Complete characterization of post-selected quantum statistics using weak measurement tomography

Holger F. Hofmann*

Graduate School of Advanced Sciences of Matter, Hiroshima University, Kagamiyama 1-3-1, Higashi Hiroshima 739-8530, Japan

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The reconstruction of quantum states from a sufficient set of experimental data can be achieved with arbitrarily weak measurement interactions. Since such weak measurements have negligible back action, the quantum state reconstruction is also valid for the postselected subensembles usually considered in weak measurement paradoxes. It is shown that postselection can then be identified with a statistical decomposition of the initial density matrix into transient density matrices conditioned by the anticipated measurement outcomes. This result indicates that it is possible to ascribe the properties determined by the final measurement outcome to each individual quantum system before the measurement has taken place. The "collapse" of the pure state wave function in a measurement can then be understood in terms of the classical "collapse" of a probability distribution as new information becomes available.

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I. INTRODUCTION

As our ability to control individual quantum systems increases, the seemingly paradoxical aspects of quantum measurement theory take on a more practical relevance. One recent example is the renewed interest in the quantum statistics of postselected weak measurements, which seem to suggest the presence of negative probabilities as the source of quantum paradoxes [1–8]. These developments in the field of weak measurement may be especially significant in the context of new experimental possibilities pioneered by quantum information related research [9–11]. However, there seems to be a certain mismatch between the conventional approach to weak measurements, which is based on the notion of measurement interaction dynamics mediated by operator observables [12–16], and the more general approach to measurements based on the measurement operators widely used in quantum information [17]. In particular, the conventional analysis seems to overemphasize the exotic and surprising aspects of the weak measurements, while the operator-based approach shows more clearly how weak measurements fit into the general framework of quantum physics [18]. As experimental weak measurements become more and more established, it may therefore be time to shift the focus away from the oddities of specific cases, toward a complete and consistent formulation of postselection effects in terms of their experimentally observable properties.

From the experimental side, quantum states and processes can be characterized by measuring their complete statistical properties, a procedure known as quantum tomography [19,20]. A significant merit of quantum tomography is that it establishes an operational approach to quantum states; that is, it defines the quantum state in terms of the experimentally accessible data. By applying quantum tomography to generalized weak measurements, it is possible to extend this operational definition to postselected subensembles of a quantum state. In the following, the general theory of quantum tomography with measurements of variable strength is formulated. It is shown that, in the limit of weak measurements, postselection partitions the initial density matrix into subensembles described by nonpositive transient density matrices. This result divides the "collapse" of the wave function into two distinct parts, one associated with the selection of the appropriate subensemble, and the other related to the actual back action of the measurement dynamics. Although the measurement back action is necessary to cover up the negative eigenvalues of the transient states, weak measurement tomography suggests that the subensemble partition has a physical meaning even before the measurement interaction takes place.

II. QUANTUM STATE TOMOGRAPHY WITH WEAK MEASUREMENTS

The starting point for the following derivation of a complete and consistent theory of weak measurement tomography is the representation of general quantum measurements by a set of operators $\{\hat{M}_m\}$ acting only on the Hilbert space of the system [17]. These operators summarize the relevant effects associated with a measurement outcome m, separating the essential properties of a quantum measurement from the technical problem of its implementation by a specific combination of system-meter interaction, meter preparation, and meter readout. In the case of Hilbert space vectors representing pure states, the application of a measurement operator $\hat{M}_m$ to a state vector changes both the length and the direction of that vector. The new direction of the state vector then represents the output state after the measurement, while the squared length represents the probability $p(m)$ of obtaining the measurement outcome m. If the quantum state is expressed in terms of the density matrix $\hat{\rho}_i$, the probability $p(m)$ is given by a product trace,

$$p(m) = \mathrm{Tr}\{\hat{M}_m\ \hat{\rho}_i \hat{M}_m^\dagger\} = \mathrm{Tr}\{\hat{M}_m^\dagger \hat{M}_m\ \hat{\rho}_i\}. \qquad (1)$$

The set of squared operators $\{\hat{M}_m^\dagger \hat{M}_m\}$ is the positive operator-valued measure (POVM) of the measurement probabilities. Since all probabilities must add up to 1, the POVM fulfills the completeness relation

$$\sum_m \hat{M}_m^\dagger \hat{M}_m = \hat{1}. \qquad (2)$$

*hofmann@hiroshima-u.ac.jp

The POVM formalism describes the general relation between experimental data and the quantum state. Specifically, Eq. (1) shows that the measurement probabilities $p(m)$ are linear combinations of the density matrix elements. If the set of relations given by Eq. (1) is invertible, the complete density matrix can be reconstructed from the available set of measurement probabilities. In a d-dimensional Hilbert space, quantum tomography can thus be performed using any combination of d^2 linearly independent measurement operators $\hat{M}_m^\dagger \hat{M}_m$.

