Near-Optimal Control of Linear Multiparameter Singularly Perturbed Systems

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Abstract—**In this note, the linear quadratic optimal control for multiparameter singularly perturbed systems is studied. The attention is focused on the design of a new near-optimal controller. The resulting controller achieves $O(\|\mu\|^{2^{i+1}})$ approximation of the optimal cost. The proposed algorithm has been numerically tested on a real physical example and produced useful results.**

Index Terms—**Generalized multiparameter algebraic Lyapunov equation, Kleinman algorithm, multiparameter singularly perturbed systems (MSPSs), near-optimal control.**

I. Introduction

The deterministic and stochastic multimodeling stability, control, filtering and dynamic games have been investigated extensively by several researchers (see, e.g., [1]–[8]). The multimodeling problems arise in large-scale dynamic systems [1], [8]. In order to obtain the optimal solution to the multimodeling problems, we must solve the multiparameter algebraic Riccati equation (MARE), which are parameterized by the small positive same order parameters ε_j, $j = 1, 2, \ldots$. Various reliable approaches to the theory of the algebraic Riccati equation (ARE) have been well documented in many literatures (see, e.g., [9]–[11]). One of the approaches is the invariant subspace approach based on the Hamiltonian matrix [10]. However, such an approach is not adequate to the multiparameter singularly perturbed systems (MSPSs) since the dimension of the required workspace to carry out the calculations for the Hamiltonian matrix is twice the dimension of the original full system. As another disadvantage, there is no guarantee of symmetry for the solution of the ARE when the ARE is known to be ill conditioned [10].

A popular approach to deal with the MSPS is the two-time-scale design method (see, e.g., [1]–[6], and [12]). In particular, in [2] and [6], a resulting near-optimal controller has been proven to have the property of a performance which is $O(\|\mu\|)$ (where $\|\mu\|$ denotes the norm of the vector $\mu := [\varepsilon_1 \ \varepsilon_2 \ \cdots]$) close to the optimal performance for the standard and nonstandard MSPS. However, when the parameters ε_j are not small enough, it is known from [7] and [8] that an $O(\|\mu\|)$ accuracy is very often not sufficient. More recently, in [20] and [22], the recursive algorithms for solving the MARE and the generalized multiparameter algebraic Lyapunov equation (GMALE) have been developed. However, there exists the drawback that the recursive algorithm converges only to the approximation solution [19] since the convergence of the recursive algorithm depend on the zero-order solutions. On the other hand, the exact slow–fast decomposition method for solving the MARE has been proposed in [7] and [8]. However, these results are restricted to the MSPS such that the Hamiltonian matrices for the fast subsystems have no eigenvalues in common (see, e.g., [8, Assumption 5]). Thus, we cannot apply the technique proposed in [7] and [8] to the practical system, such as the Pareto optimal strategy of a multiarea power system [1]. Furthermore, thus far, the loss of performance between the optimal control and the resulting controller which is based on the exact decomposition technique has not been investigated.

In this note, we study the linear-quadratic regulator problem for the MSPS. Our main result shows that a high-order approximate control can be constructed by making use of a Kleinman algorithm [9]. The resulting high-order approximate control can achieve a performance which is $O(\|\mu\|^{2^{i+1}})$ (where i denotes the iterations) close to the optimal performance. Moreover, as a special case when the parameters ε_j are unknown, we can obtain an ε_j-independent controller. Using this controller, we can achieve a performance which is $O(\|\mu\|^2)$ close to the optimal performance. It is worth to note that the $O(\|\mu\|^2)$ near-optimality is proved for the first time to the optimal control problem of the MSPS [2], [6]. As another important feature, this note presents an important improvement on some of the results of [7] and [8] in the sense that ones need no assumption that the Hamiltonian matrices for the fast subsystems have no eigenvalues in common. Hence, the resulting control is applicable to more realistic MSPS. In fact, it is shown that for numerical example our proposed algorithm is applicable to the wider class of the MSPS compared with the exact slow-fast decomposition technique.

Notation: The superscript T denotes matrix transpose. I_n denotes the $n \times n$ identity matrix. $\|\cdot\|$ is any appropriate matrix norm. $\mathrm{block\text{-}diag}$ denotes the block diagonal matrix. $\mathrm{vec}M$ denotes the column vector of the matrix M [13]. $\otimes$ denotes the Kronecker product. U_{lm} denotes a permutation matrix in the Kronecker matrix sense [13] such that $U_{lm}\mathrm{vec}M = \mathrm{vec}M^T$, $M \in \mathbf{R}^{l\times m}$.

II. Multiparameter Singularly Perturbed Systems

We consider the linear time-invariant MSPS

$$\begin{aligned} \dot{x}_0(t) &= A_{00}x_0(t) + A_{01}x_1(t) + A_{02}x_2(t) + B_{01}u_1(t) \\ &\quad + B_{02}u_2(t), \qquad x_0(0) = x_0^0 \qquad (1a)\\ \varepsilon_1\dot{x}_1(t) &= A_{10}x_0(t) + A_{11}x_1(t) + \varepsilon_3A_{12}x_2(t) \\ &\quad + B_{11}u_1(t), \qquad x_1(0) = x_1^0 \qquad (1b)\\ \varepsilon_2\dot{x}_2(t) &= A_{20}x_0(t) + \varepsilon_4A_{21}x_1(t) + A_{22}x_2(t) \\ &\quad + B_{22}u_2(t), \qquad x_2(0) = x_2^0 \qquad (1c)\end{aligned}$$

where $x_j \in \mathbf{R}^{n_j}$, $j = 0, 1, 2$ are the state vectors, $u_j \in \mathbf{R}^{m_j}$, $j = 1, 2$ are the control inputs. All the matrices are constant matrices of appropriate dimensions. The parameters ε_1 and ε_2 are two small positive singular perturbation parameters of the same order of magnitude such that [1]–[8]

$$0 < k_1 \le \alpha \equiv \frac{\varepsilon_1}{\varepsilon_2} \le k_2 < \infty. \qquad (2)$$

The parameters ε_3 and ε_4 are two weak coupling between the fast subsystems. Note that the fast state matrices A_{jj}, $j = 1, 2$ may be singular. In the optimal control of the above MSPS, the performance criterion is given by

$$\begin{aligned} J &= \tfrac{1}{2}\int_0^\infty z^T(t)z(t)\,dt \\ z(t) &= C\begin{bmatrix} x_0(t)\\ x_1(t)\\ x_2(t)\end{bmatrix} + D\begin{bmatrix} u_1(t)\\ u_2(t)\end{bmatrix} = Cx(t) + Du(t) \qquad (3)\end{aligned}$$

Manuscript received September 16, 2001; revised April 17, 2002. Recommended by Associate Editor F. M. Callier.

