The Guaranteed Cost Control for Uncertain Nonlinear Large-scale Stochastic Systems via State and Static Output Feedback

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Abstract

The guaranteed cost control (GCC) problem involved in decentralized robust control of a class of uncertain nonlinear large-scale stochastic systems with high-order interconnections is considered. After determining the appropriate conditions for the stochastic GCC controller, a class of decentralized local state feedback controllers is derived using the linear matrix inequality (LMI). The extension of the result of the study to the static output feedback control problem is discussed by considering the Karush-Kuhn-Tucker (KKT) conditions. The efficiency of the proposed design method is demonstrated on the basis of simulation results.

Key words: large-scale stochastic systems, high-order interconnections, exponentially mean-square stable (EMSS), guaranteed cost control (GCC), linear matrix inequality (LMI), static output feedback.

1. Introduction

The problem of robust control design for uncertain nonlinear systems continues to be a topic of research. Indeed, there are many strategic approaches that are adopted in nonlinear control engineering. A framework for the design of sliding-mode-based controllers for multi-input affine nonlinear systems has been considered (2). The problem of stabilization of nonlinear systems subject to uncertainty and constraints has been studied, and a robust model predictive controller has been proposed (3).

In the past few decades, problems involved in the decentralized robust control of large-scale interconnected systems with parameter uncertainties have been widely studied, and some solution approaches have been developed. For example, several schemes have been developed for the design of robust decentralized dynamic output controllers for use in linear interconnected systems with unknown nonlinear interconnections that satisfy quadratic constraints (1).

Stochastic control problems governed by Itô's differential equation have become a popular research topic in the past few decades. Robust stabilization of uncertain stochastic systems with time delays has been considered (4; 5). The stochastic H_∞ control problem with state- and control-dependent noise has been investigated (6). The stochastic LQ control problem with state- and control-dependent noise has also been studied (7). However, the LQ control for a class of nonlinear large-scale interconnected stochastic systems with norm-bounded time-varying parameter uncertainties has not yet been dealt with.

Recent advances in the study of the linear matrix inequality (LMI) have allowed re-examination of the guaranteed cost control (GCC) (12; 14) for large-scale interconnected systems (15; 16). Although the results are feasible for deterministic uncertainties, the problem of guaranteed cost stabilization for uncertain nonlinear large-scale interconnected stochastic systems remains to be solved.

Various advanced nonlinear control techniques have been used for the excitation control of power systems. In particular, a dynamical model of the same form as a large-scale nonlinear system has been developed for the ith machine that is subjected to excitation control (8; 9; 10). However, stochastic uncertainty has not been considered in this model. Meanwhile, adaptive state-feedback stabilization of high-order stochastic systems with nonlinear parameterization has been investigated (11). Although this model seems to have been included in the models of general nonlinear systems, decentralized GCC is still an interesting topic that is worthy of further study. Therefore, the GCC problem for a class of uncertain nonlinear large-scale stochastic systems with high-order interconnections is investigated.

This study is an extension of the previous studies in this field (8; 15; 16) in the sense that nonlinear large-scale interconnected systems are included in the standard Wiener process as stochastic systems. Furthermore, it should be noted that although the previously reported results (8; 15; 16) have been obtained for first-order polynomials, the interconnections considered in those studies are bounded by a polynomial-type nonlinearity. The representation of nonlinearities by high-order polynomials and stochastic processes would allow us to design a precise robust controller for the wider class. Indeed, since there are some stochastic uncertainties in practical large-scale systems (17; 22), it is necessary to consider stochastic nonlinear systems that are governed by Itô's differential equation. After defining the GCC problem for nonlinear large-scale interconnected stochastic systems, conditions suitable for the existence of decentralized robust feedback controllers are determined derived by using the Lyapunov stability criterion such that the abovementioned uncertain stochastic systems can become exponentially mean-square stable (EMSS). The matrix inequality and LMI design approach play an important role in the determination of suitable conditions and the construction of guaranteed cost controllers. It is worth mentioning that although high-order nonlinearity and stochastic uncertainty are both included, EMSS and an adequate performance bound are guaranteed. Another important feature is that the static output feedback GCC problem is solved by using the KKT conditions. A necessary condition for the optimization of the guaranteed cost is derived in the form of cross-coupled algebraic Riccati equations. Finally, numerical examples are included to demon-

strate the efficiency of the design algorithm.
Notation: The notations used in this paper are fairly standard. The superscript T denotes the matrix transpose. **block diag** denotes the block diagonal matrix. **Tr** denotes the trace of matrix. I_n denotes the $n \times n$ identity matrix. $\|\cdot\|$ denotes its Euclidean norm for a matrix. E denotes the expectation. $\lambda_{\min}$ and $\lambda_{\max}$ denote the minimum and the maximum eigenvalues for a matrix, respectively.

