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Pareto Optimal Strategy for Stochastic Weakly Coupled Large Scale Systems With State Dependent System Noise

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***Abstract*—This note is concerned with the decentralized infinite horizon stochastic Pareto-optimal static output feedback strategy for a class of weakly coupled systems with state-dependent noise. First, Pareto-optimal control problems are formulated using a static output feedback strategy. The necessary conditions are given by the cross-coupled stochastic algebraic Riccati-type equations (CSAREs) for proving the existence of the static output feedback strategy that minimizes the quadratic cost function. After determining the asymptotic structure of the solutions of the CSAREs, a new sequential numerical algorithm and Newton's method for solving the CSAREs are described. The resulting numerical solution is used to develop the Pareto-optimal strategy. Finally, the efficiency of the proposed algorithm is demonstrated by solving a numerical example for a practical megawatt-frequency control problem.**

***Index Terms*—Cross-coupled stochastic algebraic Riccati-type equations (CSAREs), Pareto optimal strategy, static output feedback, stochastic weakly-coupled systems.**

I. Introduction

Algebraic Riccati equations (AREs) frequently play an important role in the design of a controller for the optimal control problems for a weakly coupled large-scale interconnected system that is parameterized by a small coupling parameter ε. The existing parameter-independent approximate controller [1], which is very reliable, can be used if the small coupling parameter is unknown. However, when ε is not sufficiently small, an $O(\varepsilon)$ accuracy control is often not very sufficient, as is evident from [6]. Therefore, as long as ε is known, the AREs should be solved numerically. The problem of designing a controller for stochastic systems governed by Itô's differential equations has been the subject of many studies in the past few decades [2], [4], [12], [13]. The results published in these papers are theoretically elegant and practically reliable; however, in these studies, the stochastic static output feedback control has not been taken into consideration. From a practical point of view, the output feedback control is extremely attractive since states are not always available for feedback. The use of the output feedback control affords a flexible and simple design for the implementation of the controller. It is well known that coupled nonlinear matrix algebraic equations are necessary for obtaining the solution to this problem [3], [9]. For example, the necessary conditions derived in [3] for a static feedback controller in the presence of state- and control-dependent noise involve the cross-coupled stochastic algebraic Riccati-type equations (CSAREs). However, none of the previous studies have addressed the control problem of stochastic systems with multiple decision makers.

Similarly, control problems for large-scale deterministic systems have been the subject of many previously reported studies, where

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decisions in large-scale systems are usually made using multiple decision makers with different information sets. For example, the optimum megawatt-frequency control of multiarea electric energy systems [5] has been treated as Nash games of the weakly coupled large-scale systems with multiple decision makers [6]. A multimodel solution for multiple decision makers with different information sets has also been studied [18]. Moreover, decentralized output feedback controllers for large-scale systems have been extensively investigated (see, e.g., [19], [20]).

In this note, we investigate the decentralized Pareto-optimal static output feedback strategy for stochastic systems with state-dependent noise governed by Itô differential equations. We study the case wherein the local output measurements are the only information available for the decision makers. Our present research is related to that mentioned in [3]. However, we consider a significantly different problem. While the study mentioned in [3] deals with a regular stochastic optimal control problem with a single decision maker, we consider the problem involving multiple decision makers who use the local information from output measurements for each subsystem. Furthermore, we extend the existing results [12] to the decentralized stochastic static output feedback strategy of systems with multiple decision makers. We present the necessary conditions for the application of a Pareto-optimal strategy to a decentralized controller. This strategy set is based on the solutions of the CSAREs that consist of two stochastic algebraic Lyapunov-type equations (SALEs) and a nonlinear algebraic matrix equation. The boundedness of the solution to the CSAREs and the asymptotic structures of the solutions of CSAREs are established. The other important feature of our study is the development of a new sequential numerical algorithm and Newton's method for solving the CSAREs. Furthermore, for the latter case, the degradation of costs is estimated by applying the proposed approximate Pareto strategies. Finally, the efficiency of the proposed algorithm is demonstrated by using it for solving a numerical problem, for example, a two-area electric energy system.

Notation: The notations used in this note are fairly standard. I_n denotes an $n \times n$ identity matrix. **block diag** denotes a block diagonal matrix. $\|\cdot\|$ denotes the Euclidean norm of a matrix. E denotes the expectation. $\otimes$ denotes the Kronecker product. δ_{ij} denotes the Kronecker delta. **Tr** denotes sum of the diagonal elements of a matrix.

II. Stochastic Pareto Optimal Static Output Feedback Strategy

We now study the static Pareto-optimal control problem with state-dependent noise. We consider linear time-invariant weakly coupled large-scale stochastic systems

$$dx(t) = \left[A_\varepsilon x(t) + \sum_{k=1}^{N} B_{k\varepsilon} u_k(t)\right] dt + \sum_{k=1}^{N} A_{k\varepsilon} x(t) dw_k(t)$$

$$x(0) = x^0 \tag{1a}$$

$$y_i(t) = C_i x(t) = C_{ii} x_i(t), \quad i = 1, \dots, N \tag{1b}$$

where

$$A_\varepsilon := \begin{bmatrix} A_{11} & \varepsilon A_{12} & \cdots & \varepsilon A_{1N} \\ \varepsilon A_{21} & A_{22} & \cdots & \varepsilon A_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon A_{N1} & \varepsilon A_{N2} & \cdots & A_{NN} \end{bmatrix}$$

$$A_{i\varepsilon} := \begin{bmatrix} \varepsilon^{1-\delta_{i1}} A_{i11} & \varepsilon A_{i12} & \cdots & \varepsilon A_{i1N} \\ \varepsilon A_{i21} & \varepsilon^{1-\delta_{i2}} A_{i22} & \cdots & \varepsilon A_{i2N} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon A_{iN1} & \varepsilon A_{iN2} & \cdots & \varepsilon^{1-\delta_{iN}} A_{iNN} \end{bmatrix}$$

$$B_{i\varepsilon} := \left[\varepsilon^{1-\delta_{1i}} B_{1i}^T \quad \varepsilon^{1-\delta_{2i}} B_{2i}^T \quad \cdots \quad \varepsilon^{1-\delta_{Ni}} B_{Ni}^T\right]^T$$

$$C_i := \left[0 \quad \cdots \quad 0 \quad C_{ii} \quad 0 \quad \cdots \quad 0\right].$$

