series expansion at $\varepsilon = 0$ as follows:

$$\Xi = \sum_{k=0}^{\infty}\frac{\varepsilon^k}{k!}\begin{bmatrix} \Xi_{11}^{(k)}(0) & \varepsilon\Xi_{21}^{(k)\mathrm{T}}(0) \\ \Xi_{21}^{(k)}(0) & \Xi_{22}^{(k)}(0)\end{bmatrix}.$$

Substituting $\Xi_{lm}^{(k)}(0) = 0, k = 0, 1, 2, j-1$ into Ξ, it is straightforward to verify that (14). This is the required result. ■

Now, let us prove Theorem 1.

Proof: The proof of (12b) has been given in [21]. Thus, we will prove the quadratic convergence property corresponding to (12a) and the existence of the unique solution of the ARE (1). We first prove that under the assumptions in Theorem 1 the algorithm (10) converges to the desired solution of (3a) with (12a). The proof is done by using mathematical induction and Lemma 3. Subtracting (3a) from (10), it is easy to derive

$$\begin{aligned}
&\left(\bar{A}^\gamma - S^\gamma P^{(i)}\right)^{\mathrm{T}}\left(P^{(i+1)} - P\right) + \left(P^{(i+1)} - P\right)^{\mathrm{T}}\left(\bar{A}^\gamma - S^\gamma P^{(i)}\right) \\
&\qquad = -\left(P^{(i)} - P\right)^{\mathrm{T}}S^\gamma\left(P^{(i)} - P\right).
\end{aligned}$$

When $i = 0$ for the above equations, by noting that $\|P^{(0)} - P\| = O(\varepsilon)$ based on the Lemma 2, we have $\left(\bar{A}^\gamma - S^\gamma P^{(0)}\right)^{\mathrm{T}}(P^{(1)} - P)$ $+(P^{(1)} - P)^{\mathrm{T}}\left(\bar{A}^\gamma - S^\gamma P^{(i)}\right) = O(\varepsilon^2)$. Using

$$\bar{A}^\gamma - S^\gamma P^{(0)} = \begin{bmatrix} \bar{D}_1 & \bar{D}_2 + O(\varepsilon) \\ \bar{D}_3 & \bar{D}_4 + O(\varepsilon)\end{bmatrix}$$

and the known results that $\bar{D}_4$ and $\bar{D}_0 = \bar{D}_1 - \bar{D}_2\bar{D}_4^{-1}\bar{D}_3$ are stable, it follows from Lemma 3 that $\|P^{(1)} - P\| = O(\varepsilon^2)$. We now assume that $\|P^{(N)} - P\| = O\left(\varepsilon^{2^N}\right)$. Using this assumption, we conclude that

$$\left(\bar{A}^\gamma - S^\gamma P^{(N)}\right)^{\mathrm{T}}\left(P^{(N+1)} - P\right) + \left(P^{(N+1)} - P\right)^{\mathrm{T}}\left(\bar{A}^\gamma - S^\gamma P^{(N)}\right) = -O\left(\varepsilon^{2^{N+1}}\right).$$

Thus, using Lemma 3, we have $\|P^{(N+1)} - P\| = O\left(\varepsilon^{2^{N+1}}\right)$. Consequently, the (12a) holds for all $i \in \mathbf{N}$. Secondly, we show that there exists the unique solution for the ARE (1). This proof is equivalent to the proof of existence of the unique solution for the GARE (3a). Thus, the proof follows directly by applying Newton–Kantorovich theorem (see [16, p. 155]) for the GARE (3a). We now observe that function $\mathcal{F}_1(P)$ is differentiable on a convex set $\mathcal{D}$. Using the fact that

$$\nabla\mathcal{F}_1(P) = \left(\bar{A}^\gamma - S^\gamma P\right)^{\mathrm{T}} \otimes I_n + I_n \otimes \left(\bar{A}^\gamma - S^\gamma P\right)^{\mathrm{T}}$$

