

On a generalization of concavity by some aggregation functions

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Abstract—In this paper, we discuss the generalization of concavity on the subclass of the set of all membership functions, each function belonging to the subclass characterizes one convex fuzzy set respectively. The generalization is based on conjunctive aggregation functions. And the properties are investigated.

I. INTRODUCTION

The extensive applications of quasi-concave functions are found in economics and optimization. In fact, several generalizations of the quasi-concavity of functions have been introduced and studied (see [3] and references therein). In this paper, we focus on some class of functions from the n -dimensional Euclidean space \mathbb{R}^n into the unit interval $[0, 1]$. Such functions can be regarded as membership functions of fuzzy sets on \mathbb{R}^n . Therefore, it is natural to use terminologies of fuzzy theory. In general, the quasi-concavity of functions is defined by using the minimum operation (see Definition 8). As one of generalizations of quasi-concave membership functions, Ramík and Vlach ([3]–[5]) used a triangular norm instead of the minimum operation.

On the other hand, aggregation functions are very important for a generalization of operations on fuzzy sets. And they are studied widely (see [1]–[3] and references therein).

In this paper, we apply some conjunctive aggregation function instead of the minimum operation. So, this means that our proposing generalization is a further one of the above Ramík and Vlach's approaches.

II. AGGREGATION FUNCTIONS

In this section, we investigate properties of aggregation functions (You can find the details of them in [1]–[3], [6], [7]).

Throughout this paper, \mathbb{N} is the set of all natural numbers, $m \in \mathbb{N}$ is fixed, $I = \{1, \dots, m\}$ and $I_k = \{1, \dots, k\}$, $k \in \mathbb{N}$. For $a, b \in \mathbb{R}$, we put $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$, $[a, b[= \{x \in \mathbb{R} : a \leq x < b\}$, $]a, b] = \{x \in \mathbb{R} : a < x \leq b\}$ and $]a, b[= \{x \in \mathbb{R} : a < x < b\}$.

First, we remember the definition of aggregation functions.

Definition 1. Let $k \in \mathbb{N}$ and $G_k : [0, 1]^k \rightarrow [0, 1]$. For $k = 1$, G_1 is called an aggregation function if $G_1(x) = x$, $x \in [0, 1]$.

For $k \geq 2$, G_k is called an aggregation function if the following two axioms are satisfied: (G1) if $x_i, y_i \in [0, 1]$, $x_i \leq y_i$, $i \in I_k$, then $G_k(x_1, \dots, x_k) \leq G_k(y_1, \dots, y_k)$ (monotonicity) and (G2) $G_k(0, \dots, 0) = 0$ and $G_k(1, \dots, 1) = 1$ (boundary condition).

Next, we define the features of aggregation functions.

Definition 2. Let $k \in \mathbb{N}$ and $G_k : [0, 1]^k \rightarrow [0, 1]$ be an aggregation function.

- (i) G_k is said to be conjunctive if $G_k(x_1, \dots, x_k) \leq \min\{x_1, \dots, x_k\}$ for any $x_i \in [0, 1]$, $i \in I_k$.
- (ii) G_k is said to be strongly monotone increasing if $x_i, y_i \in [0, 1]$, $x_i \leq y_i$, $i \in I_k$ and $x_j < y_j$ for some $j \in I_k$ implies $G_k(x_1, \dots, x_k) < G_k(y_1, \dots, y_k)$.
- (iii) G_k is said to be strictly monotone increasing if $x_i, y_i \in [0, 1]$, $x_i < y_i$, $i \in I_k$ implies $G_k(x_1, \dots, x_k) < G_k(y_1, \dots, y_k)$.
- (iv) G_k is said to be idempotent if $G_k(x, \dots, x) = x$ for any $x \in [0, 1]$.

In special case,

- (v) G_2 is said to be commutative if $G_2(x_1, x_2) = G_2(x_2, x_1)$ for any $x_i \in [0, 1]$, $i \in I_2$.
- (vi) G_2 is said to be associative if $G_2(x_1, G_2(x_2, x_3)) = G_2(G_2(x_1, x_2), x_3)$ for any $x_i \in [0, 1]$, $i \in I_3$.