$x_i(t) \in \Re^{n_i}$, $x(t) := [\, x_1^T(t) \ \cdots \ x_N^T(t)\,]^T$, $i = 1, \dots, N$ represents the ith state vector. $u_i(t) \in \Re^{m_i}$, $i = 1, \dots, N$, represents the ith control vector. $y_i(t) \in \Re^{l_i}$, $i = 1, \dots, N$, represents the ith output measurement vector. $w_i(t) \in \Re$, $i = 1, \dots, N$, is a 1-D standard Wiener process defined on a filtered probability space [4], [12], [13]. It is natural that each subsystem has a different 1-D standard Wiener process for the weakly coupled systems under consideration. In fact, the parameter-independent subsystems (11b) that will be given later with different Wiener processes for each i can be obtained by using the representation of multiplicative noise. Furthermore, stochastic systems with multiplicative noise, which are widely used to represent system dynamics, arise in many control problems and have been extensively considered in the past [16], [17]. In the above mentioned systems, ε is a relatively small coupling parameter that connects the linear system with the other subsystems[1]. It should be noted that in the separation of the subsystems that are connected by the weakly coupled interconnections, δ_{ij} can play an important role in describing the dominant part of the subsystems. As suggested in [16], it is customary to relieve its dependence by assuming that the initial state x_0 is a random variable with a covariance matrix $E[x(0)x^T(0)] = I_{\bar{n}}$, $\bar{n} := \sum_{k=1}^{N} n_k$.

For large-scale systems, it is generally impossible to incorporate many feedback loops into the controller design, and such an incorporation, even if possible, is very expensive. These difficulties have motivated researchers to study the decentralized control theory, where each subsystem is controlled independently using its locally available information. Therefore, in this note, we make a realistic assumption that each decision maker can only know the local output measurements. In other words, we consider the following static output feedback strategy in this note

$$u_i(t) := F_i C_i x(t) = F_i C_{ii} x_i(t), \quad i = 1, \dots, N. \tag{2}$$

The main purpose of this note is to establish a static output feedback strategy and analyze its reliability. We assume that decision makers will design control strategies based on locally available information. The design specification of the ith decision maker is expressed in terms of a cost function J_i. We consider the situation where the decision makers prefer to cooperate with one another. In other words, they attempt to find the Pareto-optimal strategies. This implies that no variation in the Pareto-optimal strategy can decrease the costs of all the decision makers [7], [8].

The cost function for each strategy subset is defined by

$$J_i = E \int_0^\infty \left[x^T(t) Q_{i\varepsilon} x(t) + u_i^T(t) R_i u_i(t)\right] dt \tag{3}$$

where $i = 1, \dots, N$, $Q_{ii} = Q_{ii}^T \ge 0 \in \Re^{n_i \times n_i}$ with

$$Q_{i\varepsilon} = Q_{i\varepsilon}^T = \begin{bmatrix} \varepsilon^{1-\delta_{i1}} Q_{i1} & \varepsilon Q_{i12} & \cdots & \varepsilon Q_{i1N} \\ \varepsilon Q_{i12}^T & \varepsilon^{1-\delta_{i2}} Q_{i2} & \cdots & \varepsilon Q_{i2N} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon Q_{i1N}^T & \varepsilon Q_{i2N}^T & \cdots & \varepsilon^{1-\delta_{iN}} Q_{iN} \end{bmatrix} \ge 0 \in \Re^{n \times n}$$

and $R_i = R_i^T > 0 \in \Re^{m_i \times m_i}$.

A Pareto solution is a set $(u_1, u_2, \dots, u_N)$ that minimizes

$$J = \sum_{k=1}^{N} \gamma_k J_k, 0 < \gamma_k < 1, \sum_{k=1}^{N} \gamma_k = 1 \tag{4}$$

for some γ_k, $k = 1, \dots, N$ [7], [8].

[1]In many cases, it is possible to show that ε is precisely known. In this note, we assume that ε is precisely known.

The optimal linear quadratic regulator problem is a special case of this problem when the decision makers agree on a choice of γ_k, $k = 1, \ldots, N$ as weight factors.

The following basic assumption mentioned in [12] is introduced.

Assumption 1: Stochastic system (1) is mean-square stabilizable by the static output feedback (2).

In order to develop the necessary conditions for this problem, F_i, $i = 1, \ldots, N$ must be restricted to the following set $\mathbf{F}_i := \{F_i \in \Re^{m_i \times l_i} |$ there exists a unique positive definite symmetric matrix X_{ii} that satisfies the following parameter-independent stochastic algebraic Lyapunov-type equation (SALE):

$$X_{ii}(A_{ii} + B_{ii}F_iC_{ii}) + (A_{ii} + B_{ii}F_iC_{ii})^T X_{ii} + A_{iii}^T X_{ii} A_{iii} + \gamma_i \left(C_{ii}^T F_i^T R_i F_i C_{ii} + Q_{ii} \right) = 0, i = 1, \ldots, N.\}. \tag{5}$$

It should be noted that there exists a unique positive definite symmetric matrix X_{ii} if and only if $I_{n_i} \otimes (A_{ii} + B_{ii}F_iC_{ii})^T + (A_{ii} + B_{ii}F_iC_{ii})^T \otimes I_{n_i} + A_{iii}^T \otimes A_{iii}^T$ is nonsingular [21]. Moreover, SALE (5) will be derived later in (9).

In order to guarantee the stability of the full-order closed-loop system and the existence of solutions for the parameter-independent subsystems, F_i is necessarily restricted to the set $\mathbf{F}_i$.

Using the feedback strategy of (2) and the assumption that $E[x(0)x^T(0)] = I_{\bar{n}}$, it is immediately found that the closed-loop stochastic system is exponentially mean square stable (EMSS) [2], [4] and that the integral portion of J satisfies the relation

$$J = \mathbf{Tr}[P_\varepsilon] \tag{6}$$

if there exists a unique solution for the following SALE of P_ε

$$\begin{aligned} &\mathcal{F}(\varepsilon, P_\varepsilon, F_{1\varepsilon}, \ldots, F_{N\varepsilon}) \\ &= P_\varepsilon \hat{A}_\varepsilon + \hat{A}_\varepsilon^T P_\varepsilon + \sum_{k=1}^N A_{k\varepsilon}^T P_\varepsilon A_{k\varepsilon} \\ &\quad + \sum_{k=1}^N \gamma_k C_k^T F_{k\varepsilon}^T R_k F_{k\varepsilon} C_k + Q_\varepsilon = 0 \end{aligned} \tag{7}$$

where $\hat{A}_\varepsilon := A_\varepsilon + \sum_{k=1}^N B_{k\varepsilon}F_{k\varepsilon}C_k$ and $Q_\varepsilon := \sum_{k=1}^N \gamma_k Q_{k\varepsilon}$.