where $\otimes$ denotes Kronecker product [19] and $\mathcal{F}_1(P) = \bar{A}^{\gamma\mathrm{T}}P + P^{\mathrm{T}}\bar{A}^\gamma - P^{\mathrm{T}}S^\gamma P + Q$, $\nabla\mathcal{F}_1(P) = \partial\mathrm{vec}\mathcal{F}_1(P)/\partial(\mathrm{vec}P)^{\mathrm{T}}$, where vec denotes an ordered stack of the columns of its matrix [19], we have

$$\|\nabla\mathcal{F}_1(P_1) - \nabla\mathcal{F}_1(P_2)\| \le \bar{\gamma}\|P_1 - P_2\|$$

where $\bar{\gamma} = 2\|S^\gamma\|$. Moreover, using the fact that

$$\nabla\mathcal{F}_1(P^{(0)}) = \begin{bmatrix} \bar{D}_1 & \bar{D}_2 + O(\varepsilon) \\ \bar{D}_3 & \bar{D}_4 + O(\varepsilon) \end{bmatrix}^{\mathrm{T}} \otimes I_n + I_n \otimes \begin{bmatrix} \bar{D}_1 & \bar{D}_2 + O(\varepsilon) \\ \bar{D}_3 & \bar{D}_4 + O(\varepsilon) \end{bmatrix}^{\mathrm{T}}$$

it follows that $\nabla\mathcal{F}_1(P^{(0)})$ is nonsingular because $\bar{D}_4$ and $\bar{D}_0$ are stable. Therefore, there exists $\bar{\beta}$ such that $\left\|\left[\nabla\mathcal{F}_1(P^{(0)})\right]^{-1}\right\| \equiv \bar{\beta}$. On the other hand, since $\mathcal{F}_1\left(P^{(0)}\right) = O(\varepsilon)$, there exists $\bar{\eta}$ such that $\left\|[\nabla\mathcal{F}_1(P^{(0)})]^{-1}\right\| \cdot \left\|\mathcal{F}_1(P^{(0)})\right\| = O(\varepsilon) \equiv \bar{\eta}$. Thus, there exists $\bar{\alpha}$ such that $\bar{\alpha} \equiv \bar{\beta}\bar{\gamma}\bar{\eta} < 2^{-1}$ because of $\bar{\eta} = O(\varepsilon)$. Now, let us define

$$\begin{aligned} t^* &\equiv \frac{1}{\bar{\gamma}\bar{\beta}}\left[1 - \sqrt{1 - 2\bar{\alpha}}\right] \\ &= \frac{1}{2\|S^\gamma\| \cdot \|[\nabla\mathcal{F}(P^{(0)})]^{-1}\|}\left[1 - \sqrt{1 - 2\bar{\alpha}}\right]. \end{aligned}$$

Clearly, $\mathcal{S} \equiv \left\{P : \|P - P^{(0)}\| \le t^*\right\}$ is in the convex set $\mathcal{D}$. In the sequel, since $\|P^* - P^{(0)}\| = O(\varepsilon)$ holds for small ε, we have shown that P^* is the unique solution in $\mathcal{S}$. Therefore, the proof is completed. ■

Remark 3: The algorithm (10) which is based on the Kleinman algorithm might facilitate new approach to the singularly perturbed ARE with indefinite sign quadratic term, that is, conceptually simpler and numerically more efficient than those previously used in [7] and [15]. Moreover, by applying the results of this paper, we can get rather easily the solution for various singularly perturbed ARE with indefinite sign quadratic term.

Remark 4: Note that our proposed method is not a straightforward extension to the continuous-time case of the methods given in [22], [23]. First, it is most different from [22], [23] that our convergence proof of the iterative algorithm (10) is based on Newton–Kantorovich theorem. Moreover, how to select the initial condition is quite different. That is, we choose the initial condition as solutions of the reduced-order AREs close to the exact solutions using the property of singularly perturbed system. As a result, while the recursion in [22] has exponential convergence property, the new iterative algorithm achieves the quadratic convergence property stronger than exponential convergence property. Secondly, we have newly proved the existence of the unique solution for the GARE with indefinite sign quadratic term.

