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BUILDING A TWO AXES PROCESS MODEL OF UNDERSTANDING MATHEMATICS*

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Abstract

The purpose of this study is to make clear what kind of characteristics a model of understanding mathematics should have so as to be useful and effective in mathematics education. The models of understanding presented in preceding papers are classified into two large categories, i. e. "aspect model" and "process model". Focusing on the process of understanding mathematics, reflective thinking plays an important role to develop children's understanding, or to progress children's thinking from one level to a higher level of understanding. As a theoretical framework, a process model consisting of two axes is presented for further studies. The vertical axis in the model implies levels of understanding and the horizontal axis implies learning stages. At any level of understanding, there are three stages, i.e. intuitive, reflective and analytic stage.

INTRODUCTION

The word "understanding" is frequently used in descriptions of aims of teaching mathematics in the Course of Study (Ministry of Education, 1989) and in the teaching practices of mathematics in Japan. Placing emphasis on children's understanding is obviously desirable in mathematics education, but what it means is not clear. Moreover, what mathematics teachers should do to help children develop their understanding of mathematics has not been made sufficiently clear. These are essen-

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tial and critical problems need to be addressed.

The key to the solution of these problems, in my opinion, is to capture what it means for children to understand mathematics and to make clear the mechanism which enables children's understanding of mathematics to develop. In other words, what we need is to "understand" understanding.

It is, however, not an easy task and we need our great effort to do it. In fact, as Hirabayashi (1987) describes, the American history of research in mathematics education seems to be the struggling with interpretations of understanding. The problem of understanding is still a main issue being addressed by some researchers, especially those from cognitive psychology in PME. As a result of their work, various models of understanding as the frameworks for describing aspects or processes of children's understanding of mathematics have been developed (Skemp, 1976, 1979, 1982; Byers and Herscovics, 1977; Davis, 1978; Herscovics and Bergeron, 1983, 1984, 1985, 1988; Pirie and Kieren, 1989a, 1989b).

The purpose of this study is to address what kind of characteristics a model of understanding should have so as to be useful and effective in mathematics education. In order to achieve this purpose, in this paper, past research related to models of understanding mathematics are summarized and the fundamental conception of understanding mathematics is described. Then, basic components substantially common to the process models of understanding mathematics are discussed. Finally, I present as a theoretical framework a process model consisting of two axes, called "a two axes process model" of understanding mathematics.

FUNDAMENTAL CONCEPTION OF UNDERSTANDING MATHEMATICS

What do we mean by understanding? According to Skemp (1971), to understand something means to assimilate it into an appropriate schema (p. 43). Haylock (1982) answers this question in the following way: a simple but useful model for discussing understanding in mathematics is that to understand something means to make (cognitive) connections (p. 54). These explanations of understanding are (cognitive) psychological and imply that to understand something is to cognitively connect it to a previous understanding which is called a schema or a cognitive structure. We could say that a schema or cognitive structure is a model of a nerve net in the brain of human beings. In this sense, to understand something is substantially an individual, internal (mental) activity.

Moreover, comparing the Piagetian cognitive structures with the Kantian schemata and categories, Dubinsky and Lewin (1986) theorize that the Piagetian cognitive structures are constructed from the outset and undergo systematic changes of increasing differentiation and hierarchic integration (p. 59). This suggests that understanding as defined above is not a static activity as all-or-nothing but a complex dynamic phenomenon which could change in accordance with the construction and reconstruction of cognitive structures.

Therefore, accepting the conception of understanding mathematics as an internal (mental) dynamic activity, we necessarily need some methods to externalize children's understanding of mathematics. A retrospective method, an observation method, an interview method, and a combination of these methods are promising and useful methods for externalizing understanding. It is, however, impossible to directly see understanding when defined as a mental activity. Therefore, we need some theoretical framework. According to the definition of model by Gentner (1983), the theoretical framework for making clear aspects or processes of understanding mathematics could be called a model which has a mental activity of understanding as its prototype. In that sense, any model is indispensable for making clear understanding and the significance of building a model can be found in this point.

As mentioned in the previous section, various models of understanding mathematics have been proposed and presented in the preceding papers. These models are, for example, including a discrimination of "relational and instrumental understanding" (Skemp, 1976), "a tetrahedral model" (Byers and Herscovics, 1977), "a 2×3 matrix model" (Skemp, 1979), "a 2×4 matrix model" (Skemp, 1982), "a constructivist model" (Herscovics and Bergeron, 1983), "a two-tiered model" (Herscovics and Bergeron, 1988) and "a transcendent recursive model" (Pirie and Kieren, 1989b).

