COUNTING THE NUMBER OF BOUNDED DOMAINS SEPARATED BY HYPERPLANES

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Abstract

When some hyperplanes $H_1, \dots, H_m$ of the n-dimensional Euclidean space $\mathbf{R}^n$ are given in general position, Schläfli has determined the number of the bounded connected components in $\mathbf{R}^n - \cup_{i=1}^m H_i$, the complementary set of the union of the hyperplanes. It is equal to the binomial coefficient $\binom{m-1}{n}$, which is also equal to the number of vertices which are the intersections of n hyperplanes in $H_1, \dots, H_{m-1}$. Although Schläfli's proof is implicit and intuitive, the fact reflects an interesting aspect concerning configurations of hyperplanes. We clarify how the condition of general position works, and re-prove the fact in all of its details.

INTRODUCTION

Assume that several numbers of lines, each triad of which surrounds a triangle and just two of which meet at each intersection, are given in the plane. Then, choose one line l among them arbitrarily, and count all intersections which are not on l. Then, we see that the number of such intersections is equal to the number of the bounded domains which are formed by removing all given lines from the plane. Generalizing this fact to higher dimensions, Schläfli [Sc] has shown the following:

Theorem A. *If the hyperplanes $H_1, \dots, H_m$ of $\mathbf{R}^n$ are in general position, then the number of bounded domains of $\mathbf{R}^n - \cup_{i=1}^m H_i$ is equal to* $\binom{m-1}{n}$.

The purpose of this paper is to clarify how the condition of general position works and to give a detailed proof of Theorem A. The proof in [Sc] is implicit and intuitive in a sense, and how the condition of general position works is not necessarily clear at least in the case of $n \geq 3$. On the other hand, the result is interesting. $\binom{m-1}{n}$ is the number of vertices, the intersections of n hyperplanes in $H_1, \ldots, H_{m-1}$, and Theorem A means that the number of the bounded domains of $\mathbf{R}^n - \cup_{i=1}^m H_i$ is equal to the number of such vertices. Also, Theorem A relates the configuration of hyperplanes to combinatrics (cf. [Po], [Co], [Gr], [Im]), and it has several applications (e.g. [AK]). Therefore, the theorem deserves to be shown in all of its details with clarifying the point where the condition of general position works.

The notations and the terminologies related to the theorem are explained as follows:
(A1) $\mathbf{R}^n$ denotes the n-dimensional Euclidean space $\{\mathbf{x} = (x_1, \ldots, x_n) \mid x_i$ is a real number for $1 \leq 1 \leq n\}$. A subset $X \subset \mathbf{R}^n$ is said to be bounded if X is included in an n-dimensional disc $\{\mathbf{x} \in \mathbf{R}^n \mid \| \mathbf{x} - \mathbf{0} \| \leq r\}$ with some radius $r > 0$, where $\mathbf{0}$ is the origin of $\mathbf{R}^n$. If X is not bounded, we call X unbounded. For given bounded sets, a finite union of them and subsets of them are all bounded. We denote the empty set by ϕ, which is also bounded, and we say X is nonempty if $X \neq \phi$.
(A2) *A hyperplane H of* $\mathbf{R}^n$ is a set $H = \{\mathbf{x} \in \mathbf{R}^n \mid \mathbf{a} \cdot \mathbf{x} = b\}$ for some $\mathbf{a} \in \mathbf{R}^n - \{\mathbf{0}\}$ and a real number b. Here, $\mathbf{a}$ denotes a normal vector of H, and $\mathbf{a} \cdot \mathbf{x}$ the inner product of $\mathbf{a}$ and $\mathbf{x}$. A hyperplane H of $\mathbf{R}^n$ is an $(n-1)$-dimensional affin space, and $H^+ = \{\mathbf{x} \in \mathbf{R}^n \mid \mathbf{a} \cdot \mathbf{x} > \mathrm{b}\}$ and $H^- = \{\mathbf{x} \in \mathbf{R}^n \mid \mathbf{a} \cdot \mathbf{x} < \mathrm{b}\}$ are the open half spaces separated by H. The space $\mathbf{R}^n - \cup_{i=1}^m H_i$ is the disjoint union of connected components each of which is the intersection of half spaces separated by the hyperplanes $H_1, \ldots, H_m$. If a connected component of $\mathbf{R}^n - \cup_{i=1}^m H_i$ is nonempty and bounded, then we call it *a bounded domain.*
(A3) Hyperplanes $H_1, \ldots, H_m$ of $\mathbf{R}^n$ are said to be *in general position* if they satisfy the following (a) or (b):