In order to clarify the existence of P_ε of (7), we now investigate the asymptotic structure of the solution and establish the existence condition that is confirmed by reduced-order and parameter-independent calculations.

Since A_ε and $B_{i\varepsilon}$ include ε, the solutions P_ε of the SALE (7)—if the solutions exist-should contain ε. On the basis of this fact, the solution of SALE (7) is assumed to have the following structure [10], [11]

$$P_\varepsilon := \begin{bmatrix} P_{11} & \varepsilon P_{12} & \cdots & \varepsilon P_{1N} \\ \varepsilon P_{12}^T & P_{22} & \cdots & \varepsilon P_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon P_{1N}^T & \varepsilon P_{2N}^T & \cdots & P_{NN} \end{bmatrix} \in \Re^{\bar{n} \times \bar{n}}. \tag{8}$$

Substituting these matrices into SALE (7), setting $\varepsilon = 0$, and partitioning SALE (7), the following reduced-order SALE (9) is obtained, where $\bar{P}_{ii}$ and $\bar{F}_i, i = 1, \ldots, N$ are the zeroth-order solutions of SALE (7) as $\varepsilon = 0$

$$\bar{P}_{ii}(A_{ii} + B_{ii}\bar{F}_iC_{ii}) + (A_{ii} + B_{ii}\bar{F}_iC_{ii})^T \bar{P}_{ii} + A_{iii}^T \bar{P}_{ii} A_{iii} + \gamma_i \left(C_{ii}^T \bar{F}_i^T R_i \bar{F}_i C_{ii} + Q_{ii} \right) = 0. \tag{9}$$

The asymptotic expansion of SALE (7) for $\varepsilon = 0$ is described by the following lemma.

Lemma 1: Suppose that $\bar{F}_i \in \mathbf{F}_i$. There exists a small constant σ_1^* such that for all $\varepsilon \in (0, \sigma_1^*)$, SALE (7) admits a unique positive definite solution P_ε^* that can be expressed as

$$P_\varepsilon^* = \bar{P} + O(\varepsilon) \tag{10}$$

where $\bar{P} = \mathbf{block\ diag}(\ \bar{P}_{11} \ \cdots \ \bar{P}_{NN}\ )$.

Proof: This can be proved by applying the implicit function theorem to SALE (7). In order to do so, it is sufficient to show that the corresponding Jacobian is nonsingular at $\varepsilon = 0$. It should be noted that $\bar{F}_i \in \mathbf{F}_i$ if and only if $I_{n_i} \otimes (A_{ii} + B_{ii}\bar{F}_iC_{ii})^T + (A_{ii} + B_{ii}\bar{F}_iC_{ii})^T \otimes I_{n_i} + A_{iii}^T \otimes A_{iii}^T$ is nonsingular. Since the abovementioned relation is similar to that mentioned in [11], it is omitted. ■

It follows that the closed-loop stochastic system (1a) with $u_i(t) = \bar{F}_iC_{ii}x_i(t)$ is EMSS because SALE (7) admits a unique positive definite solution [2]. Moreover, it is easy to verify whether the behavior of the closed-loop stochastic system (1a) for a small value of ε can be stated using the following observation.

Observation 1: If $u_i(t) = F_iC_{ii}x_i(t), i = 1, \ldots, N$ are chosen to satisfy the condition $F_i \in \mathbf{F}_i$, then, for all t, there exists a positive scalar δ^* such that the following approximations hold for all all $\varepsilon \in (0, \delta^*)$:

$$x_i(t) = \tilde{x}_i(t) + O(\varepsilon) \tag{11a}$$

$$d\tilde{x}_i(t) = [A_{ii} + B_{ii}F_iC_{ii}\tilde{x}_i(t)]dt + A_{iii}\tilde{x}_i(t)dw_i(t). \tag{11b}$$

It should be noted that subsystems (11b) are obtained from (1) by setting $\varepsilon = 0$.

The necessary conditions for the Pareto optimality will be obtained in terms of the SALEs.

Theorem 1: Let us assume that $F_i \in \mathbf{F}_i$ solves the static output feedback Pareto-optimal control problem. Then, it is necessary that there exist symmetric positive definite solutions P_ε and S_ε that satisfy SALE (7) and SALE (12a), respectively; $F_{i\varepsilon}$ is obtained using (12b)

$$\begin{aligned} &\mathcal{G}(\varepsilon, S_\varepsilon, F_{1\varepsilon}, \ldots, F_{N\varepsilon}) \\ &= S_\varepsilon \hat{A}_\varepsilon^T + \hat{A}_\varepsilon S_\varepsilon + \sum_{k=1}^N A_{k\varepsilon} S_\varepsilon A_{k\varepsilon}^T + I_{\bar{n}} = 0 \end{aligned} \tag{12a}$$

$$\begin{aligned} &\mathcal{H}_i(\varepsilon, P_\varepsilon, S_\varepsilon, F_{1\varepsilon}, \ldots, F_{N\varepsilon}) \\ &= \gamma_i R_i F_{i\varepsilon} C_i S_\varepsilon C_i^T + B_{i\varepsilon}^T P_\varepsilon S_\varepsilon C_i^T = 0, \quad i = 1, \ldots, N. \end{aligned} \tag{12b}$$

Proof: The result can be proved by using the Lagrange multiplier approach. First, the closed-loop cost with the static output feedback controller $u_i(t) = F_{i\varepsilon}y(t) = F_iC_{ii}x_i(t)$ is obtained using the relation $J = \mathbf{Tr}[P_\varepsilon]$, where P_ε is the solution of SALE (7). Let us consider the Hamiltonian $\mathcal{L}$

$$\begin{aligned} &\mathcal{L}(\varepsilon, P_\varepsilon, S_\varepsilon, F_{1\varepsilon}, \ldots, F_{N\varepsilon}) \\ &= \mathbf{Tr}[P_\varepsilon] + \mathbf{Tr}[\mathcal{F}(\varepsilon, P_\varepsilon, F_{1\varepsilon}, \ldots, F_{N\varepsilon})S_\varepsilon] \end{aligned} \tag{13}$$

where S_ε is a symmetric positive definite matrix of Lagrange multipliers. The necessary conditions for $F_{i\varepsilon}$ to be optimal can be found by setting $\partial\mathcal{L}/P_\varepsilon$ and $\partial\mathcal{L}/F_{i\varepsilon}$ to zero and solving the resulting (12b) simultaneously for $F_{i\varepsilon}$. ■

Remark 1: It should be noted that Theorem 1 only gives the necessary conditions for a controller to be optimal. It is likely that the controllers obtained are not Pareto optimal even if the solutions of (7) and (12) exist.