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Digital Object Identifier 10.1109/TAC.2002.805676

where

$$C = \begin{bmatrix} C_{10} & C_{11} & 0 \\ C_{20} & 0 & C_{22} \end{bmatrix}$$

$$C^T C = Q = \begin{bmatrix} Q_{00} & Q_{01} & Q_{02} \\ Q_{01}^T & Q_{11} & 0 \\ Q_{02}^T & 0 & Q_{22} \end{bmatrix}$$

$$D^T D = R = \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix} > 0, \qquad C^T D = 0.$$

It is well known [2], [7] that the solution of the linear quadratic control problem (1) and (3) is given by

$$u_{\rm opt}(t) = \begin{bmatrix} u_{1{\rm opt}}(t) \\ u_{2{\rm opt}}(t) \end{bmatrix} = -R^{-1} B_\varepsilon^T P_\varepsilon x(t) \tag{4}$$

where P_ε satisfies the MARE

$$A_\varepsilon^T P_\varepsilon + P_\varepsilon A_\varepsilon - P_\varepsilon S_\varepsilon P_\varepsilon + Q = 0 \tag{5}$$

with

$$A_\varepsilon = \begin{bmatrix} A_{00} & A_{01} & A_{02} \\ \varepsilon_1^{-1} A_{10} & \varepsilon_1^{-1} A_{11} & \varepsilon_1^{-1} \varepsilon_3 A_{12} \\ \varepsilon_2^{-1} A_{20} & \varepsilon_2^{-1} \varepsilon_4 A_{21} & \varepsilon_2^{-1} A_{22} \end{bmatrix} \in \mathbf{R}^{N \times N}$$

$$B_\varepsilon = \begin{bmatrix} B_{01} & B_{02} \\ \varepsilon_1^{-1} B_{11} & 0 \\ 0 & \varepsilon_2^{-1} B_{22} \end{bmatrix} \in \mathbf{R}^{N \times M}$$

$$S_\varepsilon = B_\varepsilon R^{-1} B_\varepsilon^T = \begin{bmatrix} S_{00} & \varepsilon_1^{-1} S_{01} & \varepsilon_2^{-1} S_{02} \\ \varepsilon_1^{-1} S_{01}^T & \varepsilon_1^{-2} S_{11} & 0 \\ \varepsilon_2^{-1} S_{02}^T & 0 & \varepsilon_2^{-2} S_{22} \end{bmatrix} \in \mathbf{R}^{N \times N}$$

$$N := n_0 + n_1 + n_2, \qquad M := m_1 + m_2.$$

Moreover, the optimal cost is given by

$$J_{\rm opt} = \tfrac{1}{2} x^T(0) P_\varepsilon x(0). \tag{6}$$

III. Multiparameter Algebraic Riccati Equation

Before we present the near-optimal controller, we first introduce some useful results for the MARE (5). A solution P_ε of the MARE (5), if it exists, must contain the parameters ε_j, $j = 1, 2$ because the matrices A_ε and B_ε contain the ε_j^{-1}-order parameters. Taking this fact into account, we look for the solution P_ε of the MARE (5) with the structure [1]–[8]

$$P_\varepsilon = \begin{bmatrix} P_{00} & \varepsilon_1 P_{10}^T & \varepsilon_2 P_{20}^T \\ \varepsilon_1 P_{10} & \varepsilon_1 P_{11} & \sqrt{\varepsilon_1 \varepsilon_2} P_{21}^T \\ \varepsilon_2 P_{20} & \sqrt{\varepsilon_1 \varepsilon_2} P_{21} & \varepsilon_2 P_{22} \end{bmatrix} \in \mathbf{R}^{N \times N} \tag{7}$$

where $P_{00} = P_{00}^T$, $P_{11} = P_{11}^T$ and $P_{22} = P_{22}^T$.

The optimal control for the MSPS will be studied under the following basic assumptions [6]. In particular, it should be noted that we need to generalize the assumption about stabilizability and detectability, so that they can be applied to the nonstandard MSPS.

Assumption 1: The limit of α exists as ε_1 and ε_2 tend to zero [1]–[8], that is

$$\overline{\alpha} = \lim_{\substack{\varepsilon_2 \to 0^+ \\ \varepsilon_1 \to 0^+}} \alpha. \tag{8}$$

Assumption 2: The triples (A_{jj}, B_{jj}, C_{jj}), $j = 1, 2$ are stabilizable and detectable.

Assumption 3:

$$\operatorname{rank} \begin{bmatrix} sI_{n_0} - A_{00} & -A_{01} & -A_{02} & B_{01} & B_{02} \\ -A_{10} & -A_{11} & 0 & B_{11} & 0 \\ -A_{20} & 0 & -A_{22} & 0 & B_{22} \end{bmatrix} = N \tag{9a}$$

$$\operatorname{rank} \begin{bmatrix} sI_{n_0} - A_{00}^T & -A_{10}^T & -A_{20}^T & C_{10}^T & C_{20}^T \\ -A_{01}^T & -A_{11}^T & 0 & C_{11}^T & 0 \\ -A_{02}^T & 0 & -A_{22}^T & 0 & C_{22}^T \end{bmatrix} = N \tag{9b}$$

where $\mathrm{Re}[s] \ge 0$, $s \in \mathbf{C}$, $N = n_0 + n_1 + n_2$.