(a) $m \leq n$, and $\cap_{i=1}^m H_i$ is an $(n-m)$-dimensional affin space;
(b) $m \geq n+1$, and $\cap_{j=1}^{n+1} H_{i_j} = \phi$ for any $1 \leq i_1 < \cdots < i_{n+1} \leq m$, and $\cap_{j=1}^n H_{i_j}$ is a one point set for each $1 \leq i_1 < \cdots < i_n \leq m$.

(A4) $\binom{m-1}{n}$ denotes the binomial coefficient whose value is

$$(m-1)!\,/(\mathrm{n}!\,(m-n-1)!) = \Pi_{i=m-n}^{m-1} i \,/\, \Pi_{j=1}^{n} j.$$

The paper is organized as follows: In the next section, we prepare the fundamental proposition (Proposition B), and prove Theorem A by assuming it.The third section is devoted to prove Proposition B. In the process, we prepare Proposition C further to clarify the steps of the proof, and also we give Lemmas $1-3$ which explain how the condition of general position is used.

COUNTING

We denote by $B(m, n)$ for $m \geq 1$ and $n \geq 1$ the number of the bounded domains of $\mathbf{R}^n - \cup_{i=1}^{m} H_i$ for m distinct hyperplanes $H_1, \ldots, H_m$ of $\mathbf{R}^n$ which are in general position. Furthermore, set $B(m,0) = 1$ and $B(0,n) = 0$ for any $m, n \geq 1$ and $B(0, 0) = 0$. Then, we have the following:

Proposition B. *Let $m \geq 1$ and $n \geq 1$. Then,*

(i) $B(m, 1) = m - 1$;

(ii) $B(i, n) = 0$ *for* $1 \leq i \leq n$;

(iii) $B(n+1, n) = 1$;

(iv) $B(m, n) = B(m-1, n) + B(m-1, n-1)$.

We prove the proposition in the next section, and prove Theorem A first.

Proof of Theorem A. Set $D(k, n) = B(k + n, n)$ for $k \geq 0$ and $n \geq 0$, and $D(k, n) = 0$ for $k < 0$ or $n < 0$. Then, we can consider the generating function of $\{D(k, n)\}$ as follows:

$$g_k(x) = \sum_{n=0}^{\infty} D(k, n) x^n \in \boldsymbol{Z}[[x]] \quad \text{for } k \geq 1.$$

Hence, $g_1(x) = \sum_{n=0}^{\infty} x^n = 1/(1-x)$ by Proposition B(iii). For $k \geq 2$, by applying Proposition B(iv), we have

$$\begin{aligned} g_k(x) &= \sum_{n=0}^{\infty} (D(k-1, n) + D(k, n-1)) x^n \\ &= \sum_{n=0}^{\infty} D(k-1, n) x^n + \sum_{n=0}^{\infty} D(k, n-1) x^n \\ &= g_{k-1}(x) + x g_k(x). \end{aligned}$$

Thus, $g_k(x) = (1-x)^{-1} g_{k-1}(x) = (1-x)^{1-k} g_1(x) = (1-x)^{-k}$

$$= \sum_{n=0}^{\infty} \binom{n+k-1}{n} x^n$$

for any $k \geq 1$. Therefore, we have $D(k, n) = \binom{n+k-1}{n}$, which gives the required equality $B(m, n) = D(m-n, n) = \binom{m-1}{n}$. □

Instead of the generating function $g_k(x)$ in the above proof, we can also consider the following generating function of $\{D(k, n)\}$:

$$h_n(x) = \sum_{k=1}^{\infty} D(k,n)x^{k-1}.$$

Then, $h_1(x) = \Sigma_{k=1}^{\infty} kx^{k-1}$ by Proposition B(i). Proceeding just the same way as above and applying Proposition B(ii)(iv), we get the equality

$$D(k,n) = \sum_{i=1}^{k} i \binom{n+k-2-i}{k-i}.$$

Hence, we have the following collorary as a by-product.