Remark 2: The stochastic static output feedback Pareto-optimal strategy under consideration cannot be treated using the approach mentioned in [3] because of the existence of multiple decision makers. In fact, SALE (7) and (12) obtained are significantly different from those mentioned in [3].

Observation 2: If full-state information is available, i.e., $C_i := I_{n_i}$, and S_ε is nonsingular, then, according to (12b)

$$F_{i\varepsilon} = -(\gamma_i R_i)^{-1} B_{i\varepsilon}^T P_\varepsilon. \tag{14}$$

On the other hand, the following SARE (15) can be obtained by substituting $F_{i\varepsilon}$ into (7)

$$P_\varepsilon A_\varepsilon + A_\varepsilon^T P_\varepsilon + \sum_{k=1}^N A_{k\varepsilon}^T P_\varepsilon A_{k\varepsilon} - P_\varepsilon U_\varepsilon P_\varepsilon + Q_\varepsilon = 0 \tag{15}$$

where $U_\varepsilon := \sum_{k=1}^N \gamma_k^{-1} B_{k\varepsilon} R_k^{-1} B_{k\varepsilon}^T$.

If $C_i S_\varepsilon C_i^T$ is nonsingular, then, (12b) may be solved for $F_{i\varepsilon}$ to obtain

$$F_{i\varepsilon} = -(\gamma_i R_i)^{-1} B_{i\varepsilon}^T P_\varepsilon S_\varepsilon C_i^T \left(C_i S_\varepsilon C_i^T\right)^{-1}. \tag{16}$$

In the remaining part of this section, we will discuss the asymptotic structure of S_ε and $F_{i\varepsilon}$ for proposing the Pareto-optimal strategy set.

Lemma 2: If $\bar{F}_i \in \mathbf{F}_i$, there exists a small constant σ_2^* such that for all $\varepsilon \in (0,\ \sigma_2^*)$, SALE (12a) and the linear (12b) admit the positive definite solution S_ε^* and feedback gain $F_{i\varepsilon}^*$ that can be expressed as

$$S_\varepsilon^* = \bar{S} + O(\varepsilon), F_{i\varepsilon}^* = \bar{F}_i + O(\varepsilon) \tag{17}$$

where $\bar{S} = \mathbf{block\ diag}(\bar{S}_{11} \ \ \cdots \ \ \bar{S}_{NN})$

$$\bar{S}_{ii}(A_{ii} + B_{ii}\bar{F}_i C_{ii})^T + (A_{ii} + B_{ii}\bar{F}_i C_{ii})\bar{S}_{ii} + A_{iii}^T \bar{S}_{ii} A_{iii} + I_{n_i} = 0 \tag{18a}$$

$$\gamma_i R_i F_i C_{ii} \bar{S}_{ii} C_{ii}^T + B_{ii}^T \bar{P}_{ii} \bar{S}_{ii} C_{ii}^T = 0. \tag{18b}$$

Without loss of generality, as an additional technical assumption, we suppose that F_i is confined to the following set:
$\mathbf{L}_i := \{F_i \in \mathbf{F}_i \mid C_{ii}\bar{S}_{ii}C_{ii}^T > 0$, where $\bar{S}_{ii}$ satisfies (18a). $\}$.

The positive definiteness condition holds, for example, when $\bar{S}_{ii}$ is positive definite, and C_{ii} has full row rank. In this case, $\bar{F}_i$ can be written as

$$\bar{F}_i = -(\gamma_i R_i)^{-1} B_{ii}^T \bar{P}_{ii} \bar{S}_{ii} C_{ii}^T \left(C_{ii} \bar{S}_{ii} C_{ii}^T\right)^{-1}. \tag{19}$$

III. Numerical Algorithms for Solving SALES

The Pareto-optimal strategy $F_{i\varepsilon}$ of (16) can be obtained by solving SALE (7) and (12). We now propose two numerical approaches for designing Pareto-optimal strategy.

First, let us consider the following new iterative algorithm

$$P_\varepsilon^{(n+1)} \hat{A}_\varepsilon^{(n)} + \hat{A}_\varepsilon^{(n)T} P_\varepsilon^{(n+1)} + \sum_{k=1}^N A_{k\varepsilon}^T P_\varepsilon^{(n+1)} A_{k\varepsilon} + \sum_{k=1}^N \gamma_k C_k^T F_{k\varepsilon}^{(n)T} R_k F_{k\varepsilon}^{(n)} C_k + Q_\varepsilon = 0 \tag{20a}$$

$$S_\varepsilon^{(n+1)} \hat{A}_\varepsilon^{(n)T} + \hat{A}_\varepsilon^{(n)} S_\varepsilon^{(n+1)} + \sum_{k=1}^N A_{k\varepsilon} S_\varepsilon^{(n+1)} A_{k\varepsilon}^T + I_{\bar{n}} = 0 \tag{20b}$$

$$F_{i\varepsilon}^{(n+1)} = F_{i\varepsilon}^{(n)} - \alpha \left[(\gamma_i R_i)^{-1} B_{i\varepsilon}^T P_\varepsilon^{(n+1)} S_\varepsilon^{(n+1)} C_i^T \times (C_i S_\varepsilon^{(n+1)} C_i^T)^{-1} + F_{i\varepsilon}^{(n)}\right] \tag{20c}$$

where $\hat{A}_\varepsilon^{(n)} := A_\varepsilon + \sum_{k=1}^N B_{k\varepsilon} F_{k\varepsilon}^{(n)} C_k,\ n = 0, 1, \ldots$ and $\alpha \in (0,\ 1]$ is chosen to ensure the minimum is not overshot, that is, $J^{(n+1)} = \mathbf{Tr}[P_\varepsilon^{(n+1)}] < J^{(n)} = \mathbf{Tr}[P_\varepsilon^{(n)}]$. Moreover, the matrices $F_{i\varepsilon}^{(0)},\ i = 1, \ldots, N$ are chosen as the initial conditions such that the reduced-order closed-loop system $dx_i(t) = [A_{ii} + B_{ii} F_i^{(0)} C_{ii}] x_i(t) + A_{iii} x_i(t) dw_i(t)$ is EMSS.