Using similar proofs in [5], [20], [21], the following lemmas can be proved.

Lemma 1: Under Assumptions 1–3, there exist a matrix $B_s \in \mathbf{R}^{n_0 \times M}$ and a matrix C_s with the same dimension as $[\,C_{10}^T \quad C_{20}^T\,]^T$ such that $S_s = B_s R^{-1} B_s^T$, $Q_s = C_s^T C_s$. Moreover, the triple (A_s, B_s, C_s) is stabilizable and detectable.

Lemma 2: Under Assumptions 1–3, there exists a small σ^* such that for all $\|\mu\| \in (0, \sigma^*)$ the MARE (5) admits a symmetric positive–semidefinite stabilizing solution P_ε, which can be written as

$$P_\varepsilon = \begin{bmatrix} \overline{P}_{00} + O(\|\mu\|) & \varepsilon_1 \left(\overline{P}_{10} + O(\|\mu\|)\right)^T & \varepsilon_2 \left(\overline{P}_{20} + O(\|\mu\|)\right)^T \\ \varepsilon_1 \left(\overline{P}_{10} + O(\|\mu\|)\right) & \varepsilon_1 \left(\overline{P}_{11} + O(\|\mu\|)\right) & \sqrt{\varepsilon_1 \varepsilon_2} O(\|\mu\|) \\ \varepsilon_2 \left(\overline{P}_{20} + O(\|\mu\|)\right) & \sqrt{\varepsilon_1 \varepsilon_2} O(\|\mu\|) & \varepsilon_2 \left(\overline{P}_{22} + O(\|\mu\|)\right) \end{bmatrix} \tag{10}$$

where

$$A_s^T \overline{P}_{00} + \overline{P}_{00} A_s - \overline{P}_{00} S_s \overline{P}_{00} + Q_s = 0 \tag{11a}$$

$$\overline{P}_{j0} = [\,\overline{P}_{jj} \quad -I_{n_j}\,] T_{jj}^{-1} T_{j0} \begin{bmatrix} I_{n_0} \\ \overline{P}_{00} \end{bmatrix} \tag{11b}$$

$$A_{jj}^T \overline{P}_{jj} + \overline{P}_{jj} A_{jj} - \overline{P}_{jj} S_{jj} \overline{P}_{jj} + Q_{jj} = 0 \tag{11c}$$

$$T_s = T_{00} - T_{01} T_{11}^{-1} T_{10} - T_{02} T_{22}^{-1} T_{20} = \begin{bmatrix} A_s & -S_s \\ -Q_s & -A_s^T \end{bmatrix}$$

$$T_{00} = \begin{bmatrix} A_{00} & -S_{00} \\ -Q_{00} & -A_{00}^T \end{bmatrix} \quad T_{0j} = \begin{bmatrix} A_{0j} & -S_{0j} \\ -Q_{0j} & -A_{j0}^T \end{bmatrix}$$

$$T_{j0} = \begin{bmatrix} A_{j0} & -S_{0j}^T \\ -Q_{0j}^T & -A_{0j}^T \end{bmatrix} \quad T_{jj} = \begin{bmatrix} A_{jj} & -S_{jj} \\ -Q_{jj} & -A_{jj}^T \end{bmatrix}$$

$$j = 1, 2.$$

IV. Iterative Algorithm

In this note, we develop an elegant and simple algorithm which converges globally to the positive–semidefinite solution of the MARE (5). The algorithm uses 1) the GMALE, which has to be solved iteratively, and 2) is based on the Kleinman algorithm [9]. We propose the following algorithm for solving the MARE (5):

$$\left(A - SP^{(i)}\right)^T P^{(i+1)} + P^{(i+1)T} \left(A - SP^{(i)}\right) + P^{(i)T} SP^{(i)} + Q = 0, \qquad i = 0, 1, 2, \ldots \tag{12}$$

$$P_\varepsilon^{(i)} = \Phi_\varepsilon P^{(i)} = P^{(i)T} \Phi_\varepsilon$$

$$P^{(i)} = \begin{bmatrix} P_{00}^{(i)} & \varepsilon_1 P_{10}^{(i)T} & \varepsilon_2 P_{20}^{(i)T} \\ P_{10}^{(i)} & P_{11}^{(i)} & \frac{1}{\sqrt{\alpha}} P_{21}^{(i)T} \\ P_{20}^{(i)} & \sqrt{\alpha} P_{21}^{(i)} & P_{22}^{(i)} \end{bmatrix}$$

$$A = \Phi_\varepsilon A_\varepsilon, \qquad S = \Phi_\varepsilon S_\varepsilon \Phi_\varepsilon$$

with the initial condition

$$P^{(0)} = \begin{bmatrix} \overline{P}_{00} & \varepsilon_1 \overline{P}_{10}^T & \varepsilon_2 \overline{P}_{20}^T \\ \overline{P}_{10} & \overline{P}_{11} & 0 \\ \overline{P}_{20} & 0 & \overline{P}_{22} \end{bmatrix} \tag{13}$$

where $\Phi_\varepsilon = \text{block-diag}(\, I_{n_0} \;\; \varepsilon_1 I_{n_1} \;\; \varepsilon_2 I_{n_2}\,)$ and $\overline{P}_{pq}$, $pq = 00, 10, 20, 11, 22$ are defined by (11).

The algorithm (12) has the feature given in the following theorem.