Let $k \in \mathbb{N}$ and $G_k : [0, 1]^k \rightarrow [0, 1]$ be an aggregation function. If G_k is strongly monotone increasing, then G_k is strictly monotone increasing. Then G_1 is strongly monotone increasing if and only if G_1 is strictly monotone increasing. Note that there does not exist any strongly monotone increasing conjunctive aggregation function.

One of important classes of aggregation functions $G_2 : [0, 1]^2 \rightarrow [0, 1]$ is the class of triangular norms defined as follows.

Definition 3. A triangular norm (t -norm for short) is a binary operation T on $[0, 1]$, that is, a function $T : [0, 1]^2 \rightarrow [0, 1]$, such that for any $x_i \in [0, 1]$, $i \in I_4$ the following four axioms

are satisfied:

$$T(x_1, x_2) = T(x_2, x_1) \text{ (commutativity),} \quad (\text{T1})$$

$$T(x_1, T(x_2, x_3)) = T(T(x_1, x_2), x_3) \text{ (associativity),} \quad (\text{T2})$$

$$T(x_1, x_2) \leq T(x_3, x_4) \text{ if } x_1 \leq x_3, x_2 \leq x_4 \text{ (monotonicity)} \quad (\text{T3})$$

and

$$T(x_1, 1) = x_1 \text{ (boundary condition).} \quad (\text{T4})$$

Example 1. Important examples of t -norms are the minimum T_M and the drastic product T_D defined by

$$T_M(x_1, x_2) = \min\{x_1, x_2\},$$

$$T_D(x_1, x_2) = \begin{cases} T_M(x_1, x_2) & \text{if } \max\{x_1, x_2\} = 1, \\ 0 & \text{otherwise.} \end{cases}$$

T_M is the strongest t -norm and T_D is the weakest t -norm. Namely, $T_D \leq T \leq T_M$ for any t -norm T .

Let T be a t -norm. If we define $T_1 : [0, 1] \rightarrow [0, 1]$ as $T_1(x) = x, x \in [0, 1]$, and $T_2 = T$, and $T_k : [0, 1]^k \rightarrow [0, 1]$ as

$$T_k(x_1, \dots, x_{k-1}, x_k) = T(T_{k-1}(x_1, \dots, x_{k-1}), x_k)$$

for $k \geq 3$, then each $T_k, k \in \mathbb{N}$ is an aggregation function. In the remainings, whenever we consider a t -norm as an aggregation function, we assume that the aggregation function is generated as above formula. Note that each t -norm is a conjunctive aggregation function.

Definition 4. Let $k \in \mathbb{N}$, $G_k : [0, 1]^k \rightarrow [0, 1]$ and $G'_2 : [0, 1]^2 \rightarrow [0, 1]$ be aggregation functions. We say that G_k dominates G'_2 if $G_k(G'_2(x_1, y_1), \dots, G'_2(x_k, y_k)) \geq G'_2(G_k(x_1, \dots, x_k), G_k(y_1, \dots, y_k))$ for any $x_i, y_i \in [0, 1], i \in I_k$. And it is denoted $G_k \gg G'_2$.

An aggregation function $G_1 : [0, 1] \rightarrow [0, 1]$ and a t -norm T_M dominate any aggregation function $G'_2 : [0, 1]^2 \rightarrow [0, 1]$. Complete characterization of the class of all aggregation functions which dominate T_M is given in [6].

Lemma 1. Let $k \in \mathbb{N}$, and $G_2 : [0, 1]^2 \rightarrow [0, 1]$ be an aggregation function. We define aggregation functions $G_k^j : [0, 1]^k \rightarrow [0, 1], j \in I_2$ as $G_1^j(x) = x, x \in [0, 1], j \in I_2$ when $k = 1$, and $G_2^j = G_2, j \in I_2$ when $k = 2$, and

$$G_k^1(x_1, \dots, x_k) = G_2(G_{k-1}^1(x_1, \dots, x_{k-1}), x_k), \quad (1)$$

$$G_k^2(x_1, \dots, x_k) = G_2(x_1, G_{k-1}^2(x_2, \dots, x_k)) \quad (2)$$

for $x_i \in [0, 1], i \in I_k$ when $k \geq 3$.

Then the following statements hold.

