STUDENTS' REPRESENTATIONS OF FRACTIONS IN A REGULAR ELEMENTARY SCHOOL MATHEMATICS CLASSROOM

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Abstract

The study reported in this paper investigates students' representations of fractions in a regular elementary school mathematics classroom where students' construction of mathematical knowledge is emphasized in the process of teaching and learning mathematics based on a constructive approach. This paper focuses on an analysis of students' representations of fractions as they work on the fraction comparison tasks and justify their solutions in a collective classroom activity. The importance of setting a problematic situation and encouraging students to make various representations for their meaningful learning mathematics is exemplified. Some implications for teacher's activity and school mathematics curriculum are also suggested.

THEORETICAL BACKGROUND OF THE STUDY

The study reported in this paper makes a part of our research project on establishing a theory for planning and practicing mathematics class that enables students to actively construct mathematical knowledge. Nakahara (1993) has proposed a so-called "constructive approach" and established the lesson process model in the constructive approach that consists of such five steps of teaching and learning activities as being conscious, being operational, being mediate, being reflective, and making agreement. From a different perspective, Koyama (1996) has analyzed an elementary school mathematics class in Japan and showed that the process of teaching and learning mathematics in the classroom actually developed in the line with the horizontal axis, i.e. three learning stages of the intuitive, reflective, and analytical that are set up in the "two-axes process model" of understanding mathematics

(Koyama, 1992).

PURPOSE AND METHOD OF THE STUDY

As a result of the Rational Number Projects (Carpenter, Fennema, and Romberg, 1993), it is shown that representations, translations among them, and transformations within them play several important roles in mathematical learning and problem solving. Lesh, Post, and Behr (1987) notes that "the term representations here is interpreted in a naive and restricted sense as external (and therefore observable) embodiments of students' internal conceptualizations —although this external/internal dichotomy is artificial" (p.33). Moreover, as Goldin and Passantino (1996) notes, students' external representations of mathematical ideas permit us to conjecture or infer their internal representations and conceptual understanding of the ideas concerned.

On the other hand, it has been difficult for students to understand fractions as mathematical ideas and construct meanings of fractions (cf. Lesh, Behr, and Post, 1987; Post, Cramer, Behr, Lesh, and Harel, 1993; Watanabe, Reynolds, and Lo, 1995; Goldin and Passantino, 1996). Post, Cramer, Behr, Lesh, and Harel (1993) especially criticizes the instructional emphasis on developing procedural skill for fraction and the divorce of operations from their meanings, and suggests us as follows: "Fraction order and equivalence ideas are fundamentally important concepts. They form the framework for understanding fractions and decimals as quantities that can be operated on in meaningful ways" (p.340).

We have the accumulated important information on students' representations and conceptions of fractions by means of performance tests, task-based interviews, or a combination of them (cf. Carpenter, Fennema, and Romberg, 1993; Goldin and Passantino, 1996). We, however, do not have enough information on students' representations of fractions that they make and use to understand fractions and construct meanings of fractions in a regular school mathematics classroom. Therefore, the study reported in this paper focuses on an analysis of students' representations of fractions as they work on the fraction comparison tasks and justify their solutions in a collective classroom activity (cf. McClain and Cobb, 1996).

The sample episode discussed and data of students' representations analyzed in this paper are taken from a fifth-grade classroom in which the teacher, Mr. Miyamoto, has participated as a collaborating member of our research project on the constructive approach. The study reported in this paper is not such an experiment study that enables us to make valid generalizations for neither a wider population nor students in other countries, but should be regarded as one of our investigative case studies in Japan. It, however, may contribute to gain more information on students' representations of fractions in a collective classroom activity, and exemplify the importance of setting a problematic situation and encouraging students to make various representations for their meaningful learning and construction of mathematics concerned.

A REGULAR ELEMENTARY SCHOOL MATHEMATICS CLASSROOM

The classroom focused on in this paper is a fifth-grade (11 years old) classroom at the national elementary school attached to Hiroshima University in Hiroshima City, Japan. The 37 students (19 boys and 18 girls) in the classroom are heterogeneous in the same way as a typical classroom organization in Japanese elementary schools, but their average mathematical ability is higher than that of other students in the local and public schools. The teacher in the classroom, Mr. Miyamoto, has participated as a collaborating member of our research project on the constructive approach. He is an experienced and highly motivated teacher, and has a relatively deep understanding of both elementary school mathematics and his students.

