RESEARCH ON THE COMPLEMENTARITY OF INTUITION AND LOGICAL THINKING IN THE PROCESS OF UNDERSTANDING MATHEMATICS: AN EXAMINATION OF THE TWO-AXES PROCESS MODEL BY ANALYZING AN ELEMENTARY SCHOOL MATHEMATICS CLASS *

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(Received August 8, 1996)

Abstract

The purpose of this research is to demonstrate the complementarity of intuition and logical thinking in a process of understanding mathematics basing on two basic notions of mental model and reflective thinking. In this paper, we examine the validity of the so-called "two-axes process model", especially the horizontal axis consists of three learning stages by analyzing an elementary school mathematics class. Firstly, we identify some mental models of length which students have initially at the class and lead to a misjudgement or a mathematically incorrect anticipatory intuition. Secondarily, we observe how such intuition has been changed under the control of students' reflective thinking in a whole-class discussion. As a result of the protocol analysis of a class, the validity of the horizontal axis of the model is documented.

INTRODUCTION

In Japan it is one of main objectives of school mathematics education to develop student's intuition and logical thinking. To realize this objective, many mathematics educators and researchers have made extensive efforts in various ways. However, we can not say that we have satisfactorily realized the expected result. In consideration of the existing state of things, we should capture the nature of students' thinking in the teaching and learning of mathematics.

* Paper presented at the Twentieth Annual Conference of the International Group for the Psychology of Mathematics Education (PME20), University of Valencia, Spain, July 8-12, 1996.

Koyama (1988) made a theoretical study on the relationship between intuition and logical thinking from view points of both the history of mathematics development and the developmental mode of human thinking. He states, as a result of the study, that intuition and logical thinking are complementary and closely interrelated in human mathematical thinking. In other words, human thinking could developed productively and soundly only when intuition and logical thinking are in a harmonious and cooperative relation. Recognizing the such complementarity and the idea of objectification or explicitation in the van Hiele theory (van Hiele, 1958), Koyama (1992a) made clear what characteristics a model of students' understanding mathematics should have so as to be an useful and effective model in the teaching and learning of mathematics. The models of understanding mathematics presented in preceding papers are classified into two large categories, i.e."aspect model" (cf. Skemp, 1982) and "process model" (cf. Pirie & Kieren, 1989). Focusing on the process model of understanding mathematics, we recognize that reflective thinking plays an important role to develop students' understanding, or to make their thinking progress from a certain level to a higher level of understanding. Koyama (1992b, 1993) has explored basic components of students' understanding mathematics and presented the so-called "two-axes process model" of understanding as a theoretical framework for the teaching and learning of mathematics. The model consists of two axes in which the vertical axis implies some levels of understanding and the horizontal axis implies three learning stages at each level, i.e. intuitive, reflective, and analytic stage.

PURPOSE

The purpose of this research is to demonstrate the complementarity of intuition and logical thinking in a process of understanding mathematics basing on two basic notions of mental model and reflective thinking. In more concrete terms, we try to examine and identify students' mental models of a abstract and mathematical concept in regard to intuition, and observe how students think reflectively on their mental models in a whole-class discussion in regard to logical thinking. To attain the purpose, in this paper, we try to examine the validity of the two-axes process model, especially the horizontal axis of the model by analyzing an elementary school mathematics class in Japan.

THEORETICAL FRAMEWORK: THE TWO-AXES PROCESS MODEL

First of all, we must see the essence and characteristics of the two-axes process model of understanding mathematics. This model has been built as a result of the theoretical exploration in order to make the followings clear; Through what levels should students' understanding progress? How do students develop their thinking at each level of understanding? Naturally, the model consists of two axes, i.e. the vertical axis implying levels of understanding and the horizontal axis implying stages at each level.

In this model, on the horizontal axis, there is three learning stages, i.e. intuitive, reflective, and analytic stage. Those stages are originated in the work of Wittmann (1981) which emphasizes that three types of activity are necessary to develop a balance of intuitive, reflective, and formal thinking and that mathematics teaching should be modeled according to the processes of doing mathematics (p.395). Koyama (1993) have modified Wittmann's definition of three activities in order to form a horizontal axis of the two-axes process model. Those three stages are described as follows (Koyama, 1993, pp.70-71).