2. Problem Formulation

Consider a class of nonlinear large-scale interconnected stochastic systems that consist of N subsystems described by the following Itô stochastic differential equations:

$$\begin{aligned} dx_i(t) &= \Big[(A_i + \Delta A_i(t))x_i(t) + (B_i + \Delta B_i(t))u_i(t) \\ &\quad + \sum_{j=1,\ j\neq i}^{N} (G_{ij} + \Delta G_{ij}(t))g_{ij}(x_i,\ x_j)\Big]dt + A_{i1}x_i(t)dw_i(t), \quad (1) \\ x_i(0) &= x_i^0,\ i = 1, \cdots,\ N, \end{aligned}$$

where $x_i(t) \in \Re^{n_i}$ and $u_i(t) \in \Re^{m_i}$ are the state and control of the ith subsystems, respectively. $w_i(t) \in \Re$ is the one-dimensional standard Wiener process defined on a filtered probability space (6; 7). A_i, B_i and A_{i1} are constant matrices of appropriate dimensions, and G_{ij} are interconnection matrices for the interconnections between the ith subsystem and other subsystems. The unknown vector functions $g_{ij}(x_i,\ x_j) \in \Re^{l_{ij}}$ (to simplify notation, they will henceforth be written as $g_{ij}(x_i,\ x_j) = g_{ij}$) represent the high-order interconnections among the subsystems. It is assumed that the unknown vector functions $g_{ij}(x_i,\ x_j)$ are continuously differentiable on $\mathcal{G}_{ij} = \{x_i(t) \in \Re^{n_i} \mid \|x_i(t)\| < \delta_i\}$, sufficiently smooth and piecewise continuous in t (8; 15; 16). The parameter uncertainties considered here are assumed to be of the following form:

$$\begin{bmatrix} \Delta A_i(t) & \Delta B_i(t) \end{bmatrix} = D_i F_i(t) \begin{bmatrix} E_i^a & E_i^b \end{bmatrix},\ \Delta G_{ij}(t) = D_{ij} F_{ij}(t) E_{ij}, \quad (2)$$

where D_i, E_i^a, E_i^b, D_{ij} and E_{ij} are known constant real matrices of appropriate dimensions. $F_i(t) \in \Re^{p_i \times q_i}$ and $F_{ij}(t) \in \Re^{p_{ij} \times q_{ij}}$ are unknown matrix functions with Lebesgue-measurable elements and satisfy

$$F_i^T(t)F_i(t) \le I_{q_i},\ F_{ij}^T(t)F_{ij}(t) \le I_{q_{ij}}. \quad (3)$$

The following conditions are assumed to apply to the vector functions $g_{ij}(x_i,\ x_j)$ with polynomial-type nonlinearity:

Assumption 1. *There exist known constant matrices V_i and W_{ij} such that for all i, j, $t \ge 0$, $x_i \in \Re^{n_i}$ and $x_j \in \Re^{n_j}$, $\|g_{ij}(x_i,\ x_j)\| \le \sum_{k=1}^{p}[\alpha_k \|V_i x_i\| \|x_i\|^{k-1} + \beta_k \|W_{ij} x_j\| \|x_j\|^{k-1}]$, where α_k and β_k are positive scalar constants.*

It is known that $x_i(t)$ will be bounded whenever the trajectory $x_i(t)$ is confined to a compact set (21). Hence, the above assumption satisfies the inequality $\|g_{ij}(x_i,\ x_j)\| \le \sum_{k=1}^{p}[\alpha_k\gamma_i^{k-1}\|V_i x_i\| + \beta_k\gamma_j^{k-1}\|W_{ij}x_j\|] = \|\tilde{V}_i x_i\| + \|\tilde{W}_{ij}x_j\|$, where $\tilde{V}_i := \sum_{k=1}^{p}\alpha_k\gamma_i^{k-1}V_i$ and $\tilde{W}_{ij} := \sum_{k=1}^{p}\beta_k\gamma_j^{k-1}W_{ij}$ for all $\|x_i(t)\| \le \gamma_i$. It seems that the constraint assumption $\|x_i(t)\| \le \gamma_i (< \delta_i)$ is appropriate because all the trajectories have to be stable. It may be noted that if $p = 1$, $\alpha_1 = \beta_1 = 1$, Assumption 1 is equivalent to an existing assumption (8; 15; 16).

Assumption 2. *For all* $i,\ j$, $U_i := 2\sum_{j=1,\ j\neq i}^{N}(\tilde{V}_i^T\tilde{V}_i + \tilde{W}_{ji}^T\tilde{W}_{ji}) > 0$.

Associated with system (1) is the cost function

$$J = \sum_{i=1}^{N} E\left[\int_0^\infty [x_i^T(t)Q_i x_i(t) + u_i^T(t)R_i u_i(t)]dt\right], \tag{4}$$

where Q_i and R_i are positive definite symmetric matrices.

The following concept is standard in the theory of stability of stochastic systems (see, for example, (6; 19; 20) and the references given therein for further details).

Definition 1. *The stochastic system is said to be EMSS if it satisfies the following equation:*

$$E\|x(t)\|^2 \le \rho e^{-\psi(t-t_0)}E\|x(t_0)\|^2,\ \exists\rho,\ \psi > 0.$$

Lemma 1. *(19; 20) The trivial solution of a stochastic differential equation is given by the following equation:*

$$dx(t) = f(t,\ x)dt + g(t,\ x)dw(t), \tag{5}$$

where $x(t) = \begin{bmatrix} x_1^T(t) & \cdots & x_N^T(t) \end{bmatrix}^T$ *with* $f(t,\ x)$ *and* $g(t,\ x)$ *are sufficiently differentiable maps;* $x_i(t)$ *is EMSS if there exists a function* $V(x(t))$ *that satisfies the following inequalities:*

$$a_1\|x(t)\|^2 \le V(x(t)) \le a_2\|x(t)\|^2,\ a_1,\ a_2 > 0, \tag{6a}$$

$$\begin{aligned}\boldsymbol{D}V(x(t)) &:= \frac{\partial V(x(t))}{\partial x}f(t,\ x) + \frac{1}{2}\mathbf{Tr}\left[g^T(t,\ x)\frac{\partial^2 V(x(t))}{\partial x^2}g(t,\ x)\right] \\ &\le -c\|x(t)\|^2,\ c > 0 \end{aligned} \tag{6b}$$

for $x(t) \neq 0$.