In Japan the Course of Study as a national curriculum identifies the objectives and typical sequence of topics in elementary school mathematics, and teachers teach their students mathematics usually with a series of mathematics textbooks approved by the Ministry of Education as suitable textbooks. Therefore we should see the outline of the typical sequence of topics related to fraction. According to the current Course of Study (Ministry of Education, 1989), the typical sequence of topics related to fraction begins with an introduction of fractions as quantities, a basic relationship between fraction and decimal ($1/10=0.1$), and addition and subtraction with two simple fractions with a common denominator at the third grade, and then moves on as follows: fraction equivalence (e.g. $1/2=2/4$), fraction order (e.g. $1/5<3/5$, $5/7>2/7$) with a number line, and addition and subtraction with two fractions with a common denominator at the fourth grade; more general fraction equivalence (e.g. $2/3=4/6$, $12/16=3/4$), the meaning and procedure of both reduction of a fraction to the lowest terms and reduction of fractions to a common denominator, fraction order (e.g. $2/3>5/9$, $4/9<5/6$), addition and subtraction with two fractions with different denominators, fractions as operations involving two quantities (e.g. $2 \div 3 = 2/3$), relationships between fraction and decimal ($0.1 = 1/10$, $0.01 = 1/100$), and fractions as ratios at the fifth grade; multiplication and division with fractions at the sixth grade (the last grade in elementary school).

SETTING A PROBLEMATIC SITUATION

Against this curricular background, I and the teacher, Mr. Miyamoto, elaborated the lesson plan for his students in a fifth-grade classroom. The intention of the plan was to modify the sequence of topics related to fraction, before introducing formal procedures of reduction, by carefully setting a problematic situation in which students might be conscious of and actively work on the fraction comparison tasks. In order to see students' ideas and internal representations, we also decided to ask students justify their solutions by encouraging them to make and use various (external) representations of fractions and any mathematical knowledge that they had constructed.

The classroom episode described in this paper is taken from the first two lessons of successive five lessons on fractions in the Mr. Miyamoto's fifth-grade classroom in November, 1996. We decided to use three different fractions in written mathematical symbols, 4/5, 3/5, and 3/4 for setting a problematic situation at the beginning of the first lesson. These fractions were carefully chosen and might be presented to students not at the same time but one by one in the above order, with due consideration of the followings. The students had learned the simple fraction equivalence and order such as $1/2 = 2/4$, $1/5 < 3/5$, and $5/7 > 2/7$ at the fourth grade. We expected that students could easily compare two fractions with a common denominator or numerator, 4/5 vs. 3/5 and 3/5 vs. 3/4, and that they might be challenged to compare two fractions 4/5 vs. 3/4. In fact, according to the scheme of difficulty levels (Lesh, Behr, and Post, 1987, pp.50-51), the comparison of fractions 4/5 vs. 3/4 belongs to the most difficult level 3B, while the both comparisons of fractions such as 4/5 vs. 3/5 and 3/5 vs. 3/4 belong to the easiest level 1. Moreover, we chose the fraction pair as 4/5 and 3/4, because in the pair numerator and denominator are both one unit away from one, and because these fractions are easily transformed to decimals. This choice of fractions, we expected, might allow students to compare and represent fractions in various and different ways.

The process of teaching and learning in the classroom actually developed as follows. In the following protocol of the lesson, sign Tn and sign Sn mean a nth teacher's utterance and a nth student's utterance respectively.

At first, the teacher wrote the symbol 4/5 down on a blackboard and asked "What studies can you do?". A student answered "It (4/5) means four out of five candies". Then, the teacher wrote another symbol 3/5 next to 4/5 and asked the same question again. At that time, many students wanted to do computation with these fractions.

When the teacher wrote the third fraction symbol 3/4 next to 3/5 on a blackboard, some students shifted their attention to comparing those fractions as follows.

T3: Now, we have three fractions. What studies can you do?

S6: Order those fractions according to size!

S7: We want to compare fractions by changing denominator and/or numerator, for example, of four-fifths.

As expected, students answered such fraction comparison questions as 4/5 vs. 3/5 and 3/5 vs. 3/4 with, for example, the following relevant justifications.

S8 : Four-fifths is larger than three-fifths. Because three-fifths means three pieces as you divide one into five pieces, while four-fifths means four pieces as you divide one into five pieces.

S9 : Three-fourths is larger than three-fifths. Because three-fourths means three pieces as you divide one into four pieces, three-fifths means three pieces as you divide one into five pieces, and the size of one piece as you divide one into four pieces is larger.

T10: Now, please anyone say the learning task for this lesson.

S11 : Let's investigate which is larger, four-fifths (4/5) or three-fourths (3/4)!

S12 : Let's investigate which is larger when the difference between denominator and numerator is one!

S13 : I want to make a supplement to S11. Let's compare fractions with different denominators!

S14 : We need to investigate how much larger as well as which is larger.

T11 : I want to ask you justify your solutions of this learning task in more than three different ways.