Whether a POVM is invertible and therefore suitable for quantum tomography does not depend on the precision of the measurement. It is therefore possible to reconstruct the quantum state from arbitrarily weak measurements. To illustrate this point, it is useful to formulate the inversion procedure for POVMs with variable strength ϵ. The strength or weakness of a measurement can be quantified directly by the closeness of the measurement operators to multiples of the identity operator $\hat{1}$. A convenient way of representing a variably measurement strength ϵ in the formalism is

$$\hat{M}_m^\dagger \hat{M}_m = w_m(\hat{1} + \epsilon \hat{S}_m)$$
$$\text{with} \quad \sum_m w_m = \frac{1}{1+\epsilon} \quad \text{and} \quad \sum_m w_m \hat{S}_m = \frac{\hat{1}}{1+\epsilon}. \tag{3}$$

The measurement probabilities in Eq. (1) can then be expressed in terms of the expectation values of a set of self-adjoined operators $\hat{S}_m$ that is independent of the measurement strength ϵ. For quantum state reconstruction, a set of d^2 linearly independent operators $\{\hat{S}_m\}$ defines an operator expansion of the density matrix in terms of the set of reciprocal operators $\{\hat{\Lambda}_m\}$ with $\mathrm{Tr}\{\hat{S}_n \hat{\Lambda}_m\} = \delta_{n,m}$. The density matrix is then given by

$$\hat{\rho}_i = \sum_m \mathrm{Tr}\{\hat{S}_m \hat{\rho}_i\} \hat{\Lambda}_m, \tag{4}$$

where the coefficients of the expansion are related to the measurement probabilities $p(m)$ by

$$\mathrm{Tr}\{\hat{S}_m \hat{\rho}_i\} = \frac{1}{\epsilon w_m}[p(m) - w_m]. \tag{5}$$

Equations (4) and (5) express the density matrix in terms of the experimentally obtained measurement probabilities $p(m)$. Thus, quantum tomography can provide a definition of the density matrix that is based only on empirical properties of the system.

III. QUANTUM STATES OF POSTSELECTED ENSEMBLES

Conventional quantum tomography is a one-way readout process in which the quantum state is discarded after the measurement. However, the measurement operators $\hat{M}_m$ also provide a description of the quantum state after the measurement. It is therefore possible to consider the effects of a subsequent measurement with outcomes f, described by another POVM $\{\hat{\Pi}_f\}$. The joint probability of obtaining first m and then f in the measurements is given by

$$p(m, f|i) = \mathrm{Tr}\{\hat{\Pi}_f \hat{M}_m \hat{\rho}_i \hat{M}_m^\dagger\}. \tag{6}$$

Since the measurement operators $\hat{M}_m^\dagger$ and $\hat{M}_m$ are not directly multiplied, the probability $p(f|i)$ found by summing over all m is different from the product trace of $\hat{\rho}_i$ and $\hat{\Pi}_f$. This change in the probability of obtaining f from the initial state $\hat{\rho}_i$ is caused by the measurement back action associated with the measurement of m. It is therefore impossible to know whether the final result f was caused by the initial state of the system or by interaction effects related to the measurement outcome m. However, the theory of weak measurements shows how this problem can be circumvented: For very small measurement strengths ϵ, the effects of the quantum state on the measurement probabilities is linear in ϵ while the measurement back action is quadratic in ϵ. It is therefore possible to realize quantum state tomography with negligible back action.

In the limit of weak measurements ($\epsilon \ll 1$), the measurement operators $\hat{M}_m$ are approximately given by the linearized square root of the POVM,

$$\hat{M}_m \approx \sqrt{w_m}\left(\hat{1} + \frac{\epsilon}{2}\hat{S}_m\right). \tag{7}$$

In the joint probability $p(m, f|i)$ given by Eq. (6), the terms linear in ϵ are obtained by applying $\hat{S}_m$ either from the right or from the left. If quadratic terms are neglected, this is equivalent to applying the POVM from the right or from the left. Therefore, the joint probability $p(m, f|i)$ can be approximately expressed by the weak measurement POVM,

$$p(m, f|i) \approx \mathrm{Tr}\left\{\hat{\Pi}_f \, \tfrac{1}{2}(\hat{\rho}_i \hat{M}_m^\dagger \hat{M}_m + \hat{M}_m^\dagger \hat{M}_m \hat{\rho}_i)\right\}. \tag{8}$$