Intuitive Stage; Students are provided opportunities for manipulating concrete objects, or operating on mathematical concepts and relations acquired in a previous level. At this stage, they do *intuitive thinking.*

Reflective Stage; Students are stimulated and encouraged to pay attention to their own manipulating or operating activities, to be aware of them and their consequences, and to represent them in terms of diagrams, figures or language. At this stage, they do *reflective thinking.*

Analytic Stage; Students elaborate their representations to be mathematical ones using mathematical terms, verify the consequences by means of other examples or cases, or analyze the relations among consequences in order to integrate them as a whole. At this stage, they do *analytical thinkign.*

Through those three stages, not necessarily linear, students' understanding could progress from a certain level to a next higher level in the teaching and learning of mathematics. As prominent characteristics of the two-axes process model, firstly, it might be noted that the model reflects upon the complementarity of intuition and logical thinking, and that the role of reflective thinking in understanding mathematics is explicitly set in the model. Secondarily, the model could be an useful and effective one which has both *descriptive* and *prescriptive* function in the teaching and learning of mathematics. The descriptive function means that a model can describe the real aspects or processes of the growth of students' understanding mathematics. The other is the prescriptive function of a model which can suggest us, researchers or teachers of mathematics, didactical principles regarding to the followings; what kinds of didactical situation are necessary, how we should set them up, and to which direction we should guide students in order to help them develop their understanding of mathematics (Koyama, 1992a, p.181).

Those prominent characteristics of the model are, however, still expected theoretically. Therefore, we must examine both *validity* and *effectiveness* of the model in light of practices of the teaching and learning of mathematics. As a first attempt of such examination, in this paper, we will try to examine the validity of the model, especially the horizontal axis consists of three learning stages by analyzing an elementary school mathematics class.

A SKETCH OF ELEMENTARY SCHOOL MATHEMATICS CLASSES

The class to be analyzed in this paper is a part of four successive mathematics classes in a fifth grade (11 years old) classroom at the national elementary school attached to Hiroshima University in Japan. In February 1993, an elementary mathematics teacher of the classroom, Mr. Mori, planned and taught 36 students (18 boys and 18 girls) a topic named "Let's think with mathematical expressions". The students involved in those four classes are heterogeneous in the same way as a typical classroom organization in Japanese elementary schools, but their average mathematical ability is higher than that of other students in the local and public elementary schools.

In this section, firstly we see the intention of the topic held by the classroom teacher when he had planned it. Then a rough sketch is shown for an outline of four successive classes which actually developed in the classroom.

The classroom teacher, Mr. Mori, has a vision of elementary school mathematics education. Mori (1994) states it as follows: "Students' learning by solving mathematical problems is a continuous process of solving their own problems. I believe such process is an ideal form of learning elementary school mathematics that the once solution of a problem produces a more expansive problem (p.91)". He planned the topic named "Let's think with mathematical expressions" with this vision of mathematics education. The main objective of the topic is to help students appreciate thinking with mathematical expressions such as interpreting a mathematical expression expansively and insightfully.

To realize this teaching objective, he planned three sessions and four unit-hour (45 minutes) classes for the topic as follows.

First session; comparing lengths of two different semicircular roads (2 unit-hour classes)

Second session; comparing lengths of other geometrical figured roads (1 unit-hour class)

Third session; comparing areas of two different semicircular regions and summarizing the topic (1 unit-hour class)

The followings is a rough sketch of an outline of four successive classes which actually developed in his classroom. In this sketch, students' activities are focused and picked up mainly.