Theorem 2: The sequence $F_{i\varepsilon}^{(n)},\ n = 0, 1, \ldots$ in (20c) converges to a stationary point in $\mathbf{F}_i$.

Before proving the theorem, we define the following set.
$\mathbf{G}_i := \{F_{i\varepsilon} \in \Re^{m_i \times l_i} \mid \hat{A}_\varepsilon$ is Hurwitz. $\}$.

Proof: From (12b), the gradient of the Lagrangian with respect to $F_{i\varepsilon}$ is given by $L := \gamma_i R_i F_{i\varepsilon} C_i S_\varepsilon C_i^T + B_{i\varepsilon}^T P_\varepsilon S_\varepsilon C_i^T$. The inner product of the search direction $\Delta F_{i\varepsilon}$ with the gradient L is $\beta(F_{i\varepsilon}) := \mathbf{Tr}[L \Delta F_{i\varepsilon}^T]$, where $\Delta F_{i\varepsilon} := -(\gamma_i R_i)^{-1} B_{i\varepsilon}^T P_\varepsilon S_\varepsilon C_i^T (C_i S_\varepsilon C_i^T)^{-1} - F_{i\varepsilon}$. Then, we have $\beta(F_{i\varepsilon}) := -\mathbf{Tr}[\Lambda_i^T \Lambda_i] < 0$, where $\Lambda_i := (C_i S_\varepsilon C_i^T)^{-1/2}[C_i S_\varepsilon C_i^T F_{i\varepsilon}^T (\gamma_i R_i)^{1/2} + C_i S_\varepsilon P_\varepsilon B_{i\varepsilon} (\gamma_i R_i)^{-1/2}]$ if $F_{i\varepsilon} \in \mathbf{G}_i,\ i = 1, \ldots, N$ and $L \neq 0$. The continuity of the gradient implies that for each iteration, there exists some α^* sufficiently small so that (20c) is satisfied for $0 < \alpha \leq \alpha^*$. Under these conditions, the sequence $J^{(n)},\ n = 0, 1, \ldots$ with $F_{i\varepsilon}^{(n)}$ is convergent because it is monotonic and bounded. Finally, the continuity of J implies that the sequence $F_{i\varepsilon}^{(n)},\ n = 0, 1, \ldots$ is also convergent. This completes the proof of Theorem 2. ■

It should be noted that the initial stabilizing gain $F_{i\varepsilon}^{(0)}$ remains to be determined in a stochastic case; however, in a deterministic case [22], there are certain algorithms that can be used to estimate $F_{i\varepsilon}^{(0)}$.

Moreover, the convergence rate of algorithm (20) is unclear. While carrying out the computation, it is found that the computation is very sensitive because of the existence of the design parameter α. If this parameter is not chosen appropriately, the algorithm might converge to a different solution. From our past experiences, it can be concluded a sufficiently small parameter generally works well for the determination of the convergence rate of algorithm (20).

In order to improve the convergence rate and remove the design parameter α, Newton's method can also be applied

$$\begin{aligned} &P_\varepsilon^{(n+1)} \hat{A}_\varepsilon^{(n)} + \hat{A}_\varepsilon^{(n)T} P_\varepsilon^{(n+1)} + \sum_{k=1}^N A_{k\varepsilon}^T P_\varepsilon^{(n+1)} A_{k\varepsilon} \\ &\quad + \sum_{k=1}^N \left(P_\varepsilon^{(n)} B_{k\varepsilon} + \gamma_k C_k^T F_{k\varepsilon}^{(n)T} R_k\right) F_{k\varepsilon}^{(n+1)} C_k \\ &\quad + \sum_{k=1}^N C_k^T F_{k\varepsilon}^{(n+1)T} \left(P_\varepsilon^{(n)} B_{k\varepsilon} + \gamma_k C_k^T F_{k\varepsilon}^{(n)T} R_k\right)^T \\ &\quad - \sum_{k=1}^N (P_\varepsilon^{(n)} B_{k\varepsilon} F_{k\varepsilon}^{(n)} C_k \\ &\ + C_k^T F_{k\varepsilon}^{(n)T} B_{k\varepsilon}^T P_\varepsilon^{(n)} + \gamma_k C_k^T F_{k\varepsilon}^{(n)T} R_k F_{k\varepsilon}^{(n)} C_k) + Q_\varepsilon = 0 \end{aligned} \tag{21a}$$

$$\begin{aligned} &S_\varepsilon^{(n+1)} \hat{A}_\varepsilon^{(n)T} + \hat{A}_\varepsilon^{(n)} S_\varepsilon^{(n+1)} + \sum_{k=1}^N A_{k\varepsilon} P_\varepsilon^{(n+1)} A_{k\varepsilon}^T \\ &\quad + \sum_{k=1}^N (B_{k\varepsilon} F_{k\varepsilon}^{(n+1)} C_k S_\varepsilon^{(n)} + S_\varepsilon^{(n)} C_k^T F_{k\varepsilon}^{(n+1)T} B_{k\varepsilon}^T) \\ &\quad + I_n - \sum_{k=1}^N (B_{k\varepsilon} F_{k\varepsilon}^{(n)} C_k S_\varepsilon^{(n)} + S_\varepsilon^{(n)} C_k^T F_{k\varepsilon}^{(n)T} B_{k\varepsilon}^T) = 0 \end{aligned} \tag{21b}$$

$$\begin{aligned} &B_{i\varepsilon}^T P_\varepsilon^{(n+1)} S_\varepsilon^{(n)} C_i^T + \left(B_{i\varepsilon}^T P_\varepsilon^{(n)} + \gamma_i R_i F_{i\varepsilon}^{(n)} C_i\right) S_\varepsilon^{(n+1)} C_i^T \\ &\quad + \gamma_i R_i F_{i\varepsilon}^{(n+1)} C_i S_\varepsilon^{(n)} C_i^T - B_{i\varepsilon}^T P_\varepsilon^{(n)} S_\varepsilon^{(n)} C_i^T \\ &\quad - \gamma_i R_i F_{i\varepsilon}^{(n)} C_i S_\varepsilon^{(n)} C_i^T = 0 \end{aligned} \tag{21c}$$

$$P_\varepsilon^{(0)} = \bar{P}, S_\varepsilon^{(0)} = \bar{S}, F_{i\varepsilon}^{(0)} = \bar{F}_i, \quad n = 0, 1, \ldots.$$

The initial conditions for the algorithm given in (21) are obtained by solving (9), (18a), and (18b). These equations can be solved by applying Newton's method.