As Pirie and Kieren (1989b) point out, models can be classified into two large categories. The one is "aspect model" which focuses on the various kinds of understanding and the other is "process modes" which focuses on the dynamic processes of understanding. The models presented in Skemp (1976, 1979, 1982) and Byers and Herscovics (1977) belong to the former and the models in Herscovics and Bergeron (1983, 1988) and Pirie and Kieren (1989) belong to the latter. We need both aspect model and process model in order to develop children's understanding in mathematics education. These models seem to be build mainly to describe the real aspects or processes of children's understanding and they are very useful for us to grasp them.

It is, however, not sufficient to describe the real aspects or processes of children's understanding. Mathematics education, by its nature, should be organized by both teaching activity and learning activity. Therefore, a model of understanding which is useful and effective in the teaching and learning mathematics should have prescriptive as well as descriptive characteristics. Namely, the model is expected to have the prescriptive characteristic also in the sense that it can suggest us didactical principles regarding to the following questions. What kind of didactical situations are necessary and how should we set them up to help children understand mathematics? To which direction should we guide children, in developing their understanding of mathematics?

BASIC COMPONENTS OF PROCESS MODEL

In order to build a model of understanding, we must elucidate the processes of children's understanding in mathematics. In this section, focusing on a process model, we explore basic components of children's understanding. For theoretically exploring them, we examine process models of understanding (Herscovics and Bergeron, 1983, 1988; Pirie and Kieren, 1989) and a model of learning mathematics (van Hiele and van Hiele-Geldof, 1958; van Hiele, 1986).

Herscovics and Bergeron have been addressing to the difficult task of building and modifying a model of understanding in the processes of mathematical concept formation. They built "a constructivist model" of understanding mathematical concepts based on the constructivist assumption that children construct mathematical concepts. The constructivist model consists of four levels of understanding: the first one, that of intuition, a second one involving procedures, the third dealing with abstraction, and a last level, that of formalization (Herscovics and Bergeron, 1983, p. 77). Then they modified this model and presented an extended model of understanding. This extended model is called "a two-tiered model", one tier identifing three different levels of understanding of the preliminary physical concepts, the other tier identifing three distinct constituent parts of the comprehension of mathematical concepts (Herscovics and Bergeron, 1988, p. 15). Their fundamental conception underlying this model is that the understanding of a mathematical concept must rest on the understanding of the preliminary physical concepts (p. 20).

Pirie and Kieren (1989) stress that what is needed is an incisive way of viewing the whole process of gaining understanding (p.7). And they present "a transcendent recursive model" of understanding which consists of eight levels: doing, image making, image having, property noticing, formalizing, observing, structuring, and inventing. Their fundamental conception of understanding underlying the model and the important characteristic of the model are succinctly and clearly represented in the following quoted passage.

> Mathematical understanding can be characterized as levelled but non-linear. It is a recursive phenomenon and recursion is seen to occur when thinking moves between levels of sophistication. Indeed each level of understanding is contained within succeeding levels. Any particular level is dependent on the forms and processes within and, further, is constrainted by those without. (Pirie and Kieren, 1989, p. 8)

We can see that these models of understanding are process models which have prescriptive as well as descriptive characteristics and involve some levels of understanding. There is, however, an objection to the levels of understanding. In fact, examining the Herscovics and Bergeron model for understanding mathematical concepts, Sierpinska (1990) argues that what is classified here, in fact, are the levels of children's mathematical knowledge, not their acts of understanding (p. 28). This criticism is based on a different notion of understanding, that understanding is an act (of grasping the meaning) and not a process or way of knowing. It is worth notice but in my opinion there must be some levels, even if those are levels of children's mathematical knowledge, in the processes of children's understanding of mathematics. The process model of understanding mathematics should involve some hierarchical levels so as to be useful and effective in the teaching and learning mathematics.

The hierarchy of levels of understanding can be typically seen in the transcendent recursive model illustrated in Figure 1 (Pirie and Kieren, 1989, p. 8). It reminds us of the van Hieles' theory of levels of thinking in learning geometry which was presented in their doctoral dissertation (cf. van Hiele and van Hiele-Geldof, 1958). In the theory five hierarchical levels of thinking are identified and five learning stages for progressing from one level to a higher level are involved (van Hiele, 1986). We notice that these models are very similar to each other in two respects. The one is levels themselves set up and the other is the idea of progressing from a level to a outer (higher) level.