First Class

1) Teacher set up the situation: "There are two places A and B. Let's make various roads between them". Students imagined and proposed their roads. Among them, semicircular roads were adopted and two different semicircular roads were drawn on a blackboard (Figure 1). One road L was a semicircular road with the diameter AB. Another road M was a one made by two connected semicircular roads with the diameter AC and BC, where place C was located at a certain point on the segment AB.
2) Students predicted which road is shorter when comparing lengths of two roads L and M. At this point students had their own problem to be solved.
3) Students individually worked out the problem in their own ways. It must be noted that they had learned mathematical formulae for the length and area of a circle, and they know that circle ratio is about 3.14.
4) Students knew that two lengths of roads L and M are equal. Some students explained their own reasons of why two lengths are equal in the whole-class discussion. Students compared and interpreted those mathematical expressions written on a blackboard for the explanations.
5) Students compared lengths of two roads when place C had changed to be another point C' on the segment AB (Figure 2).
6) Students said their findings which they had been aware of in this class and proposed their own problems to be worked on in the next class.

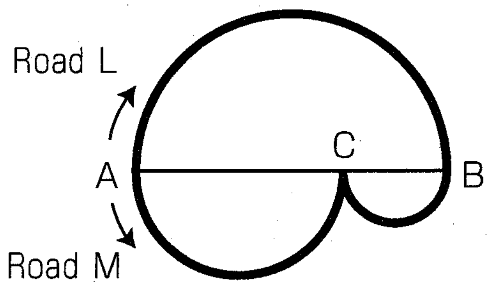

Figure 1.

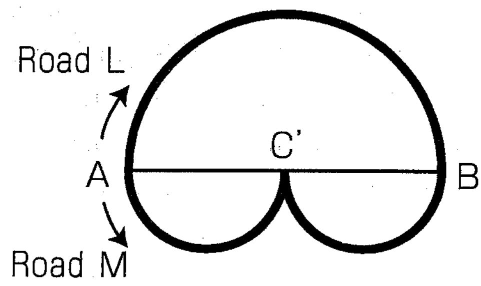

Figure 2.

Second Class

1) Students remembered what they had done in the first class.

2) Among the problems proposed at the end of the first class, students decided to work out the problem: "Compare lengths of two roads L and M when road M is changed to the one made by more than two small semicircular roads".

3) Students individually worked out the problem of comparing lengths when road M was made by three small semicircular roads (Figure 3).

4) Students presented their own solutions and compared mathematical expressions written on a blackboard in the whole-class discussion.

5) Students worked out the more general problem of comparing lengths when the number of semicircular roads of M increased (Figure 4).

6) Students said their findings which they had been aware of in this class and proposed their own problems to be worked on in the next class.

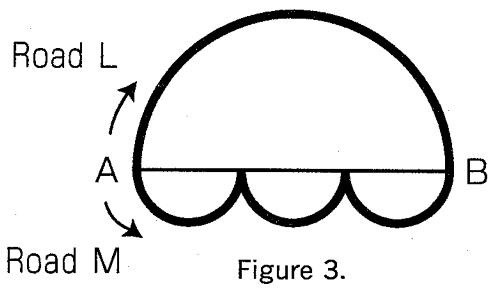

Figure 3.

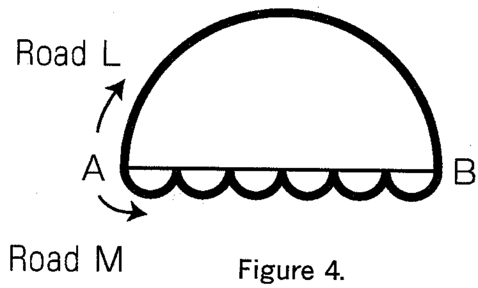

Figure 4.

Third Class

1) Students remembered what they had done in the second class.

2) Among the problems proposed at the end of the second class, students decided to work out the problem: "Seek for other geometrical figured roads which have a same rule as two semicircular roads".

3) Students individually investigated two quarter-circular roads (Figure 5).

4) Students sought for other geometrical figured roads which have the same rule by means of mathematical expressions. Students checked, for example; two equilateral triangle roads (Figure 6) and two square roads (Figure 7).

5) Students said their findings which they had been aware of in this class and proposed their own problems to be worked on in the next class.