Theorem 3: There exists a small constant $\bar{\sigma}$ such that for all $\varepsilon \in (0, \bar{\sigma}), \bar{\sigma} \leq \sigma^*$, and Newton iteration (21) converges to the exact solutions of P_ε^*, S_ε^* and $F_{i\varepsilon}^*$ with the rate of the quadratic convergence

TABLE I
ERROR PER ITERATIONS OF NEWTON'S METHOD

n	$\mathcal{E}(1.0e-02)$	$\mathcal{E}(1.0e-03)$	$\mathcal{E}(1.0e-04)$
0	$5.5838e-01$	$5.5838e-02$	$5.5838e-03$
1	$6.7248e-03$	$2.0119e-05$	$1.8658e-07$
2	$3.2017e-04$	$3.3157e-08$	$7.1895e-11$
3	$7.7601e-08$	$6.2181e-11$	
4	$5.4103e-11$		

TABLE II
NUMBER OF ITERATIONS

ε	Sequential Algorithm (20)	Newton's Method (21)
$1.0e-02$	78	4
$1.0e-03$	68	3
$1.0e-04$	57	2

rate. Moreover, the convergence solutions attain unique local solutions P_{ε}^*, S_{ε}^* and $F_{i\varepsilon}^*$ of SALE (7) and (12) in the neighborhood of the initial conditions $P_{\varepsilon}^{(0)} = \bar{P}$, $S_{\varepsilon}^{(0)} = \bar{S}$ and $F_{i\varepsilon}^{(0)} = \bar{F}_i$. That is to say, the following conditions are satisfied:

$$\left\| P_{\varepsilon}^{(n)} - P_{\varepsilon}^* \right\| = O(\varepsilon^{2^n}) \tag{22a}$$

$$\left\| S_{\varepsilon}^{(n)} - S_{\varepsilon}^* \right\| = O(\varepsilon^{2^n}) \tag{22b}$$

$$\left\| F_{i\varepsilon}^{(n)} - F_{i\varepsilon}^* \right\| = O(\varepsilon^{2^n}), \quad i = 1, \ldots, N. \tag{22c}$$

Proof: Since the proof is given directly by applying the Newton-Kantorovich theorem [14] to SALE (7) and (12) as a slight modification of the proof described in [10], it is omitted. ■

Although it has been generally shown that there exist several solutions to CSAREs [15], it should be noted that for weakly coupled systems, both positive semidefiniteness and uniqueness of the solutions are guaranteed as long as the value of ε is small.

It should be noted that since the equations used to obtain the static output feedback gain are based on the necessary conditions, the proposed sequential algorithm (20) and Newton's method (21) may converge to a local minimum. Hence, we must pay attention to the solutions obtained required.

IV. DEGRADATION OF COST PERFORMANCE

Now, we focus on the design of high-order approximate Pareto-optimal strategies, which are obtained using iterative solution (21)

$$u_i^{(n)}(t) = F_{i\varepsilon}^{(n)} C_i x(t) = F_{i\varepsilon}^{(n)} C_{ii} x_i(t), \quad i = 0, 1, \ldots, N. \tag{23}$$

Theorem 4: Let us assume that a quadratic convergence rate in (22) is attained. The high-order approximate Pareto-optimal strategies (23) provide the following relation:

$$J_i^{(n)} - J_i^* = O\left(\varepsilon^{2^n}\right) \tag{24}$$

where $J_i^{(n)} := \mathbf{Tr}[M_{i\varepsilon}]$, $J_i^* := \mathbf{Tr}[N_{i\varepsilon}]$

$$M_{i\varepsilon}\hat{A}_{\varepsilon}^{(n)} + \hat{A}_{\varepsilon}^{(n)T} M_{i\varepsilon} + \sum_{k=1}^{N} A_{k\varepsilon}^T M_{k\varepsilon} A_{k\varepsilon} + C_i^T F_{i\varepsilon}^{(n)T} R_i F_{i\varepsilon}^{(n)} C_i + Q_{i\varepsilon} = 0 \tag{25a}$$

$$N_{i\varepsilon}\hat{A}_{\varepsilon}^* + \hat{A}_{\varepsilon}^{*T} N_{i\varepsilon} + \sum_{k=1}^{N} A_{k\varepsilon}^T N_{k\varepsilon} A_{k\varepsilon} + C_i^T F_{i\varepsilon}^{*T} R_i F_{i\varepsilon}^* C_i + Q_{i\varepsilon} = 0 \tag{25b}$$

where $\hat{A}_{\varepsilon}^* := A_{\varepsilon} + \sum_{k=1}^{N} B_{k\varepsilon} F_{k\varepsilon}^* C_k$.