Theorem 1: Under Assumptions 1–3, there exists a small $\overline{\sigma}$ such that for all $\|\mu\| \in (0, \overline{\sigma})$, $\overline{\sigma} \le \sigma^*$ the iterative algorithm (12) converges to the exact solution $P_\varepsilon^* = \Phi_\varepsilon P^* = P^{*T} \Phi_\varepsilon$ with the rate of quadratic convergence, where $P_\varepsilon^{(i)} = \Phi_\varepsilon P^{(i)} = P^{(i)T} \Phi_\varepsilon$ is positive semidefinite. Moreover, zero-order solution $P^{(0)}$ is in the neighborhood of the exact solution P_ε^*. That is, the following conditions are satisfied:

$$\left\|P^{(i)} - P^*\right\| \le \frac{O\left(\|\mu\|^{2^i}\right)}{2^i \beta\gamma} = O\left(\|\mu\|^{2^i}\right), \qquad i = 0, 1, 2, \ldots \tag{14a}$$

$$\left\|P^{(0)} - P^*\right\| \le \frac{1}{\beta\gamma}\left[1 - \sqrt{1 - 2\theta}\right] \tag{14b}$$

where

$$\gamma := 2\|S\| < \infty \quad \beta := \|[\nabla \mathcal{F}(\mathcal{P}_0)]^{-1}\| \quad \theta := \beta\eta\gamma$$

with

$$\eta := \beta \cdot \|\mathcal{F}(\mathcal{P}_0)\| \quad \mathcal{F}(\mathcal{P}) := \begin{bmatrix} \text{vec}F_{00} \\ \text{vec}F_{10} \\ \text{vec}F_{20} \\ \text{vec}F_{11} \\ \text{vec}F_{21} \\ \text{vec}F_{22} \end{bmatrix}$$

$$\mathcal{G}(P) := A^T P + P^T A - P^T SP + Q = \begin{bmatrix} F_{00} & F_{10}^T & F_{20}^T \\ F_{10} & F_{11} & F_{21}^T \\ F_{20} & F_{21} & F_{22} \end{bmatrix}$$

and

$$\nabla \mathcal{F}(\mathcal{P}) := \frac{\partial \mathcal{F}(\mathcal{P})}{\partial \mathcal{P}^T} \quad \mathcal{P} = \begin{bmatrix} \text{vec}P_{00} \\ \text{vec}P_{10} \\ \text{vec}P_{20} \\ \text{vec}P_{11} \\ \text{vec}P_{21} \\ \text{vec}P_{22} \end{bmatrix} \quad \mathcal{P}_0 = \begin{bmatrix} \text{vec}\overline{P}_{00} \\ \text{vec}\overline{P}_{10} \\ \text{vec}\overline{P}_{20} \\ \text{vec}\overline{P}_{11} \\ 0 \\ \text{vec}\overline{P}_{22} \end{bmatrix}.$$

Proof: This proof is equivalent to the proof of existence of the unique solution for the following generalized multiparameter algebraic Riccati equation (GMARE)

$$A^T P + P^T A - P^T SP + Q = 0. \tag{15}$$

The proof follows directly by applying Newton–Kantorovich theorem [14], [15] for the GMARE (15). We now verify that function $\mathcal{G}(P)$ is differentiable on a certain convex set $\mathcal{D}$. Using the fact that

$$\begin{aligned} \nabla \mathcal{G}(P) &:= \frac{\partial \text{vec}\mathcal{G}(P)}{\partial (\text{vec}P)^T} \\ &= [(A - SP)^T \otimes I_N] U_{NN} + I_N \otimes (A - SP)^T \\ &= (I_{N^2} + U_{NN}) \cdot [I_N \otimes (A - SP)^T] \end{aligned} \tag{16}$$

we have

$$\begin{aligned} &\|\nabla \mathcal{G}(P_1) - \nabla \mathcal{G}(P_2)\| \le \gamma \|P_1 - P_2\| \\ \Rightarrow\ &\|\nabla \mathcal{F}(\mathcal{P}_1) - \nabla \mathcal{F}(\mathcal{P}_2)\| \le \gamma \|\mathcal{P}_1 - \mathcal{P}_2\| \end{aligned}$$

where $\gamma = 2\|S\|$. Moreover, using the following result established in [22]:

$$\det \nabla \mathcal{F}(\mathcal{P}_0) = \prod_{j=1}^{5} \det J_{jj} \cdot \det\left[I_{n_0} \otimes D_0^T + D_0^T \otimes I_{n_0}\right] + O(\|\mu\|)$$

where

$$\begin{aligned} J_{jj} &:= D_{jj}^T \otimes I_{n_0} \quad J_{33} := I_{n_1} \otimes D_{11}^T + D_{11}^T \otimes I_{n_1} \\ J_{44} &:= \sqrt{\alpha} D_{22}^T \otimes I_{n_1} + \frac{1}{\sqrt{\alpha}} I_{n_2} \otimes D_{11}^T \\ J_{55} &:= I_{n_2} \otimes D_{22}^T + D_{22}^T \otimes I_{n_2} \\ D_0 &:= D_{00} - D_{01} D_{11}^{-1} D_{10} - D_{02} D_{22}^{-1} D_{20} \\ D_{00} &:= A_{00} - S_{00} \overline{P}_{00} - S_{01} \overline{P}_{10} - S_{02} \overline{P}_{20} \\ D_{0j} &:= A_{0j} - S_{0j} \overline{P}_{jj} \quad D_{j0} := A_{j0} - S_{0j}^T \overline{P}_{00} - S_{jj} \overline{P}_{j0} \\ D_{jj} &:= A_{jj} - S_{jj} \overline{P}_{jj}, \qquad j = 1, 2 \end{aligned}$$

it is shown that there exists a small $\overline{\sigma}$ such that for sufficiently small parameter $\|\mu\| \in (0, \overline{\sigma})$, $\overline{\sigma} \le \sigma^*$, $\nabla \mathcal{F}(\mathcal{P}_0)$ is nonsingular because D_{11}, D_{22} and $D_0 = A_s - S_s \overline{P}_{00}$ are stable under Assumptions 2 and 3 (see, e.g., [1, Th. 1]). Therefore, there exists β such that $\|[\nabla \mathcal{F}(\mathcal{P}_0)]^{-1}\| \equiv \beta$. On the other hand, we verify that $\|\mathcal{F}(\mathcal{P}_0)\| = O(\|\mu\|)$ because $\overline{A}^T P^{(0)} + P^{(0)T} \overline{A} - P^{(0)T} S^{(0)} P^{(0)} + Q = 0$, $A = \overline{A} + O(\|\mu\|)$. Hence, there exists η such that $\|[\nabla \mathcal{F}(\mathcal{P}_0)]^{-1}\| \cdot \|\mathcal{F}(\mathcal{P}_0)\| \equiv \eta = O(\|\mu\|)$. Thus, there exists θ such that $\theta \equiv \beta\gamma\eta < 2^{-1}$ because $\eta = O(\|\mu\|)$. Using the Newton–Kantorovich theorem, the strict error estimate is given by (14a). Furthermore, since $\|P^* - P^{(0)}\| = O(\|\mu\|)$ holds for the small ε_j, $j = 1, \ldots, 4$, we show that P^* is the unique solution in the subset $\mathcal{S} \equiv \{P: \|P - P^{(0)}\| \le (\beta\gamma)^{-1}[1 - \sqrt{1-2\theta}]\}$. On the other hand, using (14a), we have