Based on reference (12), we define the GCC for the nonlinear large-scale interconnected stochastic systems with deterministic uncertainties.

Definition 2. *A decentralized controller $u_i(t) = K_i x_i(t)$ is said to be the GCC with cost matrix $P_i > 0$ for the uncertain nonlinear large-scale interconnected stochastic systems (1) and the cost function (4) if the closed-loop systems are EMSS and the closed-loop value of the cost function (4) satisfies the bound $J \leq \mathcal{J}$ for all admissible uncertainties.*

Although there are several definitions of stable in the context of stochastic systems (19), the EMSS has been adopted after considering the feasibility of the obtained solution and the application of the solution to practical systems.

The objective of this note is to design a decentralized linear time-invariant guaranteed cost controller $u_i(t) = K_i x_i(t),\ i = 1, \cdots,\ N$ for the nonlinear large-scale interconnected stochastic systems (1) with uncertainties (2) and (3).

3. Preliminary Result

Now, a sufficient condition for the existence of the state-feedback guaranteed cost controller for the uncertain nonlinear large-scale interconnected stochastic systems (1) is established.

Theorem 1. *Consider the nonlinear large-scale interconnected stochastic systems (1) with uncertainties (2) and (3) under Assumptions 1 and 2. If there exists a symmetric positive definite matrix $P_i \in \Re^{n_i \times n_i}$ such that the matrix inequality (7) holds, the controllers $u_i(t) = K_i x_i(t)$, $i = 1, \cdots,\ N$ are the guaranteed cost controller.*

$$\Lambda_i = \begin{bmatrix} \Xi_i & P_i\tilde{G}_{i1} & \cdots & P_i\tilde{G}_{i(i-1)} & P_i\tilde{G}_{i(i+1)} & \cdots & P_i\tilde{G}_{iN} \\ \tilde{G}_{i1}^T P_i & -I_{l_{i1}} & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \tilde{G}_{i(i-1)}^T P_i & 0 & \cdots & -I_{l_{i(i-1)}} & 0 & \cdots & 0 \\ \tilde{G}_{i(i+1)}^T P_i & 0 & \cdots & 0 & -I_{l_{i(i+1)}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \tilde{G}_{iN}^T P_i & 0 & \cdots & 0 & 0 & \cdots & -I_{l_{iN}} \end{bmatrix} < 0, \quad (7)$$

where $\Lambda_i \in \Re^{\bar{N} \times \bar{N}}$, $\bar{N} = n_i + \sum_{j=1,\ j \neq i} l_{ij}$, $\Xi_i := \tilde{A}_i^T P_i + P_i \tilde{A}_i + A_{i1}^T P_i A_{i1} + U_i + \bar{R}_i$, $\tilde{A}_i := \bar{A}_i + D_i F_i(t) \bar{E}_i$, $\tilde{G}_{ij} := G_{ij} + D_{ij} F_{ij}(t) E_{ij}$, $\bar{A}_i := A_i + B_i K_i$, $\bar{E}_i := E_i^a + E_i^b K_i$ and $\bar{R}_i := Q_i + K_i^T R_i K_i$.

In other words, the closed-loop systems are EMSS for $\|x_i(0)\| < \delta_i$, and the corresponding value of the cost function (4) satisfies the following inequality (8) for all admissible uncertainties $F_i(t)$ and $F_{ij}(t)$.

$$J < \sum_{i=1}^{N} E[x_i^T(0) P_i x_i(0)]. \quad (8)$$

Remark 1. *Note that $\tilde{G}_{ii}^T P_i$, $P_i \tilde{G}_{ii}$ and $-I_{l_{ii}}$ are not the row, column, and diagonal entries, respectively, in the matrix Λ_i.*

Proof: Combining the guaranteed cost controller $u_i(t) = K_i x_i(t)$ with (1) gives a closed-loop uncertain stochastic system of the following form:

$$dx_i(t) = \left[\tilde{A}_i x_i(t) + \sum_{j=1,\ j\neq i}^{N} \tilde{G}_{ij} g_{ij}\right] dt + A_{i1} x_i(t) dw_i(t). \tag{9}$$

Now, assume that there exists the symmetric positive definite matrix $P_i > 0,\ i = 1,\ ...\ , N$ such that the matrix inequality (7) holds for all admissible uncertainties. In order to prove that the closed-loop uncertain stochastic system (9) is EMSS, let us define the following Lyapunov function candidate (10):

$$V(x(t)) = x^T(t)\mathbf{P}x(t) = \sum_{i=1}^{N} x_i^T(t) P_i x_i(t) > 0, \tag{10}$$

where $\mathbf{P} := \textbf{block diag}\begin{pmatrix} P_1 & \cdots & P_N \end{pmatrix}$