Through the above extracted discussion, students and the teacher in this classroom posed the main learning task: investigating which fraction 4/5 or 3/4 is larger and how much larger, and justifying own solutions in more than three different ways. This mutually agreed task shows us that our choice of three different fractions 4/5, 3/5, and 3/4 and presentation of these fractions to students not at the same time but one by one effectively functioned for setting the problematic situation where students could be conscious of the learning task to be challenged.

STUDENTS' REPRESENTATIONS OF FRACTIONS

The students individually had worked on the task for about 15 minutes. During students' individual work, the teacher had circulated among students, helping some students with their works and noting down some students' typical and different ways of justification. Finally he asked each of five students present one of their ways of justification on a large white paper and put it on a blackboard. In this section, we will focus on students' representations of fractions that students made, used, and wrote or drew on their work-sheets as they worked on the fraction comparison task and justified their solutions. The term "representations" here is interpreted as the external and belongs to the five distinct types of representation systems (Lesh, Post, and Behr, 1987, p. 34). We will also focus on students' explanations and discussions among students.

Decimal Type: This representation type is characterized as transforming fractions to decimals (Figure 1). 12 out of 37 students made this type of representation.

S15 : I transform these two fractions to decimals. Four-fifths is 0.8 and three-fourths is 0.75. When I compare these two decimals, 0.8 is larger than 0.75 by 0.05. So, four-fifths is larger than three-fourths by one-twentieth.

Transform 4/5 to a decimal, $4 \div 5 = 0.8$ Transform 3/4 to a decimal, $3 \div 4 = 0.75$ Because 0.8 is larger than 0.75, 4/5 is larger.

Figure 1. Decimal Type

T13: Do you agree with S15?

S16: I do not understand why 4/5 is transformed to $4 \div 5$.

S17 : Because four - fifths means four pieces as you divide one into five pieces and $5 \div 4$ was larger than one, I think, $4 \div 5$ is right.

T14: S17 made an additional explanation to S15.

S18 : If we take 1/2 as an example, we can transform it to a decimal by $1 \div 2 \times 1$, that is $1 \div$ denominator $\times$ numerator.

T15: Do you agree with S18? Any comment?

S19 : Because 4/5 means four pieces as you divide one into five pieces, $1 \div 5$ is 0.2, and four pieces of 0.2, 0.2×4, is 0.8. So, I think 0.8 is right.

This type is possible because that the students in this classroom had already learned decimals and that both fractions 4/5 and 3/4 are relatively easy for students to transform to decimals. But, as S16 posed a question, the reason of why 4/5 can be transformed to $4 \div 5$ had not yet learned formally in this classroom. Nevertheless, S17, S18, and S19 tried eagerly to explain in their own ways by using the constructed knowledge.

Remainder Type: This representation type is characterized as noticing that smaller remainder means the subtracted is larger (Figure 2). 3 out of 37 students made this type.

$1-4/5=1/5$ $1-3/4=1/4$

The smaller remainder means the fact that the subtracted is closer to 1.
When I compare 1/5 with 1/4, 1/5 is smaller. So, 4/5 is closer to 1.

Figure 2. Remainder Type

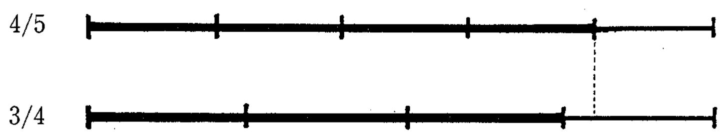
S20: I find out remainders, 1−4/5 is 1/5 and 1−3/4 is 1/4. Because the smaller remainder means the fact that the subtracted is closer to one, and 1/5 is smaller than 1/4, so, 4/5 is larger than 3/4.

S21: I want to make an additional explanation. Because in case of four-fifths we divide one as a whole into five pieces, here I think, one as a whole means five-fifths.

S23: I have a comment on the explanation of S20 about remainder. I consider it in the case of "which is closer to ten, eight or seven?". Because 10−8=2, 10−7=3, and that eight is closer to ten in which the remainder is smaller, the idea of S20 is right.

When this type of representation was explained, many students admired it as a fine one. Although the possibility of using this idea depends on fractions to be compared and this type of representation is not enough to know how much larger, we might say that three students who made this type have a good number sense and relevant meanings of fractions as a result of their learning experiences.

Line-Segment Picture Type: This representation type is characterized as drawing a line-segment picture (Figure 3). 19 out of 37 students made this type of representation.