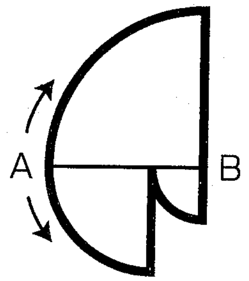

Figure 5.

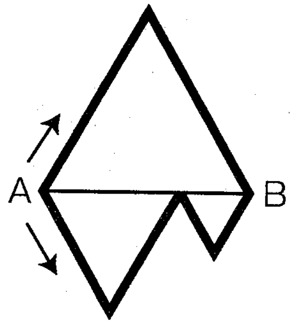

Figure 6.

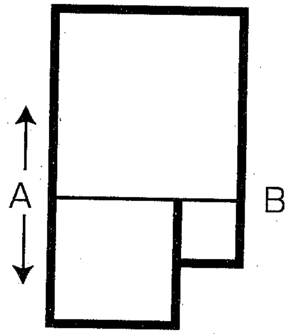

Figure 7.

Fourth Class

1) Among the problems proposed at the end of the third class, students decided to work out the problem:"Compare areas of regions encircled by two semicircular roads (Figure 8)".

2) Students individually worked out the problem with their own predictions.

3) Some students explained their solutions of the problem.

4) Students thought about how the area of region encircled by the road M changes when a point C moves from A to B on the segment AB.

5) Students represented the change of area in a graph.

6) Students read and interpreted the graph and explained their own findings about the change of area in the whole-class discussion.

7) Students looked back what they had done in all four classes and summarized the content of the topic named "Let's think with mathematical expressions".

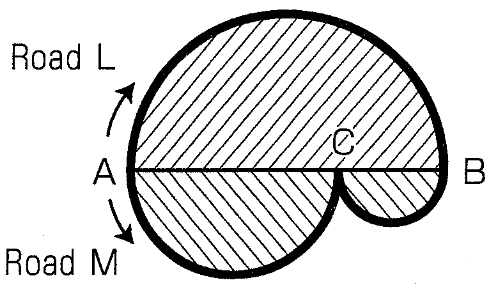

Figure 8.

DISCUSSION BY THE PROTOCOL ANALYSIS OF A CLASS

Four successive classes of the topic actually developed as shown in the above sketch. In this section, by analyzing the protocol of a class mainly in the first session, firstly we try to examine and identify students' mental models of length which lead to a misjudgement or a mathematically incorrect anticipatory intuition. Then we observe how their initial intuition has been changed under the control of students' reflective thinking in the whole-class discussion. Being based on this analysis of a class, we examine the validity of the horizontal axis, i.e. three learning stages of the two-axes process model of understanding mathematics.

Identification of Students' Mental Models of Length

In the first class, after teacher's setting up a learning situation and students' discussion about mathematical problems to be solved, the process of teaching and learning actually developed as follows. In the following protocol of a class, sign T and sign Sn mean a teacher's utterance and a nth student's utterance respectively.

T: Today, we will try to work out the problem of comparing lengths of two semicircular roads L and M (Figure 1). How do you predict which is shorter, road L or road M?

S11: The length of road M is longer than that of road L, because the road M is bent at a point C.

S12: The road M encircles a smaller area than the road L does, so the length of road M is shorter than that of road L.

S13: The length of road M is shorter than that of road L, because the road M is closer to the straight line AB.

Those three students' utterances of their prediction allow us to identify their mental models of length which they have initially at the class as products of their previous experiences of learning length. S11 has a mental model like that when the both ends of two lines are trued up, a curved line is longer than a straight line as shown in Figure 9. S13 has a similar mental model to that of S11 like that because the shortest line between two points is a straight line, a line closer to the straight line is shorter as shown in Figure 10. On the other hand, noticing area, S12 has a different kind of mental model like that the length of a closed geometrical figure is proportional to the area of it as shown in Figure 11.

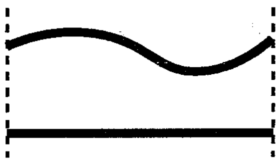

Figure 9.

Figure 10.

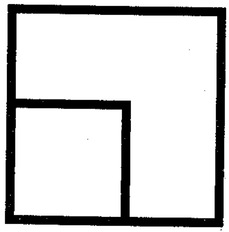

Figure 11.