First, since $\lambda_{\min}(\mathbf{P})\|x(t)\|^2 \leq V(x(t)) \leq \lambda_{\max}(\mathbf{P})\|x(t)\|^2$, condition (6a) is satisfied. Further, by using formula (6b), the stochastic differential DV(x(t))is derived as

$$\begin{aligned}\boldsymbol{D}V(x(t)) \;=\; & \sum_{i=1}^{N}\Bigg\{ x_i^T(t)(\tilde{A}_i^T P_i + P_i \tilde{A}_i) x_i(t) \\ & + \sum_{j=1,\ j\neq i}^{N} \left[g_{ij}^T \tilde{G}_{ij}^T P_i x_i(t) + x_i^T(t) P_i \tilde{G}_{ij} g_{ij}\right] + x_i^T(t) A_{i1}^T P_i A_{i1} x_i(t)\Bigg\}.\end{aligned}$$

Since

$$\begin{aligned}& \sum_{i=1}^{N} \sum_{j=1,\ j\neq i}^{N} (2x_i^T \tilde{V}_i^T \tilde{V}_i x_i + 2x_i^T \tilde{W}_{ji}^T \tilde{W}_{ji} x_i - g_{ij}^T g_{ij}) \\ = & \sum_{i=1}^{N} \sum_{j=1,\ j\neq i}^{N} (2x_i^T \tilde{V}_i^T \tilde{V}_i x_i + 2x_j^T \tilde{W}_{ij}^T \tilde{W}_{ij} x_j - g_{ij}^T g_{ij}),\end{aligned}$$

it follows that

$$\begin{aligned}& \boldsymbol{D}V(x(t)) \\ = & \sum_{i=1}^{N}\Bigg\{ z_i^T(t)\Lambda_i z_i(t) - x_i^T(t)\bar{R}_i x_i(t) \\ & - \sum_{j=1,\ j\neq i}^{N} (2x_i^T \tilde{V}_i^T \tilde{V}_i x_i + 2x_j^T \tilde{W}_{ij}^T \tilde{W}_{ij} x_j - g_{ij}^T g_{ij})\Bigg\},\end{aligned}$$

where $z_i(t) = \begin{bmatrix} x_i^T(t) & g_{i1}^T & \cdots & g_{iN}^T \end{bmatrix}^T \in \Re^{\bar{N}}$ and Ξ_i and Λ_i are given in (7).

It is easy to verify whether the inequality $2x_i^T\tilde{V}_i^T\tilde{V}_i x_i + 2x_j^T\tilde{W}_{ij}^T\tilde{W}_{ij}x_j \geq g_{ij}^T g_{ij}$ holds under Assumption 1. It immediately follows from the above inequality that (11) holds.

$$\boldsymbol{D}V(x(t)) < -\sum_{i=1}^{N} x_i^T(t)\bar{R}_i x_i(t) < -\lambda_{\min}(\mathbf{R})\|x(t)\|^2 < 0, \tag{11}$$

where $\mathbf{R} := \textbf{block diag}\begin{pmatrix} \bar{R}_1 & \cdots & \bar{R}_N \end{pmatrix}$.

Hence, $V(x(t))$ is a Lyapunov function for the closed-loop uncertain stochastic system (9). Therefore, since the inequality holds

$$E\|x(t)\|^2 \leq \frac{\lambda_{\max}(\mathbf{P})}{\lambda_{\min}(\mathbf{P})} E\|x(0)\|^2 \exp\left[-\frac{\lambda_{\min}(\mathbf{R})}{\lambda_{\max}(\mathbf{P})} t\right],$$

the closed-loop uncertain stochastic system (9) is EMSS. Moreover, applying Itô's formula yields

$$\begin{aligned} dV(x(t)) &= \boldsymbol{D}V(x(t))dt + 2\sum_{i=1}^{N} x_i^T(t)A_{i1}^T P_i x_i(t)dw_i(t) \\ &< -\sum_{i=1}^{N} x_i^T(t)\bar{R}_i x_i(t)dt + 2\sum_{i=1}^{N} x_i^T(t)A_{i1}^T P_i x_i(t)dw_i(t). \end{aligned} \tag{12}$$

Furthermore, by integrating both sides of the inequality (12) from 0 to T and by using the initial conditions, the following inequality is derived

$$E[V(x(T))] - E[V(x(0))] < -\sum_{i=1}^{N} E\left[\int_0^T x_i^T(t)\bar{R}_i x_i(t)dt\right]. \tag{13}$$

Since the closed-loop uncertain stochastic system (9) is EMSS, that is, $\lim_{T\to\infty} E\|x(T)\|^2 \to 0$, $V(x(T)) \to 0$ holds. Thus the following inequality holds:

$$J = \sum_{i=1}^{N} E\left[\int_0^\infty x_i^T(t)\bar{R}_i x_i(t)dt\right] < E[V(x(0))] = \sum_{i=1}^{N} E[x_i^T(0)P_i x_i(0)] = \mathcal{J}.$$

The proof of Theorem 1 is completed. ■

It appears that analysis process is similar to the first-order polynomials. However, the high-order nonlinearity can be treated by using the results of this study. In particular, bounded conditions simplify the analysis. Generally speaking, it is difficult to obtain the sufficient condition for the existence of the stability controller if a polynomial-type nonlinearity is considered. On the other hand, changing the polynomial-type nonlinearity into first-order polynomials and following the existing approach is helpful for obtaining the required controller (8; 15; 16). This feature of changing the nonlinearity into first-order polynomials is an important contribution of this note.