- (i) If " G_2 is commutative and associative" or " $G_2 \gg G_2$ ", then $G_k^j \gg G_2, j \in I_2$.
- (ii) If G_2 is strongly monotone increasing, then $G_k^j, j \in I_2$ have the same property.
- (iii) If G_2 is strictly monotone increasing, then $G_k^j, j \in I_2$ have the same property.

III. GENERALIZATION OF QUASI-CONCAVITY

In this section, we propose the various generalizations of quasi-concave membership functions based on conjunctive aggregation functions. Then their properties are derived.

A. G -quasi-concavity

First, we remember the basic definitions.

Definition 5. Let $X \neq \emptyset \subset \mathbb{R}^n, f : \mathbb{R}^n \rightarrow \mathbb{R}$ and $\alpha \in \mathbb{R}$. Then $U_X(f, \alpha) = \{\mathbf{x} \in X : f(\mathbf{x}) \geq \alpha\}$ is called the level set of f on X .

Definition 6. Let $X \neq \emptyset \subset \mathbb{R}^n, \mu : \mathbb{R}^n \rightarrow [0, 1]$. Then $\text{Core}_X(\mu) = \{\mathbf{x} \in X : \mu(\mathbf{x}) = 1\}$ is called the core of μ on X .

Definition 7. Let $X \neq \emptyset \subset \mathbb{R}^n$ and $\mu : \mathbb{R}^n \rightarrow [0, 1]$. Then μ is said to be level-closed, level-bounded and level-compact on X , respectively, if the level set $U_X(\mu, \alpha)$ is closed, bounded and compact for any $\alpha \in]0, 1]$.

Now, recall that the standard definitions of quasi-concavity of functions.

Definition 8. Let $X \subset \mathbb{R}^n$ be a non-empty convex set, and $f : \mathbb{R}^n \rightarrow \mathbb{R}$.

- (i) f is said to be quasi-concave on X if

$$f(\lambda \mathbf{x} + (1 - \lambda) \mathbf{y}) \geq \min\{f(\mathbf{x}), f(\mathbf{y})\}$$

for any $\mathbf{x}, \mathbf{y} \in X$ and any $\lambda \in]0, 1[$.

- (ii) f is said to be strictly quasi-concave on X if f is quasi-concave on X and

$$f(\lambda \mathbf{x} + (1 - \lambda) \mathbf{y}) > \min\{f(\mathbf{x}), f(\mathbf{y})\} \quad (3)$$

for any $\mathbf{x}, \mathbf{y} \in X, f(\mathbf{x}) \neq f(\mathbf{y})$ and any $\lambda \in]0, 1[$.

- (iii) f is said to be strongly quasi-concave on X if (3) holds for any $\mathbf{x}, \mathbf{y} \in X, \mathbf{x} \neq \mathbf{y}$ and any $\lambda \in]0, 1[$.

Then let $X \neq \emptyset \subset \mathbb{R}^n, f : \mathbb{R}^n \rightarrow \mathbb{R}$, and

$$\mathbb{L}_X(\mathbf{x}, \mathbf{y}) = \{t \in \mathbb{R} : \mathbf{x} + t(\mathbf{y} - \mathbf{x}) \in X\}.$$

for $\mathbf{x}, \mathbf{y} \in X, \mathbf{x} \neq \mathbf{y}$. We define $f_{\mathbf{x}, \mathbf{y}} : \mathbb{R} \rightarrow \mathbb{R}$ as

$$f_{\mathbf{x}, \mathbf{y}}(t) = \begin{cases} f(\mathbf{x} + t(\mathbf{y} - \mathbf{x})) & \text{if } t \in \mathbb{L}_X(\mathbf{x}, \mathbf{y}), \\ 0 & \text{otherwise.} \end{cases}$$

Definition 9. Let $X \subset \mathbb{R}^n$ be a non-empty convex set, and $f : \mathbb{R}^n \rightarrow \mathbb{R}$. f is said to be (resp. strictly, strongly) quasi-concave on X from $\mathbf{y} \in X$, if $f_{\mathbf{x}, \mathbf{y}}$ is (resp. strictly, strongly) quasi-concave on $\mathbb{L}_X(\mathbf{x}, \mathbf{y})$ for any $\mathbf{x} \in X, \mathbf{x} \neq \mathbf{y}$.