We now summarize a perturbation analysis of the GALE (10). Setting ε to zero and using Kronecker products, the GALE (10) can be written as

$$\begin{aligned} \mathcal{V}\begin{bmatrix} \mathrm{vec}\bar{P}_{11}^{(i+1)} \\ \mathrm{vec}\bar{P}_{21}^{(i+1)} \\ \mathrm{vec}\bar{P}_{22}^{(i+1)} \end{bmatrix} &= \begin{bmatrix} \mathrm{vec}\bar{Q}_{11} \\ \mathrm{vec}\bar{Q}_{12} \\ \mathrm{vec}\bar{Q}_{22} \end{bmatrix} \\ \begin{bmatrix} \bar{\mathcal{A}}_{11} & \bar{\mathcal{A}}_{12} \\ \bar{\mathcal{A}}_{21} & \bar{\mathcal{A}}_{22} \end{bmatrix} &= \bar{A}^\gamma - S^\gamma \begin{bmatrix} \bar{P}_{11}^{(i)} & 0 \\ \bar{P}_{21}^{(i)} & \bar{P}_{22}^{(i)} \end{bmatrix} \\ \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} \\ \bar{Q}_{12}^{\mathrm{T}} & \bar{Q}_{22} \end{bmatrix} &= \begin{bmatrix} \bar{P}_{11}^{(i)} & \bar{P}_{21}^{(i)\mathrm{T}} \\ 0 & \bar{P}_{22}^{(i)} \end{bmatrix} S^\gamma \begin{bmatrix} \bar{P}_{11}^{(i)} & 0 \\ \bar{P}_{21}^{(i)} & \bar{P}_{22}^{(i)} \end{bmatrix} + Q \end{aligned}$$

where $\mathcal{V}$ is given at the bottom of the page. $U_{n_1n_1}$ denotes a permutation matrix in Kronecker matrix sense [19] and $\mathrm{vec}\bar{P}_{lm}^{(i+1)}$ denotes an ordered stack of the columns of $P_{lm}^{(i+1)}$ when $\varepsilon = 0$. It can be shown, after some algebra, that the determinant of $\mathcal{V}$ is expressed as $\det\mathcal{V} = \det\left[(I_{n_2} \otimes \bar{\mathcal{A}}_{22}^{\mathrm{T}})U_{n_2n_2} + \bar{\mathcal{A}}_{22}^{\mathrm{T}} \otimes I_{n_2}\right] \cdot \det\left(I_{n_1} \otimes \bar{\mathcal{A}}_{22}^{\mathrm{T}}\right) \cdot \det\left[(I_{n_1} \otimes \bar{\mathcal{A}}_0^{\mathrm{T}})U_{n_1n_1} + \bar{\mathcal{A}}_0^{\mathrm{T}} \otimes I_{n_1}\right]$, where $\bar{\mathcal{A}}_0 = \bar{\mathcal{A}}_{11} - \bar{\mathcal{A}}_{12}\bar{\mathcal{A}}_{22}^{-1}\bar{\mathcal{A}}_{21}$. Obviously, $\bar{\mathcal{A}}_{22}$ and $\bar{\mathcal{A}}_0$ are nonsingular matrices. Thus, there exists $\mathcal{V}^{-1}$. Therefore, the condition number [16] of $\mathcal{V}$, that is, $K(\mathcal{V}) = \|\mathcal{V}\| \cdot \|\mathcal{V}^{-1}\|$ is given by $K(\mathcal{V}) = O(1)$. Since $K(\mathcal{V})$ is not large, the matrix $\mathcal{V} + O(\varepsilon)$ is well-conditioned for small ε.

We next give the convergence theorem of the algorithm (11) by similar argument corresponding to the algorithm (10).

Theorem 2: Under the Assumptions 1–3, if we select a parameter $\gamma > \bar{\gamma} = \max\{\gamma_{1s}, \gamma_{1f}, \gamma_{2s}, \gamma_{2f}\}$, then the new iterative algorithm (11) converges to the exact solution W^* of the GARE (4a) with the rate of quadratic convergence. The unique bounded solution $W^{(i)}$ of the GARE (4a) is in the neighborhood of the exact solution W^*. Furthermore, $W_\varepsilon^{(i)} = W^{(i)\mathrm{T}}\Pi_\varepsilon^{-\mathrm{T}} = \Pi_\varepsilon^{-1}W^{(i)}$ is positive semidefinite and $\hat{A}_\varepsilon^\gamma - W_\varepsilon^{(i)}R_\varepsilon^\gamma$ is stable for all i, $i = 0, 1, 2, \ldots$.