$$
\begin{bmatrix}
\Psi_i & X_iA_{i1}^T & L_i^T & G_{i1} & 0 & \cdots & G_{iN} & 0 & X_i & Y_i^T & X_i \\
A_{i1}X_i & -X_i & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
L_i & 0 & -\mu_i I_{q_i} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
G_{i1}^T & 0 & 0 & -I_{l_{i1}} & E_{i1}^T & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & E_{i1} & -\varepsilon_i I_{s_{i1}} & \cdots & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
G_{iN}^T & 0 & 0 & 0 & 0 & \cdots & -I_{l_{iN}} & E_{iN}^T & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & E_{iN} & -\varepsilon_i I_{s_{iN}} & 0 & 0 & 0 \\
X_i & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -Q_i^{-1} & 0 & 0 \\
Y_i & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & -R_i^{-1} & 0 \\
X_i & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & -U_i^{-1}
\end{bmatrix} < 0, \quad (14)
$$

where $\Psi_i := A_iX_i + B_iY_i + (A_iX_i + B_iY_i)^T + \mu_i D_i D_i^T + H_i$,
$L_i := E_i^a X_i + E_i^b Y_i$, $H_i := \sum_{j=1,\ j\neq i}^N \varepsilon_i D_{ij} D_{ij}^T$.

4. Main Result

We are now in a position to derive a sufficient condition for the existence of the GCC.

4.1. LMI Conditions

Theorem 2. *Under Assumptions 1 and 2, suppose there exist the constant positive parameters $\mu_i > 0$ and $\varepsilon_i > 0$ such that for all $i = 1, \ldots, N$, the LMI (14) have the symmetric positive definite matrices $X_i > 0 \in \Re^{n_i \times n_i}$ and a matrix $Y_i \in \Re^{m_i \times n_i}$.*

If these conditions are satisfied, the decentralized linear state feedback controllers

$$u_i(t) = K_i x_i(t) = Y_i X_i^{-1} x_i(t), \ i = 1, \ldots, N, \quad (15)$$

are the guaranteed cost controllers, and

$$J < \sum_{i=1}^{N} E[x_i^T(0) X_i^{-1} x_i(0)] \quad (16)$$

is the guaranteed cost.

Proof: First, premultiply and postmultiply both sides of the LMI (14) by

$$\textbf{block diag} \left[P_i \ I_{n_i} \ I_{q_i} \ I_{l_{i1}} \ I_{s_{i1}} \cdots I_{l_{iN}} \ I_{s_{iN}} \ I_{n_i} \ I_{m_i} \ I_{n_i} \right],$$

and let $X_i := P_i^{-1}$ and $Y_i := K_i P_i^{-1}$. Then, using the Schur complement (18), the LMI (14) holds if and only if

$$\mathcal{F}_i := \begin{bmatrix} \Gamma_i & A_{i1}^T & P_i G_{i1} & \cdots & P_i G_{iN} \\ A_{i1} & -P_i^{-1} & 0 & \cdots & 0 \\ G_{i1}^T P_i & 0 & \Theta_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ G_{iN}^T P_i & 0 & 0 & \cdots & \Theta_N \end{bmatrix} < 0, \tag{17}$$

where $\Gamma_i := \bar{A}_i^T P_i + P_i \bar{A}_i + U_i + \bar{R}_i + \mu_i P_i D_i D_i^T P_i + P_i H_i P_i + \mu_i^{-1} \bar{E}_i^T \bar{E}_i$, $\Theta_j := \varepsilon_i^{-1} E_{ij}^T E_{ij} - I_{l_{ij}}$.

Using a standard matrix inequality for all admissible uncertainties (2) and (3), the following matrix inequality is derived:

$$\begin{aligned} 0 > \mathcal{F}_i \quad & \geq \begin{bmatrix} \bar{A}_i^T P_i + P_i \bar{A}_i + U_i + \bar{R}_i & A_{i1}^T & P_i G_{i1} & \cdots & P_i G_{iN} \\ A_{i1} & -P_i^{-1} & 0 & \cdots & 0 \\ G_{i1}^T P_i & 0 & -I_{l_{i1}} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ G_{iN}^T P_i & 0 & 0 & \cdots & -I_{l_{iN}} \end{bmatrix} \\ & \quad + \bar{\mathbf{D}} F_i(t) \bar{\mathbf{E}}^T + \bar{\mathbf{E}} F_i^T(t) \bar{\mathbf{D}}^T + \mathbf{D}_i \mathbf{F}_i(t) \mathbf{E}_i + \mathbf{E}_i^T \mathbf{F}_i^T(t) \mathbf{D}_i^T \\ & = \mathcal{L}_i, \end{aligned} \tag{18}$$

where

$$\begin{aligned} & \bar{\mathbf{D}} := \begin{bmatrix} (P_i D_i)^T & O \end{bmatrix}^T, \ \bar{\mathbf{E}} := \begin{bmatrix} \bar{E}_i & O \end{bmatrix}^T, \\ & \mathbf{D}_i := \begin{bmatrix} 0 \ \ 0 \ \ P_i D_{i1} & \cdots & P_i D_{iN} \\ O & & \end{bmatrix}, \\ & \mathbf{F}_i(t) := \textbf{block diag} \begin{pmatrix} 0 & 0 & F_{i1}(t) & \cdots & F_{iN}(t) \end{pmatrix}, \\ & \mathbf{E}_i := \textbf{block diag} \begin{pmatrix} 0 & 0 & E_{i1} & \cdots & E_{iN} \end{pmatrix}. \end{aligned}$$