The first similarity can be recognized more clearly by illustrating the van Hiele model in Figure 2 (Koyama, 1988a). In fact, ignoring somewhat the difference in the scope and domain of learning mathematics, each of the two levels indicated by a thick circle in Figure 1 correspond to the level in Figure 2 as follows:

(Doing, Image Making) ↔ [1] (Concrete Object*[2], Geometrical Figure)
(Image Having, Property Noticing) ↔ (Geometrical Figure*, Property)
(Formalizing, Observing) ↔ (Property*, Proposition)
(Structuring, Inventing) ↔ (Proposition*, Logic)

The second similarity is more important in a process model than the first, because it is concerned with the crucial idea of developing children's understanding of mathematics. The idea of developing children's understanding in the Pirie and Kieren model in *recursion*, whereas, in the van Hiele model it is *objectification* or *explicitation*. These ideas seem to be substantially similar and might be stated, in other words, as reflective abstraction or reflective thinking. We can say that in the process of understanding mathematics *reflective thinking* plays an important role to develop children's understanding, or to progress their thinking from a level to a higher level of understanding. Therefore these models suggest to us that a process model should have learning stages involving reflective thinking.

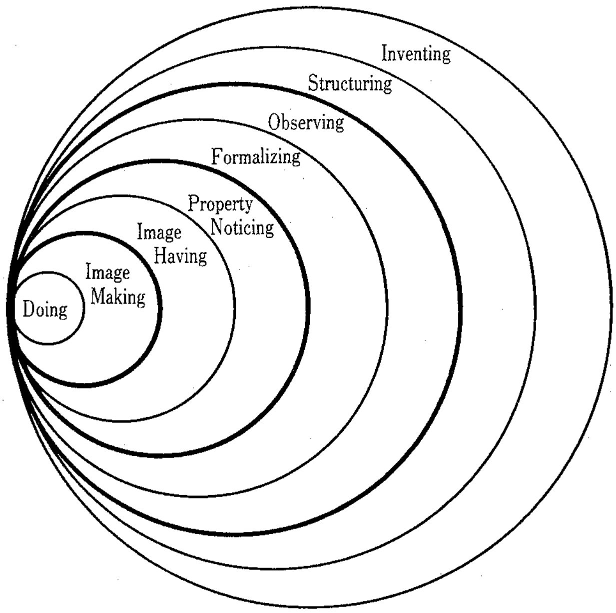

Fig. 1

[1]The sign ↔ indicates the correspondence between levels.
[2]The sign * indicates an object of thinking in each level.

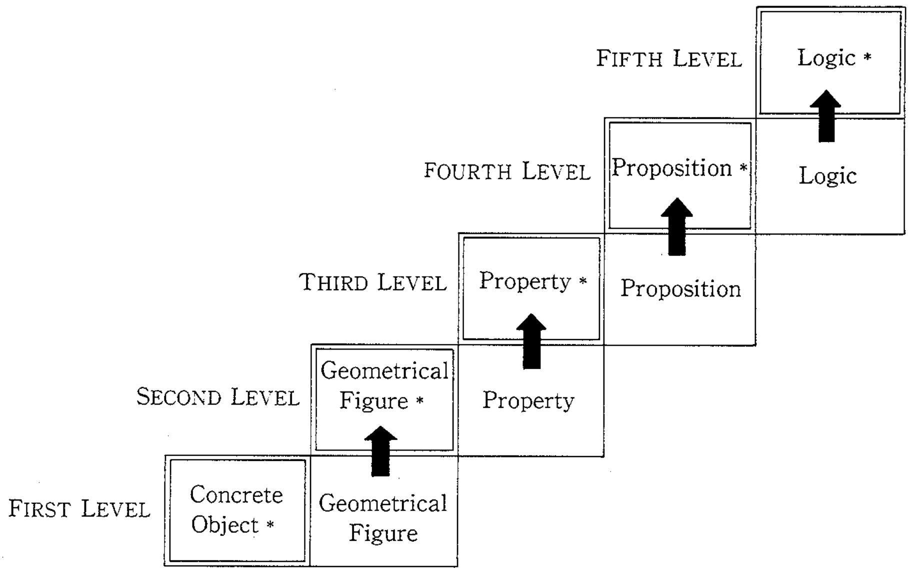

Fig. 2

The sign ↑ indicates the objectification of a way of unconscious thinking.