Collorary. $$\binom{m-1}{n} = \sum_{i=1}^{m-n} i \binom{m-2-i}{m-n-i},$$

where we regard $\binom{l}{0} = 1$ *for any* $l \in \boldsymbol{Z}$.

MAIN STEPS

When m distinct hyperplanes H_i $(1 \leq i \leq m)$ of $\mathbf{R}^n$ are given, we will denote the hyperplanes by

$$H_i = \{\mathbf{x} \in \mathbf{R}^n \mid \mathbf{a}_i \cdot \mathbf{x} = b_i\} \text{ for } 1 \leq i \leq m .$$

Here, $\mathbf{a}_i \in \mathbf{R}^n - \{\mathbf{0}\}$ is the normal vector of H_i and b_i is a real number. Let ϵ_i denote the sign $+$ or $-$. Then, the open half space $H_i^{\epsilon_i}$ separated by H_i is given as

$$H_i^{\epsilon_i} = \{ \mathbf{x} \in \mathbf{R}^n \mid \epsilon_i \mathbf{a}_i \cdot \mathbf{x} > \epsilon_i b_i \} .$$

First, we show the following lemma which yields Proposition B(i).

Lemma 1. *Assume that the hyperplanes* $H_1, \ldots ,H_m$ *are in general position.*

(i) *If* $m < n$, *then* $\cap_{i=1}^{m} H_i^{\epsilon_i}$ *is either empty or unbounded for any* ϵ_i. *Furthermore, if* $\cap_{i=1}^{m} H_i^{\epsilon_i}$ *is nonempty, it contains a line.*

(ii) *If* $m = n$, *then* $\cap_{i=1}^{n} H_i^{\epsilon_i}$ *is nonempty and unbounded for any* ϵ_i.

Proof. Concerning (i), we assume that $K = \cap_{i=1}^{m} H_i^{\epsilon_i}$ has an element $\mathbf{y}_0$. Then, the orthogonal complement in $\mathbf{R}^n$ of the vector space generated by the normal vectors $\{\mathbf{a}_1,\ldots,\mathbf{a}_m\}$ has an element $\mathbf{v} \neq \mathbf{0}$ because rank$(\mathbf{a}_1,\ldots,\mathbf{a}_m) = m < n$, and the line $l = \{\mathbf{y}_0 + t\mathbf{v} \mid t \in \mathbf{R}\}$ is a subset of K since $\epsilon_i \mathbf{a}_i \cdot (\mathbf{y}_0 + t\mathbf{v}) > \epsilon_i b_i$ for $1 \leq i \leq m$. The line l is unbounded, and thus K is unbounded, which establishes (i). Under the assumption of (ii), $\cap_{i=1}^{n} H_i$ is a one point set $\{\mathbf{x}_0\}$, and, for any ϵ_i, there is a unique $\mathbf{z} \neq \mathbf{0}$ with $\mathbf{a}_i \cdot \mathbf{z} = \epsilon_i 1$ for $1 \leq i \leq n$. Then, the half line $l_+ = \{\mathbf{x}_0 + t\mathbf{z} \mid t > 0\}$ is a subset of $\cap_{i=1}^{n} H_i^{\epsilon_i}$, because $\epsilon_i \mathbf{a}_i \cdot (\mathbf{x}_0 + t\mathbf{z}) = \epsilon_i b_i + t > \epsilon_i b_i$ for $1 \leq i \leq n$. Therefore, $\cap_{i=1}^{n} H_i^{\epsilon_i}$ is nonempty and unbounded, as required. □