Finally, using the Schur complement (18) for $\mathcal{L}_i < 0$ results in $\Lambda_i < 0$. Hence, by Theorem 1, the closed-loop stochastic systems are EMSS. This completes the proof of the theorem. ■

4.2. Design Procedure

In this subsection, the design procedure for achieving the GCC is discussed. Since the LMI (14) has a solution set $(\mu_i,\ \varepsilon_i\ X_i,\ Y_i)$, various efficient convex optimization algorithms can be applied for designing the GCC. Moreover, the solutions to these algorithms represent the set of guaranteed cost controllers. This parameterized representation can be used to design the guaranteed cost controllers; this minimizes the value of the guaranteed cost for the closed-loop uncertain nonlinear large-scale interconnected stochastic systems. Consequently, solving the following optimization problem allows us to determine the optimal bound:

$$\mathcal{D}_0 : \min_{\sum_i \mathcal{X}_i} \sum_{i=1}^{N} \gamma_i^2 E[Z_i] = \mathcal{J},\ \mathcal{X}_i \in (\mu_i,\ \varepsilon_i\ X_i,\ Y_i,\ Z_i) \tag{19}$$

such that the LMI (14) and

$$\begin{bmatrix} -Z_i & I_{n_i} \\ I_{n_i} & -X_i \end{bmatrix} < 0. \tag{20}$$

It follows that

$$J < \sum_{i=1}^{N} E[x_i^T(0)X_i^{-1}x_i(0)] \leq \sum_{i=1}^{N} \|x_i(0)\|^2 E[X_i^{-1}] \leq \min_{\sum_i \mathcal{X}_i} \sum_{i=1}^{N} \gamma_i^2 E[Z_i] = \mathcal{J}. \tag{21}$$

Thus, minimizing $\sum_{i=1}^{N} \gamma_i^2 E[Z_i]$ would yield the minimum value $\mathcal{J}$ of the guaranteed cost for uncertain nonlinear large-scale interconnected stochastic systems (1).

It should be noted that the original optimization problem involving the guaranteed cost (21) can be divided into the following simplified optimization problems (22) because each optimization problem (22) is independent of other LMIs. Hence, it is possible to solve the optimization problems (22) for each independent subsystem.

$$\begin{aligned} \mathcal{J} &= \min_{\sum_i \mathcal{X}_i} \left(\sum_{i=1}^{N} \gamma_i^2 E[Z_i] \right) \\ &= \sum_{i=1}^{N} \left(\min_{\mathcal{X}_i} \gamma_i^2 E[Z_i] \right),\ \mathcal{D}_i :\ \min_{\mathcal{X}_i} \gamma_i^2 E[Z_i],\ i = 1,\ \ldots,N. \end{aligned} \tag{22}$$

A procedure for designing the guaranteed cost controller is given below.

Step 1. Starting with any γ_i, calculate $\tilde{V}_i$ and $\tilde{W}_{ij}$.

Step 2. Find $\mathcal{X}_i$ such that the LMIs (14) and (20) are feasible. If the LMIs (14) and (20) are not feasible, decrease the design parameter γ_i and return to Step 1. If γ_i is less than a given value that corresponds to a prescribed computational accuracy, stop and declare that the GCC has failed.. Otherwise, proceed Step 3.

Step 3. Minimize $\gamma_i^2 E[Z_i]$ over the $\mathcal{X}_i$ that satisfy the LMIs (14) and (20).

Step 4. If a solution exists, obtain the gain matrix $K_i = Y_i X_i^{-1}$ for each subsystem as well as the cost bound.

5. GCC Via Static Output Feedback

The static output feedback problem is one of the most important problems. The implementation of GCC using static output feedback was first proposed in (13). Despite the reliable result obtained in (13), there still remains an important problem that should be solved analytically-the stochastic case; this case has not been investigated. Another difficulty that is faced when solving the GCC using the static output feedback is the nonconvexity of the solution set. In order to obtain a feasible solution set, the KKT conditions are considered for the optimization of the guaranteed cost.

In this section, the controls $u_i(t)$ are restricted to static output feedback

$$u_i(t) := F_i y_i(t),\ y_i(t) := C_i x_i(t),\ i = 1,\ \dots ,N, \tag{23}$$

where $y_i(t) \in \Re^{r_i}$ are the outputs of the ith subsystems. C_i are constant matrices of appropriate dimensions.

The application of the control laws (23) to uncertain nonlinear large-scale interconnected stochastic systems (1) by using a technique similar to the techniques presented in Theorems 1 and 2 provides a bound on the closed-loop cost function.

$$J \le \sum_{i=1}^{N} E[x_i^T(0) W_i x_i(0)], \tag{24}$$

where W_i is the solution of the stochastic algebraic Riccati inequality (SARI)

$$\begin{aligned} &\boldsymbol{F}(\mu_i,\ \varepsilon_i\ W_i,\ F_i) \\ &:= \hat{A}^T W_i + W_i \hat{A} + A_{i1}^T W_i A_{i1} + U_i + \hat{R}_i + \mu_i W_i D_i D_i^T W_i + \mu_i^{-1} \hat{E}_i^T \hat{E}_i \\ &\quad + \sum_{j=1,\ j\neq i}^{N} W_i \left[\varepsilon_i D_{ij} D_{ij}^T + G_{ij}(\varepsilon_i^{-1} E_{ij}^T E_{ij} - I_{l_{ij}}) G_{ij}^T\right] W_i \le 0, \end{aligned} \tag{25}$$

where $\hat{A}_i := A_i + B_i F_i C_i$, $\hat{E}_i := E_i^a + E_i^b F_i C_i$ and $\hat{R}_i := Q_i + C_i^T F_i^T R_i F_i C_i$.

