MULTI-WORLD PARADIGM IN MATHEMATICAL LEARNING (1): AN ANALYSIS OF MATHEMATICS CLASSES

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Abstract

The recent researches of mathematical education are under significant influence of three thoughts including radical constructivism, interactionism, and socioculturism. While they are mutually conflicting theoretically, it should be unavoidable to integrate and coordinate the three thoughts in order to explain the realities of mathematical learning. Nakahara (1997a) calls this view "multi-world paradigm" in mathematical learning. We illustrate two analyses of mathematical learning based on the multi-world paradigm and consider it theoretically. Multi-world paradigm suggests the very important direction for research in teaching and learning mathematics forward because educational phenomena are often complex and can't be reduced to a single theory.

INTRODUCTION

Lately, the mathematical education is under significant influence of three thoughts including radical constructivism, interactionism, and socioculturism and a lot of studies are being conducted on the mathematical education based on each of such paradigms or combinations thereof. They have a common aspect: all of three thoughts view that children's activities play very important roles in mathematical learning, and they position social interactions as an important mean in learning.

However, on the other hand, they have distinct differences in any of the view of cognition, language, and learning. In particular, difference in the view of the nature of cognition is a matter of principle that is mutually incompatible, and the integration of those thoughts is considered theoretically impossible.

MULTI -WORLD PARADIGM

Table 1 shows comparisons among radical constructivism, interactionism and socioculturism from three view points. While they are mutually conflicting theoretically, it is

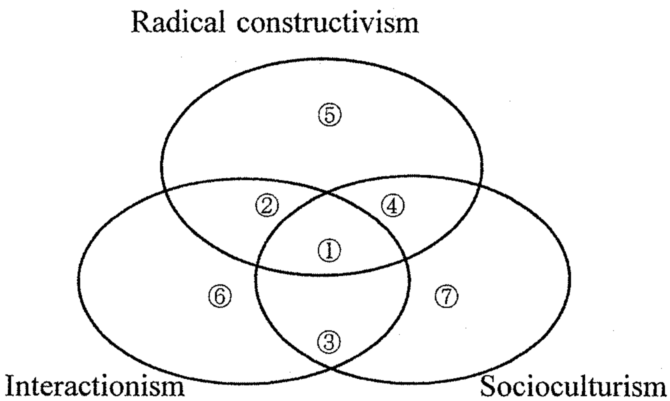
Table 1 The Comparisons among Radical Constructivism, Interactionism and Socioculturism

View points		Radical constructivism	Interactionism	Socioculturism
View of cognition	Nature of cognition	Construction by individuals	Construction by community	Enculturation
	Aims of cognition	Viability	Common cognition	Enculturation by community
	Nature of knowledge	Subjective	Inter-subjective	Social
View of learning	Nature of learning	Sense making by individuals	Sense making by community	Enculturation by community
	Motivation to learn	Cognitive conflict	Social interactions	Participation to cultural practices
	Important methodology	Social interactions, Reflexive thinking	Social interactions	Social interactions Use of tools
	Dependence on	Cognitive structures of individuals	Members in the community	Cultural situation in the community
View of language	Functions or roles	Means of thought expression	Containing a process of interpretation	Medium of cultural transmission
	Inner or Outer	Inner language to outer language	Unity with the activities	Outer language to inner language
Teacher's roles		Learning supporter	Mediator between personal and social meanings	Expert in the community

considered unavoidable to integrate and coordinate the three thoughts in order to explain the realities of learning in fair manner. To simplify the thoughts such points of view, the three thoughts may be regarded as attempts to understand the learning with the learner, other people (teacher and other learners), and culture (contents to learn) placed as the groundwork of their theories. Those three factors are considered fundamental in the learning activities. Focusing on one of those fundamental factors results in the division into radical constructivism, interactionism, and socioculturism. Considering that the learning involves the three significant factors, however, a flexible paradigm is required from a practical point of view that would coordinate them, be mutually complementing, and be beyond common notions.

Such view is comparable to the principle of complementarity and multi-world interpretation in quantum mechanics, which are theories devised to interpret practically observed motions of quantum, while containing theoretical conflict within them. The studies of mathematics learning by Cobb and Bauersfeld (1995) should be considered as have been suggested based on such standpoint. Nakahara (1997a) calls the above-shown view the multi-world paradigm in mathematical learning. It contains the three worlds of the radical constructivism, interactionism, and socioculturism as shown Fig 1.

Then we show practical study of mathematical learning based on the multi-world paradigm.