Let $\{V_l \mid 1 \leq l \leq h\}$ be the set of connected components of $\mathbf{R}^n - \cup_{i=1}^{m-1} H_i$. If $V_l \cap H_m = \phi$, then $V_l - H_m = V_l$ obviously. Remark that $\mathbf{R}^n - \cup_{i=1}^{m} H_i = (\mathbf{R}^n - \cup_{i=1}^{m-1} H_i) - H_m = \cup_{l=1}^{h} (V_l - H_m)$. If $V_l \cap H_m \neq \phi$, then $V_l - H_m = W_{l,1} \cup W_{l,2}$ for some nonempty connected open sets $W_{l,1}$ and $W_{l,2}$ with $W_{l,1} \cap W_{l,2} = \phi$. Under these notations, we have the following proposition which is the key to prove Proposition B.

Proposition C. *Assume that $H_1,\dots,H_m$ are in general position and satisfy $V_l \cap H_m \neq \phi$ for some $1 \leq l \leq h$.*

(i) *The necessary and sufficient condition for V_l to be bounded is that $V_l \cap H_m$, $W_{l,1}$ and $W_{l,2}$ are all bounded.*

(ii) *If V_l is unbounded and $V_l \cap H_m$ is bounded, then one of $W_{l,1}$ and $W_{l,2}$ is bounded and another is unbounded.*

(iii) *If both V_l and $V_l \cap H_m$ are unbounded, then both $W_{l,1}$ and $W_{l,2}$ are unbounded.*

The proofs of (i) and (iii) are easy. But, in order to make the steps clear, we first complete the proof of Proposition B by assuming Proposition C.

Proof of Proposition B. (i) is clear since m hyperplanes are m points of $\mathbf{R}^1$, and all (ii) − (iv) hold for $n = 1$. (ii) follows from Lemma 1.

We apply the double induction on $n \geq 1$ and $m \geq 1$ for the proof of (iii) and (iv), and prove them simultaneously. As for the first steps of the induction, we can use (i) and (ii). Then, we have to prove (iii) and (iv) for given $n \geq 2$ and $m \geq n+1$ by assuming that (iii) holds for any $1 \leq n' < n$ and (iv) holds for any (n', m') with $1 \leq n' < n$ and $m' \geq 1$ and for any (n, m') with $1 \leq m' < m$.

Set $K_i = H_i \cap H_m$ for $1 \leq i \leq m-1$. Then, identifying H_m with $\boldsymbol{R}^{n-1}$, we can regard $\{K_i \mid 1 \leq i \leq m-1\}$ as hyperplanes of H_m. Remark that $K_1, \dots ,K_{m-1}$ are in general position. Thus, by the inductive hypothesis, the number of the bounded domains of $H_m - \cup_{i=1}^{m-1} K_i$ is equal to $B(m-1, n-1)$, and also the number of the bounded domains of $\mathbf{R}^n - \cup_{i=1}^{m-1} H_i$ is equal to $B(m-1, n)$.

We have set $\mathbf{R}^n - \cup_{i=1}^{m-1} H_i = \cup_{l=1}^{h} V_l$ for the connected components $\{V_l\}$, and thus $B(m-1, n)$ is equal to the number of V_l which are bounded. Let c be the number of V_l for which $V_l \cap H_m$ is nonempty and bounded. Then, by Proposition C, we have $B(m, n) = B(m-1, n) + c$. However, since $H_m - \cap_{i=1}^{m-1} K_i = H_m \cap (\mathbf{R}^n - \cup_{i=1}^{m-1} H_i) = \cup_{l=1}^{h} H_m \cap V_l$, we have $c = B(m-1, n-1)$. Hence, we obtain the required equality $B(m, n) = B(m-1, n) + B(m-1, n-1)$ of (iv), which also give (iii) by taking $m = n+1$ and using (ii) and the inductive hypothesis. $\square$

Before the proof of Proposition C, we prepare two lemmas. In the first lemma, we assume that the hyperplanes $H_1, \ldots, H_{n+1}$ of $\mathbf{R}^n$ are in general position, and set $K_i = H_i \cap H_{n+1}$ for $1 \leq i \leq n$ and $\{\mathbf{f}_{n+1}\} = \cap_{i=1}^{n} H_i$. Also, we denote by $\mathbf{f}_{n+1} \star X$ the open join which is the union of all open line segments connecting $\mathbf{f}_{n+1}$ and points of X, where an open line segment l connecting points $\mathbf{a}$ and $\mathbf{b}$ means $l = \{\mathbf{x} = (1-t)\,\mathbf{a} + t\mathbf{b} \in \mathbf{R}^n \,|\, 0 < t < 1\}$.