Now, let us consider the following optimization problem.

$$\mathcal{E}_0 : \min_{\sum_i \mathcal{Y}_i} \sum_{i=1}^{N} \gamma_i^2 E[W_i] = \boldsymbol{J},\ \mathcal{Y}_i \in (\mu_i,\ \varepsilon_i\ W_i,\ F_i) \tag{26}$$

such that the SARI (25).

As discussed in the previous section, the original optimization problem for the guaranteed cost (26) can be decomposed into the reduced optimization problems (27).

$$\boldsymbol{J} = \sum_{i=1}^{N} \left(\min_{\mathcal{Y}_i} \gamma_i^2 E[W_i] \right),\ \mathcal{E}_i :\ \min_{\mathcal{Y}_i} \gamma_i^2 E[W_i],\ i = 1,\ \dots ,N \tag{27}$$

such that the SARI (25).

Theorem 3. *Let us consider the problem of suboptimization of $\mathcal{E}_i$. If v_i is a local minimum that satisfies the constraint qualification* [1]*, then there exist*

[1] When the gradients of the active inequality and equality constraints are linearly independent at v_i, it is called constraint qualification (23).

solutions $W_i \geq 0$, $S_i > 0$ and constant positive parameters $\mu_i > 0$ and $\varepsilon_i > 0$ that satisfy the cross-coupled algebraic Riccati equations.

$$\boldsymbol{F}(\mu_i,\ \varepsilon_i,\ W_i,\ F_i) = 0, \tag{28a}$$

$$S_i \tilde{A}_i^T + \tilde{A}_i S_i + A_{i1} S_i A_{i1}^T + I_{n_i} = 0, \tag{28b}$$

$$(B_i^T W_i + \mu_i^{-1} E_i^{bT} E_i^a) S_i C_i^T + (R_i + \mu_i^{-1} E_i^{bT} E_i^b) F_i C_i S_i C_i^T = 0, \tag{28c}$$

$$\mathbf{Tr}[S_i W_i D_i D_i^T W_i] - \mu_i^{-2} \mathbf{Tr}[S_i \hat{E}_i^T \hat{E}_i] = 0, \tag{28d}$$

$$\mathbf{Tr}\left[S_i W_i \left(\sum_{j=1,\ j\neq i}^{N} D_{ij} D_{ij}^T\right) W_i\right] - \varepsilon_i^{-2} \mathbf{Tr}\left[S_i W_i \left(\sum_{j=1,\ j\neq i}^{N} G_{ij} E_{ij}^T E_{ij} G_{ij}^T\right) W_i\right] = 0. \tag{28e}$$

where $v_i = (\mu_i,\ \varepsilon_i,\ [\mathrm{vec} W_i]^T,\ [\mathrm{vec} F_i]^T,\ [\mathrm{vec} S_i]^T)$,

$$\tilde{A}_i := \hat{A}_i + \mu_i D_i D_i^T W_i + \sum_{j=1,\ j\neq i}^{N} \left[\varepsilon_i D_{ij} D_{ij}^T + G_{ij}(\varepsilon_i^{-1} E_{ij}^T E_{ij} - I_{l_{ij}}) G_{ij}^T\right] W_i.$$

Proof: This can easily be proved by the direct use of the KKT conditions (23). Let us consider the Lagrangian $\boldsymbol{L}$.

$$\boldsymbol{L}(\mu_i,\ \varepsilon_i,\ W_i,\ F_i) = \mathbf{Tr}\ [W_i] + \mathbf{Tr}\ [S_i \boldsymbol{F}(\mu_i,\ \varepsilon_i,\ W_i,\ F_i)], \tag{29}$$

where S_i is a symmetric positive definite matrix of Lagrange multipliers.

The KKT conditions for F_i to be optimal can be found by setting $\partial \boldsymbol{L}/\partial \mu_i$, $\partial \boldsymbol{L}/\partial \varepsilon_i$, $\partial \boldsymbol{L}/\partial W_i$ and $\partial \boldsymbol{L}/\partial F_i$ equal to zeros. Using the obtained KKT conditions, we can derive (28). ■

In order to obtain the optimal solutions v_i, a numerical method based on Newton's method can be used to solve these sets of equations (17).