After all, we identify two such basic components of a process model as hierarchical levels and learning stages. In the next section, a process model with these two basic components is presented as a theoretical framework for developing children's understanding in the teaching and learning of mathematics.

A TWO AXES PROCESS MODEL

In order to build a process model which can prescribe as well as describe how the process of children's understanding of mathematics should progress, we must give serious consideration to the following questions. Through what levels should children's understanding progress? How do children develop their thinking in each level of understanding?

Relating to the first question, as already discussed, levels involved in the Pirie and Kieren model and the van Hiele model can be regarded as possible answers. In mathematical thinking process, logical thinking and intuition are complementary and closely interrelated. The interaction between logical thinking and intuition in a level has the energy to make another interaction in a higher level possible. Focusing on, for example, the long term process of understanding, we could identify the hierarchical levels in accordance with the characteristic which each object of intuition has

(Koyama, 1988b).

First Level; Intuition of *mathematical entities*
Second Level; Intuition of *properties of those entities*
Third Level; Intuition of *relations of those properties*
Fourth Level; Intuition of *relations of those propositions*

Although we need to examine these levels and modify them in accordance with mathematical concepts intended in the teaching and learning mathematics, they form a vertical axis of the process model of understanding.

Relating to the second question, learning stages involved in the van Hiele theory (van Hiele, 1986) and in Dienes theory (Dienes, 1960, 1963, 1970) are very suggestive. On the one hand, in the van Hiele theory five stages in the learning process leading to a higher level are discerned; information, guided orientation, explicitation, free orientation, and integration (van Hiele, 1986, pp. 53-54). On the other hand, in the Dienes theory six stages in the mathematics learning are offered based on four principles, the dynamic, constructivity, mathematical variability and perceptual variability principle (Dienes, 1960, p. 44); free play, rule-bound play, exploration of isomorphic structure, representation, symbolization and formalization (Dienes, 1963, 1970). The stages in these two models can be roughly corresponded as follows: information to free play, guided orientation to rule-bound play, explicitation to exploration of isomorphic structure and representation, free orientation to symbolization, and integration to formalization respectively.

According to Wittmann (1981), these corresponding stages are classified into three categories. He emphasizes that three types of activities are necessary in order to develop a balance of intuitive, reflective and formal thinking, based on the assumption that mathematics teaching should be modelled according to the processes of doing mathematics (Wittmann, 1981, p. 395). I have modified Wittmann's definitions of three activities a little in order to form a horizontal axis of the process mode. At any level of understanding, there are three stages, i.e. intuitive, reflective, and analytic stage.

Intuitive Stage; Children are provided opportunities for manipulating concrete objects, or operating on mathematical concepts or relations acquired in a previous level. At this stage they do *intuitive thinking*.

Reflective Stage; Children are stimulated and encouraged to pay attention to their own manipulating or operating activities, to be aware of them and their consequences, and to represent them in terms of diagrams, figures or language. At this stage they do *reflective thinking*.

Analytic Stage; Children elaborate their representations to be mathematical ones using mathematical terms, verify the consequences by means of other examples or cases, or analyze the relations among consequences in order to integrate them as a whole. At this stage they do *analytical thinking* and at the end they could progress their understanding to a next higher level.

Through these three stages (not necessarily linear) children's understanding could progress from one level to a higher level in the teaching and learning of mathematics. As a result, the process model of understanding consists of two axes, called "a two axes process model". In a two axes process model the vertical axis is formed by four hierarchical levels of understanding and the horizontal axis is formed by three stages in any level.

BY WAY OF CONCLUSION

A model of understanding mathematics should have prescriptive as well as descriptive characteristics so as to be useful and effective in mathematics education as the integration of teaching and learning activities. Based on the assumption, basic components common to the process models presented in preceding research were explored and two basic components, i.e. *hierarchical levels* of understanding and *learning stages* for developing, were identified. By using these components as its two axes, *a two axes process model* was built to elucidate the process of children's understanding in mathematics education.

The validity of this model can be assured indirectly to some extent by corroborative evidences in preceding research related to models of understanding. But the model is a theoretical one and a means to an end. Therefore, by using this model, to grasp the real processes of children's understanding in the teaching and learning certain mathematical concepts and to elabolate or modify it is left as an important task.

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