①: Division for the activities which refers to C, I, and S
②: Division for the activities which refers to C, and I
③: Division for the activities which refers to I, and S
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⑤: Division for the activities which refers to C
⑥: Division for the activities which refers to I
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Fig. 1 The multi-world paradigm

THE ILLUSTRATIONS OF MULTI-WORLD PARADIGM

In the observations and analyses of children's practical activities in mathematical learning, various phases which could be explained by each of the three thoughts are frequently seen at the same class.

First Example of a Class: the Average Speed of the Walks

By way of an example, let's consider a class to solve a problem as set forth below (Nakahara,1997).

⟨Problem⟩

> John walks every day from his home in town K to the school in town H. Yesterday he walked at the speed of 5 kilometers an hour to the point M which situated midway between his home and the school, and at 10 kilometers for the remaining half of the distance to the school. And today, as usual, he hopes to depart his home at the same time as he did yesterday, and to arrive at the school also at the same time. He, however, plans to walk from his home to the school at a constant speed instead of increasing speed midway. At what speed should he walk to school today?
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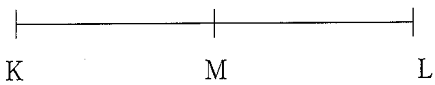
> 
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> Fig. 1

⟨Initial responses⟩

A: (solution assuming the distance as 40 kilometers)

$20 \div 5 = 4$ $\quad$ $20 \div 10 = 2$ $\quad$ $40/6 = 6\frac{2}{3}$

B: $(5+10)/2 = 7.5$

⟨Interactions by children⟩

C1: Assuming the distance as 40 kilometers, the required time with B is represented as (40 / 7.5) which does not agree with 6 hours which is obtained by the method A. That indicates B is not right.

C2: You cannot deny it. In some cases, 7.5 may be right...

C3: The ratio of speeds of 5 to 10 translates into 1 to 2. When the distance is the same, ratio of elapsed times is reversed to 2 to 1.

C4: Then, the time required for covering this part is 2 hours and 1 hour for that part, which gives an equation of a total distance $5\times2+10\times1=x\times3$. Dividing both side by 3 gives the speed value x of 6.

C5: A peculiar view!

C6: A is assuming a constant distance, while B a constant time. If the distance is constant, time will vary because speed is different. This indicates B is not right.

C7: This may be corrected.

C8: I assumed this distance as 20 kilometers and this as 10 kilometers. Since the speed of 5 kilometers an hour was made for two hours, this may be put as $5+5$. And since this part is covered at the speed of 10 kilometers an hour, the distance is covered by a single hour. Then, the solution may be obtained by

C: $(5+5+10)/3=6\frac{2}{3}$

Analysis of this Class Work

In the above class work, two kinds of solutions, A and B, were first presented by the children. These solutions were entirely of the children's original who constructed them. Consequently, children's activities in this phase of the class could be grasped and explained by the view of radical constructivism.

As seen in the interactions by children, a variety of opinions about a solution B were exchanged and eventually transformed into a solution C shown above, thus arriving at a classroom consensus. In this case, the solution was produced by the interactions among children, resulting in the consensus through mutual agreement. Consequently, children's activities in this phase could be grasped and explained by interactionism.

Further, the children in the class gradually arrived at a final conclusion that the alternatives like solution A and solution C were possibly for the question involved with mean of two velocities. The situation could be grasped that the children acquired the existing mathematics culture through participation in the class and therefore the aspect was appreciated as enculturation by children's selves. Accordingly, viewed from the result of the study, the situation could be explicable on the basis of socioculturism.

Thus a single class work has a variety of phases requiring deliberate allocations of the three "ism" for a given situation, since the independent use of any one, the isms, be it radical constructivism, interactionism, or socioculturism, may fail to satisfactorily analyze and account for various learning situations in a class. Moreover, looking closely into the formation of the solution C deemed accountable in terms of interactionism mentioned earlier, it turned out that a boy named Yukio substantially took the initiation in the series of interactions in contriving the solution C. In other words, it was Yukio who constructed the solution C. In this case, the situation could be accounted for on the basis of radical constructivism as well. Still more, it should be reappraised as enculturation, because it was something of an existing methodology.

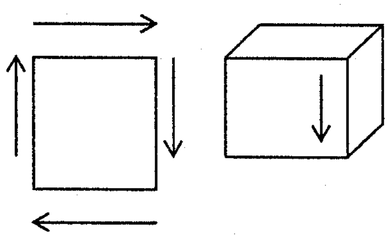
In this way, an individual phase of a class could be attributed to either radical constructivism, interactionism or socioculturism, whichever is of the greatest significance, depending on the viewpoints and circumstances concerned.