Proof: Since the proof of Theorem 2 is performed by a dual argument of Theorem 1, it is omitted. ■

As a result of applying the idea of the Kleinman algorithm, we have managed to replace the computation of the GAREs (3) and (4) which contain the small parameter ε with a sequence of the GALEs (10) and (11).

Now, we consider a method for solving the GALEs (10) and (11). Since the algorithm for solving the GALE (10) is virtually identical to the GALE (11), we give only the algorithm of the GALE (10). In order to reduce the dimension of the workspace, a new algorithm for solving the GALE which is based on the fixed point algorithm is established. Let us consider the following GALE (16), in a general form:

$$\Lambda^{\mathrm{T}}X + X^{\mathrm{T}}\Lambda + V = 0 \tag{16}$$

$$\mathcal{V} = \begin{bmatrix} \left(I_{n_1} \otimes \bar{\mathcal{A}}_{11}^{\mathrm{T}}\right)U_{n_1n_1} + \bar{\mathcal{A}}_{11}^{\mathrm{T}} \otimes I_{n_1} & \left(I_{n_1} \otimes \bar{\mathcal{A}}_{21}^{\mathrm{T}}\right)U_{n_1n_2} + \bar{\mathcal{A}}_{21}^{\mathrm{T}} \otimes I_{n_1} & 0 \\ \left(I_{n_1} \otimes \bar{\mathcal{A}}_{12}^{\mathrm{T}}\right)U_{n_1n_1} & \left(I_{n_1} \otimes \bar{\mathcal{A}}_{22}^{\mathrm{T}}\right)U_{n_1n_2} & \bar{\mathcal{A}}_{21}^{\mathrm{T}} \otimes I_{n_2} \\ 0 & 0 & \left(I_{n_2} \otimes \bar{\mathcal{A}}_{22}^{\mathrm{T}}\right)U_{n_2n_2} + \bar{\mathcal{A}}_{22}^{\mathrm{T}} \otimes I_{n_2} \end{bmatrix}.$$

where X is the solution of the GALE (16) and Λ and V are known matrices defined by

$$X = \begin{bmatrix} X_{11}(\varepsilon) & \varepsilon X_{21}(\varepsilon)^{\mathrm{T}} \\ X_{21}(\varepsilon) & X_{22}(\varepsilon) \end{bmatrix} := P^{(i+1)}$$
$$\Lambda = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{bmatrix} := \bar{A}^{\gamma} - S^{\gamma} P^{(i)}$$
$$V = \begin{bmatrix} V_{11} & V_{12} \\ V_{12}^{\mathrm{T}} & V_{22} \end{bmatrix} := P^{(i)\mathrm{T}} S^{\gamma} P^{(i)} + Q$$
$$X_{11} = X_{11}^{\mathrm{T}} \quad \Lambda_{11} \quad V_{11} = V_{11}^{\mathrm{T}} \in \mathbf{R}^{n_1 \times n_1}$$
$$X_{22} = X_{22}^{\mathrm{T}} \quad \Lambda_{22} \quad V_{22} = V_{22}^{\mathrm{T}} \in \mathbf{R}^{n_2 \times n_2}.$$

In order to solve the GALE (16) in a iterative method, we need another assumption.

Assumption 4: The matrix Λ_{22} is nonsingular and $\Lambda_0 \equiv \Lambda_{11} - \Lambda_{12}\Lambda_{22}^{-1}\Lambda_{21}$ and Λ_{22} are stable.