Proof: Subtracting (25b) from (25a) and using the result of (22), we obtain $L_{i\varepsilon} = M_{i\varepsilon} - N_{i\varepsilon}$ satisfies $L_{i\varepsilon}\hat{A}_{\varepsilon}^{(n)} + \hat{A}_{\varepsilon}^{(n)T} L_{i\varepsilon} + \sum_{k=1}^{N} A_{k\varepsilon}^T L_{k\varepsilon} A_{k\varepsilon} + O(\varepsilon^{2^n}) = 0$. Without loss of generality, it is assumed that the following structure holds:

$$L_{i\varepsilon} := \begin{bmatrix} L_{i11} & \varepsilon L_{i12} & \cdots & \varepsilon L_{i1N} \\ \varepsilon L_{i12}^T & L_{i22} & \cdots & \varepsilon L_{i2N} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon L_{i1N}^T & \varepsilon L_{i2N}^T & \cdots & L_{iNN} \end{bmatrix} \in \Re^{\bar{n} \times \bar{n}}. \tag{26}$$

Using the implicit function theorem under the condition $\bar{F}_i \in \mathbf{F}_i$, it can be shown that there exists a neighborhood of $\varepsilon = 0$ and a unique function $L_{i\varepsilon} := \bar{L}_i + O(\varepsilon)$, where $\bar{L}_i = \mathbf{block\ diag}(\bar{L}_{i11} \ \cdots \ \bar{L}_{iNN})$. Substituting $\bar{L}_i$ into (26) and letting $\varepsilon = 0$, $\bar{L}_{ijj}$, $j = 1, \ldots, N$ is satisfied $\bar{L}_{ijj}(A_{ii} + B_{ii}\bar{F}_i C_{ii}) + (A_{ii} + B_{ii}\bar{F}_i C_{ii})^T \bar{L}_{ijj} + A_{iii}^T \bar{L}_{ijj} A_{iii} = 0$. Then, since $\bar{F}_i \in \mathbf{F}_i$, $I_{n_i} \otimes (A_{ii} + B_{ii}\bar{F}_i C_{ii})^T + (A_{ii} + B_{ii}\bar{F}_i C_{ii})^T \otimes I_{n_i} + A_{iii}^T \otimes A_{iii}^T$ is nonsingular. Hence, $\bar{L}_{ijj} = 0$ for all i. Consequently, we have $L_{i\varepsilon} = O(\varepsilon)$. Subsequent iterations of the above-mentioned steps result in $L_{i\varepsilon} = O(\varepsilon^{2^n})$. This immediately leads to the desired result. ■

The proposed Pareto-optimal strategy brings the following reliability and usefulness. The strategy set can be computed with the reduced-order dimension. Moreover, the feedback uses the information regarding the local output measurement only.

V. NUMERICAL EXAMPLE

In order to demonstrate the efficiency of the stochastic Pareto-optimal strategies, we present the results obtained for the megawatt-frequency control problem of multiarea electric energy systems. The model is based on the multistage decomposition of two interconnected areas [5]. The system matrices with slight modifications are given in the equation shown at the bottom of the next page.

In order to verify the exactitude of the solution, the remainder after each iteration is computed for several values of ε by substituting $P_{\varepsilon}^{(n)}$, $S_{\varepsilon}^{(n)}$ and $F_{i\varepsilon}^{(n)}$ into SALE (7) and (12) by using Newton's method (21). Table I shows the errors $\mathcal{E}(\varepsilon)$ per iteration for various values of ε, where $\mathcal{E}(\varepsilon) := \|\mathcal{F}, \mathcal{G}, \mathcal{H}_1, \mathcal{H}_2\|$. It should be noted that when $\varepsilon = 0.01$, Newton's method (21) converges to the exact solution with an accuracy of $\mathcal{E}(\varepsilon) < 1.0e-10$ after four iterations. Hence, it can be observed from Table I that Newton's method (21) attains quadratic convergence.

The proposed sequential algorithm (20) can be applied using the same data (remainder after each iteration) with $\alpha = 0.2$. It is evident from Table II that the sequential algorithm (20) requires a large number of iterations. As a result, it is concluded that even though the convergence rate of this algorithm is unclear, the convergence speed is slow.

Using the design procedure and setting $\varepsilon = 0.01$, Pareto-optimal strategies can be given by

$$F_{1\varepsilon} = [-1.5102 \quad -2.1539 \quad -4.6490e-01]$$
$$F_{2\varepsilon} = [-2.4886 \quad -4.4284 \quad -1.9800].$$

These strategies would function as a Pareto-optimal strategy set that delivers a good performance such that the relation (24) is attained.

Finally, we evaluate the costs using the high-order approximate Pareto-optimal strategies (23). The values of the cost functional per iteration are listed in Table III, where $\eta_i = |J_i^{(n)} - J_i^*|/\varepsilon^{2^n}$. It can

TABLE III
APPROXIMATE AND OPTIMAL VALUES FOR THE COST

n	$J_1^{(n)}$	$J_2^{(n)}$	η_1	η_2
0	7.74019899	2.44638667	$1.12230196e-04$	$1.66943344e-04$
1	7.74019595	2.44639017	$1.91094524e-02$	$5.17186243e-02$
2	7.74019788	2.44638499	1.16292558	1.13458785
3	7.74019787	2.44638500	$3.26849658e+03$	$3.23296945e+03$
4	7.74019787	2.44638500	0.00000000	0.00000000
Optimal	$J_1^* = 7.74019787$	$J_2^* = 2.44638500$	–	–

be seen that the simulation results are consistent with the statement of Theorem 4.

VI. CONCLUSION

In this note, the static output feedback Pareto-optimal strategy for a stochastic system governed by Itô differential equations is developed. First, the necessary conditions for the decentralized controllers to be Pareto-optimal strategies are derived. The boundedness of the solution to the CSAREs and their asymptotic structures are then established. Secondly, a new sequential numerical algorithm for solving the reduced-order CSAREs is developed for the first time. The following conclusions can be drawn from this study: 1) The strategy set can be computed successively; 2) The strategies are based on the decentralized control technique; 3) Since the optimal strategy can be implemented using the local output measurements, the results can be practically applied in a realistic manner. These features will help realize a novel design technique for a controller that would be simple to implement.