Lemma 2. *When $H_1, \ldots, H_{n+1}$ are in general position and $\cap_{i=1}^{n+1} H_i^{\epsilon_i}$ is nonempty, we have $\cap_{i=1}^{n+1} H_i^{\epsilon_i} = \mathbf{f}_{n+1} \star \cap_{i=1}^{n} K_i^{\epsilon_i}$, where $K_i^{\epsilon_i} = H_i^{\epsilon_i} \cap H_{n+1}$ for $1 \leq i \leq n$.*

Proof. Set $\cap_{1 \leq i \neq j \leq n+1} H_i = \{\mathbf{f}_j\}$ for $1 \leq j \leq n+1$. Then, $\{\mathbf{f}_j \,|\, 1 \leq j \leq n+1\}$ is affinary independent. In fact, if $\Sigma_{j=1}^{n+1} r_j \mathbf{f}_j = 0$ for some real numbers r_j with $\Sigma_{j=1}^{n+1} r_j = 0$, then $0 = \Sigma_{j=1}^{n+1} r_j \mathbf{a}_k \cdot \mathbf{f}_j = r_k d_k + \Sigma_{1 \leq j \neq k \leq n+1} r_j b_k = r_k (d_k - b_k)$ for some d_k with $d_k \neq b_k$, and thus $r_k = 0$. Therefore, $\{\mathbf{f}_j \,|\, 1 \leq j \leq n+1\}$ is the set of vertices of an n-dimensional simplex $\triangle$, and $\{\triangle \cap H_i \,|\, 1 \leq i \leq n+1\}$ is the set of the $(n-1)$-dimensional faces of $\triangle$. If we take ϵ_i to satisfy $H_i^{\epsilon_i} \supset \mathrm{Int}(\triangle)$, where $\mathrm{Int}(\triangle)$ is the set of the interior points of the simplex $\triangle$, then it holds $\cap_{i=1}^{n+1} H_i^{\epsilon_i} = \mathrm{Int}(\triangle)$. Just the same way, if $\triangle'$ is the $(n-1)$-dimensional simplex with vertices $\{\mathbf{f}_j \,|\, 1 \leq i \leq n\}$, then $\mathrm{Int}(\triangle') = \cap_{i=1}^{n} K_i^{\epsilon_i}$. Since $\mathrm{Int}(\triangle) = \mathbf{f}_{n+1} \star \mathrm{Int}(\triangle')$, we have the required result. $\square$

Lemma 3. *Assume that the hyperplanes $H_1, \ldots, H_k$ of $\mathbf{R}^n$ are in general position and $\cap_{i=1}^{k} H_i^{\epsilon_i}$ is nonempty and bounded. Then, $k \geq n+1$, and we can choose some $1 \leq i_1 < \cdots < i_{n+1} \leq k$ for which $\cap_{j=1}^{n+1} H_{i_j}^{\epsilon_{i_j}}$ is nonempty and bounded.*

Proof. $k \geq n+1$ follows from Lemma 1. For simplicity, we put $A = \cap_{i=1}^{k} H_i^{\epsilon_i}$, which is nonempty and bounded by hypothesis. We prove the assertion by the double induction on $n \geq 1$ and $k \geq n+1$. In the case of $n = 1$, it is clearly true for any $k \geq 2$. Also, in the case of $k = n+1$, the assertion is true trivially. Thus, for given $n \geq 2$ and $k \geq n+2$, we assume that the assertion holds for any (n', k') with $1 \leq n' < n$ and $k' \geq 1$ and for any (n, k') with $1 \leq k' < k$, and prove it in the case of (n, k).