Note that the assumption 4 is satisfied for all $i \in \mathbf{N}$ because $\bar{D}_4$ and $\bar{D}_0$ are stable. The GALE (16) can be partitioned into

$$\Lambda_{11}^{\mathrm{T}} X_{11} + X_{11}\Lambda_{11} + \Lambda_{21}^{\mathrm{T}} X_{21} + X_{21}^{\mathrm{T}}\Lambda_{21} + V_{11} = 0 \tag{17a}$$
$$X_{11}\Lambda_{12} + X_{21}^{\mathrm{T}}\Lambda_{22} + \Lambda_{21}^{\mathrm{T}} X_{22} + \varepsilon\Lambda_{11}^{\mathrm{T}} X_{21}^{\mathrm{T}} + V_{12} = 0 \tag{17b}$$
$$\Lambda_{22}^{\mathrm{T}} X_{22} + X_{22}\Lambda_{22} + \varepsilon\left(\Lambda_{12}^{\mathrm{T}} X_{21}^{\mathrm{T}} + X_{21}\Lambda_{12}\right) + V_{22} = 0. \tag{17c}$$

The fixed point algorithm for solving (17) is given by

$$\Lambda_{22}^{\mathrm{T}} X_{22}^{(i+1)} + X_{22}^{(i+1)}\Lambda_{22} + \varepsilon\left(\Lambda_{12}^{\mathrm{T}} X_{21}^{(i)\mathrm{T}} + X_{21}^{(i)}\Lambda_{12}\right) + V_{22} = 0 \tag{18a}$$
$$\begin{aligned}\Lambda_0^{\mathrm{T}} X_{11}^{(i+1)} + X_{11}^{(i+1)}\Lambda_0 - \varepsilon\Lambda_{21}^{\mathrm{T}}\Lambda_{22}^{-\mathrm{T}} X_{21}^{(i)}\Lambda_0 \\ - \varepsilon\Lambda_0^{\mathrm{T}} X_{21}^{(i)\mathrm{T}}\Lambda_{22}^{-1}\Lambda_{21} + \Lambda_{21}^{\mathrm{T}}\Lambda_{22}^{-\mathrm{T}} V_{22}\Lambda_{22}^{-1}\Lambda_{21} \\ - V_{12}\Lambda_{22}^{-1}\Lambda_{21} - \Lambda_{21}^{\mathrm{T}}\Lambda_{22}^{-\mathrm{T}} V_{12}^{\mathrm{T}} + V_{11} = 0\end{aligned} \tag{18b}$$
$$X_{21}^{(i+1)} = -\Lambda_{22}^{-\mathrm{T}}\left(\Lambda_{12}^{\mathrm{T}} X_{11}^{(i+1)} + X_{22}^{(i+1)}\Lambda_{21} + \varepsilon X_{21}^{(i)}\Lambda_{11} + V_{12}^{\mathrm{T}}\right)$$
$$X_{21}^{(0)} = 0 \quad i = 0, 1, 2, 3, \ldots. \tag{18c}$$

The following theorem indicates the convergence of the algorithm (18).

Theorem 3: The algorithm (18) converges to the exact solution X_{lm} of (17) with the rate of convergence of $O(\varepsilon^i)$, that is

$$\left\|X_{lm}^{(i)} - X_{lm}\right\| = O(\varepsilon^i)$$
$$i = 1, 2, 3, \ldots \quad lm = 11, 21, 22. \tag{19}$$

Proof: The proof is done by using the mathematical induction. When $i = 0$ for the (18), the solutions $X_{lm}^{(1)}$ are equivalent to the first order approximations X_{lm} corresponding to the small parameters ε for the (17). It follows from these equations that $\left\|X_{lm}^{(1)} - X_{lm}\right\| = O(\varepsilon)$, $lm = 11, 21, 22$. When $i = N\ (N \geq 1)$, we assume that $\left\|X_{lm}^{(N)} - X_{lm}\right\| = O(\varepsilon^N)$. Subtracting (18) from (17) and using the above assumptions, we arrive at the following equations:

$$\Lambda_{22}^{\mathrm{T}}\left(X_{22}^{(N+1)} - X_{22}\right) + \left(X_{22}^{(N+1)} - X_{22}\right)\Lambda_{22} + O(\varepsilon^{N+1}) = 0$$
$$\left(X_{11}^{(N+1)} - X_{11}\right)\Lambda_{12} + \left(X_{21}^{(N+1)} - X_{21}\right)^{\mathrm{T}}\Lambda_{22} + \Lambda_{21}^{\mathrm{T}}\left(X_{22}^{(N+1)} - X_{22}\right) + O(\varepsilon^{N+1}) = 0$$
$$\Lambda_0^{\mathrm{T}}\left(X_{11}^{(N+1)} - X_{11}\right) + \left(X_{11}^{(N+1)} - X_{11}\right)\Lambda_0 + O(\varepsilon^{N+1}) = 0.$$