$$P_\varepsilon^{(i)} = \begin{bmatrix} \overline{P}_{00}+O(\|\mu\|) & \varepsilon_1\left(\overline{P}_{10}+O(\|\mu\|)\right)^T & \varepsilon_2\left(\overline{P}_{20}+O(\|\mu\|)\right)^T \\ \varepsilon_1\left(\overline{P}_{10}+O(\|\mu\|)\right) & \varepsilon_1\left(\overline{P}_{11}+O(\|\mu\|)\right) & \sqrt{\varepsilon_1\varepsilon_2}O(\|\mu\|)^T \\ \varepsilon_2\left(\overline{P}_{20}+O(\|\mu\|)\right) & \sqrt{\varepsilon_1\varepsilon_2}O(\|\mu\|) & \varepsilon_2\left(\overline{P}_{22}+O(\|\mu\|)\right) \end{bmatrix}.$$

Since $\overline{P}_{00} \ge 0$, $\overline{P}_{11} \ge 0$, and $\overline{P}_{22} \ge 0$, $P_\varepsilon^{(i)}$ is positive semidefinite as long as $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ by using the Schur complement [16]. Therefore, the proof is completed. ■

We must solve the GMALE (12) with the dimension N larger than the dimension n_j, $j = 0, 1, 2$ compared with the exact decomposition technique [7], [8]. Thus, in order to reduce the dimension of the workspace, a new algorithm for solving the GMALE (12), which is based on the fixed point algorithm, is established. Let us consider the following GMALE (17), in a general form:

$$\Lambda^T Y + Y^T \Lambda + U = 0 \tag{17}$$

where Y is the solution of the GMALE (17), and Λ and U are known matrices defined by

$$Y = \begin{bmatrix} Y_{00} & \varepsilon_1 Y_{10}^T & \varepsilon_2 Y_{20}^T \\ Y_{10} & Y_{11} & \frac{1}{\sqrt{\alpha}}Y_{21}^T \\ Y_{20} & \sqrt{\alpha}Y_{21} & Y_{22} \end{bmatrix}$$
$$\Lambda = \begin{bmatrix} \Lambda_{00} & \Lambda_{01} & \Lambda_{02} \\ \Lambda_{10} & \Lambda_{11} & \mathcal{E}\Lambda_{12} \\ \Lambda_{20} & \mathcal{E}\Lambda_{21} & \Lambda_{22} \end{bmatrix} \quad U = \begin{bmatrix} U_{00} & U_{01} & U_{02} \\ U_{01}^T & U_{11} & \mathcal{E}U_{12} \\ U_{02}^T & \mathcal{E}U_{12}^T & U_{22} \end{bmatrix}$$
$$Y_{00} = Y_{00}^T \quad Y_{11} = Y_{11}^T \quad Y_{22} = Y_{22}^T$$
$$U_{00} = U_{00}^T \quad U_{11} = U_{11}^T \quad U_{22} = U_{22}^T$$
$$Y_{00},\, \Lambda_{00},\, U_{00} \in \mathbf{R}^{n_0\times n_0} \quad Y_{11},\, \Lambda_{11},\, U_{11} \in \mathbf{R}^{n_1\times n_1}$$
$$Y_{22},\, \Lambda_{22},\, U_{22} \in \mathbf{R}^{n_2\times n_2} \quad \mathcal{E} = \sqrt{\varepsilon_1\varepsilon_2}.$$

In order to solve the GMALE (17) corresponding to the iterative algorithm (12), we need another assumption.

Assumption 4: Λ_{11}, Λ_{22} and $\Lambda_0 := \Lambda_{00} - \Lambda_{01}\Lambda_{11}^{-1}\Lambda_{10} - \Lambda_{02}\Lambda_{22}^{-1}\Lambda_{20}$ are stable.

We propose the following algorithm (18) for solving the GMALE (17):

$$\sqrt{\alpha}Y_{21}^{(i+1)T}\Lambda_{22} + \frac{1}{\sqrt{\alpha}}\Lambda_{11}^T Y_{21}^{(i+1)T} + \varepsilon_1 Y_{10}^{(i)}\Lambda_{02} + \varepsilon_2\Lambda_{01}^T Y_{20}^{(i)T} + \mathcal{E}\left(Y_{11}^{(i)}\Lambda_{12} + \Lambda_{21}^T Y_{22}^{(i)}\right) + \mathcal{E}U_{12} = 0 \tag{18a}$$

$$\Lambda_{11}^T Y_{11}^{(i+1)} + Y_{11}^{(i+1)}\Lambda_{11} + \varepsilon_1\left(\Lambda_{01}^T Y_{10}^{(i)T} + Y_{10}^{(i)}\Lambda_{01} + \Lambda_{21}^T Y_{21}^{(i)} + Y_{21}^{(i)T}\Lambda_{21}\right) + U_{11} = 0 \tag{18b}$$

$$\Lambda_{22}^T Y_{22}^{(i+1)} + Y_{22}^{(i+1)}\Lambda_{22} + \varepsilon_2\left(\Lambda_{02}^T Y_{20}^{(i)T} + Y_{20}^{(i)}\Lambda_{02} + \Lambda_{12}^T Y_{21}^{(i)T} + Y_{21}^{(i)}\Lambda_{12}\right) + U_{22} = 0 \tag{18c}$$