In [3], [5], the quasi-concavity is generalized by using the concept of star-shaped sets.

Definition 10. Let $X \subset \mathbb{R}^n$.

- (i) X is said to be star-shaped from $\mathbf{y} \in X$ if

$$\mathbb{I}(\mathbf{x}, \mathbf{y}) = \{\mathbf{z} \in \mathbb{R}^n : \mathbf{z} = \mathbf{x} + \lambda(\mathbf{y} - \mathbf{x}), \lambda \in [0, 1]\} \subset X$$

for any $\mathbf{x} \in X$.

- (ii) The set of all points in X from which X is star-shaped is called the kernel of X , and it is denoted by $\text{Ker}(X)$.
- (iii) X is said to be star-shaped if $\text{Ker}(X) \neq \emptyset$ or $X = \emptyset$.

Definition 11. Let $X \subset \mathbb{R}^n$ be a non-empty star-shaped set, and $f : \mathbb{R}^n \rightarrow \mathbb{R}$. f is said to be star-shaped on X if $U_X(f, \alpha)$ is star-shaped for any $\alpha \in \mathbb{R}$.

Let $X \subset \mathbb{R}^n$ be a non-empty convex set, and $f : \mathbb{R}^n \rightarrow \mathbb{R}$. Since any convex set of \mathbb{R}^n is star-shaped, if f is quasi-concave on X , then f is star-shaped on X . When $n = 1$, $X \subset \mathbb{R}$ is an interval, and the class of all quasi-concave functions on X coincides with the class of all star-shaped functions on X .

In [3]–[5], the quasi-concavity of membership functions is generalized by using arbitrary t -norms instead of the minimum operation. So, We apply an conjunctive aggregation function instead of it.

Definition 12. Let $X \subset \mathbb{R}^n$ be a non-empty convex set, $G_2 : [0, 1]^2 \rightarrow [0, 1]$ be a conjunctive aggregation function, and $\mu : \mathbb{R}^n \rightarrow [0, 1]$.

- (i) μ is said to be G_2 -quasi-concave on X if

$$\mu(\lambda \mathbf{x} + (1 - \lambda)\mathbf{y}) \geq G_2(\mu(\mathbf{x}), \mu(\mathbf{y}))$$

for any $\mathbf{x}, \mathbf{y} \in X$ and any $\lambda \in]0, 1[$.

- (ii) μ is said to be strictly G_2 -quasi-concave on X if μ is G -quasi-concave on X and

$$\mu(\lambda \mathbf{x} + (1 - \lambda)\mathbf{y}) > G_2(\mu(\mathbf{x}), \mu(\mathbf{y})) \quad (4)$$

for any $\mathbf{x}, \mathbf{y} \in X, \mu(\mathbf{x}) \neq \mu(\mathbf{y})$ and any $\lambda \in]0, 1[$.

- (iii) μ is said to be strongly G_2 -quasi-concave (or strongly G -quasi-concave) on X if (4) holds for any $\mathbf{x}, \mathbf{y} \in X, \mathbf{x} \neq \mathbf{y}$ and any $\lambda \in]0, 1[$.

For the sake of simplicity, “(resp. strictly, strongly) G_2 -quasi-concave” is denoted “(resp. strictly, strongly) as G -quasi-concave” without subscript 2.

For a non-empty convex set $X \subset \mathbb{R}^n$, a conjunctive aggregation function $G_2 : [0, 1]^2 \rightarrow [0, 1]$ and $\mu : \mathbb{R}^n \rightarrow [0, 1]$, it can be seen that μ is

$$\begin{array}{ccccc} \text{strongly q.c.} & \Rightarrow & \text{strictly q.c.} & \Rightarrow & \text{q.c.} \\ \downarrow & & \downarrow & & \downarrow \\ \text{strongly } G\text{-q.c.} & \Rightarrow & \text{strictly } G\text{-q.c.} & \Rightarrow & G\text{-q.c.} \end{array}$$

on X from Definition 12, where above symbol “q.c.” means “quasi-concave”.

B. Properties of G -quasi-concavity

The following theorem shows a membership function μ is (resp. strongly, strictly) quasi-concave if and only if it is (resp. strongly, strictly) G -quasi-concave.