Second Example of a Class: a "quantity" of solid

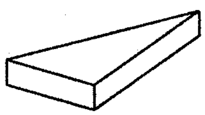
The next classroom discourse also shows such complex feature in mathematics teaching and necessity for complementarity. This teaching about cubic volumes was performed for 5th graders and recorded by a elementary school teacher, Kazunori Makino at May '96. The protocol is a last part of the teaching and modified a little to make it understandable. (The name of pupils are changed.)

Teacher: Anyone who think about the solid, please give your opinions.

Kenji: I add them since the surface with height makes a quantity. The surface is made of these side, so I calculate them by the length and the width, plus the height. I add 50mm , 30mm , and 9mm . 50+30+9=89. The quantity is 89.



Fumio: Can you get them about the thin triangle solid like signboard?



Kenji: Yes I can. When the widths 9mm , here, about 10cm and 5cm are given, I change the units to mm and add them, 9+100+50=159.

Satoshi: Can you get them only adding the length and the width, the height…? For, length plus width isn't area.

Kenji: The sum isn't area.

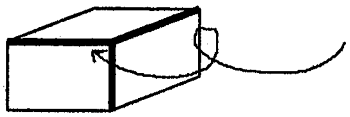
Satoshi: The sum isn't area. Though the area is 100cm^2, not equal to the sum.

Kenji: The sum isn't area.

Eisaku: Only the sum is just for 3 rows.

Kenji: I don't express the area but quantity with the sum. There are the surface and the bottom. The surface with the height makes solid. Thus I expect the sum of this and the height is solid.

Teacher:

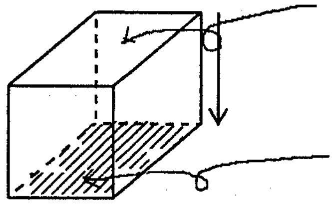


Eisaku: When each length, width and height is 10, the sum is equal to 3 rows. If 30, 10+10+10=30 means 3 rows, thus the sum doesn't fit absolutely.

Kenji: Though you call the height, I call the bottom. The bottom added to this surface can make solid.

Teacher: length width height. (He writes them on blackboard.)

Kenji: I take bottom rather than height.



Satoshi: I can understand it but it's false.

Teacher: What is false, 'Satoshi'?

Satoshi: The quantity is cubic volume, isn't it? Well, what is a volume?

Kenji: The textbook writer describes "length × width × height", and must sufficiently understand it, as 'Hideo' and 'Mayumi' said. 'Satoshi' thought the quantity is expressed by "length × width × height". It's good, but none in our classmate would read the textbook and routinely use the formula. I think a surface with bottom makes a solid and should think understandingly.

Satoshi: But it should be right.

Kenji: I am satisfied with understanding of my thought.

Satoshi: What do you get?

Kenji: This surface with bottom makes a solid and I can get the quantity. As a surface has bottom, I added them and applied the unit in the textbook.

Teacher: How do you feel about 'Kenji's' thought?

Taeko: I use "length × width × height" for a solid. I can't understand the meaning of

'Kenji's' addition.

Fumio: When 'Mariko' said volumes are numbers of $1cm^3$ cubes, I asked how you calculate them to the solid which can't consist of only $1cm^3$ cubes. But for this question, I understand the answer. Nevertheless 'Kenji's' opinion caused a panic and confused me.

Teacher: What confuses you?

Fumio: I would like to ask 'Kenji' what he gets other than area.

Eisaku: 'Kenji' said textbook writers understand what they wrote. But I think right things are written in textbook which is the book to teach. They themselves understand the thought, but ….

Yoshiko: The 'Kenji's' idea "length + width + height" is difficult to understand, though he understand it. The "length + width" aren't area, you know. I can understand you add them and height but can't understand why you add length and width. If "length × width + height", I can understand its meaning.

Satoshi: What does adding height mean?

(The classroom discourse stopped here in this step.)

Analysis of the Discourse

The teacher whose role in this classroom discourse is like a chairman. The pupils can't refute Kenji's formula, "length + width + height", which perturbs them in the social interaction. And they can't attain the counterexample that the sum of them is same but the volume different. So they didn't understand the mathematical rationality of volume formula, "length × width × height", which is the number of $1cm^3$ cubes, although most of them may know it in previous discussions or textbook. Their learning and thinking process can be described as constructivist with Piagetian model using "Conservation." In this step most children assimilate Kenji's formula to their schemata, but they can't accommodate those, and get in perturbation. The only accommodation is Yoshiko's formula, "length × width + height" as presented.