Put $V = \cap_{i=1}^{k-1} H_i^{\epsilon_i}$. If $V \cap H_k = \phi$, then $V = \cap_{i=1}^{k} H_i^{\epsilon_i} = A$ and thus we have the conclusion by the inductive hypothesis. Hence, we assume that $V \cap H_k \neq \phi$. Remark that the hyperplanes $K_i = H_i \cap H_k$ of H_k for $1 \leq i \leq k-1$ are in general position, and $\cap_{i=1}^{k-1} K_i^{\epsilon_i} = V \cap H_k \neq \phi$, where $K_i^{\epsilon_i} = H_i^{\epsilon_i} \cap H_k$. Then, by the inductive hypothesis, we can choose some $1 \leq m_1 < \cdots < m_n \leq k-1$ for which $W = \cap_{j=1}^{n} K_{m_j}^{\epsilon_{m_j}}$ is nonempty and bounded.

Let $\{\mathbf{f}\} = \cap_{j=1}^{n} H_{m_j}$. If $\mathbf{f} \in H_k^{\epsilon_k}$, then by Lemma 2 we have $\cap_{j=1}^{n} H_{m_j}^{\epsilon_{m_j}} \cap H_k^{\epsilon_k} = \mathbf{f} \star W$. But, $\mathbf{f} \star W$ is bounded since W is bounded, and thus we have the required result in this case. If $\mathbf{f} \in H_k^{-\epsilon_k}$, then similarly $\mathbf{f} \star W = \cap_{j=1}^{n} H_{m_j}^{\epsilon_{m_j}} \cap H_k^{-\epsilon_k}$ and thus $L = \cap_{i=1}^{k-1} H_i^{\epsilon_i} \cap H_k^{-\epsilon_k}$ is bounded. Then, $A \cup L = \cap_{i=1}^{k-1} H_i^{\epsilon_i}$ is nonempty and bounded. Hence, by the inductive hypothesis, we can choose some $1 \leq i_1 < \cdots < i_{n+1} \leq k-1$ for which $\cap_{j=1}^{n+1} H_{i_j}^{\epsilon_{i_j}}$ is nonempty and bounded. Thus, we have the required result. □

Now, we can complete the proof of Proposition C.

Proof of Proposition C. Since any subset of a bounded set and any finite union of bounded sets are bounded, (i) holds. Let $\overline{X}$ denote the closure of a set X with respect to the ordinary topology of $\mathbf{R}^n$. Then, X is bounded if and only if $\overline{X}$ is bounded. Concerning (iii), since we have $\overline{W_{l,i}} \supset V_l \cap H_m$ and $V_l \cap H_m$ is unbounded by the assumption, $W_{l,i}$ is unbounded for $i = 1,2$, which is the required result. Thus, only the proof of (ii) remains. By (i), either $W_{l,1}$ or $W_{l,2}$ is unbounded. We can represent V_l as $V_l = \cap_{i=1}^{m-1} H_i^{\epsilon_i}$ for some ϵ_i. Then, $V_l \cap H_m = \cap_{i=1}^{m-1} K_i^{\epsilon_i}$, where $K_i^{\epsilon_i} = H_i^{\epsilon_i} \cap H_m$ for $1 \leq i \leq m-1$. Then, applying Lemma 3 to the hyperplanes $K_i^{\epsilon_i}$ of H_m, we can choose some $1 \leq i_1 < \cdots < i_n \leq m-1$ for which $\cap_{j=1}^{n} K_{i_j}^{\epsilon_{i_j}}$ is nonempty and bounded. Let $\{\mathbf{f}\} = \cap_{j=1}^{n} K_{i_j}^{\epsilon_{i_j}}$. Then, by Lemma 2, we have $\cap_{j=1}^{n} H_{i_j}^{\epsilon_{i_j}} \cap H_m^{\epsilon_m} = \mathbf{f} \star \cap_{j=1}^{n} K_{i_j}^{\epsilon_{i_j}}$, which is bounded and includes one of $W_{l,1}$ and $W_{l,2}$ as a subset. Thus, $W_{l,1}$ or $W_{l,2}$ is bounded, and we have completed the proof. □

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