$$\Lambda_0^T Y_{00}^{(i+1)} + Y_{00}^{(i+1)}\Lambda_0 - \Lambda_{10}^T\Lambda_{11}^{-T}\Xi_{10}^{(i)} - \Xi_{10}^{(i)T}\Lambda_{11}^{-1}\Lambda_{10} - \Lambda_{20}^T\Lambda_{22}^{-T}\Xi_{20}^{(i)} - \Xi_{20}^{(i)T}\Lambda_{22}^{-1}\Lambda_{20} + U_{00} = 0 \tag{18d}$$

$$Y_{j0}^{(i+1)} = -\Lambda_{jj}^{-T}\left(\Lambda_{0j}^T Y_{00}^{(i+1)} + \Xi_{j0}^{(i)}\right), \qquad j = 1, 2 \tag{18e}$$

where

$$\Xi_{10}^{(i)} = \mathcal{E}\Lambda_{21}^T Y_{20}^{(i)} + \varepsilon_1 Y_{10}^{(i)}\Lambda_{00} + Y_{11}^{(i+1)T}\Lambda_{10} + \sqrt{\alpha}Y_{21}^{(i+1)T}\Lambda_{20} + U_{01}^T$$
$$\Xi_{20}^{(i)} = \mathcal{E}\Lambda_{12}^T Y_{10}^{(i)} + \varepsilon_2 Y_{20}^{(i)}\Lambda_{00} + Y_{22}^{(i+1)T}\Lambda_{20} + \frac{1}{\sqrt{\alpha}}Y_{21}^{(i+1)}\Lambda_{10} + U_{02}^T$$
$$Y_{10}^{(0)} = \overline{Y}_{10} \quad Y_{20}^{(0)} = \overline{Y}_{20} \quad Y_{11}^{(0)} = \overline{Y}_{11}$$
$$Y_{22}^{(0)} = \overline{Y}_{22} \quad Y_{21}^{(0)} = 0, \qquad i = 0, 1, 2, \ldots$$
$$\Lambda_0^T\overline{Y}_{00} + \overline{Y}_{00}\Lambda_0 - \Lambda_{10}^T\Lambda_{11}^{-T}U_{01}^T - U_{01}\Lambda_{11}^{-1}\Lambda_{10} - \Lambda_{20}^T\Lambda_{22}^{-T}U_{02}^T - U_{02}\Lambda_{22}^{-1}\Lambda_{20} + \Lambda_{10}^T\Lambda_{11}^{-T}U_{11}\Lambda_{11}^{-1}\Lambda_{10} + \Lambda_{20}^T\Lambda_{22}^{-T}U_{22}\Lambda_{22}^{-1}\Lambda_{20} + U_{00} = 0$$
$$\overline{Y}_{j0}^T = -\left(\overline{Y}_{00}\Lambda_{0j} + \Lambda_{j0}^T\overline{Y}_{jj} + U_{0j}\right)\Lambda_{jj}^{-1}$$
$$\Lambda_{jj}^T\overline{Y}_{jj} + \overline{Y}_{jj}\Lambda_{jj} + U_{jj} = 0, \qquad j = 1, 2.$$

The following theorem indicates the convergence of the algorithm (18).

Theorem 2: Under Assumption 4, the fixed-point algorithm (18) converges to the exact solution Y_{pq} with the rate of convergence of $O(\|\mu\|^{i+1})$, that is

$$\left\|Y_{pq}^{(i)} - Y_{pq}\right\| = O\left(\|\mu\|^{i+1}\right), \qquad i = 0, 1, 2, \ldots \quad pq = 00, 10, 20, 11, 21, 22. \tag{19}$$

Proof: Since the proof of Theorem 2 can be done by using mathematical induction similarly as in [19], it is omitted. For the fixed-point algorithm; see, e.g., [7] and [8]. ■

Using a similar technique in [18], the high-order approximate feedback controller is given. Such a linear state feedback controller is obtained by using the iterative solution of (12)

$$u_{\rm app}^{(i)}(t) = -R^{-1}B^T P^{(i)}x(t), \qquad i = 0, 1, 2, \ldots. \tag{20}$$

Theorem 3: Under Assumptions 1–3, the use of the high-order approximate control (20) results in $J_{\rm app}^{(i)}$ satisfying

$$J_{\rm app}^{(i)} = J_{\rm opt} + O\left(\|\mu\|^{2^{i+1}}\right), \qquad i = 0, 1, 2, \ldots. \tag{21}$$

Proof: When $u_{\rm app}^{(i)}$ is used, the value of the performance index is

$$J_{\rm app}^{(i)} = \tfrac{1}{2}x^T(0)W_\varepsilon^{(i)}x(0) \tag{22}$$

where $W_\varepsilon^{(i)}$ is a positive–semidefinite solution of the multiparameter algebraic Lyapunov equation (MALE)

$$\left(A_\varepsilon - S_\varepsilon P_\varepsilon^{(i)}\right)^T W_\varepsilon^{(i)} + W_\varepsilon^{(i)}(A_\varepsilon - S_\varepsilon P_\varepsilon^{(i)}) + P_\varepsilon^{(i)}S_\varepsilon P_\varepsilon^{(i)} + Q = 0. \tag{23}$$

Subtracting (5) from (23) we find that $V_\varepsilon^{(i)} = W_\varepsilon^{(i)} - P_\varepsilon$ satisfies the following MALE:

$$\left(A_\varepsilon - S_\varepsilon P_\varepsilon^{(i)}\right)^T V_\varepsilon^{(i)} + V_\varepsilon^{(i)}\left(A_\varepsilon - S_\varepsilon P_\varepsilon^{(i)}\right) + \left(P_\varepsilon - P_\varepsilon^{(i)}\right)S_\varepsilon\left(P_\varepsilon - P_\varepsilon^{(i)}\right) = 0. \tag{24}$$