Then it should be noticed that Kenji's formula is frequently used in social life, for example in the quantitative restriction of sending baggage by air plain or freight car. Thus without help of teacher as proficient, children themselves couldn't refute Kenji's additive formula and mathematically understand the multiplication formula of volume. In next step of teaching teacher should intend the interaction on socioculturism. The teacher occasionally must explain the counterexample that the addition of sides is out of proportion to volume. In Vygotskian words children work in "everyday (spontaneous) concepts (occasionally pseudoconcepts)", and also in "Zone of Proximal Development" to "scientific (theoretical) concepts". And further their discussion about the textbook as cultural tool ("objective tool") to gain mathematical knowledge is very interesting. Kenji asserts that textbook writers sufficiently understand contents and write them and that as he can understand and reason his formula, he doesn't follow the textbook routinely. His assertion makes clear that textbook is important authority for pupils to learn and think out.

THEORETICAL CONSIDERATION

Generally two types of the roles of teachers in classroom discussion are recognize. One is that the teacher should act as a chairperson and doesn't directly attach students' constructing mathematical knowledge, which is consistent with Piagetian theory as constructivism. The other is that teachers as the proficient should participate to a certain extent in constructing knowledge, which is consistent with Vygotskian theory as a sociocultural approach. Wood (1994) distinguishes 'Focusing Pattern of Interaction' influenced by a Piagetian view of learning and 'Funnel Pattern of Interaction' being reminiscent of descriptions of pedagogy provided by those proponents of Vygotsky's theory, which Bauersfeld (1988) and Voigt (1985) described. Thus the problem concerning whether constructivism, interactionism or socioculturism perspectives is not only theoretically but practically important.

The principles of radical constructivism are that the function of cognition is adaptive in the biological sense of the term, tending towards fit or viability and that cognition serves the subject's organization of the experiential world, not the discovery of an ontological reality (von Glasersfeld, 1990, pp.22-23). First of all why such solipsism principle is introduced in mathematics education? Steffe & Kieren (1994) describe that constructivists desire to inquires monism and further post-epistemology (Noddings, 1990) in order to conquer classical dualism and that they demonstrated the power of interpretative research with Erlwanger's Benny. Although radical constructivism had the proper intention and effectiveness, it had crucial problem of solipsism which it could not develop without resolving.

The research taking sociocultural perspectives in constructivism to approach the objectivity of mathematical knowledge becomes prosperous. Ernest (1991) and Bauersfeld (1992) suggest social constructivism for such research. And Cobb (1994b) advocates "theoretical pragmatism" which compares and coordinates constructivism and sociocultural perspective, studying social interaction (Cobb, Yackel & Wood,

1992) and cultural tools (Cobb, 1995). Ernest (1994) calls them "complementarist". The word complementarity itself is introduced in mathematics education by H.G.Steiner (1985).

Stephan Lerman (1996) who criticizes social constructivism and complementarity, argues about intersubjectivity in mathematics learning from sociocultural perspective with Vygotskian theory and points out the inconsistency of learning theory extending radical constructivism to social, including the intersubjectivity problem. Thus he asserts that radical constructivist could not explain children's mathematics learning sufficiently but cultural psychology which contains sociocultural perspective, can solve the problem of intersubjectivity. Moreover he claims that the complementarity used by Ernest (1994) and Cobb (1994b) is different from that of Steiner (1985). No doubt the discussion of Lerman (1996) is logical especially about intersubjective problem in mathematics learning.

Nevertheless, individual psychological researches by Piagetian constructivists who regard child as biological being have been very prosperous but cultural psychological researches by Vygotskian theorists who regard child as social and cultural being was not the main current in psychology and not necessarily plentiful because their method was complex, which now are developing furiously. But we think that constructivism as the educational viewpoint can't be excluded because it inherits the naturalism as educational philosophy which Rousseau and Pestalozzi, Dewey contributed. And the educational perspective based on only sociocultural theory, practically only funnel pattern without focusing would be incomplete.

Therefore we had better adopt complementarity position as Cobb (1994b), applying constructivism, interactionism and socioculturism in researches of mathematics teaching actively because educational phenomena are often complex and can't be explained from a single theory. These considerations show that we can get more plentiful inquiry to mathematics teaching with multi-world paradigm.

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