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A RESEARCH ON THE VALIDITY AND EFFECTIVENESS OF "TWO-AXES PROCESS MODEL" OF UNDERSTANDING MATHEMATICS AT ELEMENTARY SCHOOL LEVEL

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Abstract

The understanding mathematics in the process of teaching and learning school mathematics has been a main issue buckled down by some researchers in PME. Koyama (1992) discussed basic components that are substantially common to the process models, and presented the so-called "two-axes process model" of understanding mathematics as a useful and effective framework for mathematics teachers. This research focuses on examining the validity and effectiveness of this model. By analyzing data collected in the case study of third grade mathematics class in a national elementary school, the validity and effectiveness of the "two-axes process model" of understanding mathematics is exemplified. The teaching and learning of mathematics that enables children to understand mathematics deeply and in their meaningful way is characterized as a dialectic process of individual and social constructions. The important teacher's roles are also suggested.

INTRODUCTION

The word "understanding" is very frequently used in both the descriptions of objectives of teaching mathematics in the Course of Study (Ministry of Education, 1989) and in the mathematics teaching practices in Japan. The putting emphasis on understanding mathematics should be desirable in mathematics education, but what it does mean is not clear. Moreover, It is an essential and critical problem for us to know what mathematics teachers should do in order to help children understand mathematics. However, it also has not been made clear sufficiently.

The key for the solution of these educational problems, in my opinion, is ultimately to capture what does it mean children understand mathematics and to make clear the mechanism which enables children's understanding of mathematics develop in the teaching and learning of mathematics. In other words, it might be said to "understand" understanding. It is, however, not easy and we need our great effort to do it. The problem of understanding mathematics has been a main issue buckled down by some researchers, especially from the cognitive psychological point of view in the international group for the psychology of mathematics education (PME). As a result of their works, various models of understanding as the frameworks for describing aspects or processes of children's understanding of mathematics are presented (Skemp, 1976, 1979, 1982; Byers and Herscovics, 1977; Davis, 1978; Herscovics and Bergeron, 1983, 1984, 1985, 1988; Pirie and Kieren, 1989a, 1989b; Hiebert and Carpenter, 1992).

THEORETICAL BACKGROUND: "Two-Axes Process Model" of Understanding Mathematics

Koyama (1992) discussed basic components that are substantially common to the process models, and presented the so-called "two-axes process model" of understanding mathematics. This model of understanding mathematics consists of two axes. On the one hand, the vertical axis implies some hierarchical levels of understanding such as mathematical entities, their relations, and general relations, etc. On the other hand, the horizontal axis implies three learning stages of intuitive, reflective, and analytical at each level of understanding. Through these three stages, not necessarily linear, children's understanding is able to progress from a certain level to a next higher level in the process of teaching and learning mathematics.

Intuitive Stage: Children are provided opportunities for manipulating concrete objects, or operating mathematical concepts or relations acquired in a previous level. At this stage they do *intuitive thinking.*

Reflective Stage: Children are stimulated and encouraged to pay attention to their own manipulative or operative activities, to be aware of them and their consequences, and to represent them in terms of diagrams, figures or languages. At this stage they do *reflective thinking.*

Analytical Stage: Children elaborate their representations to be mathematical ones using mathematical terms, verify the consequences by means of other examples or cases, or analyze the relations among consequences in order to integrate them as a whole. At this stage they do *analytical thinking.*

There are two prominent characteristics in the "two-axes process model". First, it might be noted that the model reflects upon the complementarity of intuition and logical thinking, and that the role of reflective thinking in understanding mathematics is explicitly set up in the model. Second, the model could be a useful and effective one because it has both descriptive and prescriptive characteristics.

PURPOSE AND METHOD

The "two-axes process model" of understanding mathematics is expected as a useful and effective framework for mathematics teachers. In order to demonstrate the usefulness and effectiveness of the model, it is a significant task for us to examine both validity and effectiveness of the model in terms of practices of the teaching and learning of mathematics. Koyama (1996) focused on the validity of the model and demonstrated the validity of three stages at a certain level of understanding mathematics by analyzing data collected in a case study of fifth grade elementary school mathematics class in Japan.