Note that (12) is equivalent to the following MALE:

$$\left(A_\varepsilon - S_\varepsilon P_\varepsilon^{(i)}\right)^T P_\varepsilon^{(i+1)} + P_\varepsilon^{(i+1)}\left(A_\varepsilon - S_\varepsilon P_\varepsilon^{(i)}\right) + P_\varepsilon^{(i)}S_\varepsilon P_\varepsilon^{(i)} + Q = 0. \tag{25}$$

TABLE I
EXACT SOLUTION OF THE MARE (5)

$$P_{\varepsilon}=\begin{bmatrix} P_{00} & \varepsilon_1 P_{10}^T & \varepsilon_2 P_{20}^T \\ \varepsilon_1 P_{10} & \varepsilon_1 P_{11} & \sqrt{\varepsilon_1\varepsilon_2}P_{21}^T \\ \varepsilon_2 P_{20} & \sqrt{\varepsilon_1\varepsilon_2}P_{21} & \varepsilon_2 P_{22} \end{bmatrix}$$

$$P_{00}=\begin{bmatrix} 5.5462e+00 & 8.2344e-01 & 4.6229e+01 & -1.4711e-01 & 2.6640e-01 \\ 8.2344e-01 & 5.5462e+00 & -1.4711e-01 & 4.6229e+01 & -2.6640e-01 \\ 4.6229e+01 & -1.4711e-01 & 6.4114e+02 & -2.3198e+02 & 5.1277e+00 \\ -1.4711e-01 & 4.6229e+01 & -2.3198e+02 & 6.4114e+02 & -5.1277e+00 \\ 2.6640e-01 & -2.6640e-01 & 5.1277e+00 & -5.1277e+00 & 1.3178e+00 \end{bmatrix}$$

$$\varepsilon_1 P_{10}=\begin{bmatrix} 9.1182e-02 & -9.6817e-05 & 1.2538e+00 & -4.5653e-01 & 8.5347e-03 \\ 4.4721e-02 & 1.3989e-17 & 6.0236e-01 & -2.2353e-01 & 3.8104e-03 \end{bmatrix}$$

$$\varepsilon_2 P_{20}=\begin{bmatrix} -9.6817e-05 & 9.1182e-02 & -4.5653e-01 & 1.2538e+00 & -8.5347e-03 \\ 4.6754e-17 & 4.4721e-02 & -2.2353e-01 & 6.0236e-01 & -3.8104e-03 \end{bmatrix}$$

$$\varepsilon_1 P_{11}=\begin{bmatrix} 7.4660e-03 & 2.8529e-03 \\ 2.8529e-03 & 3.8886e-03 \end{bmatrix},\quad \varepsilon_2 P_{22}=\begin{bmatrix} 7.4660e-03 & 2.8529e-03 \\ 2.8529e-03 & 3.8886e-03 \end{bmatrix}$$

$$\sqrt{\varepsilon_1\varepsilon_2}P_{21}=\begin{bmatrix} -9.0046e-04 & -4.4140e-04 \\ -4.4140e-04 & -2.1649e-04 \end{bmatrix}$$

Similarly, subtracting (5) from (25) we also get the MALE

$$\left(A_\varepsilon - S_\varepsilon P_\varepsilon^{(i)}\right)^T\left(P_\varepsilon^{(i+1)}-P_\varepsilon\right)+\left(P_\varepsilon^{(i+1)}-P_\varepsilon\right)\left(A_\varepsilon - S_\varepsilon P_\varepsilon^{(i)}\right) + \left(P_\varepsilon - P_\varepsilon^{(i)}\right)S_\varepsilon\left(P_\varepsilon - P_\varepsilon^{(i)}\right)=0. \quad (26)$$

It is easy to verify that $V_\varepsilon^{(i)} = P_\varepsilon^{(i+1)} - P_\varepsilon$ because $A_\varepsilon - S_\varepsilon P_\varepsilon^{(i)}$ is stable for all $i = 0, 1, 2, \ldots$. Using Theorem 1 we obtain that

$$\left\|V_\varepsilon^{(i)}\right\| = \left\|W_\varepsilon^{(i)} - P_\varepsilon\right\| = \left\|P_\varepsilon^{(i+1)} - P_\varepsilon\right\| = O\left(\|\mu\|^{2^{i+1}}\right). \quad (27)$$

Hence, $V_\varepsilon^{(i)} = W_\varepsilon^{(i)} - P_\varepsilon = O(\|\mu\|^{2^{i+1}})$, $i = 0, 1, 2, \ldots$, which implies (21). ■

Consequently, when ε_j are known, we can get the high-order $O(\|\mu\|^{2^i})$ approximate controller which achieves $O(\|\mu\|^{2^{i+1}})$ approximation of the optimal cost by performing iterations on the reduced-order ALE (18).

In addition, we will present an important implication. If the parameter ε_j are unknown, then, in view of Theorem 3, the following corollary is easily seen by using a similar technique [17].

Corollary 1: Under Assumptions 1–3, the use of the parameter-independent controller

$$u_{\rm app}(t) = -R^{-1}B^T\tilde{P}x(t) = -R^{-1}B^T\begin{bmatrix} \overline{P}_{00} & 0 & 0 \\ \overline{P}_{10} & \overline{P}_{11} & 0 \\ \overline{P}_{20} & 0 & \overline{P}_{22} \end{bmatrix}x(t) \quad (28)$$

results in $J_{\rm app}$ satisfying

$$J_{\rm app} = J_{\rm opt} + O(\|\mu\|^2). \quad (29)$$

Proof: Since the result of Corollary 1 can be proved by using the similar technique in Theorem 3 under the fact that $P - \tilde{P} = O(\|\mu\|)$, the proof is omitted.

It is worth to note that the $O(\|\mu\|^2)$ near-optimality is proved for the first time to the optimal control problem of the MSPS [2], [6].