All those mental models can lead to a mathematically correct judgement or prediction in some cases represented in figures 9, 10, and 11. However, in case of comparing lengths of two semicircular roads worked on in their class, their mental models produced a mathematically incorrect prediction. It might be said that they can not explicitly analyze the curvature (S11), closeness (S13), and similarity (S12). In any case, we could conclude that their mental models of length which they constructed previously and had initially at the class have a negative effect on their anticipatory intuition (Fischbein, 1987; Koyama, 1991) without any explicit analysis of their mental models.

Examination of the Validity of Three Learning Stages

Next we will observe how their initial intuition has been changed under the control of their reflective thinking in the whole-class discussion. After students' predicting lengths, the process of teaching and learning actually developed as follows.

T: You have different predictions and your own reasons. Which is longer, road L or road M? Let's make it clear. Work out the problem in your own ways and write it down on notebooks.

S14: I can not do, because we have no information about the length of AB.

T: Do you need to know the actual length?

SS: (Many students say "Yes", but some students say "No".)

T: If you need to know it, use that AB is 10cm and AC is 6cm.

SS: (Students individually work out the problem by using the mathematical formula for a length of circle which they know.)

T: OK! Present your own work to your classmates. Anyone?

S15: I calculated the lengths as follows. Two answers are equal.

Road L; $10 \times 3.14 \div 2 = 15.7$ *Road M;* $6 \times 3.14 \div 2 = 9.42$

$4 \times 3.14 \div 2 = 6.28$

$9.42 + 6.28 = 15.7$

S16: I can calculate the length of road M with one mathematical expression like *this.*

Road M; $6 \times 3.14 \div 2 + 4 \times 3.14 \div 2 = 15.7$

S17 : I can do it more easily by using parentheses like this. Two answers are equal.

Road M; $(6+4) \times 3.14 \div 2 = 15.7$

S20 : We do not need to calculate the lengths. The sum of AC and CB is equal to AB (looking at Figure 1), and we can see it apparently that both mathematical expressions for road L and road M is $10 \times 3.14 \div 2$*. So we can say that the lengths of two roads are equal.*

T : You have explained your works with your own reasons well. All of you seem to understand your classmates' explanations and be convinced them.

S21: Wait, Mr.! I have another idea. I used alphabetic letters. I thought about the problem when let the length of AB, AC, and BC be a, c, and b respectively. Then we can easily see that lengths of two roads are equal because two mathematical expressions are same like this.

Road L; $a \times 3.14 \div 2$ *Road M;* $b \times 3.14 \div 2 + c \times 3.14 \div 2$

$= (b+c) \times 3.14 \div 2$

$= a \times 3.14 \div 2$

In this whole-class discussion, with the explanation of S15 as a turning point, students in this classroom reflect on their own calculating and thinking process and represent it in their own terms using mathematical expressions. This examination of the protocol allows us to conjecture that students do reflective thinking in their own ways. At this point, we should pay attention to the fact: S20 and S21 are explicitly aware that the mathematical expressions for lengths of two roads are same, while S15, S16, and S17 put their eyes on only that two answers are equal. In other words, for S15, S16, and S17 a mathematical expression is mere a thinking method to calculate an answer for comparing lengths, but for S20 and S21 the mathematical expression itself is a thinking object. This difference must be significant from a view point of the level of understanding mathematics, because, as van Hiele (1958) suggests us, the objectification could push students' understanding of mathematics up to a mathematically higher level.

In fact, the explanation of S21 stimulates other students and directs their understanding of this problem to a higher level, i.e. an understanding of the essential and mathematical structure of this problem.

T : It is a great idea. S21 used alphabetic letters. What can you see about the mathematical expressions explained by S21? Anyone?

S22: It does not depend on the actual lengths of AC and BC.

S23: They are expressed using alphabetic letters, so the lengths of two roads are equal even when a point C moves on the segment AB.

T: Is it true when a point C is close to the point A?

S24: Yes! As far as a point C is on the segment AB, two lengths are always equal.