The purpose of this research is to examine the validity and effectiveness of this model more closely by analyzing data collected in the case study of third grade mathematics class at the national elementary school attached to Hiroshima University. This school has two classrooms at each grade from the first to the sixth. Children in a classroom are heterogeneous in the same way as a typical classroom organization in Japanese elementary schools, but their average mathematical ability is higher than that of other children in the local and public schools. The mathematics teacher involved in this research is a collaborating member of our collaborative research project on mathematics education at Hiroshima University. He is an experienced and highly motivated, and has a relatively deep understanding of both elementary school mathematics and children.

A CASE STUDY OF ELEMENTARY SCHOOL MATHEMATICS CLASS

The data was collected in the case study of third grade (9 years old) mathematics class. There were 37 children (20 boys and 17 girls) in the classroom and the mathematics teacher was Mr. Wakisaka. The researcher and teacher made the case study in the mathematics class for introducing fractions to third graders. It was our main objective of the class that children were aware of the possibility of representing fractional parts by an idea of division into equal parts through such activities as placing over and folding fractional parts.

Making a Plan based on the "Two-Axes Process Model"

Therefore, using the "two-axes process model" of understanding mathematics, we planned the class in due consideration of both teaching materials of fraction and the actual state of children in the classroom as follows.

Firstly, we decide to make children use a fictitious unit named "*gel*" as a restriction so that they may construct various ways of representing fractional parts. Secondly, we give children two different fractional parts being based on the fact that as a result of our prior investigation the presentation of not one but two different fractional parts is effective for children be aware of representing fractional parts by making a comparison between them. Moreover, we use 3/5 cup of juice and 2/5 cup of juice as two fractional parts and two rectangular figures of them drawn on a sheet of paper in order to make it possible for children to notice the sum of them or the difference between them, and to place over and fold them. Thirdly, we put great emphasis on making a connection of children's various ways in representing fractional parts. In a whole-classroom discussion, we ask children to report their own solution for representing fractional parts, and encourage them to examine and refine their solutions by discussing about the similarities and differences among them, and then to share the value of various ways of representing fractional parts with others in the classroom.

Putting the Plan into Practice

The process of teaching and learning in the classroom progressed actually as follows. In the following protocol of the class, sign Tn and sign Cn mean the nth teacher's utterance and the nth child's utterance respectively.

Firstly, in order to make an emphasis on the process of abstracting fractional parts, the teacher Mr. Wakisaka told a story using two different bottles of juice, poured each of them into four same-sized cups, and then introduced rectangular figures to represent the volume of them. The teacher planned to make children use a fictitious unit named "*gel*" as a measure of volume. Therefore, he began to tell a fantastic story to his children. [Setting a Problematic Situation]

T1: *Today, let's make a space travel!*

CC: *Yes! Let's go!*

T&CC: *Four, three, two, one; fire!* (For a while)

T2: *Now, we have arrived at a jellyfish-planet. Jellyfish-aliens welcome and give us two different bottles of juice.* (He showed two bottles of juice. He poured each of them into four same-sized cups, and then he introduced rectangular figures to represent them. He prepared 1 3/5 cup of blue-colored juice and 12/5 cup of red-colored juice. But children were given no detail information about their fractional parts.)

T3: *Let's inform jellyfish-aliens of the volume! You should pay attention to the fact that jellyfish-aliens use a unit named "gel" as a measure of volume. But they understand such numerals as 0, 1, 2, 3, 4,...*

CC: *Only gel?*

T4: *Yes! So jellyfish-aliens can't understand units of measure with which you are familiar.*