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$$A_{11} = \begin{bmatrix} 0 & 0.315 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1.888 & -0.0498 & 6 & 0 \\ 0 & 0 & 0 & -3.333 & 3.333 \\ 0 & 0 & -13.9 & 0 & -33.333 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} -3.15 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 18.88 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A_{21} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 18.88 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1.888 & -0.0498 & 6 & 0 \\ 0 & 0 & -3.333 & 3.333 \\ 0 & -13.9 & 0 & -33.333 \end{bmatrix}$$

$$A_{111} = \mathbf{block\ diag}(0 \quad 0 \quad 0.00249 \quad 0 \quad 0), \quad A_{222} = \mathbf{block\ diag}(0 \quad 0.00249 \quad 0 \quad 0)$$

$$A_{112} = A_{121} = A_{122} = A_{211} = A_{212} = A_{221} = 0, \quad B_{11}^T = [0 \quad 0 \quad 0 \quad 0 \quad 33.333], \quad B_{22}^T = [0 \quad 0 \quad 0 \quad 33.333]$$

$$B_{12} = B_{21} = 0, \quad C_{11} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad C_{22} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Q_1 = \mathbf{block\ diag}(I_5 \quad \varepsilon I_4), \quad Q_2 = \mathbf{block\ diag}(\varepsilon I_5 \quad I_4), \quad R_1 = R_2 = 0.1, \ \gamma_1 = \gamma_2 = 0.5.$$

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On Constrained Steady-State Regulation: Dynamic KKT Controllers

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***Abstract*—This technical note presents a solution to the problem of regulating a general nonlinear dynamical system to an economically optimal operating point. The system is characterized by a set of exogenous inputs as an abstraction of time-varying loads and disturbances. The economically optimal operating point is implicitly defined as a solution to a given constrained convex optimization problem, which is related to steady-state operation. The system outputs and the exogenous inputs represent respectively the decision variables and the parameters in the optimization problem. The proposed solution is based on a specific dynamic extension of the Karush–Kuhn–Tucker optimality conditions for the steady-state related optimization problem, which is conceptually related to the continuous-time Arrow–Hurwicz–Uzawa algorithm. Furthermore, it can be interpreted as a generalization of the standard output regulation problem with respect to a constant reference signal.**

***Index Terms*—Complementarity systems, constraints, convex optimization, optimal control, steady-state.**

I. Introduction

In many production facilities, the optimization problem reflecting economical benefits of production is associated with *steady-state operation* of the system. The control action is required to maintain the production in an optimal regime in spite of various disturbances, and to efficiently and rapidly respond to changes in demand. Furthermore, it is desirable that the system settles in a steady-state that is optimal for novel operating conditions. The vast majority of control literature is focused on regulation and tracking with respect to known setpoints or trajectories, while coping with different types of uncertainties and disturbances in both the plant and its environment. Typically, setpoints are determined off-line by solving an appropriate optimization problem and they are updated in an open-loop manner. The increase of the frequency with which the economically optimal setpoints are updated can result in a significant increase of economic benefits accumulated in time. If the time-scale on which economic optimization is performed approaches the time-scale of the underlying physical system, i.e., of the plant dynamics, dynamic interaction in between the two has to be considered. Economic optimization then becomes a challenging control problem, especially since it has to cope with inequality constraints that reflect the physical and security limits of the plant [1].

In this technical note, we consider the problem of regulating a general nonlinear dynamical system to an implicitly defined economically optimal operating point. The considered dynamical system is characterized by a set of exogenous inputs as an abstraction of time-varying loads and disturbances acting on the system. Economic optimality is defined through a convex constrained optimization problem with system outputs as decision variables, and with the values of exogenous inputs as parameters in the optimization problem. A similar problem has already been considered in [1], see also the references therein, where the authors propose a solution that uses penalty and barrier functions to deal with inequality constraints. We propose a novel solution based on a specific dynamic extension of the Karush–Kuhn–Tucker (KKT) optimality conditions, which is conceptually related to the continuous-time Arrow–Hurwicz–Uzawa algorithm [2]. The proposed feedback controller belongs to the class of complementarity systems (CS), which was formally introduced in 1996 by Van der Schaft and Schumacher [3] (see also [4] and [5]) and have become an extensive topic of research in the hybrid systems community.

Nomenclature: For a matrix $A \in \mathbb{R}^{m\times n}$, $[A]_{ij}$ denotes the element in the ith row and jth column of A. For a vector $x \in \mathbb{R}^n$, $[x]_i$ denotes the ith element of x. A vector $x \in \mathbb{R}^n$ is said to be nonnegative (nonpositive) if $[x]_i \geq 0([x]_i \leq 0)$ for all $i \in \{1,\ldots n\}$, and in that case we write $x \geq 0$ ($x \leq 0$). The nonnegative orthant of $\mathbb{R}^n$ is defined by $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n | x \geq 0\}$. The operator $\mathrm{col}(\cdot,\ldots,\cdot)$ stacks its operands into a column vector, and $\mathrm{diag}(\cdot,\ldots,\cdot)$ denotes a square matrix with its operands on the main diagonal and zeros elsewhere. For $u, v \in \mathbb{R}^k$ we write $u \perp v$ if $u^\top v = 0$. We use the compact notational form $0 \leq u \perp v \geq 0$ to denote the complementarity conditions $u \geq 0, v \geq 0, u \perp v$. The matrix inequality $A \succ B$ means A and B are Hermitian and $A - B$ is positive definite. For a scalar-valued differentiable function $f : \mathbb{R}^n \to \mathbb{R}$, $\nabla f(x)$ denotes its gradient at $x = \mathrm{col}(x_1,\ldots,x_n)$ and is defined as a *column vector*, i.e., $\nabla f(x) \in \mathbb{R}^n$, $[\nabla f(x)]_i = (\partial f)/(\partial x_i)$. For a vector-valued differentiable function $f : \mathbb{R}^n \to \mathbb{R}^m$, $f(x) = \mathrm{col}(f_1(x),\ldots,f_m(x))$, the Jacobian at $x = \mathrm{col}(x_1,\ldots,x_n)$ is the matrix $Df(x) \in \mathbb{R}^{m\times n}$ and is defined by $[Df(x)]_{ij} = (\partial f_i(x))/(\partial x_j)$. For a vector valued function $f : \mathbb{R}^n \to \mathbb{R}^m$, we will use $\nabla f(x)$ to denote the transpose of the Jacobian, i.e., $\nabla f(x) \in \mathbb{R}^{n\times m}$, $\nabla f(x) \triangleq Df(x)^\top$, which is consistent with the gradient notation ∇f when f is a scalar-valued function. With a slight abuse of notation we will often use the same symbol to denote a signal, i.e., a function of time, as well as possible values that the signal may take at any time instant.

II. Problem Formulation

In this section, we formally present the constrained steady-state optimal regulation problem considered in this technical note. Furthermore, we list several standing assumptions, which will be instrumental in the subsequent sections. Consider a dynamical system

$$\dot{x} = f(x, w, u) \tag{1a}$$

$$y = g(x, w) \tag{1b}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the control input, $w(t) \in \mathbb{R}^{n_w}$ is an exogenous input, $y(t) \in \mathbb{R}^m$ is the measured output, f :

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