Thus, using the standard properties of the algebraic Lyapunov equation (ALE) [24], we have $\left\|X_{lm}^{(N+1)} - X_{lm}\right\| = O(\varepsilon^{N+1})$. Consequently, the (19) holds for all $i \in \mathbf{N}$. This completes the proof of the theorem concerned with the fixed point algorithm. ■

The existing method [7] can obtain the solution by solving the ALEs of lower dimensions which are the same as the slow and fast subsystems. However, in order to obtain the exact solution of the sign indefinite ARE, ones need the same workspace for calculating the inverse matrix (see [7], equation (3.3)). On the other hand, the resulting algorithm is very useful because our proposed algorithm has only to solve the ALEs of lower dimensions. Moreover, note that the algorithm (18a) is quite different from the recursive algorithm [15]. As another important feature, since our proposed algorithm is the quadratic convergence, while the recursive algorithm is the linear convergence [15], the resulting algorithm is also efficient.

In the rest of this section, we will present an important implication. If the state information is available for feedback, then the following corollary is easily seen in view of Theorem 1.

Corollary 1: Assume that $C_1^{\mathrm{T}} D_{12} = 0$ and $D_{12}^{\mathrm{T}} D_{12} = I_p$. Under the assumptions 1–3, the approximate feedback gain $K^{(i)} = -\left[B_1^{\mathrm{T}}\ B_2^{\mathrm{T}}\right] P^{(i)}$ guarantees the performance level $\left\|\left(C_1 + D_{12}K^{(i)}\right)\left(sI_n - A_\varepsilon - B_\varepsilon K^{(i)}\right)^{-1} G_\varepsilon\right\|_\infty < \gamma + O(\varepsilon^{2^i})$, where $P^{(i)}$ is defined in the statement of Theorem 1.

Proof: It can be carried out via a similar technique used in [7] and [20]. ■

IV. Numerical Example

In order to demonstrate the efficiency of our proposed algorithm, we have run a simple numerical example. The system matrix is given by

$$A_{11} = \begin{bmatrix} 0 & 0.4 \\ 0 & 0 \end{bmatrix} \quad A_{12} = \begin{bmatrix} 0 & 0 \\ 0.345 & 0 \end{bmatrix}$$
$$A_{21} = \begin{bmatrix} 0 & -0.524 \\ 0 & 0 \end{bmatrix} \quad A_{22} = \begin{bmatrix} 0 & 0.262 \\ 0 & -1 \end{bmatrix}$$
$$B_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$G_1 = \begin{bmatrix} 1.0 & 0 \\ 0 & 1.0 \end{bmatrix} \quad G_2 = \begin{bmatrix} 0.2 & 0.1 \\ 1.2 & 0.5 \end{bmatrix}$$
$$D_{12}^{\mathrm{T}} = [0 \quad 0 \quad 0 \quad 0 \quad 1] \quad D_{21} = [1 \quad 0.5]$$
$$C_{11}^{\mathrm{T}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad C_{12}^{\mathrm{T}} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$
$$C_{21} = [1 \quad 0] \quad C_{22} = [1 \quad 1].$$

The numerical results are obtained for small parameter $\varepsilon = 10^{-4}$. Since $\det A_{22} = 0$, the system is nonstandard singularly perturbed systems. The four basic quantities for the system are $\gamma_{1f} = \gamma_{2f} = 2.255\,156$, $\gamma_{1s} = \gamma_{2s} = 7.359\,056$. Thus, for every boundary value $\gamma > \bar{\gamma} = \max\{\gamma_{1f}, \gamma_{1s}, \gamma_{2f}, \gamma_{2s}\} = 7.359\,056$, the AREs (6a), (6c), (7a) and (7c) have the positive semidefinite stabilizing solutions. On the other hand, by using MATLAB, the minimum value of $\hat{\gamma}$ such that there exists the dynamic feedback controller is $\hat{\gamma} = 7.468\,750$.