Since the POVM fulfills the completeness relation given by Eq. (2), the final measurement probabilities $p(f|i)$ are not changed by the measurement of m and

$$p(f|i) = \sum_m p(m, f|i) = \mathrm{Tr}\{\hat{\Pi}_f \hat{\rho}_i\}. \tag{9}$$

Hence the measurement back action is negligible and the measurement results m merely identify the state conditioned by both initial preparation and final measurement. The conditional probability can then be written in terms of a transient density matrix $\hat{R}_{if}$, such that

$$p(m|i, f) = \frac{p(m, f|i)}{p(f|i)} = \mathrm{Tr}\{\hat{M}_m^\dagger \hat{M}_m \hat{R}_{if}\}. \tag{10}$$

According to Eqs. (8) and (9), this transient density matrix is given by

$$\hat{R}_{if} = \frac{1}{2\mathrm{Tr}\{\hat{\rho}_i \hat{\Pi}_f\}}(\hat{\rho}_i \hat{\Pi}_f + \hat{\Pi}_f \hat{\rho}_i). \tag{11}$$

Experimentally, the transient density matrix can be reconstructed using the procedure outlined in the previous section for conventional density matrices. In this case, Eq. (5) defines the weak values of $\hat{S}_m$, and $\hat{R}_{if}$ is obtained from the operator expansion given in Eq. (4), where the coefficients are now given by these weak values. Significantly, the expectation values of $\hat{S}_m$ are never obtained by strong measurements. Even in the conventional nonpostselected case, the expectation value of $\hat{S}_m$ is determined from small changes in the probability m. Therefore, there is a fundamental similarity between the way conventional quantum state tomography can obtain predictions for strong measurements from sets of weak measurements and the reconstruction of weak values from postselected measurements. If quantum tomography is accepted as an unambiguous operational definition of the density matrix in terms of the measurement statistics of m for any well-defined ensemble of quantum systems, the consistency of quantum measurement theory requires that the self-adjoint operator $\hat{R}_{if}$ is the proper statistical representation of the subensemble defined by the conditions i and f.

IV. ANTICIPATORY DECOMPOSITION

The operator $\hat{R}_{if}$ correctly predicts the outcomes of all (real) weak values that can be obtained between i and f. In this sense, it is similar to the two-state vector formalism of weak measurements [21] and its extension to mixed states [22]. However, the derivation from tomography ensures that $\hat{R}_{if}$ is a self-adjoined operator with real eigenvalues and a trace of one. It thus corresponds to a conventional density matrix, except for the possibility of negative eigenvalues. Moreover, the definition of $\hat{R}_{if}$ applies equally well to projective measurements of pure states and to noisy measurements of mixed states. The present analysis therefore bridges the gap between the classical limit and the extreme quantum limit, revealing similarities of quantum and classical statistics that tend to be obscured by the state vector formalism. Specifically, the transient density matrix $\hat{R}_{if}$ is the quantum mechanical representation of conditional probabilities $p(m|i, f)$, just as the density matrix $\hat{\rho}_i$ is the quantum mechanical representation of the probabilities $p(m|i)$. The relation between the total density matrix $\hat{\rho}_i$ and the set of transient density matrices conditioned by the final measurement outcomes f is therefore given by the weighted sum over all possible outcomes f,

$$\hat{\rho}_i = \sum_f p(f|i)\,\hat{R}_{if}. \tag{12}$$

This decomposition of the density matrix explains why the weak measurement statistics can be measured before the final measurement f has taken place: The measurement of f simply identifies a subensemble $\hat{R}_{if}$ that is already included in the total ensemble $\hat{\rho}_i$. It is therefore possible to decompose $\hat{\rho}_i$ into $\hat{R}_{if}$ in *anticipation* of the measurement of f. Such an anticipatory decomposition indicates that the measurement outcome f can already be ascribed to the quantum systems before the measurement has taken place.

V. WAVE-FUNCTION COLLAPSE WITHOUT MEASUREMENT BACK ACTION

In classical physics, the statistical rules for anticipatory decompositions are straightforward, since the state $\hat{\rho}$ is just a conventional probability distribution over all microscopic configurations of the system. In the quantum case, it is usually assumed that the "collapse" of a pure state caused by a projective measurement is fundamentally different from such a probability update, because it includes the effects of decoherence. However, weak measurement tomography shows that the "collapse" can be divided into two parts, one associated with a classical subensemble selection (that is, a Bayesian probability update) and the second one associated with the actual physical interaction that results in the decoherence. It is therefore possible to identify the set of quantum systems that produce a specific measurement result f with a uniquely defined subensemble of the total density matrix $\hat{\rho}_i$.