CC: *Do you mean we can't use units such as dl or ml?*

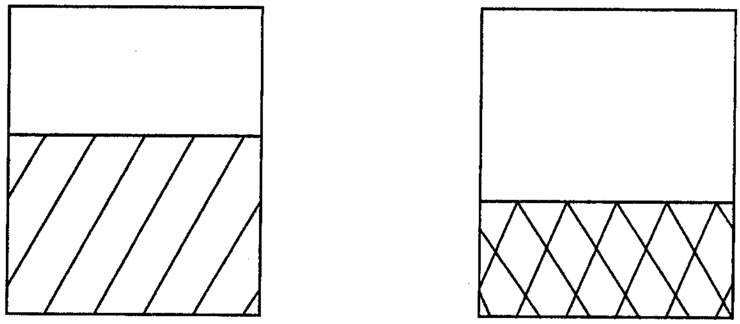
T5: *Yes, I do. You can't use those units on the earth.*

C1: *How should we do to inform them of the volume? If we could use such units as dl and ml, it would be easy. But it is really difficult.*

C2: *We had better to make another convenient unit named "bal."*

T6: *Inform them of the volume of juice somehow.*

In the next step, the teacher gave each child a sheet of paper on which two rectangular figures for two fractional parts of juice were drawn as shown in Figure 1. Children individually did manipulative or operative activities on their own way using the paper for a while. Some children measured the length, while some children cut two fractional parts out of a sheet of paper, placed one figure over another one, and then folded them. During children's activities, the teacher was observing it. [Intuitive Stage]



Blue-colored Juice (FPB) Red-colored Juice (FPR)

Figure1. Rectangular figures for fractional parts

The following shows some types of solution for representing fractional parts of blue-colored juice (FPB) and red-colored juice (FPR) that children invented as a result of their activities at the intuitive stage.

(1) FPB is half and just a little more. FPR is just a little less than half.

(2) The length of FPB is 4*cm* 7*mm*, so it is 4*gel* and 7*gel.*

(3) We had better to make another unit named *deci-gel* as a smaller unit than *gel.*

(4) FPB is less than one *gel,* so it is zero *gel.*

(5) FPB is zero *gel* and 47 *deci-gels.*

(6) The sum of FPB and FPR is one *gel.*

(7) When I divide the whole length of a rectangular figure by the unit of 1cm, it can be divided into eight equal parts. FPR is equal to just three parts, so it is zero point three *gels.*

(8) When I place FPR over FPB and fold them by noticing the difference between them, FPB is equal to three parts and FPR is equal to two parts.

(9) When I divide the whole length of a rectangular figure into ten equal parts, FPB is equal to six parts and FPR is equal to four parts.

In the following step, the teacher asked children to report their own solution for representing fractional parts, and organized a whole-classroom discussion in order to encourage them to examine and refine their solutions. During the discussion, children have paid their attention to their own manipulative or operative activities and represented their own solution in terms of figures or daily-life and mathematical language already learned.

[Reflective Stage]

T7: *Now, we inform jellyfish-aliens of the volume of fractional parts.*

C3: *I can say that FPB is half and just a little more and that FPR is just a little less than half.*

C4: *I have a question. I think the expression such as just a little moreess is not clear.*

T8: *Indeed, so it is. But, how should we do?*

C5: *May I use centimeter?*

T9: *How do you think?*

CC: *We can't use centimeter.*

T10: *Yes, you are right. We can use only gel as a unit. Is there another idea?*

C6: *I have a good idea. FPB is 4gel and 7gel. FPR is 3gel and 1gel.*

CC: *What does it mean?*

T11: *Anyone who can explain this idea?* (For a while)

C7: *Oh, I see. I'm sure that he measured the length with a rule.*

CC: *Yes, we agree with you. The length of FPB is 4cm 7mm, so he changed both units of cm and mm to gel.*

T12: *I see, so it is...*

C8: *Why do you use the same unit gel? They, cm and mm, are different units.*

C9: *Because we are allowed to use only gel as a unit!*

C10: *I say FPB is 4gel and 7deci-gel, because the units of cm and mm are different and we already learned other units of liter and deciliter for measure of volume.*

CC: *That is a good idea. It is easy to represent fractional parts with such a new unit of volume.*

C11: *But, I think that the expression 4gel and 7deci-gel is not reasonable, because 4gel is more than 1gel.*

C12: *So, my idea is better than it. FPB is less than 1gel, so it is zero gel, because we know that the number smaller than 1 is 0.*

C13: *Your idea of zero gel is not reasonable, because it means that there is nothing.*

C14: *Then, zero gel and 47deci-gel is a more reasonable expression.*

These children's way of representing two fractional parts seemed to be mainly based on using an idea of length and inventing new units of volume. The teacher wanted to change the line of such discussion and encouraged children to present other ideas.