Now, we choose as $\gamma = 8.0 > \bar{\gamma}$ to design the controller. We give the following solutions of the AREs (1) and (2) and in Table I. Table II shows the results of the errors $\left\|\mathcal{F}_1(P^{(i)})\right\|$ and $\left\|\mathcal{F}_2(W^{(i)})\right\|$ per iterations. We find that the solutions of the AREs (1) and (2) converge to the exact solution with accuracy of $\left\|\mathcal{F}_1(P^{(i)})\right\| < 10^{-14}$ and $\left\|\mathcal{F}_2(W^{(i)})\right\| < 10^{-14}$ after 2 iterative iterations, respectively. Moreover, it is interested in pointing out that the result of Table II shows that the algorithms (10) and (11) are quadratic convergence. Table III shows the results of the number of iterations required to the solution with the same accuracy of

TABLE I

$$P_\varepsilon^{(2)} = \begin{bmatrix} 1.5886644664\times 10^{1} & 7.1692654025 & 1.4777259003\times 10^{-3} & 3.2803557787\times 10^{-4} \\ 7.1692654025 & 9.5689292890 & 7.3774420477\times 10^{-4} & 1.2062616188\times 10^{-4} \\ 1.4777259003\times 10^{-3} & 7.3774420477\times 10^{-4} & 4.8953376479\times 10^{-4} & 1.0462699908\times 10^{-4} \\ 3.2803557787\times 10^{-4} & 1.2062616188\times 10^{-4} & 1.0462699908\times 10^{-4} & 2.4621105838\times 10^{-5} \end{bmatrix}.$$

$$W_\varepsilon^{(2)} = \begin{bmatrix} 1.7234972390 & -9.6975066240\times 10^{-2} & -1.5020487314 & -1.0113300267\times 10^{-1} \\ -9.6975066240\times 10^{-2} & 6.7468422739\times 10^{-1} & -1.4273792265 & 7.8173878465\times 10^{-1} \\ -1.5020487314 & -1.4273792265 & 4.8380921965 & -1.2998747057 \\ -1.0113300267\times 10^{-1} & 7.8173878465\times 10^{-1} & -1.2998747057 & 1.9265002490\times 10^{1} \end{bmatrix}.$$

TABLE II

i	$\|\mathcal{F}_1(P^{(i)})\|$	$\|\mathcal{F}_2(W^{(i)})\|$
0	1.77984×10^{-3}	3.11227×10^{-4}
1	1.58061×10^{-7}	8.84535×10^{-8}
2	6.00958×10^{-15}	6.83822×10^{-15}

TABLE III

Number of iterations such that $\|\mathcal{F}_1(P^{(i)})\| < 10^{-14}$.		
ε	recursive algorithm	Kleinman algorithm
10^{-1}	17	5
10^{-2}	9	4
10^{-3}	5	3
10^{-4}	4	2
10^{-5}	3	2
10^{-6}	2	2

$\left\|\mathcal{F}_1(P^{(i)})\right\| < 10^{-14}$ for the classical recursive algorithm [15] versus the improved iterative algorithm. It can be seen that the convergence rate of the resulting algorithm is stable for all ε since the initial conditions $P^{(0)}$ and $W^{(0)}$ are quite good. On the other hand, the classical recursive algorithm converges very slowly for the ε that is not small. However, it is important to point out that the recursive algorithm is very useful when the parameter ε is sufficiently small.

V. Conclusion

In this note, the H_∞ control problem for singularly perturbed systems has been investigated. We have shown that the Kleinman algorithm can be used well to solve the ARE under the appropriate initial condition. Comparing with [15], since our proposed algorithm is quadratic convergence, the required solution can be easily obtained up to an arbitrary order of accuracy, that is, $O\left(\varepsilon^{2^i}\right)$. Moreover, we have also presented the method for solving the GALE by means of the fixed point algorithm. It avoids high dimension and numerical stiffness. Another important feature, if we use the state information, then the high-order $O\left(\varepsilon^{2^i}\right)$ accuracy controller achieves the performance $\gamma + O\left(\varepsilon^{2^i}\right)$ compared with the existing controller [7], [20].

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