6. Numerical Example

In order to demonstrate the efficiency of our proposed control, a simple numerical example is presented. Consider the uncertain nonlinear large-scale interconnected stochastic systems (1) composed of three scalar dimensional subsystems. The system matrices and the unknown functions along with the uncertainties are given below: $A_1 = \begin{bmatrix} -2 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 \end{bmatrix}$, $A_3 = \begin{bmatrix} 0 \end{bmatrix}$, $A_{11} = \begin{bmatrix} -0.002 \end{bmatrix}$, $A_{21} = \begin{bmatrix} -0.003 \end{bmatrix}$, $A_{31} = \begin{bmatrix} 0.001 \end{bmatrix}$, $B_1 = \begin{bmatrix} 1 \end{bmatrix}$, $B_2 = \begin{bmatrix} 1.5 \end{bmatrix}$, $B_3 = \begin{bmatrix} 1.2 \end{bmatrix}$, $D_1 = \begin{bmatrix} 1 \end{bmatrix}$, $D_2 = \begin{bmatrix} 1.2 \end{bmatrix}$, $D_3 = \begin{bmatrix} 0.5 \end{bmatrix}$, $E_1^a = E_2^a = E_3^a = \begin{bmatrix} 0.1 \end{bmatrix}$, $E_1^b = E_2^b = E_3^b = \begin{bmatrix} 0.01 \end{bmatrix}$, $G_{12} = \begin{bmatrix} 0.2 \end{bmatrix}$, $G_{13} = \begin{bmatrix} 0.1 \end{bmatrix}$, $G_{23} = \begin{bmatrix} 0.4 \end{bmatrix}$, $G_{21} = \begin{bmatrix} 0.1 \end{bmatrix}$, $G_{31} = \begin{bmatrix} 0.3 \end{bmatrix}$, $G_{32} = \begin{bmatrix} 0.2 \end{bmatrix}$, $g_{ij}(x_i,\ x_j) = x_i^2 x_j^2$, $p = 1$, $D_{12} = D_{13} = D_{23} = D_{21} = D_{31} = D_{32} = \begin{bmatrix} 1 \end{bmatrix}$, $E_{12} = E_{13} = \begin{bmatrix} 0.015 \end{bmatrix}$, $E_{23} = E_{21} = \begin{bmatrix} 0.01 \end{bmatrix}$, $E_{31} = E_{32} = \begin{bmatrix} 0.03 \end{bmatrix}$. $R_i = 0.1$, $Q_i = \begin{bmatrix} 0.1 \end{bmatrix}$, $i = 1,\ 2,\ 3$.

This nonlinear large-scale stochastic system cannot be treated using the technique given in (8; 15; 16) because the interconnection term cannot be bounded by a linear combination of the state $x_i(t)$ and high-order interconnections are included. Taking the norm of $g_{ij}(x_i,\ x_j)$ yields $\|g_{ij}(x_i,\ x_j)\| = \|x_i^2 x_j^2\| \leq 0.5(\|x_i\|^4 + \|x_j\|^4)$. Hence, there exists $\tilde{V}_i = \tilde{W}_{ij} = 0.5\gamma_i^3$, where $V_i = W_{ij} = 1$, $\alpha_1 = \alpha_2 = \alpha_3 = \beta_1 = \beta_2 = \beta_3 = 0$, $\alpha_4 = \beta_4 = 1$, $i,\ j = 1,\ 2,\ 3,\ i \neq j$. The design parameter is initially set as $\gamma_i = 2$. By applying Theorem 3 and solving the corresponding optimization problem (22), the decentralized linear optimal state feedback controllers are obtained as $K_1 = \left[\ -4.6814\ \right]$, $K_2 = \left[\ -7.0928\ \right]$ and $K_3 = \left[\ -6.3544\ \right]$. Consequently, the optimal guaranteed cost of the closed-loop uncertain stochastic systems is $\mathcal{J} = 5.83979$, where $\min_{\mathcal{X}_1} J_1 = 1.85402$, $\min_{\mathcal{X}_2} J_2 = 1.87641$ and $\min_{\mathcal{X}_3} J_3 = 2.10936$.

It should be noted that in contrast to existing results (8; 15; 16), there is some stochastic uncertainty in our results; however, it is possible to construct a decentralized robust controller by using our design procedure. Therefore, the proposed design method is very useful since the resulting decentralized robust controller can be used in the control of more practical large-scale interconnected stochastic systems.

From Theorem 2, the initial states of (1) must obey the inequality $\sqrt{\lambda_{\max}(\mathbf{P})/\lambda_{\min}(\mathbf{P})}\|x_i(0)\| \leq \gamma_i = 2$. Thus, the stability region of (1) is $\|x_i(t)\| \leq 1.8750$ because $\sqrt{\lambda_{\max}(\mathbf{P})/\lambda_{\min}(\mathbf{P})} = 1.0666$.

7. Conclusions

In this paper, the GCC problem for nonlinear large-scale stochastic systems with high-order interconnections has been addressed. It should be noted that the difference between this study and past studies is that stochastic uncertainties are considered in this study (8; 9; 10; 15; 16). The consideration of the uncertainties allows us to design an accurate robust controller for a wide class of nonlinear stochastic systems. Furthermore, since the proposed design technique is based on the GCC, the transient response of the controller can be changed by adjusting the weighting matrix of the cost function.

The decentralized robust controller, which minimizes the value of the guaranteed cost for closed-loop uncertain large-scale interconnected stochastic systems, can be obtained by solving the LMI. Consequently, the resulting decentralized linear feedback controller attains EMSS and optimal cost bound for uncertain large-scale interconnected stochastic systems. Moreover, since the resulting optimization problem guarantees the convexity of the objective function under LMI constraints, the decentralized guaranteed cost controller can be easily constructed using a software such as MATLAB LMI control Toolbox. The extension of the result of this study to the static output feedback control problem has been investigated by considering the KKT conditions. Even if the LMI technique cannot be used because of the nonconvexity of the solution set resulting from the use of a static output feedback strategy, the optimal GCC can be computed numerically.

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