T: Is it true? Please explain your reason in more detail.

(The following discussions are omitted.)

We can see in the above protocol that students do think about both the meaning of alphabetic letters and the structure of mathematical expressions. In other words, students in the classroom try to represent consequences of their reflective thinking more mathematically, analyze explicitly the structure of the problem, and integrate their findings as a whole. Therefore we might say that at this point of the class students do their analytic thinking.

CONCLUSIONS AND FINAL REMARKS

As a result of this observation and protocol analysis of the class, we see that the process of teaching and learning mathematics in this classroom actually developed in the line with the horizontal axis, i.e. three learning stages of the intuitive, reflective, and analytic which are set up in the two-axes process model of understanding mathematics. Therefore, we could conclude that the validity of three stages at a certain level of understanding mathematics has been demonstrated by the analysis of an elementary school mathematics class.

By the end of the first class, students in this classroom have become to be able to control their mathematically incorrect anticipatory intuition which they had initially at the first class by the logical thinking with mathematical expressions. It is saliently demonstrated by the fact that at the beginning of the second class 34 out of 36 students could predict correctly even when the road M is changed to be made by more than two small semicircular roads. This fact allows us to insist that as a result of their learning experiences students have a fairly determined intuition supported by the logical thinking with mathematical expressions including alphabetic letters.

In this paper, we have examined the validity of the horizontal axis consisted of three learning stages by analyzing an elementary school mathematics class. In doing it, we regarded students in a classroom as a whole and observed their process of understanding mathematics. It is, however, needless to say that we must also pay attention to an individual student and his/her process of understanding mathematics. Moreover, we have to examine the *effectiveness* of the two-axes process model of understanding mathematics in a sense that we can really make a teaching plan with this model and help students develop their understanding of mathematics to be an expected and higher level. Those are difficult but important tasks to be faced and addressed in our future research.

REFERENCES

Fischbein, E. (1987), *Intuition in Science and Mathematics: An Educational Approach.* Holland: D. Reidel Publishing Company.

Koyama, M. (1988), A Study on Intuition in Mathematics Education. In Hirabayashi, I.(Ed.), *Perspectives in Mathematics Education.* Tokyo: Seibunsha, 175-193. (in Japanese)

Koyama, M. (1991), A Serial Study on Fischbein's Research into Intuition: General Characteristics of Intuitive Cognition and Didactical Implications for Mathematics Education. *Bulletin of WJASME: Research in Mathematics Education, 14,* 9-15. (in Japanese)

Koyama, M. (1992a), Models of Understanding in Mathematics Education. In Iwago, K.(Ed.), *New Development of Mathematics Education.* Tokyo: Seibunsha, 172-184. (in Japanese)

Koyama, M. (1992b), Exploring Basic Components of the Process Model of Understanding Mathematics for Building a Two-Axes Process Model. *Proceedings of the Sixteenth PME Conference,* New Hampshire, USA, *Vol.* II, 17-24.

Koyama, M. (1993), Building a Two-Axes Process Model of Understanding Mathematics. *Hiroshima Journal of Mathematics Education, 1,* 63-73.

Mori, Y. (1994), Aspects of Problem Posing and Features of the Learning Environments and Situations for Effective Instruction. *Journal of the Hiroshima University Curriculum Research and Development Association, 9,* 89-110. (in Japanese)

Pirie, S. and Kieren, T. (1989), A Recursive Theory of Mathematical Understanding. *For the Learning of Mathematics, Vol. 9,* No.3, 7-11.

Skemp, R. R. (1982), Symbolic Understanding. *Mathematics Teaching, 9,* 59-61.

van Hiele, P. M. and van Hiele-Geldof, D. (1958), A Method of Initiation into Geometry at Secondary Schools. In Freudenthal, H. (Ed.), *Report on Methods of*

Initiation into Geometry. Groningen Wolters, 67-80.

Wittmann, E. (1981), The Complementary Roles of Intuitive and Reflective Thinking in Mathematics Teaching. *Educational Studies in Mathematics Education, 12,* 389-397.

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