T13: *I see. Are there any other ideas?*

C15: *When I divided the whole length of a rectangular figure by the unit of 1cm, it was divided into eight equal parts and FPR was equal to just three parts. So FPR is zero point three gels.*

C16: *What does it mean?*

C17: *It means that FPR is less than one gel and that FPR is equal to three parts.*

C18: *I think that the expression such as FPR is zero point three gel is a similar expression as*

FPB is zero gel and 47deci-gel.

C19: *Even when we divide the whole length into ten equal parts, could you still say FPR is zero point three gels? If FPR was equal to four parts in case of ten-parts division, would you express it as zero point four gels?*

T14: *You mean that we can say FPR is zero point four gels in such a case as it is equal to four parts.*

C20: *I did not consider such case.*

T15: *Do you understand what he wanted to do?*

C21: *It is similar to my idea, but I divided the whole length of a rectangular figure into five equal parts. It is a reason that when I placed FPR over FPB and folded them by noticing the difference between them, each of them could be divided into five equal parts. Then, FPB is equal to three parts and FPR is equal to two parts.*

Finally the teacher encouraged some children to report their own way of representing fractional parts by a division into equal parts, and aimed to help all children be aware of the possibility of representing fractional parts by an idea of division into equal parts. [Analytical Stage]

C22: *When I divided the whole length into ten parts, FPB is equal to six parts and FPR is equal to four parts.*

C23: *Mr., I think there are many ways of division into equal parts.*

CC: *Yes! There are many ways.*

T16: *Oh, there are many ways?*

C24: *Yes! There are as many as we like by multiplying the number of division, for example 8, 16, and 24 or 5, 10, and 15 etc.*

T17: *So, in the next class, we will continue to investigate the way of representing fractional parts less than one gel by using this idea of division into equal parts.*

CONCLUSION

When we made the plan of this mathematics class for introducing fractions to third graders, we used the "two-axes process model" as a framework and embodied it with teaching materials of fractions in due consideration both of the objectives of the class and the actual state of children in the classroom. In the classroom, the restriction of using a fictitious unit named "*gel*" imposed on children, the choice of 3/5 cup and 2/5 cup of juice as two fractional parts, and the two drawn rectangular figures with 8cm length given to children could make it possible to set the problematic situation where they individually invent a new unit and construct various ways of representing fractional parts less than one "*gel*" [intuitive stage]. As a result of their manipulative or operative activities with those figures followed by the constructive interaction in a whole-class discussion, children by themselves examined and refined their solutions [reflective stage]. Finally, children integrated some ideas for representing fractional parts, and in more general sense they could be aware of the possibility of representing fractional parts by using an idea of division into equal parts as a result of social constructions [analytical stage].

Through this research, we find out the followings. First, we can exemplify the validity and effectiveness of the "two-axes process model" of understanding mathematics by this case study of the elementary school mathematics class. Second, we might be able to characterize such a teaching and learning of mathematics that enables children to understand mathematics deeply and in their meaningful way as the dialectic process of children's individual and social constructions in their classroom.

In order to realize such mathematics classroom, it is suggested that a teacher should make a plan of teaching and learning mathematics in the light of "two-axes process model", and embody it with teaching materials of a topic in due consideration both of the objectives and the actual state of children. The teacher also should play a role as a facilitator for the dialectic process of individual and social constructions through a discussion with children and among them. There are two important features of teacher's role in the process of teaching and learning mathematics. The one is related to children's individual construction and it is to set the problematic situation where children are able to have their own learning tasks and encourage them to have various mathematical ideas and ways individually. The other is related to children's social construction and it is to encourage and allow them to explain, share and discuss their various mathematical ideas and ways socially in the classroom.

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