To understand the significance of this result, it may be useful to reconsider some of the paradoxes of quantum mechanics in the light of these experimentally accessible facts. For instance, it is now possible to resolve the problem of double-slit interference by combining interference information with which-path information. Specifically, the initial pure state superposition $|i\rangle = (|1\rangle + |2\rangle)/\sqrt{2}$ of a particle passing through slit 1 and a particle passing through slit 2 can be decomposed into

$$\begin{aligned}\hat{R}_{i1} &= |1\rangle\langle 1| + \tfrac{1}{2}(|1\rangle\langle 2| + |2\rangle\langle 1|),\\ \hat{R}_{i2} &= |2\rangle\langle 2| + \tfrac{1}{2}(|1\rangle\langle 2| + |2\rangle\langle 1|)\end{aligned} \tag{13}$$

in anticipation of a which-path measurement. Weak measurements performed between the preparation of the superposition $|i\rangle$ and the final which-path measurements show that the path information coexists with the interference pattern of the superposition. Therefore, the double-slit interference pattern can be obtained from weak measurements even if the particle has passed through only one of the slits. Equation (14) thus shows that coherence between two alternatives does not imply that both alternatives are simultaneously realized.

Essentially, weak measurement tomography is an analysis of quantum statistics that reveals additional details about where quantum information is located before it is measured. In particular, it fills a gap left by the measurement postulate by showing that the measurement result f corresponds to properties of the system before the measurement and is not just randomly generated by a role of the dice at the time of measurement. Thus, we can solve the paradox of Schrödinger's cat by experimentally confirming that the cat already showed signs of being dead before somebody opened the box to look. In the simplified scenario where the death of the cat is either caused or not caused by a superposition of unitary operations at a well-defined instance, weak measurement tomography would clearly show that the cat was already dead during the time interval between that instance and the final measurement. In the case of entanglement paradoxes, weak measurement tomography indicates that the nonlocal collapse of entangled states merely corresponds to a Bayesian probability update in the remote system, providing a classical analogy that can explain nonclassical properties of quantum mechanics such as the impossibility of nonlocal signaling and the transfer of quantum information by classical channels in quantum teleportation [23,24].

VI. EXPERIMENTALLY OBSERVABLE NEGATIVE PROBABILITIES

As explained earlier, weak measurement tomography identifies the quantum statistics of systems with well-defined initial and final properties. This means that the available information about each system can be more precise than the uncertainty limit allows. Such super-certain information finds its quantum statistical expression in the nonpositive transient density matrices $\hat{R}_{if}$ that summarize the results of all possible weak measurements between i and f. The negative eigenvalues of $\hat{R}_{if}$ represent weak values that exceed the eigenvalue bounds of positive density matrices, providing a consistent framework for the resolution of quantum paradoxes by weak measurements [4–8].

Quantitative descriptions of quantum paradoxes such as Bell's inequalities [25], Leggett-Garg inequalities [4–6], or contextuality inequalities [26] are usually formulated in terms of precise measurements performed on different representatives of the same state. To connect this conventional formulation with weak measurements, it is necessary to show that the joint probabilities determined in weak measurements provide a unique and measurement-independent definition of joint probabilities in quantum systems. In particular, the joint probabilities should not depend on the order in which the result is obtained [27]. For a pair of strong measurements $\{\hat{\Pi}_f\}$ and $\{\hat{\Phi}_g\}$, the joint probability from a weak measurement of g followed by a strong measurement of f should therefore be equal to the joint probability of a weak measurement of f followed by a strong measurement of g,

$$\begin{aligned} p(f,g|i) &= p(f|i)\,\mathrm{Tr}\{\hat{\Phi}_g\hat{R}_{if}\} \\ &= p(g|i)\,\mathrm{Tr}\{\hat{\Pi}_f\hat{R}_{ig}\}. \end{aligned} \quad (14)$$