The least common multiple of 5 and 4 is 20.
$20\div5=4, 4\times4=16.$ $20\div4=5, 3\times5=15.$ $16/20-15/20=1/20.$

Figure 3. Line-Segment Picture Type

S24: I draw this picture. The dotted line in the picture shows that four-fifths is larger than three-fourths and the difference between them. I use the least common multiple of five and four, that is twenty, to change denominators to the common. Because *four-fifths is equal to sixteen-twentieths and three-fourths is equal to fifteenh-twentieths, the difference is one-twentieth.*

In this type, there was a variety of students' representation. For example, some students represented these two fractions by the line-segment picture of 10 units or 20 units length with or without written language explanations. It, however, should be noted that all students who made a line-segment picture drew two equal length line-segments to represent the whole. Moreover, some students began noticing the similarity and difference between the decimal type and the line-segment type as follows.

S27: I think that the ideas of S15 (Figure 1) and S24 (Figure 3) are similar except for the difference between decimal and fraction.

S28: The unit in case of S15 is 0.1, and the unit in case of S24 is 1/20.

***Thin-Rectangle Picture Type*:** This representation type is characterized as drawing a thin-rectangle picture (Figure 4). 28 out of 37 students made this type of representation.

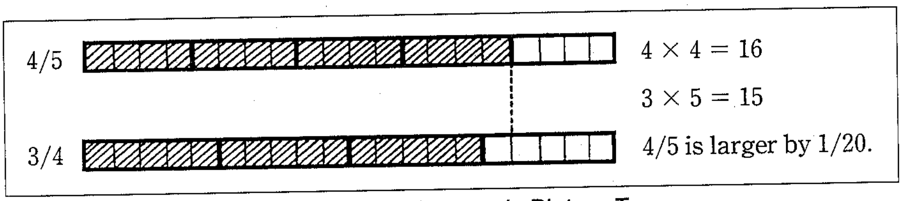


Figure 4. Thin-Rectangle Picture Type

S29: I draw this picture. Because the least common multiple of two denominators five and four is twenty, I divide sections into more small sections. In case of four-fifths, I divide each section into four small sections, and I get 4 × 4, that is 16, small sections. In case of three-fourths, I divide each section into five small sections, and I get 3 × 5, that is 15, small sections. Because 16−15=1, it means that four-fifths is larger than three-fourths by one-twentieth.

There was also a variety of students' representation in this type and we can see same examples and point out same things as the mentioned above in case of the line-segment picture type except for difference between thin-rectangles and line-segments. When S29 was explained, S30 pointed out similarity and difference between the line-segment picture type and the thin-rectangle picture type as follows.

S30: This idea of S29 (Figure 4) and that of S24 (Figure 3) are the same. We note the difference between them only in their pictures. One is line segments and another is thin rectangles.

Equivalent Fraction Type: This representation type is characterized as making equivalent fractions (Figure 5). 31 out of 37 students made this type of representation.

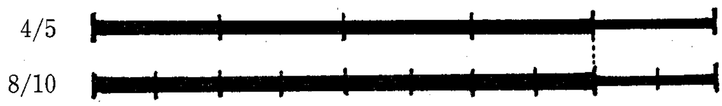
4/5 = 8/10 = 12/15 = 16/20. 3/4 = 6/8 = 9/12 = 12/16 = 15/20.
16/20−15/20 = 1/20

Figure 5. Equivalent Fraction Type

S31: I find out fractions that are equal to each of four-fifths and three-fourths like this (Figure 5). The difference is one-twentieth because that four-fifths is equal to sixteen-twentieths and three-fourths is equal to fifteen-twentieths.

S32: I do not understand. Why do you multiply same number, for example two, to both denominator and numerator?

S33: Because the size of a whole is fixed. I will show you it by this picture.



This type was most popular in this classroom and often used with the line-segment picture or thin-rectangle picture type. The reason of the fact is that the students had learned simple fraction equivalence at the fourth grade and the least common multiple of two natural numbers before this lesson at the fifth grade, and that it is included in the learning task in this lesson to know how much larger. Therefore, if the teacher had not asked students justify their solutions in more than three different ways, students' representations might have converged at this type and their active discussion nor meaningful learning might have not occurred.

CONCLUSION

The study reported in this paper exemplifies the importance of setting a problematic situation in which students are able to be conscious of their own tasks and encouraging students to make various representations for their meaningful learning mathematics. Especially in case of learning fractions at the fifth grade, the choice of different fractions (4/5, 3/5, and 3/4) and the presentation of these fractions one by one effectively functioned for setting such a situation. The teacher's activity of encouraging and allowing his students to make, explain, discuss their various representations (Decimal Type, Line-Segment Picture Type, Thin-Rectangle Picture Type, Equivalent Fraction Type, etc.) played an important role for their meaningful learning of fractions.

This study also suggests, at least for school mathematics curriculum in Japan, the possibility of changing the sequence of topics related to fraction that is identified in the curriculum by carefully setting a problematic situation in which students might be conscious of and actively work on the fraction comparison tasks, before introducing formal procedures of reduction of fraction(s).

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