V. NUMERICAL EXAMPLE

In order to demonstrate the efficiency of our proposed algorithm, we have run a numerical example. The system matrix is given as a modification of [1, Appendix A]

$$A_{00}=\begin{bmatrix} 0 & 0 & 4.5 & 0 & 1 \\ 0 & 0 & 0 & 4.5 & -1 \\ 0 & 0 & -0.05 & 0 & -0.1 \\ 0 & 0 & 0 & -0.05 & 0.1 \\ 0 & 0 & 32.7 & -32.7 & 0 \end{bmatrix}$$

$$A_{01}=\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0.1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\quad A_{02}=\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0.1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$A_{10}=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.4 & 0 & 0 \end{bmatrix}\quad A_{20}=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.4 & 0 \end{bmatrix}$$

$$A_{11}=A_{22}=\begin{bmatrix} -0.05 & 0.05 \\ 0 & -0.1 \end{bmatrix}$$

$$B_{01}=B_{02}=[0 \quad 0 \quad 0 \quad 0 \quad 0]^T \quad B_{11}=B_{22}=\begin{bmatrix} 0 \\ 0.1 \end{bmatrix}$$

$$Q = \operatorname{diag}(1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 0.5 \quad 0.5 \quad 0.5 \quad 0.5)$$
$$R = \operatorname{diag}(20 \quad 20).$$

TABLE II
ERROR PER ITERATIONS

i	$\|\mathcal{G}(P^{(i)})\|$
0	$1.2916e-00$
1	$1.4350e-02$
2	$7.9244e-05$
3	$2.5918e-12$

TABLE III
ERROR $\|\mathcal{G}(P)\|$

$\varepsilon_1=\varepsilon_2$	Kleinman algorithm	MATLAB
$10e-2$	$5.6646e-11$	$2.4062e-10$
$10e-3$	$1.4637e-11$	$1.2534e-09$
$10e-4$	$3.4385e-11$	$4.4533e-08$
$10e-5$	$9.3788e-12$	$1.5047e-06$
$10e-6$	$1.3580e-11$	$5.7044e-04$
$10e-7$	$2.8934e-12$	$4.6487e-04$
$10e-8$	$1.9069e-11$	$3.1307e-01$

TABLE IV
CPU TIME [s]

$\varepsilon_1=\varepsilon_2$	Kleinman algorithm	MATLAB
$10e-2$	$2.5620e-01$	$2.6600e-02$
$10e-3$	$8.7500e-02$	$2.6600e-02$
$10e-4$	$6.4100e-02$	$2.5000e-02$
$10e-5$	$5.6300e-02$	$2.8100e-02$
$10e-6$	$4.6900e-02$	$2.5000e-02$
$10e-7$	$4.5300e-02$	$2.6600e-02$
$10e-8$	$3.9100e-02$	$2.9700e-02$

TABLE V
LOSS OF THE COST FUNCTIONAL

i	$E[J_{\rm app}^{(i)}]$	$E[J_{\rm app}^{(i)}]-E[J_{\rm opt}]$	η [%]
0	$1.2631e-03$	$3.2818e-07$	$2.5988e-02$
1	$1.2628e-03$	$5.1635e-11$	$4.0889e-06$
2	$1.2628e-03$	$5.2042e-18$	$4.1212e-13$
3	$1.2628e-03$	$8.6736e-19$	$6.8686e-14$
optimal	$1.2628e-03$	–	–

The small parameters are chosen as $\varepsilon_1=\varepsilon_2=0.001$ and $\varepsilon_3=\varepsilon_4=0$. Note that we cannot apply the technique proposed in [7] and [8] to the MSPS since the Hamiltonian matrices T_{jj}, $j=1,2$ have eigenvalues in common. We give a solution of the MARE (5) in Table I. We find that the algorithm (12) converges to the exact solution with accuracy of $\|\mathcal{G}(P^{(i)})\|<10^{-10}$ after three iterations. In order to verify the exactitude of the solution, we calculate the remainder per iteration by substituting $P_\varepsilon^{(i)}$ into the MARE (5). In Table II, we present results for the error $\|\mathcal{G}(P^{(i)})\|$ per iterations. It can be seen that the initial guess (13) for the algorithm (12) is quite good.

In order to verify the exactitude of the solution, we substitute the obtained reference solution $P_\varepsilon^{\rm sch}$ by using the function `are` of MATLAB into the MARE (5). We find that the remainder is $\|\mathcal{G}(\Phi_\varepsilon^{-1}P_\varepsilon^{\rm sch})\|=1.2534e-09$. For different values of ε_1 and ε_2, the remainder of the algorithm (12) versus MATLAB are given by Table III. From Table III, we see that the resulting algorithm of this note is very useful. In Table IV, we give the results of the CPU time when we have run the new method versus MATLAB. The CPU time represents the average based on the computations of ten runs. From Table IV, even if the iterative algorithm (12) takes a lot of CPU time in case of not very small value of the singular perturbation parameter, our algorithm can obtain the exact solution.

Finally, we evaluate the costs using the high-order controller (20). We assume that the initial conditions are zero mean independent random vectors with covariance matrix

$$E[x(0)x^T(0)]=10^{-4}{\rm diag}\,(1\quad 1\quad 0.01\quad 0.01\quad 1\quad 1\quad 1\quad 1\quad 1).$$

Letting $\varepsilon_1=\varepsilon_2=0.001$, the average value of the cost functional per iterations are given in Table V, where $\eta=((E[J_{\rm app}^{(i)}]-E[J_{\rm opt}])/E[J_{\rm opt}])\times 100$. Table V verifies that $u_{\rm app}^{(i)}(t)$ has improved the cost functional $J_{\rm app}^{(i)}$ as the number of iterations increases. Furthermore, when the perturbation parameters values are not known exactly, the parameter-independent controller (28) with $i=0$ may result in benefiting instead of having loss.

VI. CONCLUSION

In this note, we have studied the near-optimal control problem associated with the MSPS. The main contribution of this note is to propose the new algorithm for solving the MARE. We have shown that our proposed algorithm which is based on the Kleinman algorithm has the quadratic convergence property. In addition, we have also presented a new numerical method for solving the GMALE by a fixed point algorithm. Finally, we have shown that the resulting $O(\|\mu\|^{2^i})$ accuracy controller achieves the cost $J_{\rm opt}+O(\|\mu\|^{2^{i+1}})$.

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