Weak measurement tomography can confirm that these two results are indeed equal. This means that the transient density matrices $\hat{R}_{if}$ and $\hat{R}_{ig}$ describe the same statistical correlations between the measurement of f and the measurement of g. We can therefore conclude that weak measurement tomography provides a consistent definition of joint probabilities for measurements that cannot be performed jointly. In terms of the POVMs, this joint probability reads

$$p(f,g|i) = \mathrm{Tr}\left\{\tfrac{1}{2}(\hat{\Phi}_g\hat{\Pi}_f + \hat{\Pi}_f\hat{\Phi}_g)\,\hat{\rho}_i\right\}. \quad (15)$$

It should be emphasized that each of these joint probabilities can be measured directly in an appropriate weak measurement. Equation (15) thus provides a consistent definition of joint probability based directly on the measurement operators for a pair of well-defined strong measurements. This is different from the assumptions used in the construction of quasi-probabilities such as the Wigner function, where the probabilities are defined in terms of observables and not in terms of the measurement operators. This means that the Wigner function and other quasi-probabilities should not be interpreted directly as a joint probability of the projections on position and momentum eigenstates. Instead, all quasi-probabilities represent specific assumptions about the uncertainty statistics involved in finite-resolution measurements of the observables [28,29]. The present results indicate that such assumptions may not provide a consistent description of joint probabilities for pairs of precisely defined strong measurements. It is therefore important to distinguish carefully between the mathematical construction of quasi-probabilities and their actual empirically valid meaning [30]. On the other hand, the negative probabilities predicted by Eq. (15) are a natural consequence of quantum statistics, required by the consistency of weak and strong results. Thus, the validity of the results with respect to the specific pair of measurement operators is not a matter of interpretation, even though the precise meaning of negative joint probabilities for outcomes that never occur jointly may be difficult to understand.

VII. IMPLICATIONS OF UNCERTAINTY FOR INDIVIDUAL QUANTUM SYSTEMS

Obviously, negative probabilities cannot be interpreted as relative frequencies of actual measurements. Nevertheless, they can be derived from the relative frequencies of weak measurements that consistently give the same results as the corresponding strong measurements. To reconcile the strangeness of negative probabilities with the empirical sense of reality justified by reproducible measurement results, it is important to remember that the validity of an anticipatory decomposition depends on the performance of the actual measurement. If an alternative measurement is performed instead, the subensembles need to be "reshuffled" before the correct decomposition is applied. For practical purposes, contradictions are avoided because the reality of an individual system is determined by only one of the possible measurements. The resolution of quantum paradoxes by negative probabilities is therefore based on the difference between the total statistical ensemble and its individual representatives. The reality of the representative is given by precise values of i and f (or i and g), while the ensemble properties are described by the transient density matrix. In the case of the subensemble defined by i and f, this transient density matrix is given by $\hat{R}_{if}$, which may have negative eigenvalues since it can only be observed in weak measurements. Equations (14) and (15) show that the joint negative probabilities predicted by weak measurement tomography of i and f are consistent with the joint negative probabilities obtained from i and g. The result can therefore be summarized in terms of a uniquely defined joint probability (15) that expresses the relation between strong measurements of f and of g directly in terms of their POVM operators. However, the negative eigenvalues of this operator clearly indicate that individual realities must be restricted to the actual measurement outcomes caused by the respective system. Weak measurement tomography thus decides quantum paradoxes in favor of locality and causality but against the notion of a nonempirical realism that attempts to provide a description of individual quantum objects beyond the uncertainty limited reality of its individual effects.

VIII. CONCLUSIONS

In conclusion, weak measurement tomography reveals a striking consistency of the quantum measurement formalism with classical statistics by defining an unambiguous partition of the total ensemble described by $\hat{\rho}_i$ into well-defined subensembles $\hat{R}_{if}$ representing the future measurement outcomes f. The reconstruction of quantum states by weak measurements thus provides empirical evidence that the selection of a measurement outcome f does not eliminate the coherences between f and other outcomes. In particular, weak measurements can show that particles moving only through slit 1 of a double-slit experiment carry the complete interference pattern of the initial state with them until the physical interaction of the final measurement randomizes the phase relation. Empirical evidence thus favors a statistical interpretation of quantum mechanics that assigns reality to individual measurement outcomes even before the measurement is performed. Quantum paradoxes can then be explained in terms of the negative probabilities described by the nonpositive transient density matrices $\hat{R}_{if}$. As was shown here, these transient density matrices uniquely define the joint probabilities between the measurement results f and the possible outcomes of other measurements g. The negative values of such joint probabilities demonstrate that no consistent simultaneous assignment of both g and f is possible. Weak measurements can thus provide experimental proof that quantum paradoxes arise from the fallacious imposition of nonempirical realities beyond the uncertainty limits restricting the observable effects of individual systems.

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