Theorem 1. Let $X \subset \mathbb{R}$ be a non-empty convex set, $G_2 : [0, 1]^2 \rightarrow [0, 1]$ be a conjunctive aggregation function satisfying $G_2 \geq T_D$, and $\mu : \mathbb{R} \rightarrow [0, 1]$ satisfying $\text{Core}_X(\mu) \neq \emptyset$. If μ is (resp. strictly, strongly) G -quasi-concave on X then μ is (resp. strictly, strongly) quasi-concave on X .

The following examples show that neither the condition “ $G_2 \geq T_D$ ” nor the condition “ $\text{Core}_X(\mu) \neq \emptyset$ ” in Theorem 1 can be eliminated.

Example 2. (i) Set $X = \mathbb{R}$, and define a conjunctive aggregation function $G_2 : [0, 1]^2 \rightarrow [0, 1]$ as

$$G_2(x, y) = \begin{cases} 1 & \text{if } x = y = 1, \\ 0 & \text{otherwise} \end{cases}$$

for $x, y \in [0, 1]$, and define $\mu : \mathbb{R} \rightarrow [0, 1]$ as

$$\mu(x) = \begin{cases} 1 & \text{if } x = 0, \\ \frac{1}{3} \sin x + \frac{1}{2} & \text{otherwise} \end{cases}$$

for $x \in \mathbb{R}$. In this case, $G_2 \not\geq T_D$ and $\text{Core}_X(\mu) = \{0\} \neq \emptyset$. Since μ is strongly G -quasi-concave, strictly G -quasi-concave and also G -quasi-concave on X . However, μ is not quasi-concave on X . Thus, μ is neither strictly quasi-concave on X nor strongly quasi-concave on X .

- (ii) Set $X = \mathbb{R}$ and $G_2 = T_D$, and define $\mu : \mathbb{R} \rightarrow [0, 1]$ as $\mu(x) = \frac{1}{3} \sin x + \frac{1}{2}$ for $x \in \mathbb{R}$. In this case, $G_2 \geq T_D$ and $\text{Core}_X(\mu) = \emptyset$. Since μ is strongly G -quasi-concave on X , μ is strictly G -quasi-concave on X and also quasi-concave on X . However, μ is not quasi-concave on X . Thus, μ is neither strictly quasi-concave on X nor strongly quasi-concave on X .

The following Theorems 2, 3 give us the relationship among the variety of quasi-concavity, star-shapedness.

Theorem 2 (Theorem 4.34 in [3]). Let $X \subset \mathbb{R}^n$ be a non-empty convex set, and $\mu : \mathbb{R}^n \rightarrow [0, 1]$. Then the following statements are equivalent.

- (i) $\text{Core}_X(\mu) \subset \bigcap_{\alpha \in [0, 1]} \text{Ker}(U_X(\mu, \alpha))$.
- (ii) μ is quasi-concave on X from \bar{x} for any $\bar{x} \in \text{Core}_X(\mu)$.
- (iii) μ is T_D -quasi-concave on X .

Theorem 3. Let $X \subset \mathbb{R}^n$ be a non-empty convex set, and $\mu : \mathbb{R}^n \rightarrow [0, 1]$. Then the following statements hold.

- (i) If μ is strictly (resp. strongly) T_D -quasi-concave on X , then μ is strictly (resp. strongly) quasi-concave on X from \bar{x} for any $\bar{x} \in \text{Core}_X(\mu)$.
- (ii) Assume that $\mu(\mathbf{x}) > 0, \mathbf{x} \in X$. If μ is strictly (resp. strongly) quasi-concave on X from \bar{x} for any $\bar{x} \in \text{Core}_X(\mu)$, then μ is strictly (resp. strongly) T_D -quasi-concave on X .

The following example shows that the condition “ $\mu(\mathbf{x}) > 0, \mathbf{x} \in X$ ” in Theorem 3 (ii) and 4 (ii) can not be eliminated.

Example 3. Let $\mathbf{x} = (x, y) \in \mathbb{R}^2$, and set $X = \{\mathbf{x} \in \mathbb{R}^2 : |x| + |y| \leq 1\}$. Define $\tilde{\mu} : \mathbb{R}^2 \rightarrow [0, 1]$ as $\tilde{\mu}(\mathbf{x}) = \max\{1 - |x| - |y|, 0\}$ and $\mu : \mathbb{R}^2 \rightarrow [0, 1]$ as

$$\mu(\mathbf{x}) = \begin{cases} 1 - \frac{1}{2}(|x| + |y|) & \text{if } \mathbf{x} \in [-1, 1] \times \{0\}, \\ \tilde{\mu}(\mathbf{x}) & \text{otherwise.} \end{cases}$$

In this case, $\text{Core}_X(\mu) = \{0\}$ and not “ $\mu(\mathbf{x}) > 0, \mathbf{x} \in X$ ”. Since μ is strongly quasi-concave on X from $\mathbf{0}$, μ is the same. On the other hand, if we put $\mathbf{y} = (1, 0), \mathbf{z} = (0, 1)$, then $\mu(\mathbf{y}) = \frac{1}{2} \neq \mu(\mathbf{z}) = 0$ and $\mu(\frac{1}{2}\mathbf{y} + \frac{1}{2}\mathbf{z}) = 0 = T_D(\mu(\mathbf{y}), \mu(\mathbf{z}))$. Thus, μ is neither strictly nor strongly T_D -quasi-concave on X .

Theorem 4. *Let $X \subset \mathbb{R}^n$ be a non-empty convex set, $\mu : \mathbb{R}^n \rightarrow [0, 1]$ satisfying $\text{Core}_X(\mu) \neq \emptyset$ and $G_2 : [0, 1]^2 \rightarrow [0, 1]$ be a conjunctive aggregation function satisfying $G_2 \geq T_D$. Assume that μ is G -quasi-concave on X . Then μ is star-shaped on X .*

Example 2 shows that neither the condition “ $\text{Core}_X(\mu) \neq \emptyset$ ” nor the condition “ $G_2 \geq T_D$ ” in Theorem 4 can be eliminated.

C. Aggregation of G -quasi-concave functions

Now, several G -quasi-concave membership functions are aggregated by using an aggregation function, as the result, we obtain a new aggregated membership function. We investigate properties of such aggregated membership functions.

Let $\mu_i : \mathbb{R}^n \rightarrow [0, 1], i \in I$, and $G_m : [0, 1]^m \rightarrow [0, 1]$ be an aggregation function. Then we define a function $\psi : \mathbb{R}^n \rightarrow [0, 1]$ as

$$\psi(\mathbf{x}) = G_m(\mu_1(\mathbf{x}), \dots, \mu_m(\mathbf{x})), \quad \mathbf{x} \in \mathbb{R}^n. \quad (5)$$

Theorem 5. *Let $X \subset \mathbb{R}^n$ be a non-empty convex set, $G'_2 : [0, 1]^2 \rightarrow [0, 1]$ be a conjunctive aggregation function satisfying $G'_2 \geq T_D$, $G_m : [0, 1]^m \rightarrow [0, 1]$ be an aggregation function, and $\mu_i : \mathbb{R}^n \rightarrow [0, 1], i \in I$. Moreover, assume that $\bigcap_{i \in I} \text{Core}_X(\mu_i) \neq \emptyset$ and that $\mu_i, i \in I$ are G' -quasi-concave on X . Then a function $\psi : \mathbb{R}^n \rightarrow [0, 1]$ defined by (5) is star-shaped on X .*

Theorem 6. *Let $X \subset \mathbb{R}^n$ be a non-empty convex set, $G'_2 : [0, 1]^2 \rightarrow [0, 1]$ be a conjunctive aggregation function and $\mu_i : \mathbb{R}^n \rightarrow [0, 1], i \in I$. Assume that $\mu_i, i \in I$ are G' -quasi-concave on X . Let $G_m : [0, 1]^m \rightarrow [0, 1]$ be an aggregation function satisfying $G_m \gg G'_2$. Then a function $\psi : \mathbb{R}^n \rightarrow [0, 1]$ defined by (5) is G' -quasi-concave on X .*

From Theorem 6 and Lemma 1, we have the following corollary.

Corollary 1. *Let $X \subset \mathbb{R}^n$ be a non-empty convex set, $G_2 : [0, 1]^2 \rightarrow [0, 1]$ be a conjunctive aggregation function, and $\mu_i : \mathbb{R}^n \rightarrow [0, 1], i \in I$. Assume that $\mu_i, i \in I$ are G -quasi-concave on X . Then a function $\psi_1 : \mathbb{R}^n \rightarrow [0, 1]$ defined as $\psi_1(\mathbf{x}) = T_M(\mu_1(\mathbf{x}), \dots, \mu_m(\mathbf{x})), \mathbf{x} \in \mathbb{R}^n$ is G -quasi-concave on X . If “ G_2 is commutative and associative” or “ $G_2 \gg G_2$ ”, then functions $\psi_2^j : \mathbb{R}^n \rightarrow [0, 1], j \in I_2$ defined as $\psi_2^j(\mathbf{x}) = G_m^j(\mu_1(\mathbf{x}), \dots, \mu_m(\mathbf{x})), \mathbf{x} \in \mathbb{R}^n, j \in I_2$ are G -quasi-concave on X , where $G_m^j : [0, 1]^m \rightarrow [0, 1], j \in I_2$ are aggregation functions defined by (1) and (2).*

For strict G -quasi-concavity and strong G -quasi-concavity of a function ψ defined by (5), we obtain the properties like as Theorem 6.

Theorem 7. *Let $X \subset \mathbb{R}^n$ be a convex set, $G'_2 : [0, 1]^2 \rightarrow [0, 1]$ be a conjunctive aggregation function, and $\mu_i : \mathbb{R}^n \rightarrow [0, 1], i \in I$. Assume that $\mu_i, i \in I$ are strictly G' -quasi-concave on X . Let $G_m : [0, 1]^m \rightarrow [0, 1]$ be an aggregation function. Assume that G_m is strongly monotone increasing and that $G_m \gg G'_2$. Then $\psi : \mathbb{R}^n \rightarrow [0, 1]$ defined by (5) is strictly G' -quasi-concave on X .*

Theorem 8. *Let $X \subset \mathbb{R}^n$ be a non-empty convex set, $G'_2 : [0, 1]^2 \rightarrow [0, 1]$ be a conjunctive aggregation function, and $\mu_i : \mathbb{R}^n \rightarrow [0, 1], i \in I$. Assume that $\mu_i, i \in I$ are strongly G' -quasi-concave on X . Let $G_m : [0, 1]^m \rightarrow [0, 1]$ be an aggregation function. Assume that G_m is strictly monotone increasing and that $G_m \gg G'_2$. Then a function $\psi : \mathbb{R}^n \rightarrow [0, 1]$ defined by (5) is strongly G' -quasi-concave on X .*

From Theorem 8 and Lemma 1, we have the following corollary.

Corollary 2. *Let $X \subset \mathbb{R}^n$ be a non-empty convex set, $G_2 : [0, 1]^2 \rightarrow [0, 1]$ be a conjunctive aggregation function, and $\mu_i : \mathbb{R}^n \rightarrow [0, 1], i \in I$.*

Assume that $\mu_i, i \in I$ are strongly G -quasi-concave on X . Then a function $\psi_1 : \mathbb{R}^n \rightarrow [0, 1]$ defined as $\psi_1(\mathbf{x}) = T_M(\mu_1(\mathbf{x}), \dots, \mu_m(\mathbf{x})), \mathbf{x} \in \mathbb{R}^n$ is strongly G -quasi-concave on X .

Assume also that G_2 is strictly monotone increasing. If “ G_2 is commutative and associative” or “ $G_2 \gg G_2$ ”, then functions $\psi_2^j : \mathbb{R}^n \rightarrow [0, 1], j \in I_2$ defined as $\psi_2^j(\mathbf{x}) = G_m^j(\mu_1(\mathbf{x}), \dots, \mu_m(\mathbf{x})), \mathbf{x} \in \mathbb{R}^n, j \in I_2$ are strongly G -quasi-concave on X , where $G_m^j : [0, 1]^m \rightarrow [0, 1], j \in I_2$ are aggregation functions defined by (1) and (2).

IV. CONCLUSIONS.

In this paper, the quasi-concavity of membership functions was generalized by using conjunctive aggregation functions instead of the minimum operation. Our generalized quasi-concavity was called G -quasi-concavity. Then the properties of G -quasi-concavity of membership functions are investigated.

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